THE PRIMAL-DUAL LEARNING AUGMENTED ALGORITHM FOR PARKING PERMIT PROBLEM WITH THREE PERMIT TYPES

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ABSTRACT

We consider a parking permit problem with three permit types. We propose a randomized primal-dual algorithm, and a learning-augmented modification for it. We prove consistency and robustness bounds for this modification.

1 INTRODUCTION

Online algorithms make decisions using only the information available at the moment, without being able to account for information revealed later. Because of this uncertainty the algorithms are usually analysed for the worst-case scenario, resulting in pessimistic quality metrics.

In the last decade studies have used machine-learning based predictions to improve online algorithms' quality. This new approach is made possible with the latest achievements of artificial intelligence and machine learning methods, enabling predictions of input data for optimization problems. This additional information can improve online algorithms, as is evident with semi-online algorithms. However, semi-online algorithms assume total correctness of this additional data. This assumption often cannot be achieved in practice. On the opposite, predictions are rarely guaranteed to be correct. Thus, prediction accuracy should not be relied upon when designing prediction-augmented algorithms. The algorithms are instead required to be consistent and robust, i.e. - if the prediction is close to accurate, the found solution must be "close" to either the best offline solution or to the optimal solution, - if the prediction is arbitrarily inaccurate, found solution should be "close" to the classic online solution.

The difficulty in designing such an algorithm arises in finding a balance between these qualities. Following the prediction blindly can lead to a bad solution. On the other hand, if the algorithm doesn't trust the prediction at all, it cannot benefit from a good prediction. This approach was first described in Lykouris & Vassilvitskii (2021) and Medina & Vassilvitskii (2017), where such algorithms were named "learning-augmented" or "prediction-augmented". Similar algorithms were later developed for other combinatorial optimization problems, such as the ski rental problem Angelopoulos et al. (2020); Gollapudi & Panigrahi (2019); Kumar et al. (2018); Wang et al. (2020), scheduling problems Kumar et al. (2018); Bamas et al. (2020); Bampis et al. (2022); Evripidis et al. (2023); Lindermayr & Megow (2022); Wei & Zhang (2020) and many others.

Our paper considers the parking permit problem, which was first proposed in Meyerson (2005), and is a generalization of the ski rental problem. We propose a randomized primal-dual algorithm for the restricted version of the problem with three permit types, and present a bound for its' competitive ratio. We also propose parametrized learning-augmented algorithms with two prediction types, and show the consistency and robustness bounds.

1.1 PROBLEM DEFINITION

We consider the following problem, denoted as $\mathcal{P}_3$. There are three parking permit types. For permit type $k, k = 1..3$ its cost C_k and duration D_k are known. For each k the entire time interval of the problem consists of disjoint intervals of length D_k. Permit is only valid during the time interval it was purchased on. W.l.o.g. we suppose that

- $C_1 = 1, D_1 = 1,$

- $C_2 = B, D_2 = d$, we designate this time interval as a "week",
- $C_3 = A, D_3 = nd$, we designate this time interval as a "year".

The schedule consists of days, some of which are marked as rainy. The days are revealed to the algorithm one at a time. If the new day is rainy and no purchased permit covers it, algorithm must choose, which type of permit to purchase. Purchased permits cannot be refunded. Algorithm has to find the set of permits with the minimum total cost.

An instance I of the problem $\mathcal{P}_3$ is defined by concrete values of A, B, d, n and the schedule of rainy days. Denote the cost of algorithm's solution with $ALG(I)$ and the cost of optimal solution with $OPT(I)$ for the instance I. Value $R(A) = \max\limits_{I}(\frac{ALG(I)}{OPT(I)})$ is called the competitive ratio of the algorithm.

1.2 Prediction model

In our work,we consider a new prediction model. At the start the algorithm is advised, whether it should buy the year-long permit. At the start of each week the algorithm is advised, whether it should buy the permit covering this week. We denote the entire set of these advice as the prediction Π. Thus, if the prediction is perfectly accurate, we can compute the cost of optimal solution of instance I of the problem $\mathcal{P}_3$. As this cost depends on the prediction, we denote it as $c(\Pi)$.

