Technical Appendices and Supplementary Material

A Proof for 6-approximation on Pseudometric-weighted CC with LP-UMVD-PIVOT

With the parameters $\alpha = \frac{1}{3}$ and $\beta = 0$, the corresponding rounding functions are as follows:

$$f^+(x) = f^-(x) = \begin{cases} 0, & x < \frac{1}{3}; \\ x, & \frac{1}{3} \leq x \leq \frac{2}{3}; \\ 1, & x > \frac{2}{3}. \end{cases} \tag{16}$$

Since the formulas for $e.cost$ and $e.lp$ in the pseudometric-weighted CC are compatible with those in the classical CC, and weights w affect $\mathcal{C}$ linearly, the following lemma presented by Chawla et al. [16] remains applicable.

Lemma 3 (Lemma 5 of [16])**.** *Suppose f^+ is a monotonically non-decreasing piecewise convex function; f^- is a monotonically non-decreasing piecewise concave function. Then $\mathcal{C} \geq 0$ for all possible configurations if it holds whenever:*

1. *the triangle inequality is tight for (x_{uv}, x_{vw}, x_{wu}); or*
2. *each value of x_e ($e \in \{uv, vw, wu\}$) belongs to the endpoint of the domain for some piece of the rounding function f^s, where s is the sign of e.*

Therefore, it suffices to show $6 \cdot LP - ALG \geq 0$ in either of the two cases outlined above.

We introduce another useful lemma:

Lemma 4. *If $x_{uv}, x_{vw}, x_{wu} \in \left[\frac{1}{\alpha}, 1 - \frac{1}{\alpha}\right]$, then $\alpha \cdot LP - ALG \geq 0$.*

Proof.

$$\begin{aligned} \alpha \cdot e.lp_w - e.cost_w \geq\ & \alpha(1 - p_{vw}p_{wu}) \min\{x, 1-x\} \\ & - \max\{p_{vw}(1 - p_{wu}) + (1 - p_{vw})p_{wu}, (1 - p_{vw})(1 - p_{wu})\} \\ \geq\ & \alpha(1 - p_{vw}p_{wu}) \cdot \frac{1}{\alpha} - (1 - p_{vw}p_{wu}) = 0. \end{aligned}$$

Same inequality holds for other vertices, hence $\alpha \cdot LP - ALG \geq 0$. □

For notational simplicity, define $(x, y, z) := (x_{uv}, x_{vw}, x_{wu})$ and *w.l.o.g.* $x \leq y \leq z$.

A.1 Triangle inequality is tight.

In this case, $z = x + y$. As the rounding functions are defined piecewise, we can partition the entire space of possible configuration of (x, y) into a finite number of regions (labeled I through VI in Figure 2) where the behaviors of p_{uv}, p_{vw}, p_{wu} are well identified in each.

Since $f^+ = f^-$, the value of p_e is independent of the sign of e, therefore the formula for both $e.cost_w$ and $e.lp_w$ are independent of the sign of the edges vw and wu. Therefore, once the region and the sign of the edge uv are fixed, the formula for $e.cost_w$ and $e.lp_w$ are fully resolved as a function of x, y, z, as shown in Table 1. By the implication of Lemma 2, it suffices to verify all possible configurations of the following procedure:

1. **Select any region**: This results in 12 corresponding cells, with 6 possible cases.
2. **Select any 2 columns**: This results in 8 corresponding cells, with 3 possible cases.
3. **For each column, select the sign**: This results in 4 corresponding cells, with 4 possible cases.
4. **The following must hold**: $\alpha \cdot (e.lp_a + e.lp_b) - (e.cost_a + e.cost_b) \geq 0$, where a, b are the vertices corresponding to the selected columns.

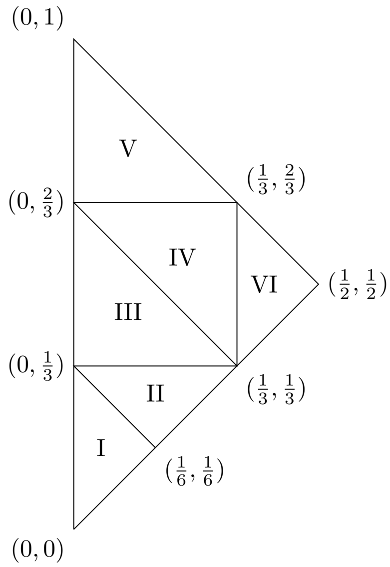

Figure 2: Configuration of regions induced from (16).

Table 1: Formula for $e.cost$ and $e.lp$ for each configuration.

		$x_{uv} = x$		$x_{vw} = y$		$x_{wu} = z = x + y$	
Region	Sign	$e.cost_w$	$e.lp_w$	$e.cost_u$	$e.lp_u$	$e.cost_v$	$e.lp_v$
I	'+'	0	x	0	y	0	z
	'−'	1	$1-x$	1	$1-y$	1	$1-z$
II	'+'	z	x	z	y	0	z
	'−'	$1-z$	$1-x$	$1-z$	$1-y$	1	$1-z$
III	'+'	$y+z-2yz$	$(1-yz)x$	z	y	y	z
	'−'	$(1-y)(1-z)$	$(1-yz)(1-x)$	$1-z$	$1-y$	$1-y$	$1-z$
IV	'+'	$1-y$	$(1-y)x$	1	y	y	z
	'−'	0	$(1-y)(1-x)$	0	$1-y$	$1-y$	$1-z$
V	'+'	0	0	1	y	1	z
	'−'	0	0	0	$1-y$	0	$1-z$
VI	'+'	$1-y$	$(1-y)x$	$1-x$	$(1-x)y$	$x+y-2xy$	$(1-xy)z$
	'−'	0	$(1-y)(1-x)$	0	$(1-x)(1-y)$	$(1-x)(1-y)$	$(1-xy)(1-z)$

A.1.1 Region I, II, V

All formulas are affine; therefore, checking $\alpha\cdot(e.lp_a+e.lp_b)-(e.cost_a+e.cost_b)\geq 0$ at the extremal points is sufficient. Table 2 shows extremal points of the regions with the value of $6\cdot e.cost - e.lp$ for each configuration. In each region with the point, adding any two values from different columns results in a nonnegative value, validating the nonnegativity throughout the region with sign configuration.

A.1.2 Region III

We can also further optimize procedures 3 and 4: Compute $\alpha \cdot e.lp_a - e.cost_a$ for each region, sign, and vertex configuration. For a given region and vertex, if one sign configuration dominates another, choose the smaller one.

For example, consider $6 \cdot e.lp_w - e.cost_w$ in Table 1. Since $x \leq \frac{1}{2}$, $(1-yz)(1-x) \geq (1-yz)x$. For $e.cost_w$, $y+z-2yz \geq (1-y)(1-z)$ since

$$y+z-2yz-(1-y)(1-z) = \frac{1-(2-3y)(2-3z)}{3} \geq 0$$

Table 2: Value of $6 \cdot e.lp - e.cost$ for each extremal point of the region.

Region	(x, y)	Sign	$6 \cdot e.lp_w - e.cost_w$	$6 \cdot e.lp_u - e.cost_u$	$6 \cdot e.lp_v - e.cost_v$
I	$(0, 0)$	'+'	0	0	0
		'−'	5	5	5
	$(0, \frac{1}{3})$	'+'	0	2	2
		'−'	5	3	3
	$(\frac{1}{6}, \frac{1}{6})$	'+'	1	1	2
		'−'	4	4	3
II	$(0, \frac{1}{3})$	'+'	$-1/3$	$5/3$	2
		'−'	$16/3$	$10/3$	3
	$(\frac{1}{6}, \frac{1}{6})$	'+'	$2/3$	$2/3$	2
		'−'	$13/3$	$1/3$	3
	$(\frac{1}{3}, \frac{1}{3})$	'+'	$4/3$	$4/3$	4
		'−'	$11/3$	$11/3$	1
V	$(0, \frac{2}{3})$	'+'	0	3	3
		'−'	0	2	2
	$(0, 1)$	'+'	0	5	5
		'−'	0	0	0
	$(\frac{1}{3}, \frac{2}{3})$	'+'	0	3	5
		'−'	0	2	0

since $y, z \in [1/3, 2/3]$ within the region. Therefore, $6 \cdot e.lp_w - e.cost_w$ is smaller with the sign '+'.

Now, consider the minimum value possible for the formula above within the region. Replacing $x = z - y$,

$$6 \cdot e.lp_w - e.cost_w = -6yz^2 + 6y^2z + 2yz - 7y + 5z.$$

Fixing y, the formula is concave of z; therefore, it is minimized in $z = y$ or $z = 2/3$, resulting in $2y^2 - 2y$ and $4y^2 - \frac{25}{3}y + \frac{10}{3}$ each. Within the range $y \in [1/3, 2/3]$, the minimum value of each is $-1/2$ in $y = 1/2$ and $-4/9$ in $y = 2/3$. Thus, $6 \cdot e.lp_w - e.cost_w \geq -1/2$.

