Sample complexity of Schrödinger potential estimation

Anonymous Author(s)
Affiliation
Address
email

Abstract

We address the problem of Schrödinger potential estimation, which plays a crucial role in modern generative modelling approaches based on Schrodinger bridges and stochastic optimal control for SDEs. Given a simple prior diffusion process, these methods search for a path between two given distributions ρ_0 and ρ_T requiring minimal efforts. The optimal drift in this case can be expressed through a Schrödinger potential. In the present paper, we study generalization ability of an empirical Kullback-Leibler (KL) risk minimizer over a class of admissible log-potentials aimed at fitting the marginal distribution at time T. Under reasonable assumptions on the target distribution ρ_T and the prior process, we derive a non-asymptotic high-probability upper bound on the KL-divergence between ρ_T and the terminal density corresponding to the estimated log-potential. In particular, we show that the excess KL-risk may decrease as fast as $\mathcal{O}(\log n/n)$ when the sample size n tends to infinity even if both ρ_0 and ρ_T have unbounded supports.

1 Introduction

The Schrödinger Bridge problem (SBP) originates from a question posed by Erwin Schrödinger in 1932 [Schrödinger, 1932], seeking the most likely evolution of a probability distribution between two given endpoint distributions while minimizing relative entropy with respect to a prior stochastic process. This problem has deep connections with optimal transport [Leonard, 2014] and stochastic control [Dai Pra, 1991]. In its simplest continuous-time form, one aims to construct a so-called *Schrödinger Markov process* whose joint begin-end distribution $\pi(\mathrm{d}x, \mathrm{d}z)$ has the representation

$$\pi(\mathrm{d}x, \mathrm{d}z) = \mathsf{Q}(z, T \mid x, 0)\, \nu_0(\mathrm{d}x)\, \nu_T(\mathrm{d}z), \tag{1}$$

where $\mathsf{Q}(z, T \mid x, 0)$ is the transition kernel of a reference Markov process, and ν_0, ν_T are unknown "boundary potentials" to be determined. The desired marginals $\pi(\mathrm{d}x, \mathbb{R}^d)$ and $\pi(\mathbb{R}^d, \mathrm{d}z)$ are given, and one seeks ν_0 and ν_T that reproduce these marginals. In the rest of the paper, we assume that both $\pi(\mathrm{d}x, \mathbb{R}^d)$ and $\pi(\mathbb{R}^d, \mathrm{d}z)$ are absolutely continuous with respect to the Lebesgue measure and denote the corresponding densities by ρ_0 and ρ_T^*, respectively. Classical existence proofs for the SBP date back to Fortet [1940] (in 1D) and Beurling [1960], with a modern fixed-point approach in [Chen et al., 2016]. Recent extensions to the case of noncompactly supported marginal distributions can be found in [Conforti et al., 2024] and [Eckstein, 2025]. Recently, the problem attracted attention of machine learners in the context of generative modelling (see, for instance, [Tzen and Raginsky, 2019, De Bortoli et al., 2021, Shi et al., 2023, Korotin et al., 2024, Gushchin et al., 2024a, Rapakoulias et al., 2024] to name a few). It follows from Theorem 3.2 in [Dai Pra, 1991] that the optimal Markov process X_t^* solving the Schrödinger problem with marginals (ρ_0, ρ_T^*) can be constructed as a solution of the following SDE:

$$\mathrm{d}X_t^* = \left(b(X_t^*, t) + \sigma(X_t^*, t)\sigma(X_t^*, t)^\top \nabla \log h(X_t^*, t)\right)\, \mathrm{d}t + \sigma(X_t^*, t)\, \mathrm{d}W_t, \quad X_0 \sim \rho_0,$$

where

$$h(w,t) = \int_{\mathbb{R}^d} \mathsf{Q}(y,T \mid w,t)\, \nu_T(\mathrm{d}y)$$

and Q is the transition density of the reference (or base) diffusion process

$$\mathrm{d}X_t = b(X_t,t)\,\mathrm{d}t + \sigma(X_t,t)\,\mathrm{d}W_t, \quad X_0 \sim \rho_0.$$

The transition density Q^* of the reciprocal process X_t^* can be obtained from Q via the so-called Doob's h–transform:

$$\mathsf{Q}^*(y,T \mid x,t) = \mathsf{Q}(y,T \mid x,t)\, \frac{h(y,T)}{h(x,t)}. \tag{2}$$

This is precisely the law of the base process conditioned by the function h (see [Jamison, 1974]). In many presentations of the Schrödinger Bridge problem, one takes a very simple reference process (for instance, a Brownian motion) so that its transition kernel is straightforward to write down (see, for example, [Pooladian and Niles-Weed, 2024] and [Baptista et al., 2024]). However, there are several practical and theoretical advantages to considering more general (potentially higher-dimensional, or with domain constraints, or with a non-trivial drift/diffusion) reference processes.

In the present paper, we are interested in estimation of the Schrödinger potential ν_T from n i.i.d. samples $Y_1,\dots,Y_n \sim \rho_T^*$. Given a class of log-potentials Ψ, we study generalization ability of an empirical risk minimizer

$$\widehat{\psi} \in \operatorname*{argmin}_{\psi\in\Psi} \left\{ -\frac{1}{n}\sum_{i=1}^{n} \log\left(\int_{\mathbb{R}^d} \mathsf{Q}(Y_i,T \mid x,0)\, \frac{h_\psi(Y_i,T)}{h_\psi(x,0)}\, \rho_0(x)\mathrm{d}x \right) \right\}, \tag{3}$$

where

$$h_\psi(x,t) = \int \mathsf{Q}(y,T \mid x,t)\, e^{\psi(y)}\,\mathrm{d}y.$$

Let us note that, in view of (2),

$$\rho_T^\psi(y) = \int_{\mathbb{R}^d} \mathsf{Q}(y,T \mid x,0)\, \frac{h_\psi(y,T)}{h_\psi(x,0)}\, \rho_0(x)\mathrm{d}x$$

is the marginal endpoint probability density of a diffusion process X_t^ψ corresponding to Doob's h_ψ-transform:

$$\mathrm{d}X_t^\psi = \Big(b(X_t^\psi,t) + \sigma(X_t^\psi,t)\sigma(X_t^\psi,t)^\top \nabla \log h_\psi(X_t^\psi,t) \Big)\,\mathrm{d}t + \sigma(X_t^\psi,t)\,\mathrm{d}W_t, \quad X_0 \sim \rho_0.$$

In other words, the estimate $\widehat{\psi}$ minimizes empirical Kullback-Leibler (KL) divergence between the actual target ρ_T^* and the marginal densities ρ_T^ψ over the class of admissible log-potentials Ψ. That is, we chose the log-potential ψ that makes the transformed reference diffusion hit the observed terminal law, and measure error only through KL of the marginals. Because h_ψ is used inside the Doob factor, the learnt potential is compatible with a single Markov process; one never risks obtaining mutually inconsistent forward/backward potentials. The method combines the full problem (the marginals, transition densities, and the potential function) into one single optimization framework. By doing so, it aims to directly minimize the objective of matching the marginals at time T without separating the problem into smaller subproblems. In contrast, the Sinkhorn algorithm, commonly used for optimal transport problems, approaches the problem by iteratively updating the potentials in a decoupled manner. At each iteration, a simpler least squares problem appears, which is linear in one potential function given that another one is fixed from the previous iteration. The Sinkhorn algorithm alternates between updating the potential functions to match the marginals of the distributions and adjusting the transport plan until convergence. We refer to Pooladian and Niles-Weed [2024], Chiarini et al. [2024] for recent results. The primary advantage of the Sinkhorn approach is its computational efficiency. By decoupling the optimization process into simpler, linear problems, the Sinkhorn method can handle large-scale problems effectively. This iterative procedure allows for faster updates, and it has become a popular method for many optimal transport applications, see Genevay et al. [2018], March and Henry-Labordere [2023] However, the approach presented in this paper differs in that it does not separate the problem into independent steps. Instead, it aims at solving the Schrödinger system approximately by formulating it as a single optimization problem involving Doob h-transform of the base process X parametrized by the Schrödinger potential. Unlike iterative proportional fitting (Sinkhorn), everything is learnt in one go, avoiding slow or unstable fixed-point cycles. This results in a more accurate and robust solution. The trade-off between the two methods lies in computational efficiency versus the quality of the solution. The Sinkhorn approach provides a quick and efficient solution by solving simpler problems at each iteration, but it may not achieve the best possible solution for the full problem. On the other hand, the method presented in this paper offers a more holistic approach, which could lead to a more accurate matching of the marginal distributions but might require more computational resources.

The approach presented in this paper can also be compared to methods that rely on optimization over transport maps, see Korotin et al. [2024], Gushchin et al. [2024a]. In transport map-based approaches, the goal is to find a map $\mathcal{T}$ that transports one probability distribution to another. The optimization typically focuses on minimizing a quadratic cost functional that penalizes the difference between the target distribution and the transformed distribution under the transport map. These methods are often framed as optimal transport problems, where the map $\mathcal{T}$ is determined by solving an optimization problem that involves the marginal distributions. The advantage of optimization over transport maps lies in its clear geometric interpretation, where the transport map provides a direct way to relate the two distributions. This can lead to efficient algorithms, especially when the transport map can be parametrized in a way that allows for fast computations, such as in the case of certain neural network architectures or simple affine transformations, Rapakoulias et al. [2024].

However, transport map-based approaches are typically constrained to quadratic costs, which may limit their applicability in some cases. Specifically, quadratic cost functionals, such as the 2-Wasserstein distance, often assume a certain structure or symmetry that may not be ideal for more general or complex problems.

In contrast, the approach discussed in this paper is not limited to quadratic costs. It allows for more general cost structures and is based on minimizing the Kullback-Leibler divergence (KL-divergence), which can accommodate a wider range of problem types. This flexibility is particularly valuable when dealing with more complex distributions or when the underlying problem involves non-quadratic costs that capture other aspects of the distribution, such as entropy regularization or non-linear interactions between variables.

Contribution The main contribution of the present paper a sharper non-asymptotic high-probability upper bound on generalization error of the empirical risk minimizer $\widehat{\psi}$ defined in (3).

- Taking a multivariate Ornstein-Uhlenbeck process as the reference one, we show that (see Theorem 1), with probability at least $(1 - 2\delta)$, the excess KL-risk of the marginal endpoint density $\widehat{\rho}_T$ corresponding to $\widehat{\psi}$ satisfies the inequality
$$\mathsf{KL}(\rho^*_T, \widehat{\rho}_T) - \inf_{\psi \in \Psi} \mathsf{KL}(\rho^*_T, \rho^\psi_T) \lesssim \sqrt{\Upsilon(n,\delta) \inf_{\psi \in \Psi} \mathsf{KL}(\rho^*_T, \rho^\psi_T)} + \Upsilon(n,\delta),$$
where
$$\Upsilon(n,\delta) \lesssim \frac{\log^2 n + \log(1/\delta) \log n}{n}.$$
Here and further in the paper, the sign $\lesssim$ stands for an inequality up to a multiplicative constant. The derived upper bound has several advantages over the existing results. First, in contrast to Korotin et al. [2024], the excess risk may decrease as fast as $\mathcal{O}(\log^2 n/n)$ provided that the class of log-potentials Ψ is rich enough to approximate the target density ρ^*_T. Second, unlike theoretical guarantees for Sinkhorn-based approaches (see e.g. Pooladian and Niles-Weed [2024]), we are able to relate the endpoint marginal densities ρ^*_T and $\widehat{\rho}_T$.
- We impose very mild assumptions on the target density ρ^*_T. We only require ρ^*_T to be bounded and sub-Gaussian. On the other hand, the available convergence proofs for the Sinkhorn algorithm rely on the stronger assumption that the marginals are log-concave, see Conforti et al. [2024]. We also avoid the so-called strong density assumptions like boundedness from below often used in nonparametric statistics in the context of log-density estimation.
- The assumptions on the class of log-potentials Ψ are also reasonable. We support our claim with several examples.