If for any instance I and any prediction Π

$$\frac{ALG(I, \Pi)}{OPT(I)} \leq \beta,$$

we say that the algorithm is β-robust.

If for any instance I and any prediction Π

$$\frac{ALG(I, \Pi)}{c(\Pi)} \leq \sigma,$$

we say that the algorithm is σ-consistent.

The consistency and robustness have the following intuitive interpretation. A consistent algorithm finds an approximation for the solution of the predicted schedule, and if the schedule was predicted correctly, the algorithm approximates the optimal solution well. Robustness is independent of the prediction, and guarantees the algorithm will approximate the optimal solution regardless of prediction errors.

1.3 Related work

The ski rental problem requires you to decide whether to pay a small fee to rent skis on the current day or buy skis and ski on them on subsequent days. Unfortunately, you don't know how long you'll be skiing. It depends on the weather, health and a host of other circumstances. Thus, you are faced with the simplest online problem, first mentioned by Rudolph in the context of the work on competitive snoopy caching. By applying a simple rule: buy skis when the rental cost reaches the cost of the skis, we guarantee a 2-approximation solution in the worst case. This rule represents the best possible deterministic strategy. A randomized $\frac{e}{e-1}$-competitive algorithm was proposed in Karlin et al. (1994).

Due to the simplicity and universality of the ski rental problem, various formulations with predictions are intensively studied for it Bamas et al. (2020); Gollapudi & Panigrahi (2019); Kodialam (2019); Kumar et al. (2018); Wang et al. (2020). Note that in our work we use an approach based on the construction of a primal-dual algorithm proposed for online problems with predictions in Bamas et al. (2020).

The ski rental problem allows for many direct generalizations: the multi-shop ski rental problem et al. (2023), the parking permit problem Meyerson (2005), the Bahncard problem Fleicher (2001). In addition, the issue of renting or buying is key for such complex online problems as snoopy cashing

Karlin et al. (1988; 1994), TCP acknowledgmentBuchbinder et al. (2007), total completion time scheduling Seiden (2000) and others.

In this paper, we consider the parking permit problem proposed by Meyerson Meyerson (2005). In his paper, Meyerson considered the problem with k types of permits and presented k-competitive deterministic and $O(\log k)$-competitive randomized online algorithms. He also proved that an arbitrary deterministic algorithm has a competitive ratio of at least $k/3$, and an arbitrary randomized algorithm has a competitive ratio of at least $(\log k)/2$. Note that these results do not provide a reasonable lower bound on the competitive ratio for small values of k. For $k = 3$, Kharchenko & Kononov (2024) showed that no deterministic algorithm can obtain a competitive ratio of less than 3 and presented a parameterized deterministic prediction-augmented algorithm.

2 A PRIMAL-DUAL ONLINE ALGORITHM

In this section we present a randomized algorithm for $\mathcal{P}$. The algorithm is based on the primal-dual approach. For each day the algorithm computes fractional values x, y_i, z_{ij}, that correspond to the probabilities of purchasing respective permit types on this day. Then a random value q is obtained from the uniform distribution on $[0, 1]$. If q is in $[0, x]$, the year-long permit is purchased. If q is in $(x, x + y_i]$, the week-long permit is purchased. Else the day-long permit is purchased. Buchbinder et al. (2007) shows that the expected cost of the solution constructed with this procedure is equal to the cost of the fractional solution.

To apply the primal-dual method we formulate the parking permit problem with 3 permit types as a linear programming problem. Let D denote the set of pairs (i, j), where j is the number of a rainy day in the i-th week.

The linear programming problem is formalized as follows:

$$Ax + B\sum_{i=1}^{n} y_i + \sum_{(i,j)\in D} z_{ij} \to \min \tag{1}$$

$$x + y_i + z_{ij} \geq 1 \qquad \forall (i,j) \in D \tag{2}$$

$$x \geq 0, \quad y_i \geq 0, \quad z_{ij} \geq 0 \tag{3}$$

The dual problem is formalized as follows

$$\sum_{(i,j)\in D} \xi_{ij} \to \max \tag{4}$$

$$\sum_{(i,j)\in D} \xi_{ij} \leq A \tag{5}$$

$$\sum_{j|(i,j)\in D} \xi_{ij} \leq B \qquad \forall i \tag{6}$$

$$0 \leq \xi_{ij} \leq 1 \qquad \forall (i,j) \in D \tag{7}$$

Denote $e_T(\alpha) = (1 + \frac{1}{T})^{\alpha T}$. Note that when T increases, $e_T(\alpha)$ appoaches e^{α} from below.

