APPENDIX A: PROOF OF THEOREM 1

Assumption 1. *There exist positive constants α_1, α_2, α_3 such that for any $x, y \in \mathbb{R}^n$,*

$$|h_1^{'}(x) - h_1^{'}(y)| \leq \alpha_1 \|x-y\|, \ \ |h_2^{'}(x) - h_2^{'}(y)| \leq \alpha_2 \|x-y\|,$$
$$|h_3^{'}(x) - h_3^{'}(y)| \leq \alpha_3 \|x-y\|.$$

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + u(t). \tag{1}$$

Under Assumption 1, for any initial value $X(0) = \xi \in \mathbb{R}^2$, if $\varrho_2 \neq 0$ and $\beta > 1$, then there a.s. exists a unique global solution $X(t)$ to system (1) on $t \in [0, \infty)$.

Proof. Under Assumption 1, then, we can calculate that

$$\begin{aligned}
&X^{\mathrm{T}}(t)f(X(t))\\
=&\phi(t)h_1^{'}(\phi(t)c)c + \phi(t)h_2^{'}(\phi(t)(\tilde{\theta}(t)+c))\tilde{\theta}(t)+\\
&\phi(t)h_2^{'}(\phi(t)(\tilde{\theta}(t)+c))c+\\
&\tilde{\theta}(t)h_3^{'}(\phi(t)(\tilde{\theta}(t)+c))\phi(t)\\
\leq&[(1+\frac{1}{2}\alpha_1^2)c^2 + 2c + \frac{1}{2}]|X|^2 + (\alpha_2^2 + \frac{1}{2}\alpha_3^2)|X|^4.
\end{aligned}$$

For any bounded initial value $X(0) \in \mathbb{R}^n$, there exists a unique maximal local strong solution $X(t)$ of system (1) on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show that the solution is actually global, we only need to prove that $\tau_e = \infty$ a.s. Let k_0 be a sufficiently large positive number such that $|X(0)| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : |X(t)| \geq k\}$$

with the traditional setting $\inf \emptyset = \infty$, where $\emptyset$ denotes the empty set. Clearly, τ_k is increasing as $k \to \infty$ and $\tau_k \to \tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$, then $\tau_e = \infty$ a.s., which implies the desired result. This is also equivalent to prove that, for any $t > 0$, $\mathbb{P}(\tau_k \leq t) \to 0$ as $k \to \infty$. To prove this statement, for any $p \in (0,1)$, define a C^2-function

$$V(x) = |X(t)|^p.$$

One can obtain that $X(t) \neq 0$ for all $0 \leq t \leq \tau_e$ a.s. Thus, one can apply the Itô formula to show that for any $t \in [0, \tau_e)$,

$$\begin{aligned}
\mathrm{d}V(X(t)) =& LV(X(t))\mathrm{d}t + p\varrho_1|X(t)|^p\mathrm{d}B_1(t)\\
&+ p\varrho_2|X(t)|^{\beta+p}\mathrm{d}B_2(t),
\end{aligned}$$

where LV is defined as

$$\begin{aligned}
LV(X) =& p|X|^{p-2}X^{\mathrm{T}}f(X(t)) + \frac{p(p-1)\varrho_1^2}{2}|X|^p\\
&+ \frac{p(p-1)\varrho_2^2}{2}|X|^{2\beta+p}
\end{aligned}$$

By Assumption 1, we therefore have

$$\begin{aligned}
LV(X) \leq& \frac{p(p-1)\varrho_2^2}{2}|X|^{2\beta+p} + ((1+\frac{1}{2}\alpha_1^2)c^2 + 2c + \frac{1}{2})\\
&p|X|^{\alpha+p} + p\left(\frac{(p-1)\varrho_1^2}{2} + (\alpha_2^2 + \frac{1}{2}\alpha_3^2)\right)|X|^p.
\end{aligned}$$

Noting that $p \in (0,1)$ and $\beta > 1$ and $\varrho_2 \neq 0$, by the boundedness of polynomial functions, there exists a positive constant $\bar{H}$ such that $LV(x) \leq \bar{H}$. We therefore have

$$\begin{aligned}
\mathbb{E}V(X(t \wedge \tau_k)) &\leq \mathbb{E}|\xi|^p + \mathbb{E}\int_0^{t\wedge\tau_k} LV(X(s))\mathrm{d}s\\
&\leq \mathbb{E}|\xi|^p + \bar{H}_t\\
&=: \bar{H}_t,
\end{aligned}$$

where $\bar{H}_t$ is independent of k. By the definition of τ_k, $|X(\tau_k)| = k$, so