Paper structure The rest of the paper is organized as follows. Section 2 is devoted to a short review of related work. In Section 3, we introduce necessary definitions and notations. After that, we present our main result (Theorem 1) in Section 4 and discuss main ideas of its proof in Section 5. Rigorous derivations as well as auxiliary technical results are deferred to the supplementary material.

2 Related work

Here is a short review of methods used in the literature to compute Schrödinger potentials, including the Sinkhorn algorithm. The Schrödinger potential, which arises in optimal transport problems, represents a key component in the solution of transport problems involving marginal distributions. Over time, several methods have been proposed to compute these potentials efficiently, with applications in areas ranging from statistical mechanics to machine learning. Here, we review some of the most prominent methods used in the literature.

Sinkhorn algorithm The Sinkhorn algorithm Sinkhorn [1967] is one of the most widely used methods for computing Schrödinger potentials in the context of optimal transport. It is based on iterative scaling and aims to solve the optimal transport problem by alternating between updating two potentials ν_0 and ν_T to enforce marginal constraints. The key advantage of the Sinkhorn approach is its computational efficiency, particularly when the transport problem is framed with a quadratic cost (such as the 2-Wasserstein distance), see Pavon et al. [2021], Chen et al. [2021], Stromme [2023] for reference. In each iteration, the algorithm solves a simpler problem that involves scaling the potentials in a way that brings the marginals of the transformed distribution closer to the target. Although Sinkhorn's algorithm is efficient and widely applicable, it is often limited by its assumption of quadratic costs. Additionally, the algorithm does not directly handle more complex cost structures, such as non-quadratic costs or non-linear dynamics, which can be a limitation in some applications.

Sinkhorn bridge The Sinkhorn Bridge proposed by Pooladian and Niles-Weed [2024], provides a way to estimate the Schrödinger bridge using Sinkhorn's algorithm in an efficient manner. The key insight of this method is that the potentials obtained from the static entropic optimal transport problem can be modified to yield a natural plug-in estimator for the drift function that defines the Schrödinger bridge. However, this work does not provide bounds on the distance between marginal distributions at time $T = 1$ because there is an exploding term $(1 - \tau)^{k+2}$ as $\tau \to 1$ where k is the dimension of the underlying manifold. This term leads to a "curse of dimensionality" where the error grows rapidly as τ approaches 1, especially in high-dimensional settings. As a result, the estimation error increases significantly when attempting to estimate the Schrödinger bridge at the terminal time, making it difficult to obtain precise bounds for $T = 1$.

Dual Formulation of the Schrödinger Problem In the dual formulation of the Schrödinger problem, the Schrödinger potential is computed by solving a convex optimization problem. This approach reformulates the problem in terms of a dual objective that involves the Kullback-Leibler (KL) divergence between the target and predicted distributions. The dual problem is then solved using optimization techniques such as gradient descent or variational methods, see Zhang and Chen [2022], Tzen and Raginsky [2019] for reference. This formulation is more flexible than the Sinkhorn algorithm, as it can accommodate more general cost functions and is not limited to quadratic losses.

While the dual approach is flexible, it is often computationally more demanding than Sinkhorn's method due to the need for iterative optimization over high-dimensional spaces. This makes the dual formulation suitable for smaller or more specialized problems, but it can become computationally expensive in large-scale applications.

Approximate Solutions Using Monte Carlo Methods Monte Carlo methods, particularly those relying on reverse diffusion processes, have also been employed to approximate Schrödinger potentials. In these methods, a reverse process is simulated, and the potential is iteratively refined to minimize the discrepancy between the predicted and target marginals, see Korotin et al. [2024] for reference. These methods are often used when the problem involves complex dynamics that are difficult to capture using direct optimization techniques.

Monte Carlo methods are particularly useful when dealing with high-dimensional problems, as they allow for the sampling of large spaces. However, they can be computationally expensive and may require a significant number of samples to achieve an accurate solution.

In addition, there are approaches that rely heavily on Monte Carlo approximations of intermediate values rather than the Schrödinger potentials themselves, among which the following should be noted De Bortoli et al. [2021], Vargas et al. [2021], Peluchetti [2023].

Neural Network-Based Approaches Recent advancements in deep learning have led to the use of neural networks to approximate Schrödinger potentials. These approaches treat the potential function as a parameterized neural network and use gradient-based optimization techniques to learn the potential that best matches the marginals. The use of neural networks offers a flexible and powerful way to model complex non-linear potentials, making these methods well-suited for problems with intricate dynamics or non-quadratic costs.While neural network-based approaches are highly flexible, they require large amounts of data and computational resources to train the network, and they are often prone to overfitting if not regularized appropriately. Despite these challenges, they represent a promising direction for future research, especially when the problem at hand involves complex and high-dimensional systems. We refer to Liu et al. [2023], Wang et al. [2021] for recent results.

Iterative Markovian Fitting The Iterative Markovian Fitting (IMF) method, introduced in the recent work by Shi et al. [2023], offers an approach to solving Schrödinger Bridge (SB) problems. Unlike previous methods, such as Iterative Proportional Fitting (IPF), IMF guarantees the preservation of both the initial and terminal distributions in each iteration, which is a key advantage over IPF where these marginals are not always preserved. IMF alternates between two types of projections: Markovian projections and reciprocal projections, ensuring that the resulting distribution remains within the correct class (Markovian or reciprocal) while progressively approximating the Schrödinger Bridge. We refer to Gushchin et al. [2024b] for recent results.

In Silveri et al. [2024], the authors provide the convergence analysis for diffusion flow matching (DFM), a method used to generate approximate samples from a target distribution by bridging it with a base distribution through diffusion dynamics. Their theoretical work includes non-asymptotic bounds on the Kullback-Leibler (KL) divergence between the true target distribution and the distribution generated by the DFM model. A key insight from this paper is the incorporation of two sources of error: drift-estimation and time-discretization errors. However, while the convergence analysis offers theoretical guarantees, the statistical error is not explicitly addressed in this paper. The analysis assumes that all expectations are exact, which might not hold in practical settings where samples are finite, and statistical errors could arise due to the approximations involved in the generative process. Thus, future work will need to extend this analysis to quantify the impact of statistical approximations in finite-sample settings.

3 Preliminaries and notations

This section collects necessary definitions and notations. As we announced in the contribution paragraph, we are going to consider a multivariate Ornstein-Uhlenbeck process as a reference one. For this reason, we elaborate on its basic properties in this section.

Multivariate Ornstein-Uhlenbeck process To be more specific, we will consider the base process X_t^0 solving the SDE

$$\mathrm{d}X_t^0 = b\left(m - X_t^0\right)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}W_t, \quad 0 \leqslant t \leqslant T,$$

where $b > 0$ controls the drift rate, $m \in \mathbb{R}^d$ represents the mean-reversion level, $\Sigma \in \mathbb{R}^{d \times d}$ is a positive definite symmetric matrix, and W_t is a standard d-dimensional Wiener process. It is known that the conditional distribution of X_t^0 given $X_0^0 = x$ is Gaussian $\mathcal{N}\big(m_t(x), \Sigma_t\big)$ with

$$m_t(x) = (1 - e^{-bt})m + e^{-bt}x \quad \text{and} \quad \Sigma_t = \frac{1 - e^{-2bt}}{2b}\Sigma. \tag{4}$$

This implies that the corresponding Doob's h-transform can be expressed through the Ornstein-Uhlenbeck operator

$$\mathcal{T}_t g(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(\Sigma_t)}} \int\limits_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\|\Sigma_t^{-1/2}(y - m_t(x))\|^2\right\} g(y)\,\mathrm{d}y.$$

Indeed, it holds that $h_\psi(x,t) = \mathcal{T}_{T-t} e^{\psi(x)}$. Then, introducing

$$\mathsf{q}(y \,|\, x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(\Sigma_T)}} \exp\left\{-\frac{1}{2}\|\Sigma_T^{-1/2}(y - m_T(x))\|^2\right\},$$

we note that

$$\rho_T^\psi(y) = \int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x) e^{\psi(y)}}{\mathcal{T}_T e^{\psi(x)}}\, \rho_0(x)\,\mathrm{d}x \tag{5}$$

is the marginal density of X_T^ψ, the endpoint of a random process X_t^ψ governed by h_ψ:

$$\mathrm{d}X_t^\psi = b\left(m - X_t^\psi\right)\mathrm{d}t + \nabla \log\left(\mathcal{T}_{T-t} e^{\psi(X_t^\psi)}\right)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}W_t, \quad X_0^\psi \sim \rho_0.$$

If the Schrödinger potential ν_T admits a density e^{ψ^*} with respect to the Lebesgue measure, then the optimally controlled process X_t^* solves the SDE

$$\mathrm{d}X_t^* = b\left(m - X_t^*\right)dt + \nabla \log\left(\mathcal{T}_{T-t} e^{\psi^*(X_t^*)}\right)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}W_t, \quad X_0^* \sim \rho_0.$$

Finally, it is well known that the unique stationary (invariant) distribution of X_t^0 is Gaussian, that is, X_t^0 converges to X_∞^0 in distribution as $t \to \infty$ with $X_\infty \sim \mathcal{N}(m, \Sigma/(2b))$. Since the parameters of the limiting distribution do not depend on the starting point, $\mathcal{T}_\infty g(x) \equiv \mathcal{T}_\infty g$ is a constant.

Other notations The notation $f \lesssim g$ or $g \gtrsim f$ means that $f = \mathcal{O}(g)$. Besides, we often replace $\max\{a, b\}$ and $\min\{a, b\}$ by shorter expressions $a \vee b$ and $a \wedge b$, respectively. For any $s \geqslant 1$, the Orlicz ψ_s-norm of a random variable ξ is defined as

$$\|\xi\|_{\psi_s} = \inf\left\{u > 0 : \mathbb{E} e^{|\xi|^s/u^s} \leqslant 2\right\}.$$

Finally, given $p \geqslant 1$ and a probability density ρ, the weighted L_p-norm of a function f is defined as $\|f\|_{L_p(\rho)} = \left(\mathbb{E}_{\xi\sim\rho}|f(\xi)|^p\right)^{1/p}$. Given two probability densities $\rho_0 \ll \rho_1$ on $\mathbb{R}^d$, the Kullback-Leibler divergence between them is defined as $\mathsf{KL}(\rho_0, \rho_1) = \mathbb{E}_{\xi\sim\rho_0} \log\left(\rho_0(\xi)/\rho_1(\xi)\right)$.