Algorithm 1 is a primal-dual algorithm, which is a modification of the algorithm for the ski rental problem presented in Buchbinder et al. (2007). The algorithm starts with a zero solution to the primal problem. Each new rainy day $(i, j) \in D$ adds a new constraint $x + y_i + z_{ij} \geq 1$. To satisfy this constraint, the algorithm updates the primal and dual variables in such a way as to keep the ratio between the objective function values of the primal and dual problems as small as possible.

Theorem 1. *Algorithm 1 is* $1 + \frac{1}{(e_A(1)-1)} + \frac{1}{(e_B(1)-1)}$*-competitive.*

Proof. On each update the primal solution is feasible by definition of z_{ij}. Suppose the dual solution is not feasible. We will show that the restrictions (5), (6) are satisfied. Suppose that

$$\sum_{(i,j)\in D} \xi_{ij} > A.$$

Algorithm 1 The primal-dual online algorithm

$x \leftarrow 0, \quad y_i \leftarrow 0, \quad z_{ij} \leftarrow 0$
if schedule is empty **then**
  return x, y_i, z_{ij} as the fractional solution.
end if
Reveal the next rainy day $(i, j) \in D$.
if $x < 1$ and $\quad y_i < 1 - x$: **then** (denote the following procedure as "updating x, y_i, z_{ij}")
  $x \leftarrow (1 + \frac{1}{A})x + \frac{1}{(e_A(1)-1)A}$
  $\hat{y_i} \leftarrow (1 + \frac{1}{B})y_i + \frac{1}{(e_B(1)-1)B}$
  $y_i \leftarrow \hat{y_i}(1 - x)$
  $z_{ij} \leftarrow 1 - x - y_i$
  $\xi_{ij} \leftarrow 1$
end if

It is only possible after at least $A + 1$ updates of x. Note that after A updates

$$x = \frac{1}{(e_A(1) - 1)A} \sum_{k=0}^{A-1} \left(1 + \frac{1}{A}\right)^k = \frac{1}{(e_A(1) - 1)A} \cdot \frac{(1 + \frac{1}{A})^A - 1}{(1 + \frac{1}{A}) - 1} = 1.$$

Thus, update number $A + 1$ is not performed, as x is not less than 1, which leads to a contradiction. Similarly, for any $i = 1, \ldots, n$,

$$\sum_{j|(i,j)\in D} \xi_{ij} \leq B.$$

Thus, the dual solution is feasible.

Denote the value of target function for the primal problem as $\mathfrak{P}$, for the dual problem – as $\mathfrak{D}$. At the start of the algorithm both of these values are zero. Suppose the algorithm has just processed an arbirary rainy day. Let $\Delta\mathfrak{P}, \Delta\mathfrak{D}$ be the increments of target function for primal and dual problem respectively. We will find a bound for $\frac{\Delta\mathfrak{P}}{\Delta\mathfrak{D}}$.

$$\begin{aligned} \frac{\Delta\mathfrak{P}}{\Delta\mathfrak{D}} \leq \Delta\mathfrak{P} = A\left(\frac{x}{A} + \frac{1}{(e_A(1) - 1)A}\right) + B\left(\frac{y_i}{B} + \frac{1}{(e_B(1) - 1)B}\right) + z_{ij} = \\ = x + y_i + z_{ij} + \frac{1}{(e_A(1) - 1)} + \frac{1}{(e_B(1) - 1)} = \\ = 1 + \frac{1}{(e_A(1) - 1)} + \frac{1}{(e_B(1) - 1)}. \end{aligned} \tag{8}$$

Since the cost of a feasible solution of the dual problem is a lower bound for the cost of the optimal solution of the primal problem, we get $R \leq 1 + \frac{1}{(e_A(1)-1)} + \frac{1}{(e_B(1)-1)}$. □