Finally, consider the minimum value possible for $6 \cdot e.lp_u - e.cost_u$ and $6 \cdot e.lp_v - e.cost_v$. Again, these are affine; hence checking on extremal points is sufficient. It turns out that $6 \cdot e.lp_u - e.cost_u \geq 4/3$, minimized at $(y, z) = (1/3, 2/3)$ with '+' sign; $6 \cdot e.lp_v - e.cost_v \geq 4/3$, minimized at $(y, z) = (1/3, 2/3)$ with '−' sign.

Therefore, any objective value is greater than or equal to $4/3 - 1/2 = 5/6 \geq 0$ in the region.

A.1.3 Region IV

We can utilize the optimization method explained in A.1.2 again: Since $(1-y)x \leq (1-y)(1-x)$ and $1 - y \geq 0$, the formula with the sign '+' is smaller for $6 \cdot e.lp_w - e.cost_w$, which is $(6x - 1)(1 - y)$. This has a minimum value of $-1/3$ in $(x, y) = (0, 2/3)$.

Also, since $6z - y = 6x + 5y \geq 10/3 > 5/2$ in this region, the formula with the sign '$-$' is smaller for $6 \cdot e.lp_v - e.cost_v$, which is $5 - 6x - 5y$. This has a minimum value of $-1/3$ in $(x, y) = (1/3, 2/3)$.

The formula for $6 \cdot e.lp_u - e.cost_u$ is $6y - 1$ or $6(1 - y)$. Each of them has a minimum value 1 in $(x, y) = (1/3, 1/3)$, 2 in $(x, y) = (0, 2/3)$, respectively. Thus, $(6 \cdot e.lp_w - e.cost_w) + (6 \cdot e.lp_u - e.cost_u)$ and $(6 \cdot e.lp_u - e.cost_u) + (6 \cdot e.lp_v - e.cost_v)$ are at least $-1/3 + 1 = 2/3 \geq 0$.

Finally,

$$\begin{aligned}(6 \cdot e.lp_w - e.cost_w) + (6 \cdot e.lp_v - e.cost_v) &\geq (6x - 1)(1 - y) + (5 - 6x - 5y) \\ &= -6xy - 4y + 4 \geq -6y + 4 \geq 0,\end{aligned}$$

Making the approximation factor $\alpha = 6$ tight at $(x, y) = (1/3, 2/3)$ as well as the objective value nonnegative.

A.1.4 Region VI

As in IV, the formula with sign '+' is smaller for $6 \cdot e.lp_w - e.cost_w$, which is $(6x-1)(1-y)$. This has a minimum value of $1/3$ in $(x, y) = (1/3, 2/3)$.

The formula for $6 \cdot e.lp_u - e.cost_u$ is $(1-x)(6y-1)$ or $6(1-x)(1-y)$. Each of them has a minimum value $2/3$ in $(x, y) = (1/3, 1/3)$, $4/3$ in $(x, y) = (1/3, 2/3)$, respectively.

For $6 \cdot e.lp_v - e.cost_v$, consider the difference between those of '+' sign and '−' sign.

$$\begin{aligned}6(1-xy)(z-(1-z)) - (x+y-2xy-(1-x)(1-y)) &\geq 2(1-xy) - (-3xy+2x+2y-1)\\ &= xy - 2x - 2y + 3\\ &= (2-x)(2-y) - 1 \geq \frac{11}{9} > 0,\end{aligned}$$

since $z \geq 2/3$. Therefore, the formula with sign '−' is smaller, which is $6(1-xy)(1-z) - (1-x)(1-y)$.

Define $p = xy$, then

$$6(1-xy)(1-z) - (1-x)(1-y) = 6pz - 5z - 7p + 5,$$

with region $z \in [2/3, 1]$, $p \in \left[\frac{z}{3} - \frac{1}{9}, \left(\frac{z}{2}\right)^2\right]$. Fixing the value of z, it is a decreasing function with respect to p; fixing the value of p, it is a decreasing function with respect to z. Therefore, the function is minimized at $z = 1$, $p = \frac{1}{4}$ with the value $-\frac{1}{4}$.

Therefore, any objective value is greater than or equal to $1/3 - 1/4 = 1/12 \geq 0$ in the region.

Overall, $6 \cdot LP - ALG \geq 0$ when the triangle inequality is tight.

A.2 All coordinates belong to endpoint of segments.

There are total 4 possible LP values: 0, 1/3, 2/3, 1. Among all the possible (x, y, z) tuples, there are only 7 tuples whose triangle inequality is not tight. Moreover, 3 tuples $(x, y, z) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ among them only consist of value $1/3$ or $2/3$, thus $3 \cdot LP - ALG \geq 0$ by Lemma 4. Therefore, we only need to verify the remaining 3 tuples.

Since left-side and right-side limits of the value of rounding function at such point might differ, we have to consider all possible limits of $6 \cdot LP - ALG$ up to $2^3 = 8$ directions. Table 3 shows remaining 3 tuples and the directional limit of the value of $6 \cdot e.cost - e.lp$ for each configuration. Note that since $x \leq y \leq z$, only up to 4 out of 8 directions are required to verification. As in argument on region I, II, V of Case 1, in each point with direction specified, adding any two values with different column results in a nonnegative value, validating the nonnegativity.

This completes our case analysis.

B Proof for Theorem 1

Suppose that LP-PIVOT achieves an expected α-approximation.

We begin by considering the configuration $(x, 1-x, 1)$ with signs $(+, +, -)$ or $(+, -, -)$ and weights $(1, 0, 1)$. Then:

$$\alpha(1 - f^+(1-x))x \geq (2 - f^+(x))(1 - f^+(1-x)), \tag{17}$$

$$\alpha(1 - f^-(1-x))x \geq (2 - f^+(x))(1 - f^-(1-x)), \tag{18}$$

which implies:

$$f^+(x) \leq 2 - \alpha x \implies f^+(1-x) = f^-(1-x) = 1. \tag{19}$$

Next, consider the configuration $(0, x, x)$ with signs $(+, +, +)$ and weights $(1, 1, 0)$:

$$\alpha x \geq 3f^+(x) - 2f^+(x)^2, \tag{20}$$

which yields the bound $f^+(x) \leq \alpha x$. Therefore, for $x \in [0, 1/\alpha]$, we have

$$f^+(x) \leq \alpha x \leq 2 - \alpha x, \quad f^+(1-x) = 1.$$

Table 3: Value of $6 \cdot e.lp - e.cost$ for each points in consideration of Case 2.

(x, y, z)	Sign	$6 \cdot e.lp_w - e.cost_w$	$6 \cdot e.lp_u - e.cost_u$	$6 \cdot e.lp_v - e.cost_v$
$(\frac{2}{3} - \delta, \frac{2}{3} - \delta, 1)$	'+'	1	1	26/9
	'−'	2/3	2/3	−1/9
$(\frac{2}{3} - \delta, \frac{2}{3} + \delta, 1)$	'+'	0	1	5/3
	'−'	0	2/3	0
$(\frac{2}{3} + \delta, \frac{2}{3} + \delta, 1)$	'+'	0	0	0
	'−'	0	0	0
$(\frac{1}{3} - \delta, 1, 1)$	'+'	0	5	5
	'−'	0	0	0
$(\frac{1}{3} + \delta, 1, 1)$	'+'	0	10/3	10/3
	'−'	0	0	0
$(\frac{2}{3} - \delta, 1, 1)$	'+'	0	5/3	5/3
	'−'	0	0	0
$(\frac{2}{3} + \delta, 1, 1)$	'+'	0	0	0
	'−'	0	0	0
$(1, 1, 1)$	'+'	0	0	0
	'−'	0	0	0

Now consider the configuration $(x, x, 2x)$ with signs $(+, +, +)$ and weights $(1, 1, 0)$:

$$\alpha x(1 - f^+(x)f^+(2x)) \geq f^+(x) + f^+(2x) - 2f^+(x)f^+(2x). \tag{21}$$

Rewriting the inequality, we obtain:

$$(1 - (2 - \alpha x)f^+(x))f^+(2x) \leq \alpha x - f^+(x). \tag{22}$$

Observe that for $x \in [0, 1/\alpha)$ where $f^+(x) < 1$, the term

$$1 - (2 - \alpha x)f^+(x) \geq 1 - 2f^+(x) + f^+(x)^2 \geq 0$$

is strictly positive. Hence:

$$f^+(2x) \leq \frac{\alpha x - f^+(x)}{1 - (2 - \alpha x)f^+(x)} = \alpha x - \frac{(1 - \alpha x)^2 f^+(x)}{1 - (2 - \alpha x)f^+(x)} \leq \alpha x, \tag{23}$$

for $x \in [0, 1/\alpha)$.