4 Main result

In the present section, we discuss statistical properties of the empirical risk minimizer $\widehat{\psi}$ defined in (3). In particular, Theorem 1 provides a Bernstein-type upper bound on its excess KL-risk. We impose the following assumptions. First, as we announced before, we use the Ornstein-Uhlenbeck process X_t^0 as the reference one.

Assumption 1. *The base process X^0 solves the following SDE*

$$\mathrm{d}X_t^0 = b\left(m - X_t^0\right)\mathrm{d}t + \Sigma^{1/2}\,\mathrm{d}W_t, \quad 0 \leqslant t \leqslant T.$$

where $b > 0$, $m \in \mathbb{R}^d$, Σ is a positive definite symmetric matrix of size $d \times d$, and W is a d-dimensional Brownian motion.

Main properties of the Ornstein-Uhlenbeck process were discussed in the previous section. Second, we suppose that the target density ρ_T^* meets the following requirements.

Assumption 2. *The target distribution at time T admits a bounded density ρ_T^* with respect to the Lebesgue measure such that*

$$\rho_T^*(x) \leqslant \rho_{\max} \quad \textit{for all } x \in \mathbb{R}^d.$$

Moreover, the target distribution ρ_T^* is sub-Gaussian with variance proxy $\mathtt{v}^2$, that is,

$$\mathbb{E}_{Y\sim\rho_T^*} e^{u^\top Y} \leqslant e^{\mathtt{v}^2\|u\|^2/2} \quad \textit{for any } u \in \mathbb{R}^d. \tag{6}$$

Assumption 2 is very mild. Despite the fact that we deal with logarithmic loss, we do not require ρ_T^* to be bounded away from zero. We do not even require its support to be compact. This significantly complicates the proof of the excess KL-bound and poses nontrivial technical challenges. Let us note that the condition 6 yields that $\mathbb{E}_{Y\sim\rho_T^*}Y = 0$. However, it does not diminish generality of our setup.

The remaining assumptions concern properties of the class of log-potentials Ψ. First, we assume that admissible log-potentials $\psi(x)$ are bounded from above and behave as $\mathcal{O}(\|x\|^2)$ as x tends to infinity.

Assumption 3. *There exist non-negative constants Λ and M such that*

$$-\Lambda\left\|\Sigma^{-1/2}(x-m)\right\|^2 - M \leqslant \psi(x) \leqslant M \quad \textit{for all } x\in\mathbb{R}^d \textit{ and } \psi\in\Psi.$$

Moreover, for any $\psi\in\Psi$, it holds that $\mathcal{T}_\infty\psi = \mathbb{E}\psi(X_\infty) = 0$.

The condition $\mathcal{T}_\infty\psi = 0$ appears because of the fact that the Schrödinger potentials ν_0 and ν_T (see (1)) are defined up to a multiplicative constant. The requirement $\mathcal{T}_\infty\psi = 0$ is nothing but a normalization. Second, we assume that Ψ is parametrized by a finite-dimensional parameter $\theta\in\mathbb{R}^D$:

$$\Psi = \{\psi_\theta : \theta\in\Theta\},$$

where Θ is a subset of a D-dimensional cube $[-R,R]^D$ and each function ψ_θ maps $\mathbb{R}^d$ onto $\mathbb{R}$. We suppose that the parametrization is sufficiently smooth in the following sense.

Assumption 4. *There exists $L\geqslant 0$ such that*

$$|\psi_\theta(x)-\psi_{\theta'}(x)| \leqslant L\left(1+\|x\|^2\right)\|\theta-\theta'\|_\infty \quad \textit{for all } \theta,\theta'\in\Theta \textit{ and all } x\in\mathbb{R}^d.$$

Assumptions 3 and 4 are quite general. We provide two examples when they hold. First, in a recent paper [Korotin et al., 2024], the authors model $e^{\psi(x)}$ as a Gaussian mixture. Let $\alpha_1,\ldots,\alpha_K$ be non-negative numbers such that $\alpha_1+\ldots+\alpha_K = 1$ and consider

$$e^{\psi(x)} = e^{-C}\sum_{k=1}^K \alpha_k\varphi_{m_k,\Sigma_k}(x), \quad \text{where} \quad \varphi_{m_k,\Sigma_k}(x) = \frac{e^{-\|\Sigma_k^{-1/2}(x-m_k)\|^2/2}}{(2\pi)^{d/2}\det(\Sigma_k)^{1/2}}.$$

Here C is a normalizing constant which ensures that $\mathcal{T}_\infty\psi = 0$. In this situation, the parameter θ consists of all α_k's and all components of m_k's and Σ_k's, $k\in\{1,\ldots,K\}$. If the smallest eigenvalues of $\Sigma_1,\ldots,\Sigma_K$ are bounded away from zero uniformly over $k\in\{1,\ldots,K\}$, then $e^{\psi(x)}$ is bounded. On the other hand, if K is fixed, there is a component with a weight at least $1/K$. Without loss of generality, we assume that it is the first one. Then

$$\psi(x) \geqslant -C + \log\left(\alpha_1\varphi_{m_1,\Sigma_1}(x)\right) \geqslant -C - \log K - \frac{1}{2}\left\|\Sigma_1^{-1/2}(x-m_1)\right\|^2,$$

and we conclude that Assumption 3 is satisfied. Verification of the Assumption 4 is straightforward once we assume that the weight of each component is bounded away from zero, and the norms $\|m_k\|$, $\|\Sigma_k\|$, and $\|\Sigma_k^{-1}\|$ are bounded uniformly over $k\in\{1,\ldots,K\}$ (which is the case in [Korotin et al., 2024]). Second, Assumptions 3 and 4 will be fulfilled if one deals, for example, with a class of truncated feedforward neural networks with bounded weights and ReLU activations. It is known that (see [Schmidt-Hieber, 2020, Lemma 5]) they are Lipschitz with respect to each weight, and the Lipschitz constant grows linearly with $\|x\|$. More generally, Conforti [2024] analyzed semiconvexity properties of the Schrödinger potentials under rather mild assumptions on the marginals.

We are ready to formulate the main result of this section.

Theorem 1. *Let ρ_0 be the density of the standard Gaussian distribution $\mathcal{N}(0,I_d)$. Grant Assumptions 1, 2, 3, and 4. Assume that T is sufficiently large in a sense that*

$$bT \geqslant (5+\log d)\vee\log\left(160b\left(\mathtt{v}^2\vee 1\right)\left\|\Sigma^{-1}\right\|\right).$$

Let $\widehat{\psi}$ be defined in (3) *and let $\widehat{\rho}_T$ be the corresponding density of $X_T^{\widehat{\psi}}$. Then, for any $\delta\in(0,1/2)$, with probability at least $1-2\delta$, it holds that*

$$\mathsf{KL}(\rho_T^*,\widehat{\rho}_T) - \inf_{\psi\in\Psi}\mathsf{KL}(\rho_T^*,\rho_T^\psi) \lesssim \sqrt{\Upsilon(n,\delta)\inf_{\psi\in\Psi}\mathsf{KL}(\rho_T^*,\rho_T^\psi)} + \Upsilon(n,\delta),$$

where

$$\Upsilon(n,\delta) = (\Lambda d + M + d)\left(d + \log\frac{RLn}{\delta} + (M\vee\log\Lambda)\sqrt{d}e^{-bT}\right)\frac{D\log n}{n}.$$

The hidden constant behind $\lesssim$ depends on Σ, m, b, and $\mathtt{v}$ only.

In Theorem 1, we assume that ρ_0 is the density of $\mathcal{N}(0, I_d)$. Though it is a standard choice of initial distribution in practice, we would like to emphasize that unbounded support of ρ_0 significantly complicates the proof and makes the problem even more challenging.

The problem of Schrödinger potential estimation was also studied in [Korotin et al., 2024] and [Pooladian and Niles-Weed, 2024]. In [Korotin et al., 2024], the authors suggest an algorithm called Light Schrödinger Bridge, which is based on minimization of the empirical KL-divergence between entropic optimal transport plans. This slightly differs from our setup, since we aim to minimize empirical KL-divergence between marginal endpoint distributions. The reason is that Korotin, Gushchin, and Burnaev [2024] are motivated by the style transfer task, where the initial distribution is also unknown. In contrast, we focus on generative modelling where the initial distribution ρ_0 is available to learner. In [Korotin et al., 2024, Theorem A.1], the authors consider the case when admissible potentials are Gaussian mixtures with K components. Assuming that both initial and finite distibutions have a compact support, they prove a $\mathcal{O}(n^{-1/2})$ upper bound on the Rademacher complexity of such class. On the other hand, we allow the support of ρ_0 and ρ_T^* to be unbounded. Besides, the rate of convergence presented in Theorem 1 may be much faster than $\mathcal{O}(n^{-1/2})$ if the target distribution is close to $\{\rho_T^\psi : \psi \in \Psi\}$. In the realizable case (that is, $\rho_T^* \in \{\rho_T^\psi : \psi \in \Psi\}$) the right-hand side in Theorem 1 becomes $\mathcal{O}(\log^2 n/n)$. Finally Theorem 1 provides a high-probability upper bound on the excess risk while the result of Korotin et al. [2024] holds in expectation. In [Pooladian and Niles-Weed, 2024] the authors study properties of a plug-in Sinkhorn-based estimator. Similarly to Korotin et al. [2024], they consider the case of compactly supported initial and target measures. However, they assume that these measures are supported on smooth k-dimensional submanifolds. They derive a $\mathcal{O}(n^{-1/2} + (T-\tau)^{-k-2}n^{-1})$ bound on the squared total variation distance between *path measures* up to moment $\tau < T$. Unfortunately, the second term grows very fast when τ approaches T, and there are no guarantees whether the marginal endpoint distributions will be close to each other.

In Theorem 1, we focus on the statistical error leaving study of the approximation out of the scope of the present paper. The reason is that there are few results on properties of the true log-potential $\psi^*(x) = \log\big(\nu_T(\mathrm{d}x)/\mathrm{d}x\big)$. However, we would like to note that, according to our findings (see Lemma B.2 and (5)), if ψ^* fulfils Assumption 3, then for any $\psi \in \Psi$ and $y \in \mathbb{R}^d$

$$\log \frac{\rho_T^*(y)}{\rho_T^\psi(y)} \lesssim |\psi(y) - \psi^*(y)| + \left(\mathcal{T}_\infty|\psi - \psi^*|\right)^{1/\mathcal{K}(T)} \|\Sigma^{-1/2}(y-m)\|^{2-2/\mathcal{K}(T)} e^{\mathcal{O}(e^{-bT}\|\Sigma^{-1/2}(y-m)\|^2)},$$

where $1 \leqslant \mathcal{K}(T) \leqslant 1 + \mathcal{O}(\sqrt{d}e^{-bT})$. In the proof of Theorem 1 (see Step 4), we show that the expectation

$$\mathbb{E}_{Y\sim\rho_T^*} \left\|\Sigma^{-1/2}(Y-m)\right\|^{2-2/\mathcal{K}(T)} e^{\mathcal{O}(e^{-bT}\|\Sigma^{-1/2}(Y-m)\|^2)}$$

is finite, provided that $bT \geqslant (5 + \log d) \vee \log\left(160b\,(\mathtt{v}^2 \vee 1)\left\|\Sigma^{-1}\right\|\right)$. This allows us to relate the KL-divergence between ρ_T^* and ρ_T^ψ with the distances between the corresponding log-potentials:

$$\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) \lesssim \|\psi - \psi^*\|_{L_1(\rho_T^*)} + \left(\mathcal{T}_\infty|\psi - \psi^*|\right)^{1/\mathcal{K}(T)}.$$

5 Proof sketch of Theorem 1

In this section, we discuss main ideas used in the proof of Theorem 1. Rigorous derivations are deferred to Appendix A. Since the proof is quite long, we split it into several steps.