Remark 1. *Note that for $A \geq 4$ and $B \geq 2$ we get $R < 2.5$,, and by increasing A and B the value of R tends to $\frac{e+1}{e-1} \approx 2.16$. If $A \leq 4$ and $B \leq 2$ a deterministic 2-competitive algorithm can be derived easily.*

3 Prediction-augmented primal-dual algorithm

Suppose that the algorithm receives binary values P_3, P_2^i and real-valued parameters $\lambda, \mu \in (0, 1]$. W.l.o.g. we assume that λA, λB, $\frac{A}{\lambda}$ and $\frac{B}{\mu}$ are integer numbers.

If $P_3 = 1$, the prediction expects the year to be rainy, and the algorithm will increment x faster. In the opposite case x is incremented slower. The week permit variables y_i similarly depend on the prediction. The values λ and μ reflect the degree of mistrust regarding the yearly and the weekly predictions correspondingly. Raising these values makes the algorithm more robust to prediction mistakes, but makes less use of the provided information.

We begin the algorithm description by introducing two procedures. The first procedure is used if the year is predicted to be rainy, the other is used if the prediction doesn't recommend buying year-long permit.

Additionally, the first procedure uses counters ϕ and ν_i, $i = 1, \dots, n$ to keep track of rainy days for the year and the current week respectively.

Algorithm 2 "Rainy year" procedure

1: $x \leftarrow 0, \quad y_i \leftarrow 0, \quad \hat{y}_i \leftarrow 0, \quad z_{ij} \leftarrow 0, \quad \xi_{ij} \leftarrow 0 \quad \phi \leftarrow 0, \quad \nu_i \leftarrow 0 \quad \forall i, j$
2: **if** schedule is empty **then**
3:     return x, y_i, z_{ij} as fractional solution.
4: **end if**
5: Reveal new rainy day $(i, j) \in D$. ▷ Denote the following steps as "variables update"
6: **if** $P_2^i = 1$ **then**
7:     **if** $x < 1$ **and** $\nu_i < B$ **then**
8:         $x \leftarrow (1 + \frac{1}{A})x + \frac{1}{(e_A(\lambda)-1)A}$
9:         $\xi_{ij} \leftarrow 1$
10:         $\phi \leftarrow \phi + 1$
11:         $\nu_i \leftarrow \nu_i + 1$
12:         **if** $y_i < 1 - x$ **and** $\lambda A - \phi > B - \nu_i$ **then**
13:             $\hat{y}_i \leftarrow (1 + \frac{1}{B})\hat{y}_i + \frac{1}{(e_B(\mu)-1)B}$
14:         **end if**
15:     **end if**
16: **end if**
17: **if** $P_2^i = 0$ **and** $x < 1$ **then**
18:     **if** $y_i < 1 - x$ **then**
19:         $\hat{y}_i \leftarrow (1 + \frac{1}{B})\hat{y}_i + \frac{1}{(e_B(\frac{1}{\mu})-1)B}$
20:         $\xi_{ij} \leftarrow \mu$
21:     **end if**
22: **end if**
23: $y_i \leftarrow \hat{y}_i(1 - x)$
24: $z_{ij} \leftarrow 1 - x - y_i$
25: Go to step 2.

Note that the "Rainy day" procedure increases year-long purchase probability x on each rainy week, even if $x + y_i = 1$. This can be interpreted as confirming the year-long prediction.

The $\lambda A - \phi \geq B - \nu_i$ test in line 13 of the "Rainy day" procedure allows to skip increasing $\hat{y}_i$(and consequently y_i) if x reaches 1 before $\hat{y}_i$ reaches 1. Intuitively, if the algorithm is "set on" buying the year-long permit, it stops considering weekly permits, keeping fractional cost low.