Combining the above, we conclude:

$$f^+(x) < \frac{\alpha}{2}x \quad \text{for } x \in [0, 2/\alpha).$$

Set $x = \frac{4}{3\alpha}$. Then from (19) and (23), we have:

$$f^+\left(\frac{4}{3\alpha}\right) \leq \frac{2}{3} = 2 - \alpha x, \quad \text{so} \quad f^+\left(1 - \frac{4}{3\alpha}\right) = 1.$$

Since $f^+(x) < 1$ for $x \in [0, 2/\alpha)$, this implies:

$$\frac{2}{\alpha} \leq 1 - \frac{4}{3\alpha} \quad \Longrightarrow \quad \alpha \geq \frac{10}{3}.$$

Hence, the approximation factor must be at least $10/3$.

C Proof for Theorem 2

We can apply the same argument with Appendix A, with the partition by regions (Figure 3) and formula for $e.cost$ and $e.lp$ (Table 4) changed accordingly.

Define $(x, y, z) := (x_{uv}, x_{vw}, x_{wu})$ and *w.l.o.g.* $x \leq y \leq z$.

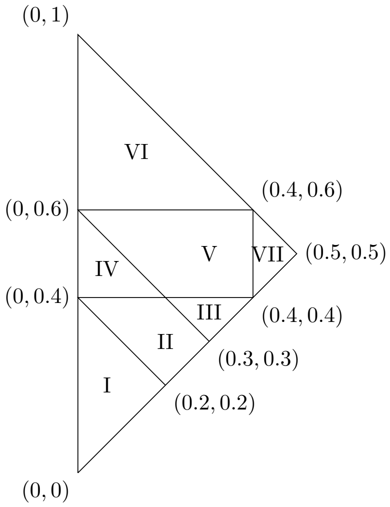

Figure 3: Configuration of regions induced from (11).

Table 4: Formula for $e.cost$ and $e.lp$ for each configuration.

		$x_{uv} = x$		$x_{vw} = y$	
Region	Sign	$e.cost$	$e.lp$	$e.cost$	$e.lp$
I	‘+’	0	x	0	y
	‘−’	1	$1-x$	1	$1-y$
II	‘+’	$\frac{5}{3}z$	x	$\frac{5}{3}z$	y
	‘−’	$1-\frac{5}{3}z$	$1-x$	$1-\frac{5}{3}z$	$1-y$
III	‘+’	1	x	1	y
	‘−’	0	$1-x$	0	$1-y$
IV	‘+’	$\frac{5}{3}y+\frac{5}{3}z-\frac{50}{9}yz$	$(1-\frac{25}{9}yz)x$	$\frac{5}{3}z$	y
	‘−’	$(1-\frac{5}{3}y)(1-\frac{5}{3}z)$	$(1-\frac{25}{9}yz)(1-x)$	$1-\frac{5}{3}z$	$1-y$
V	‘+’	$1-\frac{5}{3}y$	$(1-\frac{5}{3}y)x$	1	y
	‘−’	0	$(1-\frac{5}{3}y)(1-x)$	0	$1-y$
VI	‘+’	0	0	1	y
	‘−’	0	0	0	$1-y$
VII	‘+’	$1-\frac{5}{3}y$	$(1-\frac{5}{3}y)x$	$1-\frac{5}{3}x$	$(1-\frac{5}{3}x)y$
	‘−’	0	$(1-\frac{5}{3}y)(1-x)$	0	$(1-\frac{5}{3}x)(1-y)$

		$x_{wu} = z = x + y$	
Region	Sign	$e.cost$	$e.lp$
I	‘+’	0	z
	‘−’	1	$1-z$
II	‘+’	0	z
	‘−’	1	$1-z$
III	‘+’	0	z
	‘−’	1	$1-z$
IV	‘+’	$\frac{5}{3}y$	z
	‘−’	$1-\frac{5}{3}y$	$1-z$
V	‘+’	$\frac{5}{3}y$	z
	‘−’ $1-\frac{5}{3}y$	$1-z$	
VI	‘+’	1	z
	‘−’	0	$1-z$
VII	‘+’	$\frac{5}{3}x+\frac{5}{3}y-\frac{50}{9}xy$	$(1-\frac{25}{9}xy)z$
	‘−’	$(1-\frac{5}{3}x)(1-\frac{5}{3}y)$	$(1-\frac{25}{9}xy)(1-z)$

Table 5: Value of $\frac{10}{3} \cdot e.lp - e.cost$ for each extremal point of the region.

Region	(x, y)	Sign	$\frac{10}{3} \cdot e.lp_w - e.cost_w$	$\frac{10}{3} \cdot e.lp_u - e.cost_u$	$\frac{10}{3} \cdot e.lp_v - e.cost_v$
I	$(0, 0)$	'+'	0	0	0
		'−'	$7/3$	$7/3$	$7/3$
	$(0, \frac{2}{5})$	'+'	0	$4/3$	$4/3$
		'−'	$7/3$	1	1
	$(\frac{1}{5}, \frac{1}{5})$	'+'	$2/3$	$2/3$	$4/3$
		'−'	$5/3$	$5/3$	1
II	$(0, \frac{2}{5})$	'+'	$-2/3$	$2/3$	$4/3$
		'−'	3	$5/3$	1
	$(\frac{1}{5}, \frac{1}{5})$	'+'	0	0	$4/3$
		'−'	$7/3$	$7/3$	1
	$(\frac{1}{5}, \frac{2}{5})$	'+'	$-1/3$	$1/3$	2
		'−'	$8/3$	2	$1/3$
	$(\frac{3}{10}, \frac{3}{10})$	'+'	0	0	2
		'−'	$7/3$	$7/3$	$1/3$
III	$(\frac{1}{5}, \frac{2}{5})$	'+'	$-1/3$	$1/3$	2
		'−'	$8/3$	2	$1/3$
	$(\frac{3}{10}, \frac{3}{10})$	'+'	0	0	2
		'−'	$7/3$	$7/3$	$1/3$
	$(\frac{2}{5}, \frac{2}{5})$	'+'	$1/3$	$1/3$	$8/3$
		'−'	2	2	$-1/3$
VI	$(0, \frac{3}{5})$	'+'	0	1	1
		'−'	0	$4/3$	$4/3$
	$(0, 1)$	'+'	0	$7/3$	$7/3$
		'−'	0	0	0
	$(\frac{2}{5}, \frac{3}{5})$	'+'	0	1	$7/3$
		'−'	0	$4/3$	0

C.1 Triangle inequality is tight.

C.1.1 Region I, II, III, VI

Affine formulae; refer to Table 5. Also note that the factor $10/3$ is tight for $(0, \frac{2}{5}, \frac{2}{5})$ with $(+, +, +)$,

C.1.2 Region IV

The argument here is analogous with A.1.2: $(1 - \frac{25}{9}yz)x \leq (1 - \frac{25}{9}yz)(1 - x)$ as $x \leq \frac{1}{2}$; $\frac{5}{3}y + \frac{5}{3}z - \frac{50}{9}yz \geq (1 - \frac{5}{3}y)(1 - \frac{5}{3}z)$ as

$$\frac{5}{3}y + \frac{5}{3}z - \frac{50}{9}yz - \left(1 - \frac{5}{3}y\right)\left(1 - \frac{5}{3}z\right) = \frac{1 - (2 - 5y)(2 - 5z)}{3}$$

and $y, z \in [2/5, 3/5]$. Therefore, $\frac{10}{3} \cdot e.lp_w - e.cost_w$ is smaller with sign '+'.

Replacing $x = z - y$,

$$\frac{10}{3} \cdot e.lp_w - e.cost_w = -\frac{250}{27}yz^2 + \frac{250}{27}y^2z + \frac{50}{9}yz - 5y + \frac{5}{3}z.$$

Fixing y, this formula is concave of z, hence minimized in either $z = y$ or $z = 3/5$, which yield $\frac{50}{9}y^2 - \frac{10}{3}y$ and $\frac{50}{9}y^2 - 5y + 1$ each. Within the range $y \in [2/5, 3/5]$, the minimum value of each is $-1/9$ in $y = 2/5$ and $-1/8$ in $y = 9/20$. Thus, $\frac{10}{3} \cdot e.lp_w - e.cost_w \geq -1/8$.

Finally, $\frac{10}{3} \cdot e.lp_u - e.cost_y \geq 1/3$, minimized at $(y, z) = (2/5, 3/5)$ with '+' sign; $\frac{10}{3} \cdot e.lp_v - e.cost_v \geq 2/3$, minimized at $(y, z) = (2/5, 2/5)$ with '+' sign.

Therefore, any objective value is at least $1/3 - 1/8 = 5/24 \geq 0$ in the region.

C.1.3 Region V

The argument here is analogous with A.1.3: $\frac{10}{3} \cdot e.lp_w - e.cost_w$ is smaller with sign '$+$', which is $(\frac{10}{3}x - 1)(1 - \frac{5}{3}y)$. This has a minimum value of $-1/9$ in $(x, y) = (1/5, 2/5)$.