Step 1: log-density properties. Let us note that Assumptions 3 and 4 concern properties of log-potentials $\psi \in \Psi$ while empirical risks include marginal densities ρ_T^ψ. For this reason, before we consider the empirical process

$$\frac{1}{n}\sum_{i=1}^n \log\frac{\rho_T^*(Y_i)}{\rho_T^\psi(Y_i)} - \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right), \quad \psi \in \Psi,$$

we have to study the random variables $\log\left(\rho_T^*(Y_i)/\rho_T^\psi(Y_i)\right)$, $1 \leqslant i \leqslant n$. Using basic properties of the Ornstein-Uhlenbeck operator, we show that

$$-\log \rho_T^\psi(y) \lesssim -\psi(y) + \left\| \Sigma^{-1/2}(y-m) \right\|^2 .$$

In view of Assumption 3, this means that $-\log \rho_T^\psi(y)$ grows as fast as a quadratic function. Since the target distribution is sub-Gaussian and has a bounded density, this yields that the random variables $\log\left(\rho_T^*(Y_i)/\rho_T^\psi(Y_i)\right)$, $1 \leqslant i \leqslant n$, are sub-exponential. More specifically, applying Lemma C.3 we obtain the following upper bound on their Orlicz norm:

$$\left\| \log \frac{\rho_T^*(Y_i)}{\rho_T^\psi(Y_i)} \right\|_{\psi_1} \lesssim \Lambda d + M + d \quad \text{for all } i \in \{1, \dots, n\}.$$

Step 2: ε-net argument and Bernstein's inequality. The result obtained on the first step allows us to use concentration inequalities for sub-exponential random variables. Let us fix $\varepsilon \in (0, R)$ and let Θ_ε stand for the minimal ε-net of Θ with respect to the ℓ_∞-norm. We denote the set of corresponding log-potentials by Ψ_ε:

$$\Psi_\varepsilon = \{\psi_\theta : \theta \in \Theta_\varepsilon\}.$$

Using Bernstein's inequality for unbounded random variables (see, for instance, [Lecué and Mitchell, 2012, Proposition 5.2]) and the union bound, we obtain that

$$\left| \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) - \frac{1}{n}\sum_{i=1}^n \log \frac{\rho_T^*(Y_i)}{\rho_T^\psi(Y_i)} \right| \lesssim \sqrt{\mathrm{Var}\left(\log \frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)}\right) \frac{\log(2|\Psi_\varepsilon|/\delta)}{n}} + \frac{(\Lambda d + M + d)\log n \log(2|\Psi_\varepsilon|/\delta)}{n}$$

with probability at least $(1-\delta)$ simultaneously for all $\psi \in \Psi_\varepsilon$.

Step 3: bounding the loss variance. One of the key ingredients in the proof of Theorem 1, which allows us to hope for faster rates of convergence than $\mathcal{O}(n^{-1/2})$, is analysis of the variance of $\log\left(\rho_T^*(Y_1)/\rho_T^\psi(Y_1)\right)$, $\psi \in \Psi$. Despite the fact that the admissible log-potentials may be unbounded, we are still able to show that the class Ψ satisfies a Bernstein-type condition

$$\mathrm{Var}\left(\log \frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)}\right) \lesssim (\Lambda d + M + d)\log n \left(\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) + \frac{1}{n}\right).$$

Steps 4 and 5: from ε-net to a uniform Bernstein-type bound. The hardest and technically involved part of the proof is to show that the losses $\log\left(\rho_T^*(y)/\rho_T^\psi(y)\right)$ and $\log\left(\rho_T^*(y)/\rho_T^\phi(y)\right)$ do not differ too much, once the corresponding log-potentials ψ and ϕ are close to each other. This follows from Lemma B.2, which relies on properties of the Ornstein-Uhlenbeck operator established in and Lemma B.3. We would like to note that the unbounded support of the initial density ρ_0 significantly complicates the proof of Lemma B.2. Nevertheless, we prove that

$$\log \frac{\rho_T^\psi(y)}{\rho_T^\phi(y)} \lesssim |\psi(y) - \phi(y)| + (\mathcal{T}_\infty|\psi - \phi|)^{1/\mathcal{K}(T)} \left\|\Sigma^{-1/2}(y-m)\right\|^{2-2/\mathcal{K}(T)} e^{\mathcal{O}(e^{-bT}\|\Sigma^{-1/2}(y-m)\|^2)},$$

where $1 \leqslant \mathcal{K}(T) \leqslant 1 + \mathcal{O}(\sqrt{d}e^{-bT})$. Though the right-hand side depends exponentially on the squared norm of $\Sigma^{-1/2}(y-m)$, the coefficient $\mathcal{O}(e^{-bT})$ is quite small, which is enough for our purposes.

Steps 6 and 7: choice of ε and the final bound. The rest of the proof is quite standard. On Step 6, we choose an appropriate ε and obtain a uniform Berstein-type inequality

$$\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) - \frac{1}{n}\sum_{i=1}^n \log \frac{\rho_T^*(Y_i)}{\rho_T^\psi(Y_i)} \lesssim \sqrt{\Upsilon(n,\delta)\, \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right)} + \Upsilon(n,\delta),$$

where

$$\Upsilon(n,\delta) = (\Lambda d + M + d)\left(d + \log \frac{RLn}{\delta} + (M \vee \log \Lambda)\sqrt{d}e^{-bT}\right)\frac{D \log n}{n},$$

which holds simultaneously for all $\psi \in \Psi$ with probability at least $(1-2\delta)$. After that, we transform it into the desired excess risk bound and finish the proof.

References

R. Baptista, A.-A. Pooladian, M. Brennan, Y. Marzouk, and J. Niles-Weed. Conditional simulation via entropic optimal transport: Toward non-parametric estimation of conditional Brenier maps. *arXiv preprint arXiv:2411.07154*, 2024.

A. Beurling. An automorphism of product measures. *Ann. Math. (2)*, 72:189–200, 1960. ISSN 0003-486X. doi: 10.2307/1970151.

C. Bunne, Y.-P. Hsieh, M. Cuturi, and A. Krause. The Schrödinger bridge between gaussian measures has a closed form. In *International Conference on Artificial Intelligence and Statistics*, pages 5802–5833. PMLR, 2023.

Y. Chen, T. Georgiou, and M. Pavon. Entropic and displacement interpolation: a computational approach using the Hilbert metric. *SIAM Journal on Applied Mathematics*, 76(6):2375–2396, 2016.

Y. Chen, T. T. Georgiou, and M. Pavon. Stochastic control liaisons: Richard Sinkhorn meets Gaspard Monge on a Schrödinger bridge. *Siam Review*, 63(2):249–313, 2021.

A. Chiarini, G. Conforti, G. Greco, and L. Tamanini. A semiconcavity approach to stability of entropic plans and exponential convergence of Sinkhorn's algorithm, 2024. URL `https://arxiv.org/abs/2412.09235`.

G. Conforti. Weak semiconvexity estimates for Schrödinger potentials and logarithmic Sobolev inequality for Schrödinger bridges, 2024. URL `https://arxiv.org/abs/2301.00083`.

G. Conforti, A. Durmus, and G. Greco. Quantitative contraction rates for Sinkhorn algorithm: beyond bounded costs and compact marginals, 2024. URL `https://arxiv.org/abs/2304.04451`.

P. Dai Pra. A stochastic control approach to reciprocal diffusion processes. *Applied Mathematics and Optimization*, 23(1):313–329, 1991. doi: 10.1007/BF01445134.

V. De Bortoli, J. Thornton, J. Heng, and A. Doucet. Diffusion Schrödinger bridge with applications to score-based generative modeling. *Advances in Neural Information Processing Systems*, 34: 17695–17709, 2021.

S. Eckstein. Hilbert's projective metric for functions of bounded growth and exponential convergence of Sinkhorn's algorithm. *Probability Theory and Related Fields*, pages 1–37, 2025.

R. Fortet. Résolution d'un système d'équations de M. Schrödinger. *J. Math. Pures Appl. (9)*, 19: 83–105, 1940. ISSN 0021-7824.

A. Genevay, G. Peyré, and M. Cuturi. Learning generative models with Sinkhorn divergences. In *International Conference on Artificial Intelligence and Statistics*, pages 1608–1617. PMLR, 2018.

N. Gushchin, S. Kholkin, E. Burnaev, and A. Korotin. Light and optimal Schrödinger bridge matching. In *Forty-first International Conference on Machine Learning*, 2024a. URL `https://doi.org/10.48550/arXiv.2402.03207`.

N. Gushchin, D. Selikhanovych, S. Kholkin, E. Burnaev, and A. Korotin. Adversarial Schrödinger bridge matching. In *The Thirty-eighth Annual Conference on Neural Information Processing Systems*, 2024b. URL `https://arxiv.org/abs/2405.14449`.

B. Jamison. Reciprocal processes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 30(1):65–86, 1974.

H. Janati, B. Muzellec, G. Peyré, and M. Cuturi. Entropic optimal transport between unbalanced gaussian measures has a closed form. In *Advances in Neural Information Processing Systems*, volume 33, pages 10468–10479. Curran Associates, Inc., 2020. URL `https://proceedings.neurips.cc/paper_files/paper/2020/file/766e428d1e232bbdd58664b41346196c-Paper.pdf`.

A. Korotin, N. Gushchin, and E. Burnaev. Light Schrödinger bridge. In *The Twelfth International Conference on Learning Representations*, 2024. URL `https://doi.org/10.48550/arXiv.2310.01174`.

G. Lecué and C. Mitchell. Oracle inequalities for cross-validation type procedures. *Electronic Journal of Statistics*, 6(none):1803 – 1837, 2012. doi: 10.1214/12-EJS730. URL `https://doi.org/10.1214/12-EJS730`.

C. Leonard. A survey of the Schrödinger problem and some of its connections with optimal transport. *Discrete and Continuous Dynamical Systems*, 34(4):1533–1574, 2014.

G.-H. Liu, A. Vahdat, D.-A. Huang, E. A. Theodorou, W. Nie, and A. Anandkumar. I^2SB: image-to-image Schrödinger bridge. In *Fortieth International Conference on Machine Learning*, 2023. URL `https://doi.org/10.48550/arXiv.2302.05872`.

H. D. March and P. Henry-Labordere. Building arbitrage-free implied volatility: Sinkhorn's algorithm and variants, 2023. URL `https://arxiv.org/abs/1902.04456`.