Let $\beta = \max\{1 + \frac{1}{e_A(\lambda)-1} + \frac{1}{e_B(\mu)-1}, \frac{1}{\mu} \cdot [1 + \frac{1}{(e_A(\lambda)-1)} + \frac{1}{(e_B(1/\mu)-1)}],$
$\frac{1}{\lambda} \cdot [1 + \frac{1}{(e_A(1/\lambda)-1)} + \frac{1}{(e_B(\mu)-1)}], \frac{1}{\lambda\mu} \cdot [1 + \frac{1}{(e_A(1/\lambda)-1)} + \frac{1}{(e_B(1/\mu)-1)}]\}$.

Let $\gamma = \max\{\frac{\mu}{1-e_B(-\mu)} + \frac{\lambda}{1-e_A(-\lambda)}, \lambda(1 + \frac{1}{e_A(\lambda)-1}) + \frac{\lambda\mu}{1-e_B(-\mu)}\}$.

Theorem 2. *The described algorithm obtains a fractional solution with cost not greater than* $\min\{\beta \cdot OPT(I), \gamma \cdot c(\Pi)\}$. *There* $OPT(I)$ *denotes the cost of optimal solution for the given problem instance,* $c(\Pi)$ *denotes the cost of predicted solution.*

Proof. We will show that both the primal and dual solutions are feasible. The primal solution is feasible by definition of z_{ij}. Suppose that the dual solution is not feasible. Consider possible restriction violations.

Algorithm 3 "Clear year" procedure

$x \leftarrow 0, \quad y_i \leftarrow 0, \quad z_{ij} \leftarrow 0, \quad \hat{y}_i \leftarrow 0, \quad \xi_{ij} \leftarrow 0 \quad \forall i, j$
if schedule is empty **then**
    return x, y_i, z_{ij} as fractional solution.
end if
Reveal new rainy day $(i, j) \in D$. ▷ Denote the following steps as "variables update"
if $x < 1$ **and** $y_i < 1 - x$ **then:**
    $x \leftarrow (1 + \frac{1}{A})x + \frac{1}{(e_A(\frac{1}{\lambda})-1)A}$
    $\xi_{ij} \leftarrow \lambda$
    if $P_2^i = 1$ **then**
        $\hat{y}_i \leftarrow (1 + \frac{1}{B})\hat{y}_i + \frac{1}{(e_B(\mu)-1)B}$
    else
        $\hat{y}_i \leftarrow (1 + \frac{1}{B})\hat{y}_i + \frac{1}{(e_B(\frac{1}{\mu})-1)B}$
        $\xi_{ij} \leftarrow \mu\xi_{ij}$
    end if
    $y_i \leftarrow \hat{y}_i(1 - x)$
end if
$z_{ij} \leftarrow 1 - x - y_i$
Go to step 2.

Algorithm 4 Primal-dual prediction-augmented algorithm

if $P_3 = 1$ **then**
    Execute "Rainy year" procedure
else
    Execute "Dry year" procedure
end if

1. Suppose that $P_3 = 1$. After λA updates of x

$$x = \frac{1}{(e_A(\lambda) - 1)A} \sum_{k=0}^{\lambda A - 1} \left(1 + \frac{1}{A}\right)^k =$$
$$= \frac{1}{(e_A(\lambda) - 1)A} \cdot \frac{(1 + \frac{1}{A})^{\lambda A} - 1}{(1 + \frac{1}{A}) - 1} = 1. \quad (9)$$

If $x = 1$ the conditions in lines 7 and 17 of the "Rainy day" procedure are not satisfied and there are no more variable updates. Each update of x increments the sum $\sum\limits_{(i,j)\in D} \xi_{ij}$ at most by 1. Thus, $\sum_{(i,j)\in D} \xi_{ij} \leq A$.

Suppose that $P_3 = 0$. By reasoning similar to (9), x reaches 1 after $\frac{A}{\lambda}$ updates. Each update increases $\sum\limits_{(i,j)\in D} \xi_{ij}$ by at most λ. Thus $\sum_{(i,j)\in D} \xi_{ij} \leq A$. and the restriction (5) of the dual problem is satisfied.

2. Suppose that for some i the condition (6) is violated, i.e.

$$\sum_{j|(i,j)\in D} \xi_{ij} > B.$$

Suppose that $P_2^i = 1$. If the year is rainy, then, due to the condition $\nu_i < B$ on line 7 of the "Rainy day" procedure, at most B variables ξ_{ij} reach 1 and $\sum_{j|(i,j)\in D} \xi_{ij} \leq B$.