The formula for $\frac{10}{3} \cdot e.lp_v - e.cost_v$ is either $5/3(2x + y)$ or $7/3 - 5/3(2x + y)$, whose global minimum value is 0 in $(x, y) = (2/5, 3/5)$ with sign '$-$'.

The formula for $\frac{10}{3} \cdot e.lp_u - e.cost_u$ is either $10/3 \cdot y - 1$ or $10/3(1 - y)$, whose global minimum value is $1/3$ in $y = 2/5$ with sign '$+$'. Thus, $(\frac{10}{3} \cdot e.lp_w - e.cost_w) + (\frac{10}{3} \cdot e.lp_u - e.cost_u)$ and $(\frac{10}{3} \cdot e.lp_u - e.cost_u) + (\frac{10}{3} \cdot e.lp_v - e.cost_v)$ are at least $-1/9 + 1/3 = 2/9 \geq 0$.

Finally,

$$
\begin{aligned}
&\left(\frac{10}{3} \cdot e.lp_w - e.cost_w\right) + \left(\frac{10}{3} \cdot e.lp_v - e.cost_v\right) \\
&\quad \geq \min\left\{\left(\frac{10}{3}x - 1\right)\left(1 - \frac{5}{3}y\right) + \frac{5}{3}(2x + y), \left(\frac{10}{3}x - 1\right)\left(1 - \frac{5}{3}y\right) + \frac{7}{3} - \frac{5}{3}(2x + y)\right\} \\
&\quad = \min\left\{\left(\frac{10}{3}x - 2\right)\left(2 - \frac{5}{3}y\right) + 3, \frac{50}{9}xy + \frac{4}{3}\right\} \\
&\quad \geq \min\left\{1, \frac{4}{3}\right\} \geq 1 \geq 0.
\end{aligned}
$$

C.1.4 Region VII

The argument here is analogous with A.1.4: $\frac{10}{3} \cdot e.lp_w - e.cost_w$ is smaller with sign '$+$', which is $(\frac{10}{3}x - 1)(1 - \frac{5}{3}y)$. This has a minimum value of 0 in $(x, y) = (0.4, 0.6)$.

The formula for $\frac{10}{3} \cdot e.lp_u - e.cost_u$ is either $(1 - \frac{5}{3}x)(\frac{10}{3}y - 1)$ or $\frac{10}{3}(1 - \frac{5}{3}x)(1 - y)$, whose global minimum value is $1/9$ in $(x, y) = (2/5, 2/5), (1/2, 1/2)$ with sign '$+$'.

For $\frac{10}{3} \cdot e.lp_v - e.cost_v$,

$$
\begin{aligned}
&\frac{10}{3}\left(1 - \frac{25}{9}xy\right)(2z - 1) - \left(\frac{5}{3}x + \frac{5}{3}y - \frac{50}{9}xy - \left(1 - \frac{5}{3}x\right)\left(1 - \frac{5}{3}y\right)\right) \\
&\quad \geq 2\left(1 - \frac{25}{9}xy\right) - \left(-\frac{25}{3}xy + \frac{10}{3}x + \frac{10}{3}y - 1\right) \\
&\quad = \frac{25}{9}xy - \frac{10}{3}x - \frac{10}{3}y + 3 \\
&\quad = \left(2 - \frac{5}{3}x\right)\left(2 - \frac{5}{3}y\right) - 1 \geq \frac{1}{3} \geq 0,
\end{aligned}
$$

since $z \geq \frac{4}{5}$. Therefore, the formula with sign '$-$' is smaller, which is $\frac{10}{3}(1 - \frac{25}{9}xy)(1 - z) - (1 - \frac{5}{3}x)(1 - \frac{5}{3}y)$.

Define $p = xy$, then

$$
\frac{10}{3}\left(1 - \frac{25}{9}xy\right)(1 - z) - \left(1 - \frac{5}{3}x\right)\left(1 - \frac{5}{3}y\right) = \frac{250}{27}pz - \frac{5}{3}z - \frac{325}{27}p + \frac{7}{3},
$$

with region $z \in [4/5, 1]$, $p \in \left[\frac{2}{5}z - \frac{4}{25}, \left(\frac{z}{2}\right)^2\right]$. Fixing the value of z,it is a decreasing function with respect to p. Therefore, the function is minimized when $p = \left(\frac{z}{2}\right)^2$, which now becomes:

$$
\frac{125}{54}z^3 - \frac{325}{108}z^2 - \frac{5}{3}z + \frac{7}{3}.
$$

This function decreases in the range $z \in [4/5, 1]$, hence the minimum value is $-1/36$ in $z = 1$. Therefore, $(\frac{10}{3} \cdot e.lp_w - e.cost_w) + (\frac{10}{3} \cdot e.lp_u - e.cost_u)$ and $(\frac{10}{3} \cdot e.lp_u - e.cost_u) + (\frac{10}{3} \cdot e.lp_v - e.cost_v)$ are at least $1/9 - 1/36 = 1/12 \geq 0$.

Table 6: Value of $\frac{10}{3} \cdot e.lp - e.cost$ for each points in consideration of Case 2.

(x,y,z)	Sign	$\frac{10}{3} \cdot e.lp_w - e.cost_w$	$\frac{10}{3} \cdot e.lp_u - e.cost_u$	$\frac{10}{3} \cdot e.lp_v - e.cost_v$
$(\frac{3}{5}, \frac{3}{5}, 1)$	'+'	0	0	0
	'−'	0	0	0
$(\frac{2}{5} - \delta, 1, 1)$	'+'	0	7/3	7/3
	'−'	0	0	0
$(\frac{2}{5} + \delta, 1, 1)$	'+'	0	7/9	7/9
	'−'	0	0	0
$(\frac{3}{5}, 1, 1)$	'+'	0	0	0
	'−'	0	0	0
$(1, 1, 1)$	'+'	0	0	0
	'−'	0	0	0

Finally,

$$\begin{aligned}
&\left(\frac{10}{3} \cdot e.lp_w - e.cost_w\right) + \left(\frac{10}{3} \cdot e.lp_v - e.cost_v\right) \\
&\geq \left(\frac{10}{3}x - 1\right)\left(1 - \frac{5}{3}y\right) + \frac{10}{3}\left(1 - \frac{25}{9}xy\right)(1-z) - \left(1 - \frac{5}{3}x\right)\left(1 - \frac{5}{3}y\right) \\
&= (5x-2)\left(1 - \frac{5}{3}y\right) + \frac{10}{3}\left(1 - \frac{25}{9}xy\right)(1-z) \\
&\geq 0 \cdot 0 + \frac{10}{3} \cdot \frac{11}{36} \cdot 0 = 0,
\end{aligned}$$

since $x \geq \frac{2}{5}$, $y \leq \frac{3}{5}$, $z \leq 1$.

C.2 All coordinates belong to endpoint of segments.

There are also 4 tuples of (x, y, z) to verify, which are fully covered in Table 6. Note that there are now only single point $2/5$ where rounding functions are not continuous.

This completes our case analysis.

D Proof for Theorem 3

Suppose LP-CCC is an expected α-approximation algorithm, where $2 \leq \alpha < 4$.

Plugging $(\frac{1}{2}, \frac{1}{2}, 1)$ with $(\circ, \circ, -)$, we obtain:

$$\alpha\left(1 - f^\circ\left(\frac{1}{2}\right)\right) \geq 2\left(1 - f^\circ\left(\frac{1}{2}\right)\right) + \left(1 - f^\circ\left(\frac{1}{2}\right)\right)^2, \tag{24}$$

which implies:

$$f^\circ\left(\frac{1}{2}\right) \geq 3 - \alpha. \tag{25}$$

Next, plugging $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ with $(+, +, \circ)$ yields:

$$\frac{1 - f^+\left(\frac{1}{4}\right)^2 + 2\left(f^+\left(\frac{1}{4}\right) + f^\circ\left(\frac{1}{2}\right) - 2f^+\left(\frac{1}{4}\right)f^\circ\left(\frac{1}{2}\right)\right)}{\frac{3}{4}\left(1 - f^+\left(\frac{1}{4}\right)^2\right) + \frac{1}{2}\left(1 - f^+\left(\frac{1}{4}\right)f^\circ\left(\frac{1}{2}\right)\right)} \leq \alpha, \tag{26}$$

which can be rewritten as:

$$\left(2 + \left(\frac{\alpha}{2} - 4\right)f^+\left(\frac{1}{4}\right)\right)f^\circ\left(\frac{1}{2}\right) \leq \frac{\alpha}{4}\left(5 - 3f^+\left(\frac{1}{4}\right)^2\right) + \left(f^+\left(\frac{1}{4}\right) - 1\right)^2 - 2. \tag{27}$$

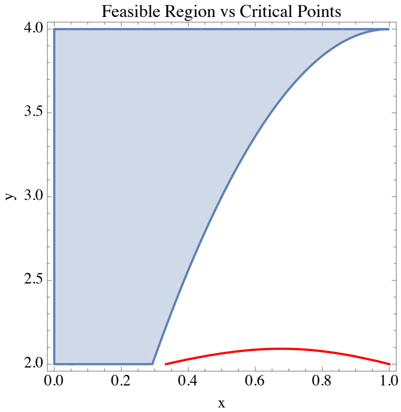

Figure 4: Comparison between the domain of g and the curve where $\frac{\partial}{\partial x} g(x, y) = 0$.