M. Pavon, G. Trigila, and E. G. Tabak. The data-driven Schrödinger bridge. *Communications on Pure and Applied Mathematics*, 74(7):1545–1573, 2021.

S. Peluchetti. Diffusion bridge mixture transports, Schrödinger bridge problems and generative modeling. *Journal of Machine Learning Research*, 24(374):1–51, 2023.

A.-A. Pooladian and J. Niles-Weed. Plug-in estimation of Schrödinger bridges. *arXiv preprint arXiv:2408.11686*, 2024.

G. Rapakoulias, A. R. Pedram, and P. Tsiotras. Go With the Flow: Fast Diffusion for Gaussian Mixture Models. Preprint. ArXiv:2412.09059v3, 2024.

P. Rigollet and J.-C. Hütter. High-dimensional statistics. Preprint. ArXiv:2310.19244, 2023.

J. Schmidt-Hieber. Nonparametric regression using deep neural networks with ReLU activation function. *The Annals of Statistics*, 48(4):1875–1897, 2020.

E. Schrödinger. Über die Umkehrung der Naturgesetze. *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse*, pages 144–153, 1932.

Y. Shi, V. De Bortoli, A. Campbell, and A. Doucet. Diffusion Schrödinger bridge matching. *arXiv preprint arXiv:2303.16852*, 2023.

M. G. Silveri, A. O. Durmus, and G. Conforti. Theoretical guarantees in KL for diffusion flow matching. In *The Thirty-eighth Annual Conference on Neural Information Processing Systems*, 2024. URL `https://openreview.net/forum?id=ia4WUCwHA9`.

R. Sinkhorn. Diagonal equivalence to matrices with prescribed row and column sums. *The American Mathematical Monthly*, 74(4):402–405, 1967. ISSN 00029890, 19300972. URL `http://www.jstor.org/stable/2314570`.

A. Stromme. Sampling from a Schrödinger bridge. In F. Ruiz, J. Dy, and J.-W. van de Meent, editors, *Proceedings of The 26th International Conference on Artificial Intelligence and Statistics*, volume 206 of *Proceedings of Machine Learning Research*, pages 4058–4067. PMLR, 25–27 Apr 2023. URL `https://proceedings.mlr.press/v206/stromme23a.html`.

B. Tzen and M. Raginsky. Theoretical guarantees for sampling and inference in generative models with latent diffusions. In *Conference on Learning Theory*, pages 3084–3114. PMLR, 2019.

F. Vargas, P. Thodoroff, A. Lamacraft, and N. Lawrence. Solving Schrödinger bridges via maximum likelihood. *Entropy*, 23(9):1134, 2021.

R. Vershynin. *High-Dimensional Probability: An Introduction with Applications in Data Science*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2018.

G. Wang, Y. Jiao, Q. Xu, Y. Wang, and C. Yang. Deep generative learning via Schrödinger bridge. In *International conference on machine learning*, pages 10794–10804. PMLR, 2021.

Q. Zhang and Y. Chen. Path integral sampler: A stochastic control approach for sampling. In *International Conference on Learning Representations*, 2022. URL `https://doi.org/10.48550/arXiv.2111.15141`.

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Contents

A Proof of Theorem 1

The proof of our main result is quite cumbersome. For this reason, we split it into several steps. We hope that a reader will find it more convenient.

Step 1: log-density properties. Before we move to the study of the empirical process

$$\frac{1}{n}\sum_{i=1}^{n}\log\frac{\rho_T^*(Y_i)}{\rho_T^{\psi}(Y_i)} - \mathsf{KL}\left(\rho_T^*, \rho_T^{\psi}\right), \quad \psi \in \Psi,$$

let us fix a log-potential $\psi \in \Psi$ and consider the corresponding marginal density ρ_T^{ψ}. Since, according to Assumption 3, $\psi(x)$ does not exceed M for all $x \in \mathbb{R}^d$, we can apply Lemma B.1 claiming that

$$\psi(y) - \log\rho_T^{\psi}(y) \leqslant \frac{2b}{1-e^{-2bT}}\left\|\Sigma^{-1/2}\big(y-m\big)\right\|^2 + \mathcal{O}(M+d).$$

This and the upper bound $\rho_T^*(y) \leqslant \rho_{\max}$ yield that

$$\log\frac{\rho_T^*(y)}{\rho_T^{\psi}(y)} \leqslant \log\rho_{\max} - \psi(y) + \frac{2b}{1-e^{-2bT}}\left\|\Sigma^{-1/2}\big(y-m\big)\right\|^2 + \mathcal{O}(M+d).$$

Since $\psi(y) \geqslant -\Lambda\|y\|^2 - M$ due to Assumption 3, we obtain that

$$\begin{aligned}\log\frac{\rho_T^*(y)}{\rho_T^{\psi}(y)} &\leqslant \Lambda\|y\|^2 + M + \frac{2b\|\Sigma^{-1}\|}{1-e^{-2bT}}\left\|\big(y-m\big)\right\|^2 + \mathcal{O}(M+d)\\ &\leqslant \left(\Lambda + \frac{4b\|\Sigma^{-1}\|}{1-e^{-2bT}}\right)\|y\|^2 + \mathcal{O}(M+d).\end{aligned} \tag{7}$$

The hidden constant in the right-hand side of (7) depends on $\rho_{\max}$. Besides, in the last line, we used the Cauchy-Schwarz inequality $\|y-m\|^2 \leqslant 2\|y\|^2 + 2\|m\|^2$. The inequality (7) ensures that the conditions of Lemma C.3 are fulfilled. Applying this lemma with $A = \Lambda + 4b\|\Sigma^{-1}\|/(1-e^{-2bt})$ and $B = \mathcal{O}(M+d)$, we obtain that

$$\left\|\log\frac{\rho_T^*(Y_i)}{\rho_T^{\psi}(Y_i)}\right\|_{\psi_1} \lesssim \Lambda d + M + d \quad \text{for all } i \in \{1,\dots,n\}, \tag{8}$$

where the hidden constant behind $\mathcal{O}(\cdot)$ depends on $\rho_{\max}$ and v^2. The bound (8) on the Orlicz norm of $\log\big(\rho_T^*(Y_i)/\rho_T^{\psi}(Y_i)\big)$, $i \in \{1,\dots,n\}$, plays a crucial role in our analysis, because it allows us to use properties of sub-exponential random variables.

Step 2: ε-net argument and Bernstein's inequality. Let $\varepsilon \in (0,1)$ be a parameter to be specified a bit later. Let Θ_ε stand for the minimal ε-net of Θ with respect to the ℓ_∞-norm and let us introduce

$$\Psi_\varepsilon = \{\psi_\theta : \theta \in \Theta_\varepsilon\}.$$

Since $\Theta \subseteq [-R,R]^D$, it is known that

$$|\Psi_\varepsilon| \leqslant |\Theta_\varepsilon| \leqslant \left(\frac{2R}{\varepsilon}\right)^D.$$

In view of (8), we can use Bernstein's inequality for unbounded random variables. According to [Lecué and Mitchell, 2012, Proposition 5.2]), for any fixed $\psi \in \Psi_\varepsilon$, with probability at least $1-\delta/|\Psi_\varepsilon|$ it holds that

$$\begin{aligned}\left|\mathsf{KL}\left(\rho_T^*, \rho_T^{\psi}\right) - \frac{1}{n}\sum_{i=1}^{n}\log\frac{\rho_T^*(Y_i)}{\rho_T^{\psi}(Y_i)}\right| &\lesssim \sqrt{\mathrm{Var}\left(\log\frac{\rho_T^*(Y_1)}{\rho_T^{\psi}(Y_1)}\right)\frac{\log(2|\Psi_\varepsilon|/\delta)}{n}}\\ &\quad + \left\|\max_{1\leqslant i\leqslant n}\left\{\log\frac{\rho_T^*(Y_i)}{\rho_T^{\psi}(Y_i)} - \mathsf{KL}\left(\rho_T^*, \rho_T^{\psi}\right)\right\}\right\|_{\psi_1}\frac{\log(2|\Psi_\varepsilon|/\delta)}{n},\end{aligned}$$

where $\lesssim$ stands for an inequality up to an absolute constant. The union bound yields that there is an event $\mathcal{E}_0$ such that $\mathbb{P}(\mathcal{E}_0) \geqslant 1 - \delta$ and

$$\begin{aligned} \left| \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) - \frac{1}{n}\sum_{i=1}^n \log\frac{\rho_T^*(Y_i)}{\rho_T^\psi(Y_i)} \right| &\lesssim \sqrt{\operatorname{Var}\left(\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)}\right)\frac{\log(2|\Psi_\varepsilon|/\delta)}{n}} \\ &\quad + \left\| \max_{1\leqslant i\leqslant n}\left\{\log\frac{\rho_T^*(Y_i)}{\rho_T^\psi(Y_i)} - \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right)\right\}\right\|_{\psi_1}\frac{\log(2|\Psi_\varepsilon|/\delta)}{n} \end{aligned}$$

simultaneously for all $\psi \in \Psi_\varepsilon$ on $\mathcal{E}_0$. Using Pisier's inequality (see, for example, [Lecué and Mitchell, 2012, p. 1827]) and the triangle inequality, one can show that

$$\begin{aligned} \left\| \max_{1\leqslant i\leqslant n}\left\{\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)} - \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right)\right\}\right\|_{\psi_1} &\lesssim \log n \left\|\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)} - \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right)\right\|_{\psi_1} \\ &\lesssim \log n \left\|\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)}\right\|_{\psi_1} + \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right)\log n. \end{aligned}$$

In view of (8), we obtain that

$$\left\| \max_{1\leqslant i\leqslant n}\left\{\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)} - \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right)\right\}\right\|_{\psi_1} \lesssim \log n\left(\Lambda d + M + d + \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right)\right).$$

On the other hand, the Kullback-Leibler divergence between ρ_T^* and ρ_T^ψ is the expectation of the random variable $\log\left(\rho_T^*(Y_1)/\rho_T^\psi(Y_1)\right)$ with a finite ψ_1-norm. This means that (see [Vershynin, 2018, Proposition 2.7.1])

$$\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) = \mathbb{E}_{Y_1\sim\rho_T^*}\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)} \leqslant \mathbb{E}_{Y_1\sim\rho_T^*}\left|\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)}\right| \lesssim \left\|\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)}\right\|_{\psi_1} \lesssim \Lambda d + M + d.$$

Then, on the event $\mathcal{E}_0$, we have

$$\begin{aligned} \left| \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) - \frac{1}{n}\sum_{i=1}^n \log\frac{\rho_T^*(Y_i)}{\rho_T^\psi(Y_i)} \right| &\lesssim \sqrt{\operatorname{Var}\left(\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)}\right)\frac{\log(2|\Psi_\varepsilon|/\delta)}{n}} \\ &\quad + \frac{(\Lambda d + M + d)\log n\log(2|\Psi_\varepsilon|/\delta)}{n} \end{aligned} \tag{9}$$

simultaneously for all $\psi \in \Psi_\varepsilon$.