Suppose that $P_2^i = 1$ and $P_3 = 0$. After μB updates of y_i we get

$$\hat{y}_i = \frac{1}{(e_B(\mu)-1)B} \sum_{k=0}^{\mu B-1} \left(1+\frac{1}{B}\right)^k = \\ = \frac{1}{(e_B(\mu)-1)B} \cdot \frac{(1+\frac{1}{B})^{\mu B}-1}{(1+\frac{1}{B})-1} = 1. \quad (10)$$

Thus, after μB updates $y_i = \hat{y}_i(1-x) = 1-x$ and there are no more variable updates. Each update increases $\sum_{j|(i,j)\in D} \xi_{ij}$ by at most λ, $\lambda \leq 1$, thus the sum is less than or equal to B.

Suppose that $P_2^i = 0$. With reasoning similar to (10), we get that y_i reaches $1-x$ after $\frac{B}{\mu}$ updates. Each update increases $\sum_{j|(i,j)\in D} \xi_{ij}$ by at most μ. Thus, $\sum_{j|(i,j)\in D} \xi_{ij} \leq B$ and the restriction (6) of the dual problem is satisfied.

Thus, the dual solution is feasible. Similar to 1 denote primal and dual increments by $\Delta\mathfrak{P}, \Delta\mathfrak{D}$. Consider possible increment combinations.

1. $P_3 = 1, P_2^i = 1$

$$\frac{\Delta\mathfrak{P}}{\Delta\mathfrak{D}} \leq \Delta\mathfrak{P} = A\left(\frac{x}{A} + \frac{1}{(e_A(\lambda)-1)A}\right) + B\left(\frac{y_i}{B} + \frac{1}{(e_B(\mu)-1)B}\right) + z_{ij} = \\ = 1 + \frac{1}{e_A(\lambda)-1} + \frac{1}{e_B(\mu)-1} = \beta_1 \quad (11)$$

2. $P_3 = 1, P_2^i = 0$

$$\frac{\Delta\mathfrak{P}}{\Delta\mathfrak{D}} \leq \frac{\Delta\mathfrak{P}}{\mu} \leq \frac{1}{\mu} \cdot \left[A\left(\frac{x}{A} + \frac{1}{(e_A(\lambda)-1)A}\right) + B\left(\frac{y_i}{B} + \frac{1}{(e_B(1/\mu)-1)B}\right) + z_{ij}\right] = \\ = \frac{1}{\mu} \cdot \left[1 + \frac{1}{(e_A(\lambda)-1)} + \frac{1}{(e_B(1/\mu)-1)}\right] = \beta_2 \quad (12)$$

3. $P_3 = 0, P_2^i = 1$

$$\frac{\Delta\mathfrak{P}}{\Delta\mathfrak{D}} \leq \frac{\Delta\mathfrak{P}}{\lambda} \leq \frac{1}{\lambda} \cdot \left[A\left(\frac{x}{A} + \frac{1}{(e_A(1/\lambda)-1)A}\right) + B\left(\frac{y_i}{B} + \frac{1}{(e_B(\mu)-1)B}\right) + z_{ij}\right] = \\ = \frac{1}{\lambda} \cdot \left[1 + \frac{1}{(e_A(1/\lambda)-1)} + \frac{1}{(e_B(\mu)-1)}\right] = \beta_3 \quad (13)$$

4. $P_3 = 0, P_2^i = 0$

$$\frac{\Delta\mathfrak{P}}{\Delta\mathfrak{D}} \leq \frac{\Delta\mathfrak{P}}{\lambda\mu} \leq \frac{1}{\lambda\mu} \cdot \left[A\left(\frac{x}{A} + \frac{1}{(e_A(1/\lambda)-1)A}\right) + B\left(\frac{y_i}{B} + \frac{1}{(e_B(1/\mu)-1)B}\right) + z_{ij}\right] = \\ = \frac{1}{\lambda\mu} \cdot \left[1 + \frac{1}{(e_A(1/\lambda)-1)} + \frac{1}{(e_B(1/\mu)-1)}\right] = \beta_4 \quad (14)$$

For each algorithm defined by λ and μ the robustness is bounded by the maximum of $\beta_i, i = 1..4$. The following image displays the maximum of the values for each λ, μ pair.