Using the known bound from [16], plugging $(0, x, x)$ with $(+, +, +)$ yields:

$$f^+(x) \leq 1 - \sqrt{1 - \alpha x}, \tag{28}$$

which implies:

$$\begin{aligned}\left(2 + \left(\frac{\alpha}{2} - 4\right) f^+\left(\frac{1}{4}\right)\right) &\geq \left(2 + \left(\frac{\alpha}{2} - 4\right)\left(1 - \sqrt{1 - \frac{\alpha}{4}}\right)\right) \\ &= 2\sqrt{1 - \frac{\alpha}{4}}\left(\left(\sqrt{1 - \frac{\alpha}{4}} - \frac{1}{2}\right)^2 + \frac{3}{4}\right) > 0,\end{aligned}$$

for any $\alpha < 4$.

Therefore, we obtain the following upper bound on $f^\circ\left(\frac{1}{2}\right)$:

$$f^\circ\left(\frac{1}{2}\right) \leq \frac{\frac{\alpha}{4}\left(5 - 3f^+\left(\frac{1}{4}\right)^2\right) + \left(f^+\left(\frac{1}{4}\right) - 1\right)^2 - 2}{2 + \left(\frac{\alpha}{2} - 4\right) f^+\left(\frac{1}{4}\right)}. \tag{29}$$

Define

$$g(x, y) := \frac{\frac{y}{4}\left(5 - 3x^2\right) + (x - 1)^2 - 2}{2 + \left(\frac{y}{2} - 4\right) x}$$

for $x \in \left[0, 1 - \sqrt{1 - \frac{y}{4}}\right]$ and $y \in [2, 4)$, to examine the maximum possible value of $f^\circ\left(\frac{1}{2}\right)$ for a given α.

The partial derivative of $g(x, y)$ with respect to x is:

$$\frac{\partial}{\partial x} g(x, y) = \frac{3y - 4}{2(8 - y)} + \frac{5y^3 - 84y^2 + 368y - 448}{2(8 - y)(4 - 8x + xy)^2}. \tag{30}$$

Within the given region, $\frac{\partial}{\partial x} g(x, y)$ is nonzero, as shown in Figure 4. Moreover, evaluating at $x = 0$ gives:

$$\left.\frac{\partial}{\partial x} g(x, y)\right|_{x=0} = \frac{5y^3 - 84y^2 + 416y - 512}{32(8 - y)} = -\frac{5y^2 - 44y + 64}{32} \geq \frac{1}{8}. \tag{31}$$

Therefore, for any $y_0 \in [2, 4)$, the function $g(x, y_0)$ is strictly increasing within the region. Consequently,

$$g(x, y_0) \leq g\left(1 - \sqrt{1 - \frac{y_0}{4}}, y_0\right),$$

which corresponds to the parabolic boundary of the region depicted in Figure 4.

Parameterizing this curve as $(1-t, 4(1-t^2))$ for $t \in \left(0, \frac{1}{\sqrt{2}}\right]$, the value of g along the curve is:

$$g(1-t, 4(1-t^2)) = \frac{(1-t^2)(2+6t-3t^2)+t^2-2}{2-2(t^2+1)(1-t)} \tag{32}$$

$$= \frac{3t^3-6t^2-4t+6}{2(t^2-t+1)}. \tag{33}$$

This is the upper bound for $f^\circ\left(\frac{1}{2}\right)$ when $\alpha = 4(1-t^2)$.

On the other hand, from (25), the lower bound is $3-\alpha = 3-4(1-t^2) = 4t^2-1$.

Therefore, the inequality

$$2(t^2-t+1)(4t^2-1) \leq 3t^3-6t^2-4t+6 \tag{34}$$

must hold.

Define:

$$h(t) := 2(t^2-t+1)(4t^2-1)-(3t^3-6t^2-4t+6) = 8t^4-11t^3+12t^2+6t-8.$$

Since $h''(t) = 96t^2-66t+24 > 0$ and $h'(0) = 6 > 0$, h is convex and increasing for $t \geq 0$.

Finally, since $h\left(\sqrt{1-\frac{2.11}{4}}\right) > 0$, we conclude that $h(t) > 0$ when $t \geq \sqrt{1-\frac{2.11}{4}}$, implying that the lower bound for $f^\circ\left(\frac{1}{2}\right)$ exceeds the upper bound when $\alpha \leq 2.11$.

Therefore, the approximation factor of LP-CCC for CCC must be greater than 2.11.

E Proof for Theorem 4

Unlike the pseudometric-weighted case, the variables x^c_{vw} and x^c_{wu} can multiplicatively affect the value of $e.lp_w(u,v)$ in (15). Therefore, Lemma 3 does not apply, and we must instead verify the inequality $\alpha \cdot LP - ALG \geq 0$ directly.

For sign configurations that do not involve a '$\circ$' (i.e., all edges have $+$ or $-$ signs), the inequality $\alpha \cdot LP - ALG \geq 0$ with $\alpha = 2.06 - \varepsilon$ is already established in [16]. Thus, it suffices to check the remaining 6 configurations involving at least one '$\circ$' sign.

As before, define $(x, y, z) := (x_{uv}, x_{vw}, x_{wu})$ and $(p_x, p_y, p_z) := (p_{uv}, p_{vw}, p_{wu})$.

E.1 $(\circ, \circ, \circ)$ Triangles

$$2 \cdot e.lp_w(u,v) \geq 2(1-p_{uw}p_{vw}) \cdot \frac{1}{2} = (1-p_{uw}p_{vw}) = e.cost_w(u,v).$$

Since the same inequality holds for all permutations of the endpoints, we conclude:

$$2 \cdot LP - ALG \geq 0.$$

E.2 $(\circ, \circ, +)$ Triangles

If $z \geq \frac{1}{2}$,

$$2 \cdot e.lp_v(w,u) - e.cost_v(w,u) \geq (1-p_xp_y) - (p_x+p_y-2p_xp_y) = (1-p_x)(1-p_y) \geq 0,$$

and by the same reasoning as in Section E.1, we also have

$$2 \cdot e.lp_w(u,v) \geq e.cost_w(u,v), \quad 2 \cdot e.lp_u(v,w) \geq e.cost_u(v,w).$$

Hence, $2 \cdot LP - ALG \geq 0$.

Otherwise, i.e., if $1 - 2z \geq 0$:

$$
\begin{aligned}
2 \cdot LP - ALG = 2 \cdot &\left((2 - (p_x + p_y)p_z) \max\left\{\frac{1}{2}, 1-x, 1-y, 1-z\right\} + (1 - p_x p_y) z \right) \\
&- (1 - p_y p_z + 1 - p_x p_z + p_x + p_y - 2p_x p_y) \\
\geq\ & 2 \cdot ((2 - (p_x + p_y)p_z)(1 - z) + (1 - p_x p_y) z) \\
&- (2 - (p_x + p_y)p_z + (1 - p_x p_y) - (1 - p_x)(1 - p_y)) \\
=\ & (1 - 2z)(2 - (p_x + p_y)p_z) + (1 - 2z)(p_x p_y - 1) + (1 - p_x)(1 - p_y) \\
=\ & (1 - 2z)\,(2 - (p_x + p_y)p_z + p_x p_y - 1) + (1 - p_x)(1 - p_y) \\
\geq\ & (1 - 2z)\,(2 - p_x - p_y + p_x p_y - 1) + (1 - p_x)(1 - p_y) \\
=\ & (1 - 2z)(1 - p_x)(1 - p_y) + (1 - p_x)(1 - p_y) \geq 0.
\end{aligned}
$$

E.3 $(\circ, \circ, -)$ Triangles

If $z \leq 1 - \frac{1}{\alpha}$, then as in Section E.2, we have $\alpha \cdot LP - ALG \geq 0$.

Otherwise, since $1 - z \leq 1/\alpha < 1/2$,

$$
\begin{aligned}
&\alpha \cdot LP - ALG \\
&= \alpha \cdot \left((2 - (p_x + p_y) z) \max\left\{\frac{1}{2}, 1-x, 1-y, 1-z\right\} + (1 - p_x p_y)(1 - z) \right) \\
&\quad - (1 - p_y z + 1 - p_x z + (1 - p_x)(1 - p_y)) \\
&= \alpha (2 - (p_x + p_y) z) \max\left\{\frac{1}{2}, 1-x, 1-y\right\} + \alpha(1 - p_x p_y)(1 - z) \\
&\quad - (2 - (p_x + p_y) z + (1 - p_x)(1 - p_y))\,.
\end{aligned}
$$

Fixing the values of x and y, the function is affine with respect to z, and its slope is given by

$$
\begin{aligned}
&- \alpha (p_x + p_y) \max\left\{\frac{1}{2}, 1-x, 1-y\right\} - \alpha(1 - p_x p_y) + (p_x + p_y) \\
&\leq \left(-\frac{\alpha}{2} + 1\right)(p_x + p_y) - \alpha(1 - p_x p_y) \leq 0.
\end{aligned}
$$

Therefore, the function is minimized when z is maximized.