Step 3: bounding the loss variance. One of the key ingredients in the proof of Theorem 1, which allows us to hope for faster rates of convergence than $\mathcal{O}(n^{-1/2})$, is analysis of the variance of $\log\left(\rho_T^*(Y_1)/\rho_T^\psi(Y_1)\right)$, $\psi \in \Psi$. On this step, we are going show that it satisfies a Bernstein-type condition

$$\operatorname{Var}\left(\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)}\right) \lesssim (\Lambda d + M + d)\log n\left(\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) + \frac{1}{n}\right) \quad \text{for all } \psi \in \Psi.$$

The proof of this fact easily follows from Lemmata C.1 and C.2 presented in Appendix C. Indeed, Lemma C.1 implies that

$$\begin{aligned} \operatorname{Var}\left(\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)}\right) &\leqslant \mathbb{E}\left(\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)}\right)^2 \\ &\leqslant 2\log(1/\omega)\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) + 2\mathbb{E}\left(\log\frac{\omega\rho_T^*(Y_1) + (1-\omega)\rho_T^\psi(Y_1)}{\rho_T^\psi(Y_1)}\right)^2 \end{aligned} \tag{10}$$

for any $\omega \in (0,1)$. On the other hand, let A and B be non-negative constants such that

$$\log \frac{\rho_T^*(y)}{\rho_T^\psi(y)} \leqslant A\|y\|^2 + B \quad \text{for all } y \in \mathbb{R}^d.$$

Note that, due to (7), we can take $A = \mathcal{O}(\Lambda)$ and $B = \mathcal{O}(M+d)$. Then, according to Lemma C.2, it holds that

$$\mathbb{E}\left(\log \frac{\omega\rho_T^*(Y_1) + (1-\omega)\rho_T^\psi(Y_1)}{\rho_T^\psi(Y_1)}\right)^2 \lesssim e^B\omega + 6^d\left(\log\frac{1}{\omega} + A\right)e^{B/(16A\mathtt{v}^2)}\omega^{1/(16A\mathtt{v}^2)}.$$

Note that the assumptions the statement of Lemma C.2 imposed on ρ_T^* are milder than we require in Assumption 2. Taking

$$\omega = e^{-B}\left(\frac{1}{n} \wedge (6^d n)^{-16A\mathtt{v}^2}\right)$$

we obtain that

$$\max\left\{e^B\omega, 6^d e^{B/(16A\mathtt{v}^2)}\omega^{1/(16A\mathtt{v}^2)}\right\} \leqslant \frac{1}{n},$$

and then

$$\log(1/\omega) = \max\left\{B + \log n, 16A\mathtt{v}^2 d\log 6 + 16A\mathtt{v}^2\log n\right\} \lesssim (\Lambda d + M + d)\log n.$$

Furthermore, such choice of ω ensures that

$$\mathbb{E}\left(\log \frac{\omega\rho_T^*(Y_1) + (1-\omega)\rho_T^\psi(Y_1)}{\rho_T^\psi(Y_1)}\right)^2 \lesssim \frac{(Ad + B + 1)\log n}{n} \lesssim \frac{(\Lambda d + M + d)\log n}{n}.$$

Summing up the last two inequalities and (10), we deduce that

$$\operatorname{Var}\left(\log\frac{\rho_T^*(Y_1)}{\rho_T^\psi(Y_1)}\right) \lesssim (\Lambda d + M + d)\log n\left(\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) + \frac{1}{n}\right). \tag{11}$$

Step 4: from ε-net to a uniform Bernstein-type bound, part 1. Substituting the variance in (9) by its upper bound (11), we observe that, on the event $\mathcal{E}_0$, simultaneously for all $\psi \in \Psi_\varepsilon$ it holds that

$$\begin{aligned}\left|\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) - \frac{1}{n}\sum_{i=1}^n \log\frac{\rho_T^*(Y_i)}{\rho_T^\psi(Y_i)}\right| &\lesssim \sqrt{(\Lambda d + M + d)\log n\left(\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) + \frac{1}{n}\right)\frac{\log(2|\Psi_\varepsilon|/\delta)}{n}} \\ &\quad + (\Lambda d + M + d)\frac{\log n\log(2|\Psi_\varepsilon|/\delta)}{n} \\ &\lesssim \sqrt{(\Lambda d + M + d)\,\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right)\frac{\log n\log(2|\Psi_\varepsilon|/\delta)}{n}} \\ &\quad + (\Lambda d + M + d)\frac{\log n\log(2|\Psi_\varepsilon|/\delta)}{n}.\end{aligned}$$

The goal of this step is to transform this upper bound to a one holding uniformly for all $\psi \in \Psi$. For this purpose, let us fix an arbitrary $\theta \in \Theta$ and let θ_ε be the closest to θ element of the ε-net Θ_ε. According to the definition of Θ_ε, this means that

$$\|\theta - \theta_\varepsilon\|_\infty \leqslant \varepsilon.$$

Let us denote the corresponding to θ and θ_ε functions by $\psi \in \Psi$ and $\psi_\varepsilon \in \Psi$, respectively. The goal of this and the next steps is to show that the differences

$$\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) - \mathsf{KL}\left(\rho_T^*, \rho_T^{\psi_\varepsilon}\right) \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^n \log\rho_T^\psi(Y_i) - \frac{1}{n}\sum_{i=1}^n\log\rho_T^{\psi_\varepsilon}(Y_i)$$

are sufficiently small. Let us first elaborate on the difference of KL-divergences and postpone the study of the empirical risks until the next step. Note that

$$\begin{aligned}\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) - \mathsf{KL}\left(\rho_T^*, \rho_T^{\psi_\varepsilon}\right) &= \mathbb{E}_{Y\sim\rho_T^*}\log\frac{\rho_T^*(Y)}{\rho_T^\psi(Y)} - \mathbb{E}_{Y\sim\rho_T^*}\log\frac{\rho_T^*(Y)}{\rho_T^{\psi_\varepsilon}(Y)} \\ &= \mathbb{E}_{Y\sim\rho_T^*}\log\frac{\rho_T^{\psi_\varepsilon}(Y)}{\rho_T^\psi(Y)}.\end{aligned}$$

We would like to recall that for any log-potential ψ the corresponding marginal density ρ_T^{ψ} has the form

$$\rho_T^{\psi}(y) = \int\limits_{\mathbb{R}^d} \frac{e^{\psi(y)}\,\mathsf{q}(y\,|\,x)}{\mathcal{T}_T e^{\psi(x)}}\,\rho(x)\mathrm{d}x.$$

Due to Assumption 4, it holds that

$$\left|\mathbb{E}_{Y\sim\rho_T^*}\psi(Y) - \mathbb{E}_{Y\sim\rho_T^*}\psi_\varepsilon(Y)\right| \leqslant L\varepsilon\left(1 + \mathbb{E}_{Y\sim\rho_T^*}\|Y\|^2\right). \tag{12}$$

For sub-Gaussian random vectors, we have

$$\mathbb{E}_{Y\sim\rho_T^*}\|Y\|^2 = \operatorname{Tr}\left(\mathbb{E}_{Y\sim\rho_T^*}YY^\top\right) \lesssim \mathtt{v}^2 d,$$

and then

$$\left|\mathbb{E}_{Y\sim\rho_T^*}\psi(Y) - \mathbb{E}_{Y\sim\rho_T^*}\psi_\varepsilon(Y)\right| \lesssim L\varepsilon d.$$

Thus, it remains to bound

$$\begin{aligned}
&\left|\mathbb{E}_{Y\sim\rho_T^*}\left(\log\frac{\rho_T^{\psi}(Y)}{\rho_T^{\psi_\varepsilon}(Y)} + \psi_\varepsilon(Y) - \psi(Y)\right)\right| \\
&= \left|\mathbb{E}_{Y\sim\rho_T^*}\log\left(\int\limits_{\mathbb{R}^d}\frac{\mathsf{q}(Y\,|\,x)}{\mathcal{T}_T e^{\psi(x)}}\,\rho(x)\mathrm{d}x\right) - \mathbb{E}_{Y\sim\rho_T^*}\log\left(\int\limits_{\mathbb{R}^d}\frac{\mathsf{q}(Y\,|\,x)}{\mathcal{T}_T e^{\psi_\varepsilon(x)}}\,\rho(x)\mathrm{d}x\right)\right|.
\end{aligned}$$

This is the hardest and the most technical part of our derivations. It relies on properties of the Ornstein-Uhlenbeck operator we establish in Lemma B.2 and Lemma B.3. However, with these lemmata at hand, the desired bound on the difference of KL-divergences becomes straightforward. Indeed, according to Lemma B.2, for any $y \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\left|\log\int\limits_{\mathbb{R}^d}\frac{\mathsf{q}(y\,|\,x)}{\mathcal{T}_T e^{\psi(x)}}\,\rho_0(x)\mathrm{d}x - \log\int\limits_{\mathbb{R}^d}\frac{\mathsf{q}(y\,|\,x)}{\mathcal{T}_T e^{\psi_\varepsilon(x)}}\,\rho_0(x)\mathrm{d}x\right| \\
&\lesssim \left(\mathcal{T}_\infty|\psi - \psi_\varepsilon|\right)^{1/\mathcal{K}(T)}\left(d^2 + \left\|\Sigma^{-1/2}(y - m)\right\|^2\right)^{1-1/\mathcal{K}(T)} \\
&\quad\cdot\exp\left\{\mathcal{O}\big(d + (M\vee\log\Lambda)\sqrt{d}e^{-bT}\big)\right\} \\
&\quad\cdot\exp\left\{3e^{-bT}\left\|\Sigma_T^{-1/2}(y - m)\right\|^2 + \mathcal{O}(e^{-bT})\right\},
\end{aligned}$$

where Σ_T is defined in (4) and

$$1 \leqslant \mathcal{K}(T) = \left(1 - e^{-2bT}\right)^{-5e^2\sqrt{d}}\cdot\exp\left\{2e^2\sqrt{d}\arcsin(e^{-bT})\right\} = 1 + \mathcal{O}\left(\sqrt{d}e^{-bT}\right).$$

Let us introduce

$$\mathcal{H}(T) = \exp\left\{\mathcal{O}\big(d + (M\vee\log\Lambda)\sqrt{d}e^{-bT}\big)\right\}. \tag{13}$$