We will now present the consistency bound. We suppose that the prediction is correct for the schedule. Suppose that $P_3 = 1$. Denote the number of rainy weeks before buying year-long permit as r_0, the number of rainy days before buying year-long permit and not included in any rainy week as d_0. Since x reaches or exceeds 1 after λA updates, and each rainy week causes exactly B updates,

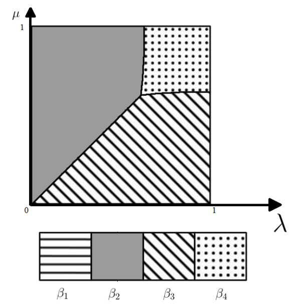

Figure 1: Robustness bound for given λ, μ

$\lambda A = r_0 B + d_0$. Each rainy week i caused μB updates of y_i. Thus, the cost added by updates of y_i is equal to

$$\mu r_0 B \left(1 + \frac{1}{e_B(\mu) - 1}\right) + d_0 \left(1 + \frac{1}{e_B(\frac{1}{\mu}) - 1}\right) = r_0 B \left(\frac{\mu e_B(\mu)}{e_B(\mu) - 1}\right) + d_0 \left(\frac{e_B(\frac{1}{\mu})}{e_B(\frac{1}{\mu}) - 1}\right)$$

Bamas et al. (2020) shows, that $1 + \frac{1}{e_B(1/\mu)-1} \leq \frac{\mu}{1-e_B(-\mu)}$. Note also that $\frac{\mu e_B(\mu)}{e_B(\mu)-1} < \frac{e_B(\frac{1}{\mu})}{e_B(\frac{1}{\mu})-1}$ and $\mu < 1$. Thus

$$r_0 B \left(\frac{\mu e_B(\mu)}{e_B(\mu) - 1}\right) + d_0 \left(\frac{e_B(\frac{1}{\mu})}{e_B(\frac{1}{\mu}) - 1}\right) \leq (r_0 B + d_0) \frac{\mu}{1 - e_B(-\mu)} \leq \frac{\mu \lambda A}{1 - e_B(-\mu)}.$$

Now we consider the cost added by the updates of x. As noted before, λA updates are performed, each increasing the cost by $1 + \frac{1}{e_A(\lambda)-1}$.

Since $P_3 = 1$ implies the optimal cost of A, we get

$$R \leq \lambda \left(1 + \frac{1}{e_A(\lambda) - 1}\right) + \frac{\lambda \mu}{1 - e_B(-\mu)}.$$

Now suppose that $P_3 = 0$. Similarly let r_0 be the number of rainy weeks, d_0 the number of rainy days during clear weeks. The optimal solution costs $r_0 B + d_0$. Similarly to the previous step, the cost added by updates of y_i is not greater than $(r_0 B + d_0)\frac{\mu}{1-e_B(-\mu)}$. Variable x is updated $r_0 B + d_0$ times, each time increasing the cost by $1 + \frac{1}{e_A(\frac{1}{\lambda})-1} \leq \frac{\lambda}{1-e_A(-\lambda)}$. Thus,

$$R \leq \frac{\mu}{1 - e_B(-\mu)} + \frac{\lambda}{1 - e_A(-\lambda)}$$

The consistency bound of the algorithm is not greater than $\max\{\frac{\mu}{1-e_B(-\mu)} + \frac{\lambda}{1-e_A(-\lambda)}, \lambda(1 + \frac{1}{e_A(\lambda)-1}) + \frac{\lambda\mu}{1-e_B(-\mu)}\}$. □

4 Conclusions

In this work, we presented a randomized primal-dual algorithm, and a randomized learning-augmented primal-dual algorithm for the parking permit problem with three types of permits. We showed that the randomized primal-dual algorithm is a R-competitive, where $R < 2.5$ and tends to $\frac{e+1}{e-1}$ as the costs A and B increase. We also obtained consistency and robust bounds for the randomized learning-augmented primal-dual algorithm.

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