Now, *w.l.o.g.*, assume $x \leq y$, and consider the following three cases:

1. $x \geq 1/2$
   $p_x, p_y \geq 0.85$ as $x, y \geq 1/2$. Applying $z = 1$,
   $$
   \begin{aligned}
   &\alpha \cdot LP - ALG \\
   &\geq \alpha (2 - p_x - p_y) \max\left\{\frac{1}{2}, 1-x, 1-y\right\} - (2 - p_x - p_y) - (1 - p_x)(1 - p_y) \\
   &= \left(\frac{\alpha}{2} - 1\right)(2 - p_x - p_y) - (1 - p_x)(1 - p_y) \\
   &\geq \left(\frac{\alpha}{2} - 1\right)(2 - p_x - p_y) - \frac{1}{4}(2 - p_x - p_y)^2 \\
   &= \frac{1}{4}(2 - (p_x + p_y))(2\alpha - 6 + (p_x + p_y)).
   \end{aligned}
   $$
   Since $6 - 2\alpha = 1.7 \leq p_x + p_y \leq 2$, $\alpha \cdot LP - ALG \geq 0$.
2. $x < 1/2$, $x + y \geq 1$
   Applying $z = 1$ again,
   $$
   \begin{aligned}
   &\alpha \cdot LP - ALG \\
   &\geq \alpha (2 - p_x - p_y) \max\left\{\frac{1}{2}, 1-x, 1-y\right\} - (2 - p_x - p_y) - (1 - p_x)(1 - p_y) \\
   &= (\alpha(1 - x) - 1)(2 - p_x - p_y) - (1 - p_x)(1 - p_y).
   \end{aligned}
   $$

Note that the function is affine with respect to p_y; therefore, the function is minimized when p_y reaches the boundary, i.e., $y = 1$ or $y = 1 - x$.
If $y = 1$, then $p_y = 1$, hence

$$
\begin{aligned}
&(\alpha(1-x)-1)(2-p_x-p_y)-(1-p_x)(1-p_y)\\
&=(\alpha(1-x)-1)(1-p_x)\\
&\geq\left(\frac{\alpha}{2}-1\right)(1-p_x)\geq 0.
\end{aligned}
$$

On the other hand, if $y = 1 - x$, then $p_y = \frac{3}{10}y + \frac{7}{10} = 1 - \frac{3}{10}x$ as $y > 1/2$, hence

$$
\begin{aligned}
&(\alpha(1-x)-1)(2-p_x-p_y)-(1-p_x)(1-p_y)\\
&=(\alpha(1-x)-1)\left(1-\frac{17}{10}x+\frac{3}{10}x\right)-\frac{3}{10}\left(1-\frac{17}{10}x\right)x\\
&=3.52\cdot\left(x-\frac{1}{2}\right)\left(x-\frac{115}{176}\right)\geq 0
\end{aligned}
$$

as $x < 1/2$ and $p_x = \frac{17}{10}x$.

3. $x + y < 1$
Applying $z = x + y$,

$$
\begin{aligned}
&\alpha\cdot LP-ALG\\
&=\alpha(2-(p_x+p_y)(x+y))\max\left\{\frac{1}{2},1-x,1-y\right\}+\alpha(1-p_xp_y)(1-x-y)\\
&\quad-(2-(p_x+p_y)(x+y)+(1-p_x)(1-p_y))\\
&=(\alpha(1-x)-1)(2-(p_x+p_y)(x+y))+\alpha(1-p_xp_y)(1-x-y)-(1-p_x)(1-p_y)\\
&\geq 0.3\cdot(\alpha(1-x)-1)+\alpha(1-p_xp_y)(1-x-y)-(1-p_x)(1-p_y).
\end{aligned}
$$

For the last inequality, since $p_x \leq 1.7x$ and $p_y \leq 1.7y$, $2-(p_x+p_y)(x+y) \geq 2-1.7(x+y)^2 \geq 0.3$.
Differentiating the function by y,

$$
\begin{aligned}
&-\alpha p_x\frac{dp_y}{dy}\cdot(1-x-y)-\alpha(1-p_xp_y)+\frac{dp_y}{dy}\cdot(1-p_x)\\
&\leq-\alpha p_x\frac{dp_y}{dy}\cdot(1-x-y)-\alpha(1-p_x)+\frac{dp_y}{dy}\cdot(1-p_x)<0
\end{aligned}
$$

since $\frac{dp_y}{dy} \leq 1.7 < \alpha$. Hence, the function is minimized when y is maximized.
Applying $y = 1 - x$, since $y = 1 - x > 1/2$, $p_y = \frac{3}{10}y + \frac{7}{10} = 1 - \frac{3}{10}x$ and thus

$$
\begin{aligned}
&0.3\cdot(\alpha(1-x)-1)+\alpha(1-p_xp_y)(1-x-y)-(1-p_x)(1-p_y)\\
&=0.3\cdot(\alpha(1-x)-1)-\left(1-\frac{17}{10}x\right)\cdot\frac{3}{10}x\\
&=0.51\left(x-\frac{1}{2}\right)\left(x-\frac{23}{17}\right)\geq 0
\end{aligned}
$$

since $x < 1/2$.

E.4 $(\circ, +, +)$ Triangles

For Sections E.4 and E.5, we utilize the following lemma, which serves as a weaker version of Lemma 3.

Lemma 5. *Let the sign configuration be such that the edge uv is the only edge labeled with a '$\circ$' among the three edges of the triangle. Suppose the rounding function f° is piecewise affine. Then $\mathcal{C} \geq 0$ for all possible configurations of (x_{uv}, x_{vw}, x_{wu}) if the inequality holds in each of the following cases:*

1. *The triangle inequality is tight for* (x_{uv}, x_{vw}, x_{wu})*;*

2. x_{uv} *lies at an endpoint of a domain interval for some piece of the function* f°*;*

3. $x_{uv} = \min\left\{\frac{1}{2}, x_{vw}, x_{wu}\right\}$.

Proof. Fix the values of x_{vw} and x_{wu}, and treat $\mathcal{C}$ as a function of x_{uv} and p_{uv}. Within this function, the term involving x_{uv} appears only in

$$e.lp_w(uv) = (1 - p_{vw}p_{wu}) \cdot \max\left\{\frac{1}{2}, 1 - x_{uv}, 1 - x_{vw}, 1 - x_{wu}\right\},$$

which is piecewise affine with two domain intervals:

$$\left[0, \min\left\{\frac{1}{2}, x_{vw}, x_{wu}\right\}\right] \quad \text{and} \quad \left[\min\left\{\frac{1}{2}, x_{vw}, x_{wu}\right\}, 1\right].$$

The remaining terms in $\mathcal{C}$ are affine in p_{uv}.

Hence, $\mathcal{C}$ is a piecewise affine function defined over two domains in the (x_{uv}, p_{uv}) space:

$$\left[0, \min\left\{\frac{1}{2}, x_{vw}, x_{wu}\right\}\right] \times [0,1] \quad \text{and} \quad \left[\min\left\{\frac{1}{2}, x_{vw}, x_{wu}\right\}, 1\right] \times [0,1].$$

Since f° is piecewise affine, substituting $p_{uv} = f^\circ(x_{uv})$ yields a piecewise affine function in x_{uv}. The breakpoints of this composition occur at:

1. the endpoints of the domain intervals of f°, and

2. the point $x_{uv} = \min\left\{\frac{1}{2}, x_{vw}, x_{wu}\right\}$ where the domain of the piecewise max function changes.

Therefore, the minimum of $\mathcal{C}$ occurs either at domain boundaries or at breakpoints—precisely the three cases specified in the lemma. $\square$

Moreover, in this particular case, an even stronger argument can be made: Examining the full formula of $\mathcal{C} = \alpha \cdot LP - ALG$ with this sign configuration,

$$\begin{aligned}
&\alpha \cdot LP - ALG \\
&= \alpha \cdot \left((1 - p_y p_z)\max\left\{\frac{1}{2}, 1-x, 1-y, 1-z\right\} + (1 - p_x p_z)y + (1 - p_x p_y)z\right) \\
&\quad - (1 - p_y p_z + p_x + p_z - 2p_x p_z + p_x + p_y - 2p_x p_y) \\
&= \alpha(1 - p_y p_z)\max\left\{\frac{1}{2}, 1-x, 1-y, 1-z\right\} + (-\alpha(y p_z + p_y z) - 2 + 2(p_y + p_z))\, p_x \\
&\quad + \alpha(y+z) - (1 + p_y + p_z - p_y p_z)
\end{aligned}$$

The first term is a monotonically decreasing function of x; the second term is a monotonically decreasing function of p_x since $-\alpha(yp_z + p_y z) - 2 + 2(p_y + p_z) \le 0$, as shown in the following Lemma:

Lemma 6. *For* f^+ *given as equation 12 and* $\alpha \in \left[\frac{1}{0.5095}, \frac{2}{3 \cdot 0.5095 - 0.19}\right] \approx [1.963, 2.988]$,

$$\alpha(xf^+(y) + f^+(x)y) + 2 - 2(f^+(x) + f^+(y)) \ge 0$$

in $[0,1]^2$.