Then, due to the Cauchy-Schwarz inequality, it holds that

$$\begin{aligned}
&\left|\mathbb{E}_{Y\sim\rho_T^*}\log\left(\int\limits_{\mathbb{R}^d}\frac{\mathsf{q}(Y\,|\,x)}{\mathcal{T}_T e^{\psi(x)}}\,\rho(x)\mathrm{d}x\right) - \mathbb{E}_{Y\sim\rho_T^*}\log\left(\int\limits_{\mathbb{R}^d}\frac{\mathsf{q}(Y\,|\,x)}{\mathcal{T}_T e^{\psi_\varepsilon(x)}}\,\rho(x)\mathrm{d}x\right)\right| \\
&\lesssim \mathcal{H}(T)\left(\mathcal{T}_\infty|\psi - \psi_\varepsilon|\right)^{1/\mathcal{K}(T)}\mathbb{E}_{Y\sim\rho_T^*}\left[\left(d^2 + \left\|\Sigma^{-1/2}(Y - m)\right\|^2\right)^{1-1/\mathcal{K}(T)}\right. \\
&\quad\left.\cdot\exp\left\{3e^{-bT}\left\|\Sigma_T^{-1/2}(Y - m)\right\|^2 + \mathcal{O}(e^{-bT})\right\}\right] \\
&\leqslant \mathcal{H}(T)\left(\mathcal{T}_\infty|\psi - \psi_\varepsilon|\right)^{1/\mathcal{K}(T)}\sqrt{\mathbb{E}_{Y\sim\rho_T^*}\left(d^2 + \left\|\Sigma^{-1/2}(Y - m)\right\|^2\right)^{2-2/\mathcal{K}(T)}} \\
&\quad\cdot\sqrt{\mathbb{E}_{Y\sim\rho_T^*}\exp\left\{12e^{-bT}\left\|\Sigma_T^{-1/2}Y\right\|^2 + 12e^{-bT}\left\|\Sigma_T^{-1/2}m\right\|^2 + \mathcal{O}(e^{-bT})\right\}}.
\end{aligned}$$

Since $24\mathtt{v}^2 e^{-bT}\|\Sigma_T^{-1}\| < 1$, the exponential moment in the right-hand side is finite and

$$\begin{aligned}\mathbb{E}_{Y\sim\rho_T^*}\exp\left\{12e^{-bT}\left\|\Sigma_T^{-1/2}Y\right\|^2\right\} &\leqslant \mathbb{E}_{Y\sim\rho_T^*}\exp\left\{12e^{-bT}\left\|\Sigma_T^{-1}\right\|\|Y\|^2\right\}\\ &\lesssim (1-24\mathtt{v}^2e^{-bT})^{-d/2}\\ &\lesssim de^{-bT}\leqslant e^{-5}.\end{aligned}$$

The last inequality is due to the fact that $bT \geqslant 5+\log d$. This yields that

$$\begin{aligned}&\left|\mathbb{E}_{Y\sim\rho_T^*}\log\left(\int\limits_{\mathbb{R}^d}\frac{\mathsf{q}(Y\,|\,x)}{\mathcal{T}_T e^{\psi(x)}}\,\rho(x)\mathrm{d}x\right)-\mathbb{E}_{Y\sim\rho_T^*}\log\left(\int\limits_{\mathbb{R}^d}\frac{\mathsf{q}(Y\,|\,x)}{\mathcal{T}_T e^{\psi_\varepsilon(x)}}\,\rho(x)\mathrm{d}x\right)\right|\\ &\lesssim \mathcal{H}(T)\left(\mathcal{T}_\infty|\psi-\psi_\varepsilon|\right)^{1/\mathcal{K}(T)}\cdot d^{\mathcal{O}(\sqrt{d}e^{-bT})}\cdot\exp\left\{6e^{-bT}\left\|\Sigma_T^{-1/2}m\right\|^2+\mathcal{O}(e^{-bT})\right\}\\ &\lesssim \mathcal{H}(T)\left(\mathcal{T}_\infty|\psi-\psi_\varepsilon|\right)^{1/\mathcal{K}(T)}.\end{aligned}$$

Let us elaborate on $\mathcal{T}_\infty|\psi-\psi_\varepsilon|$. By the definition of $\mathcal{T}_\infty$ and Assumption 4, it holds that

$$\begin{aligned}\mathcal{T}_\infty|\psi-\psi_\varepsilon| &= \mathbb{E}_{\eta\sim\mathcal{N}(m,\Sigma)}|\psi(\eta)-\psi_\varepsilon(\eta)|\\ &\leqslant L\|\theta-\theta_\varepsilon\|_\infty\,\mathbb{E}_{\eta\sim\mathcal{N}(m,\Sigma)}(1+\|\eta\|^2)\\ &\leqslant L\varepsilon\,\mathbb{E}_{\eta\sim\mathcal{N}(m,\Sigma)}(1+\|\eta\|^2).\end{aligned}\tag{14}$$

The expression in the right-hand side can be computed explicitly:

$$L\varepsilon\,\mathbb{E}_{\eta\sim\mathcal{N}(m,\Sigma)}(1+\|\eta\|^2) = L\varepsilon\left(1+\|m\|^2+\mathrm{Tr}(\Sigma)\right)\leqslant L\varepsilon\left(1+\|m\|^2+\|\Sigma\|d\right)\lesssim L\varepsilon d.$$

Hence, we showed that on the event $\mathcal{E}_1$

$$\begin{aligned}&\left|\mathbb{E}_{Y\sim\rho_T^*}\log\left(\int\limits_{\mathbb{R}^d}\frac{\mathsf{q}(Y\,|\,x)}{\mathcal{T}_T e^{\psi(x)}}\,\rho(x)\mathrm{d}x\right)-\mathbb{E}_{Y\sim\rho_T^*}\log\left(\int\limits_{\mathbb{R}^d}\frac{\mathsf{q}(Y\,|\,x)}{\mathcal{T}_T e^{\psi_\varepsilon(x)}}\,\rho(x)\mathrm{d}x\right)\right|\\ &\lesssim \mathcal{H}(T)(L\varepsilon d)^{1/\mathcal{K}(T)}.\end{aligned}$$

This, together with (12), implies that

$$\left|\mathsf{KL}\left(\rho_T^*,\rho_T^{\psi}\right)-\mathsf{KL}\left(\rho_T^*,\rho_T^{\psi_\varepsilon}\right)\right|\lesssim L\varepsilon d+de^{-bT}\,\mathcal{H}(T)(L\varepsilon d)^{1/\mathcal{K}(T)}\lesssim L\varepsilon d+\mathcal{H}(T)(L\varepsilon d)^{1/\mathcal{K}(T)}.$$

Step 5: from ε-net to a uniform Bernstein-type bound, part 2. The proof of an upper bound on the absolute value of

$$\frac{1}{n}\sum_{i=1}^n\log\rho_T^{\psi}(Y_i)-\frac{1}{n}\sum_{i=1}^n\log\rho_T^{\psi_\varepsilon}(Y_i)$$

proceeds in a similar way. Due to Assumption 4, it holds that

$$\left|\frac{1}{n}\sum_{i=1}^n\left(\psi(Y_i)-\psi_\varepsilon(Y_i)\right)\right|\leqslant\frac{L\varepsilon}{n}\sum_{i=1}^n\left(1+\|Y_i\|^2\right).$$

It is known (see the proof of Theorem 1.19 in [Rigollet and Hütter, 2023]) that the norm of a sub-Gaussian random vector satisfies the inequality

$$\mathbb{P}\left(\|Y_1\|\geqslant u\right)\leqslant 6^d\exp\left\{-\frac{u^2}{8\mathtt{v}^2}\right\}\quad\text{for all } u>0.$$

This and the union bound imply that there exists and event $\mathcal{E}_1$ of probability at least $1-\delta$ such that

$$\max_{1\leqslant i\leqslant n}\|Y_i\|^2\leqslant d\log 6+8\mathtt{v}^2\log(n/\delta)\quad\text{on }\mathcal{E}_1.\tag{15}$$

On this event, we have[1]

$$
\begin{aligned}
\left| \frac{1}{n} \sum_{i=1}^{n} (\psi(Y_i) - \psi_\varepsilon(Y_i)) \right| &\leqslant L\varepsilon \left(1 + d \log 6 + 8\mathtt{v}^2 \log^2(n/\delta)\right) \\
&\lesssim L\varepsilon \left(d + \log(n/\delta)\right). \qquad (16)
\end{aligned}
$$

As in the previous step, the study of

$$
\begin{aligned}
&\left| \frac{1}{n} \sum_{i=1}^{n} \log \frac{\rho_T^{\psi}(Y_i)}{\rho_T^{\psi_\varepsilon}(Y_i)} + \psi_\varepsilon(Y_i) - \psi(Y_i) \right| \\
&= \left| \frac{1}{n} \sum_{i=1}^{n} \left[\log \left(\int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(Y_i \,|\, x)}{\mathcal{T}_T e^{\psi(x)}} \, \rho(x) \mathrm{d}x \right) - \log \left(\int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(Y_i \,|\, x)}{\mathcal{T}_T e^{\psi_\varepsilon(x)}} \, \rho(x) \mathrm{d}x \right) \right] \right|
\end{aligned}
$$

relies on Lemma B.3. Applying this lemma and using (14), we observe that, on the event $\mathcal{E}_1$,

$$
\begin{aligned}
&\left| \frac{1}{n} \sum_{i=1}^{n} \left[\log \left(\int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(Y_i \,|\, x)}{\mathcal{T}_T e^{\psi(x)}} \, \rho(x) \mathrm{d}x \right) - \log \left(\int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(Y_i \,|\, x)}{\mathcal{T}_T e^{\psi_\varepsilon(x)}} \, \rho(x) \mathrm{d}x \right) \right] \right| \\
&\lesssim \mathcal{H}(T) \left(\mathcal{T}_\infty |\psi - \psi_\varepsilon|\right)^{1/\mathcal{K}(T)} \\
&\quad \cdot \mathbb{E}_{Y \sim \rho_T^*} \Bigg[\left(d^2 + \left\|\Sigma^{-1}\right\| \left(d \log 6 + 8\mathtt{v}^2 \log(n/\delta)\right) + \left\|\Sigma^{-1/2} m\right\|^2 \right)^{1 - 1/\mathcal{K}(t)} \\
&\quad \cdot \exp\left\{ 6 e^{-bT} \left\|\Sigma_T^{-1}\right\| \left(d \log 6 + 8\mathtt{v}^2 \log(n/\delta)\right) + 6 e^{-bT} \left\|\Sigma_T^{-1/2} m\right\|^2 + \mathcal{O}(e^{-bT}) \right\} \Bigg] \\
&\lesssim \mathcal{H}(T) (L\varepsilon d)^{1/\mathcal{K}(T)} \exp\left\{ 12 e^{-bT} \left\|\Sigma_T^{-1}\right\| \left(d \log 6 + 8\mathtt{v}^2 \log(n/\delta)\right) \right\}.
\end{aligned}
$$

This, together with (16), implies that

$$
\begin{aligned}
&\left| \frac{1}{n} \sum_{i=1}^{n} \log \rho_T^{\psi}(Y_i) - \frac{1}{n} \sum_{i=1}^{n} \log \rho_T^{\psi_\varepsilon}(Y_i) \right| \\
&\lesssim L\varepsilon \left(d + \log(n/\delta)\right) + \mathcal{H}(T) (L\varepsilon d)^{1/\mathcal{K}(T)} \exp\left\{ 12 e^{-bT} \left\|\Sigma_T^{-1}\right\| \left(d \log 6 + 8\mathtt{v}^2 \log(n/\delta)\right) \right\}.
\end{aligned}
$$

Step 6: choice of ε. Let us take

$$
\varepsilon = \frac{1}{Lnd} \wedge \frac{\left(\mathcal{H}(T)\right)^{-\mathcal{K}(T)}}{Ldn^{\mathcal{K}(T)}} \exp\left\{ -12 e^{-bT} \left\|\Sigma_T^{-1}\right\| \left(d \log 6 + 8\mathtt{v}^2 \log(n/\delta)\right) \right\}.
$$

Such a choice of ε ensures that

$$
\max\left\{ L\varepsilon d, \mathcal{H}(T) (L\varepsilon d)^{1/\mathcal{K}(T)} \exp\left\{ 12 \mathcal{K}(T) e^{-bT} \left\|\Sigma_T^{-1}\right\| \left(d \log 6 + 8\mathtt{v}^2 \log(n/\delta)\right) \right\} \right\} \leqslant \frac{1}{n}
$$

and

$$
\begin{aligned}
\log(1/\varepsilon) &\leqslant \log(Ld) + \mathcal{K}(T) \left(\log \mathcal{H}(T) + \log n + 3d e^{-bT} \log 6 + 24 \mathtt{v}^2 e^{-bT} \log(n/\delta) \right) \\
&\lesssim d + \log(Ln) + e^{-bT} \log(n/\delta) + (M \vee \log \Lambda) \sqrt{d} e^{-bT} \qquad (17) \\
&\lesssim d + \log(Ln) + e^{-bT} \log(1/\delta) + (M \vee \log \Lambda) \sqrt{d} e^{-bT}.
\end{aligned}
$$

Here we used the fact that, due to (13)

$$
\log \mathcal{H}(T) = \mathcal{O}\left(d + (M \vee \log \Lambda) \sqrt{d} e^{-bT}\right).
$$

[1] One can obtain a bit sharper bound $\mathcal{O}\big(L\varepsilon(d + \log(1/\delta))\big)$ using concentration inequalities for sums of independent sub-Gaussian random variables. However, the bound $\mathcal{O}\big(L\varepsilon(d + \log(n/\delta)\big)$ obtained in a simpler way is enough for our purposes.