$$\begin{aligned}\mathbb{P}(\tau_k \le t)k^p &\le \mathbb{P}(\tau_k \le t)V(X(\tau_k))] \\ &\le \mathbb{E}[l_{\{\tau_k \le t\}} V(X(t \wedge \tau_k))] \\ &\le \mathbb{E}V(X(t \wedge \tau_k)) \\ &\le \bar{H}_t,\end{aligned}$$

which implies that

$$\limsup_{k\to\infty} \mathbb{P}(\tau_k \le t) \le \lim_{k\to\infty} \frac{\bar{H}_t}{k^p} = 0,$$

as required. □

Appendix B: Proof of Theorem 2

Let Assumption 1 hold. Assume that $\varrho_2 \neq 0$ and $\beta > 1$. If

$$\frac{\varrho_1^2}{2} - \varphi > 0,$$

where

$$\varphi = \max_{x\ge 0}\left\{ -\frac{\varrho_2^2}{2}x^{2\beta} + (\alpha_2^2 + \frac{1}{2}\alpha_3^2)x^2 + [(1+\frac{1}{2}\alpha_1^2)c^2 + 2c + \frac{1}{2}]\right\}, \tag{2}$$

then for any $X(0) = \xi$ with sufficiently small constant $\epsilon \in (0, \varrho_1^2/2 - \varphi)$, the global solution $X(t)$ of system (1) has the property that

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{t} \le -\left(\frac{\varrho_1^2}{2} - \varphi\right) + \epsilon, \quad a.s.$$

that is, the solution of system (1) is a.s. exponentially stable.

Proof. Applying Itô formula to $\log|X(t)|$ yields

$$\begin{aligned}\log|X(t)| = \log|X(0)| &+ \int_0^t \Bigg[|X(t)|^{-2}X^{\mathrm{T}}(s)f(X(s)) \\ &- \frac{\varrho_2^2}{2}|X(s)|^{2\beta} - \frac{\varrho_1^2}{2}\Bigg]\mathrm{d}s + \int_0^t \varrho_1 \mathrm{d}B_1(t) \\ &+ \varrho_2 \int_0^t |X(s)|^{\beta}\mathrm{d}B_2(s).\end{aligned}$$

Letting $M(t) = \varrho_2 \int_0^t |X(s)|^{\beta}\mathrm{d}B_2(s)$, clearly $M(t)$ is a continuous local martingale with the quadratic variation

$$< M(t), M(t) >= \varrho_2^2 \int_0^t |X(s)|^{2\beta}\mathrm{d}s.$$

For any $\varepsilon \in (0,1)$, choose $\theta > 0$ such that $\theta\varepsilon > 1$. Then for each integer $m > 0$, the exponential martingale inequality gives

$$\mathbb{P}\left\{ \sup_{1\le t\le m}\left[M(t) - \frac{\varepsilon\varrho_2^2}{2}\int_0^t |X(s)|^{2\beta}\mathrm{d}s\right] \ge \theta\varepsilon\log m\right\} \le \frac{1}{m^{\theta\varepsilon}}.$$

Since $\sum_{m=1}^{\infty} m^{-\theta\varepsilon} < \infty$, by the well-known Borel-Cantelli lemma, there exists an $\bar{\Omega}_0 \subseteq \Omega$ with $\mathbb{P}(\bar{\Omega}_0) = 1$ such that for any $\omega \in \bar{\Omega}_0$, there exists an integer $\bar{m}(\omega)$, when $m > \bar{m}(\omega)$, and $m - 1 \le t \le m$,

$$M(t) \le \frac{\varepsilon\varrho_2^2}{2}\int_0^t |X(s)|^{2\beta}\mathrm{d}s + \theta\varepsilon\log(t+1).$$

This, together with Assumption 1, yields

$$\log|X(t)| \leq \log|\xi| + \int_0^t \Bigg[-\frac{\varrho_2^2(1-\varepsilon)}{2}|X(s)|^{2\beta} + (1+(\frac{1}{2}\alpha_1^2)c^2+2c+\frac{1}{2})|X(s)|^{\alpha} + (\alpha_2^2+\frac{1}{2}\alpha_3^2) - \frac{\varrho_1^2}{2}\Bigg]\mathrm{d}s + \int_0^t \varrho_1 \mathrm{d}B_1(t) + \theta\varepsilon\log(t+1).$$

Letting ϵ be sufficiently small, by the definition of φ in (**??**), for sufficiently small $\epsilon \in (0, \varrho_1^2/2-\varphi)$, we have

$$\log|X(t)| \leq \log|\xi| - \Bigg[(\frac{\varrho_1^2}{2}-\varphi)-\epsilon\Bigg]t + \int_0^t \varrho_1\mathrm{d}B_1(t) + \theta\varepsilon\log(t+1).$$

Applying the strong law of large number, we therefore have

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{t} \leq -(\frac{\varrho_1^2}{2}-\varphi)+\epsilon \quad a.s.$$

□