Proof. *w.l.o.g.* $x \le y$. Consider the 2 possible cases, based on the value of y.

1. $y \ge 0.5095$

In this case, $f^+(y) = 1$ and thus

$$\begin{aligned}&\alpha(xf^+(y) + f^+(x)y) + 2 - 2(f^+(x) + f^+(y))\\&= \alpha(x + f^+(x)y) - 2f^+(x),\end{aligned}$$

which is an increasing function of y.

If $x \geq 0.5095$, setting $y = x$,

$$\alpha(x + f^+(x)y) - 2f^+(x) \geq 2\alpha x - 2 \geq 0$$

since $x > 1/2$. Otherwise, setting $y = 0.5095$,

$$\alpha(x + f^+(x)y) - 2f^+(x) \geq (\alpha \cdot 0.5095 - 2)f^+(x) + \alpha x$$

is concave of x since $\alpha \cdot 0.5095 - 2 < 0$ and f^+ is convex in the range. Hence, this is minimized at either $x = 0$ or $x = 0.5095$, where the former case results in 0 and the latter case results in $2\alpha \cdot 0.5095 - 2 \geq 0$.

2. $y < 0.5095$

   As a function of y,

$$\begin{aligned}&\alpha(xf^+(y) + f^+(x)y) + 2 - 2(f^+(x) + f^+(y))\\&= (\alpha x - 2)f^+(y) + \alpha f^+(x)y + 2 - 2f^+(x)\end{aligned}$$

   is concave since $\alpha x - 2 \leq \alpha \cdot 0.5095 - 2 < 0$ and f^+ is convex in the range. Hence, this is minimized at either $y = x$ or $y = 0.5095$, where the latter case reduce to Case 1, and the former case results in

$$\begin{aligned}&(\alpha x - 2)f^+(y) + \alpha f^+(x)y + 2 - 2f^+(x)\\&= 2\left((\alpha x - 2)f^+(x) + 1\right),\end{aligned}$$

   which is $2 \geq 0$ if $x < 0.19$, or

$$2\left((\alpha x - 2)\left(\frac{x - 0.19}{0.5095 - 0.19}\right)^2 + 1\right)$$

   otherwise. This is a decreasing function when $x \in \left[0, \frac{0.19+4/\alpha}{3}\right]$, which contains $[0, 0.5095]$. Therefore, this is minimized when $x = 0.5095$, which also reduce to Case 1.

This completes the proof of the lemma. □

Therefore, as p_x is a monotonically increasing function of x, the function is minimized when x is maximized.

Now, *w.l.o.g.* $y \leq z$ and consider all 2 possible cases:

1. $y + z \geq 1$

   Applying $x = 1$,

$$\begin{aligned}&\alpha \cdot LP - ALG\\&\geq \alpha \cdot \left((1 - p_y p_z)\max\left\{\frac{1}{2}, 1 - y, 1 - z\right\} + (1 - p_y)z + (1 - p_z)y\right)\\&\quad - (3 - p_y p_z - p_z - p_y)\\&\geq \alpha \cdot \left(\frac{1}{2}(1 - p_y p_z) + (1 - p_z)y + (1 - p_y)z\right) - (3 - p_y p_z - p_z - p_y)\\&\geq \alpha \cdot \left(\frac{1}{2}(1 - p_y p_z) + (1 - p_y)\frac{p_z}{2} + (1 - p_z)\frac{p_y}{2}\right) - (3 - p_y p_z - p_z - p_y)\\&= 2.225\left(\left(\frac{6}{89}\right)^2 - \left(p_y - \frac{83}{89}\right)\left(p_z - \frac{83}{89}\right)\right).\end{aligned}$$

   Since $z \geq 1/2$, $p_z > 0.94 > \frac{83}{89}$, thus the function is decreasing with respect to p_y. The minimum value of the function is therefore 0 when $p_y = p_z = 1$.

2. $y + z < 1$

We use the similar argument with of those conducted by Chawla et al.[16] on the $(+, +, -)$ case. For simplicity, let $a = 0.19,\ b = 0.5095$.

If $y < a$, $p_y = 0$, hence

$$\begin{aligned}
&\alpha \cdot LP - ALG \\
&= \alpha \cdot ((1-y) + (1 - p_x p_z) y + z) - (1 + 2p_x + p_z - 2p_x p_z) \\
&= -\alpha p_x p_z y + \alpha(1+z) - (1 + 2p_x + p_z - 2p_x p_z)\,.
\end{aligned}$$

This is a non-increasing function of y, and thus it is sufficient to check on $y = a$.

If $z > b$, $p_z = 1$, hence

$$\begin{aligned}
&\alpha \cdot LP - ALG \\
&= \alpha \cdot ((1-p_y)(1-y) + (1-p_x)y + (1 - p_x p_y) z) - (2 - 2p_x p_y) \\
&= \alpha(1 - p_x p_y) z + \alpha((1-p_y)(1-y) + (1-p_x)y) - 2(1 - p_x p_y).
\end{aligned}$$

This is a non-decreasing function of z, and thus it is sufficient to check on $z = b$.

Now the problem reduces to only $a \le y \le z \le b$ case, where the formula for p_y and p_z are both quadratic.

We now show that the function is minimized when $y = z$ as well as $x = y + z$. Consider the parametrization $(x, y, z) = (x, t - c, t + c)$, where $t \in [a, 1/2],\ c \in [0, \min\{t - a, b - t\}]$, and $x = 2t$ since $\alpha \cdot LP - ALG$ is a monotonically increasing function of x. Then

$$\begin{aligned}
&\alpha \cdot LP - ALG \\
&= \alpha \cdot ((1 - p_y p_z)(1-t) + (1 - p_x p_z)(t - c) + (1 - p_x p_y)(t + c)) \\
&\quad - (1 + 2p_x + p_y + p_z - p_y p_z - 2p_x p_z - 2p_x p_y) \\
&\quad + \alpha(1 - p_y) c.
\end{aligned}$$

Fixing the value of t, the sum of first 2 terms is a negative even quartic function of c, where the negativity is obtained from $-\alpha(1-t) + 1 \le -\alpha(1-b) + 1 < 0$. Thus, we can write the function as

$$P(t)c^4 + Q(t)c^2 + \alpha(1 - p_y)c + R(t),$$

where $P(t) = \frac{-\alpha(1-t)+1}{(b-a)^4} < 0$. Since $\alpha(1-p_y)c \ge 0$ if $c \ge 0$, it is sufficient to show that $Q \ge 0$ and $\sqrt{\frac{Q}{-P}} \ge \min\{t - a, b - t\}$ to prove that the function is minimized when $c = 0$.

Calculating the Q, this results in

$$\begin{aligned}
Q(t) &= \frac{2}{(b-a)^2} \cdot \left((\alpha(1-t) - 1) \left(\frac{t-a}{b-a} \right)^2 + \alpha p_x (t - 2a) - (1 - 2p_x) \right) \\
&= -P(t) \cdot (t-a)^2 + \frac{2}{(b-a)^2} \cdot (-1 + (\alpha t + 2 - 2\alpha a) p_x) \\
&\ge -P(t) \cdot (t-a)^2,
\end{aligned}$$

where the final inequality is derived from

$$\frac{1}{\alpha t + 2 - 2\alpha a} \le \frac{1}{2 - \alpha a} \approx 0.628 < 0.646 = f^\circ(2 \cdot 0.19) \le p_x.$$

Hence, $Q \ge -P(t) \cdot (t-a)^2 > 0$ and therefore $\sqrt{\frac{Q}{-P}} \ge t - a \ge \min\{t - a, b - t\}$, which implies that $\alpha \cdot LP - ALG$ is minimized at $c = 0$, i.e. at $(2t, t, t)$.

Finally, plugging $(x, y, z) = (2t, t, t)$,

$$\begin{aligned}
&\alpha \cdot LP - ALG \\
&= \alpha \cdot \big((1 - p_y^2)(1 - y) + 2(1 - p_x p_y) y \big) - \big(1 + 2p_x + 2p_y - p_y^2 - 4p_x p_y \big) \\
&= \big(\alpha(1 - p_y^2)(1 - t) + 2\alpha t - 1 - 2p_y + p_y^2 \big) - (2\alpha p_y t + 2 - 4p_y) p_x.
\end{aligned}$$

The top left region of Figure 1 shows the region $(2t, f^\circ(2t))$ where the formula above is negative when f^+ is given as equation 12. Since our proposed f° doesn't intersect the region, $\alpha \cdot LP - ALG \ge 0$ if $y = z$, and thus $\alpha \cdot LP - ALG \ge 0$ whenever $a \le y \le z \le b$.

E.5 $(\circ, +, -)$ Triangles

We would prove that the lower bound $\mathcal{C}'$ of $\mathcal{C}$, which is defined as

$$\begin{aligned}
&\alpha \cdot LP - ALG \\
&= \alpha \cdot \left((1-p_y z)\max\left\{\frac{1}{2}, 1-x, 1-y, 1-z\right\} + (1-p_x p_y)(1-z) + (1-p_x z)y\right) \\
&\quad - (1 - p_y z + p_x + z - 2p_x z + (1-p_x)(1-p_y)) \\
&\geq \alpha \cdot \left((1-p_y z)\max\left\{\frac{1}{2}, 1-y, 1-z\right\} + (1-p_x p_y)(1-z) + (1-p_x z)y\right) \\
&\quad - (1 - p_y z + p_x + z - 2p_x z + (1-p_x)(1-p_y)) \triangleq \mathcal{C}'
\end{aligned}$$

is nonnegative.

Since this is now affine with respect to p_x, this further enhances the argument Lemma 5 to check only on the endpoint.

1. $y \geq 1/2$, $z \leq 1/2$
   As in E.2 and E.3, $2 \cdot LP - ALG \geq 0$.
2. $y, z \geq 1/2$
   Similar with E.4, we can deduce a stronger argument than of Lemma 5: Examining the full formula of $\mathcal{C}'$,

$$\begin{aligned}
\mathcal{C}' &= \alpha \cdot \left((1-p_y z)\max\left\{\frac{1}{2}, 1-y, 1-z\right\} + (1-p_x p_y)(1-z) + (1-p_x z)y\right) \\
&\quad - (1 - p_y z + p_x + z - 2p_x z + (1-p_x)(1-p_y)) \\
&= (-\alpha y z - \alpha p_y (1-z) - p_y + 2z)p_x \\
&\quad + \alpha(1-z) + \alpha y + p_y z - z + p_y - 2 + \alpha(1-p_y z)\max\left\{\frac{1}{2}, 1-y, 1-z\right\}.
\end{aligned}$$

   The first term is a monotonically decreasing function of p_x since

$$-\alpha y z - \alpha p_y(1-z) - p_y + 2z = (\alpha(p_y - y) + 2)z - (\alpha+1)p_y$$

   is an increasing function of z since $p_y \geq y$ in the range,

$$-\alpha y z - \alpha p_y(1-z) - p_y + 2z \leq -\alpha y - p_y + 2 \leq -\frac{\alpha}{2} - f^+\left(\frac{1}{2}\right) + 2 \approx -0.016 < 0.$$

   Hence, the function is minimized when x is maximized. Setting $x = 1$,

$$\begin{aligned}
\mathcal{C}' &\geq \alpha \cdot \left((1-p_y z)\cdot\frac{1}{2} + (1-p_y)(1-z) + (1-z)y\right) - (2 - p_y z - z) \\
&= \left(-\alpha + 1 - \alpha y + \left(\frac{\alpha}{2}+1\right)p_y\right)z + \alpha y - \alpha p_y + \frac{3}{2}\alpha - 2.
\end{aligned}$$

   Since $y \geq 1/2$ and $p_y \leq 1$,

$$-\alpha + 1 - \alpha y + \left(\frac{\alpha}{2}+1\right)p_y \leq -\alpha + 2 < 0.$$

   Hence, the function is minimized when z is maximized. Setting $z = 1$,

$$\begin{aligned}
\mathcal{C}' &\geq \frac{\alpha}{2} - 1 + \left(-\frac{\alpha}{2}+1\right)p_y \\
&= \left(\frac{\alpha}{2} - 1\right)(1-p_y) \geq 0.
\end{aligned}$$

3. $y \leq z$, $y < 1/2$

$$\begin{aligned}
\mathcal{C}' &= \alpha \cdot ((1-p_y z)(1-y) + (1-p_x p_y)(1-z) + (1-p_x z)y) \\
&\quad - (1 - p_y z + p_x + z - 2p_x z + (1-p_x)(1-p_y)) \\
&= (-\alpha(1 + p_y - p_y y - p_x p_y + p_x y) + 2p_x + p_y - 1)z \\
&\quad + \alpha(2 - p_x p_y) - 2 + p_y - p_x p_y.
\end{aligned}$$

This is a decreasing function of z since

$$\begin{aligned}&-\alpha(1+p_y-p_yy-p_xp_y+p_xy)+2p_x+p_y-1\\&=(2+\alpha p_y-\alpha y)p_x-\alpha(1+p_y-p_yy)+p_y-1\end{aligned}$$

is an increasing function of p_x since $2+\alpha p_y-\alpha y\geq 2-\frac{\alpha}{2}\geq 0$,

$$\begin{aligned}&-\alpha(1+p_y-p_yy-p_xp_y+p_xy)+2p_x+p_y-1\\&\leq-\alpha(1-p_yy+y)+1+p_y\\&=-(1-p_y)(1+\alpha y)-\alpha+2\leq-\alpha+2<0.\end{aligned}$$

Hence, the function is minimized when z is maximized. Setting $z=1$,

$$\mathcal{C}'\geq\alpha\cdot((1-p_y)(1-y)+(1-p_x)y)-(3-2p_x-2p_y+p_xp_y)\,.$$

As mentioned above, it is sufficient to check on $x=1$ and $x=1-y$.

(a) $x=1$

$$\begin{aligned}\mathcal{C}'&=\alpha\cdot(1-p_y)(1-y)-(1-p_y)\\&=(1-p_y)(\alpha-1-\alpha y)\\&\geq(1-p_y)\left(\frac{\alpha}{2}-1\right)\geq 0.\end{aligned}$$

(b) $x=1-y$

Since $x=1-y>1/2$, $p_x=\frac{3}{10}x+\frac{7}{10}=1-\frac{3}{10}y$. Therefore,

$$\mathcal{C}'=\alpha\cdot\left((1-p_y)(1-y)+\frac{3}{10}y^2\right)-\left(1+\frac{3}{10}y\right)(2-p_y)+1.$$

If $y<0.19$,

$$\begin{aligned}\mathcal{C}'&=\alpha\cdot\left(1-y+\frac{3}{10}y^2\right)-2\left(1+\frac{3}{10}y\right)+1\\&=\frac{1}{200}(129y^2-550y+230)>0.65\geq 0.\end{aligned}$$

Otherwise, i.e. $0.19\leq y<1/2<0.5095$,

$$\mathcal{C}'\approx 0.74331+2.39737y-19.7409y^2+24.0007y^3,$$

whose minimum value in the region is approximately $7.236\times10^{-6}\geq 0$ in $y\approx 0.4788$.

4. $z<y<1/2$

$$\begin{aligned}\mathcal{C}'&=\alpha\cdot((1-p_yz)(1-z)+(1-p_xp_y)(1-z)+(1-p_xz)y)\\&\quad-(1-p_yz+p_x+z-2p_xz+(1-p_x)(1-p_y))\\&=\alpha p_yz^2+(-\alpha(2+p_y-p_xp_y+p_xy)+2p_x+p_y-1)z\\&\quad+\alpha(2+y-p_xp_y)-2+p_y-p_xp_y.\end{aligned}$$

Differentiating by z,

$$\begin{aligned}&2\alpha p_yz-\alpha(2+p_y-p_xp_y+p_xy)+2p_x+p_y-1\\&\leq\alpha p_y-\alpha(2+p_y-p_xp_y+p_xy)+2p_x+p_y-1\\&=(-2\alpha+2p_x+p_y)-\alpha((1-p_x)p_y+p_xy)-1\leq 0\end{aligned}$$

since $z<1/2$ and $-2\alpha+2p_x+p_y\leq-2\alpha+3=-1.3<0$. Therefore, the function is minimized when z is maximized.

Plugging $z=y$, this now reduce to Case 3 with $z=y$ due to the continuity of $\mathcal{C}'$.

E.6 $(\circ,-,-)$ Triangles

As in E.1, $2e.lp_w(u,v)\geq e.cost_w(u,v)$. For other two terms,

$$\begin{aligned}e.lp_u(v,w)+e.lp_v(w,u)&\geq(1-p_{uv}z)(1-y)+(1-p_{uv}y)(1-z)\\&\geq(1-p_{uv})(1-y)+(1-p_{uv})(1-z)\\&=(1-p_{uv})(1-p_{vw})+(1-p_{uv})(1-p_{wu})\\&=e.cost_v(w,u)+e.cost_u(v,w).\end{aligned}$$

Hence, $2\cdot LP-ALG\geq 0$.

This completes our case analysis.