Hence, on the intersection of the events $\mathcal{E}_0$ and $\mathcal{E}_1$ (that, is with probability at least $(1-2\delta)$), for all $\psi \in \Psi$ it holds that

$$\begin{aligned}\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) - \frac{1}{n}\sum_{i=1}^n \log \frac{\rho_T^*(Y_i)}{\rho_T^\psi(Y_i)} \lesssim & \frac{de^{-bT} + \log n}{n} \\ & + \sqrt{(\Lambda d + M + d)\, \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) \frac{\log n \log(2|\Psi_\varepsilon|/\delta)}{n}} \\ & + (\Lambda d + M + d)\, \frac{\log n \log(2|\Psi_\varepsilon|/\delta)}{n}.\end{aligned}$$

Since $\log(1/\varepsilon)$ satisfies (17), we have

$$\log(|\Psi_\varepsilon|/\delta) \leqslant D \log \frac{2R}{\varepsilon\delta} \lesssim Dd + D\log\frac{RLn}{\delta} + D(M \vee \log\Lambda)\sqrt{d}e^{-bT}.$$

This bound implies that

$$\begin{aligned}&(\Lambda d + M + d)\, \frac{\log n \log(2|\Psi_\varepsilon|/\delta)}{n} \\ &\lesssim (\Lambda d + M + d)\left(d + \log\frac{RLn}{\delta} + (M \vee \log\Lambda)\sqrt{d}e^{-bT}\right)\frac{D\log n}{n} \\ &= \Upsilon(n,\delta)\end{aligned}$$

and

$$\mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right) - \frac{1}{n}\sum_{i=1}^n \log \frac{\rho_T^*(Y_i)}{\rho_T^\psi(Y_i)} \lesssim \sqrt{\Upsilon(n,\delta)\, \mathsf{KL}\left(\rho_T^*, \rho_T^\psi\right)} + \Upsilon(n,\delta)$$

simultaneously for all $\psi \in \Psi$ with probability at least $(1-2\delta)$.

Step 7: final bound. Let ψ° minimize[2] $\mathsf{KL}(\rho_T^*, \rho_T^\psi)$ over $\psi \in \Psi$ and denote the corresponding density by ρ_T°. Since $\widehat{\psi}$ minimizes the empirical risk, it holds that

$$\begin{aligned}&\mathsf{KL}\left(\rho_T^*, \widehat{\rho}_T\right) - \mathsf{KL}\left(\rho_T^*, \rho_T^\circ\right) \\ &\leqslant \mathsf{KL}\left(\rho_T^*, \widehat{\rho}_T\right) - \frac{1}{n}\sum_{i=1}^n \log\frac{\rho_T^*(Y_i)}{\widehat{\rho}_T(Y_i)} - \mathsf{KL}\left(\rho_T^*, \rho_T^\circ\right) + \frac{1}{n}\sum_{i=1}^n \log\frac{\rho_T^*(Y_i)}{\rho_T^\circ(Y_i)} \\ &\lesssim \sqrt{\Upsilon(n,\delta)\, \mathsf{KL}\left(\rho_T^*, \rho_T^\circ\right)} + \sqrt{\Upsilon(n,\delta)\, \mathsf{KL}\left(\rho_T^*, \widehat{\rho}_T\right)} + \Upsilon(n,\delta) \\ &\leqslant \sqrt{\Upsilon(n,\delta)\, \left(\mathsf{KL}\left(\rho_T^*, \widehat{\rho}_T\right) - \mathsf{KL}\left(\rho_T^*, \rho_T^\circ\right)\right)} + 2\sqrt{\Upsilon(n,\delta)\, \mathsf{KL}\left(\rho_T^*, \rho_T^\circ\right)} + \Upsilon(n,\delta).\end{aligned}$$

Solving the quadratic inequality with respect to $\left(\mathsf{KL}\left(\rho_T^*, \widehat{\rho}_T\right) - \mathsf{KL}\left(\rho_T^*, \rho_T^\circ\right)\right)^{1/2}$, we obtain that

$$\mathsf{KL}\left(\rho_T^*, \widehat{\rho}_T\right) - \mathsf{KL}\left(\rho_T^*, \rho_T^\circ\right) \lesssim \sqrt{\Upsilon(n,\delta)\, \mathsf{KL}\left(\rho_T^*, \rho_T^\circ\right)} + \Upsilon(n,\delta).$$

□

B Properties of the Ornstein-Uhlenbeck operator

This section contains auxiliary results on properties of the Ornstein-Uhlenbeck operator used in the proof of Theorem 1. The first one helps us to establish that under Assumption 3 $-\log\rho_T^\psi(y)$ grows as fast as $\mathcal{O}(\|y\|^2)$.

Lemma B.1. *Let $M \in \mathbb{R}$ and let $\psi : \mathbb{R}^d \to \mathbb{R}$ be such that*

$$\mathcal{T}_\infty \psi = 0 \quad \textit{and} \quad \psi(x) \leqslant M \quad \textit{for all } x \in \mathbb{R}^d.$$

[2] If the minimum is not attained, one should consider a minimizing sequence.

Then the density

$$\rho_T^\psi(y) = \int\limits_{\mathbb{R}^d} \frac{e^{\psi(y)} \mathsf{q}(y \,|\, x)}{\mathcal{T}_T e^{\psi(x)}} \rho_0(x) \, \mathrm{d}x,$$

corresponding to the log-potential ψ, satisfies

$$\psi(y) - \log \rho_T^\psi(y) \leqslant M + \frac{d}{2}\log(2\pi) + \frac{d}{2}\log\left(\frac{\|\Sigma\|}{2b} + e^{-2bT}\right) + \frac{2b}{1 - e^{-2bT}} \left\|\Sigma^{-1/2}(y - m)\right\|^2 + \frac{2be^{-2bT}}{1 - e^{-2bT}} \left\|\Sigma^{-1/2} m\right\|^2.$$

The proof of Lemma B.1 is deferred to Appendix B.1. Under Assumptions 2 and 3, it helps us to conclude that $\log\left(\rho_T^*(Y_1)/\rho_T^\psi(Y_1)\right)$ is a sub-exponential random variable. The next auxiliary lemma shows that $\log \rho_T^\psi(y)$ changes smoothly with respect to ψ. This is a key technical result which allows us to derive a uniform Bernstein-type inequality relating $\mathsf{KL}(\rho_T^*, \rho_T^\psi)$ and its empirical counterpart.

Lemma B.2. *Let us consider arbitrary functions $f_0 : \mathbb{R}^d \to \mathbb{R}$ and $f_1 : \mathbb{R}^d \to \mathbb{R}$ such that $\mathcal{T}_\infty f_0 = \mathcal{T}_\infty f_1 = 0$. Assume that there are some constants $M \in \mathbb{R}$, $A \geqslant 0$, and $B \geqslant \max\{M, 0\}$ such that*

$$-A\left\|\Sigma^{-1/2}(x - m)\right\|^2 - B \leqslant f_i(x) \leqslant M \quad \textit{for all } x \in \mathbb{R}^d \textit{ and } i \in \{0, 1\}.$$

Let

$$\mathsf{q}(y \,|\, x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(\Sigma_T)}} \exp\left\{-\frac{1}{2}\|\Sigma_T^{-1/2}(y - m_T(x))\|^2\right\}.$$

Suppose that $bT \geqslant (5 + \log d) \vee \log(160b\|\Sigma^{-1}\|)$. Then for any $y \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\left|\log \int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x)}{\mathcal{T}_T e^{f_1(x)}} \rho_0(x)\mathrm{d}x - \log \int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x)}{\mathcal{T}_T e^{f_0(x)}} \rho_0(x)\mathrm{d}x\right| \\
&\lesssim (\mathcal{T}_\infty|f_1 - f_0|)^{1/\mathcal{K}(T)} \left(d^2 + \left\|\Sigma^{-1/2}(y - m)\right\|^2\right)^{1 - 1/\mathcal{K}(T)} \\
&\quad \cdot \exp\left\{\mathcal{O}(d + (M + \log(A \vee B))\sqrt{d}e^{-bT})\right\} \\
&\quad \cdot \exp\left\{3e^{-bT}\left\|\Sigma_T^{-1/2}(y - m)\right\|^2 + \mathcal{O}(e^{-bT})\right\},
\end{aligned}$$

where the function $\mathcal{K}(t)$ is defined in (21). *The hidden constants behind $\lesssim$ and $\mathcal{O}$ depend on Σ, m, and b only.*

Let us note that Lemma B.2 provides a more subtle result than Lemma B.1. We provide the proof of Lemma B.2 in Appendix B.2. Unlike Lemma B.1, the proof of Lemma B.2 is quite long and technical and relies on non-trivial properties of the Ornstein-Uhlenbeck operator $\mathcal{T}_t$. In particular, it relies on the following result about asymptotic behaviour of $\mathcal{T}_t$.

Lemma B.3. *Let $f : \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$. Assume that there exists $M \in \mathbb{R}$ and some non-negative constants A, B, and α such that*

$$f(x) \leqslant M, \quad 0 \leqslant g(x) \leqslant A\left\|\Sigma^{-1/2}(x - m)\right\|^\alpha + B \quad \textit{for all } x \in \mathbb{R}^d. \tag{18}$$

Let us fix arbitrary $x \in \mathbb{R}^d$ and $t > 0$ and introduce

$$G(x) = Be^M + 2^{\alpha-1}Ae^M\left\|\Sigma^{-1/2}(x - m)\right\|^\alpha + 4^{\alpha-1}Ae^M(2b)^{-\alpha/2}\left((10\alpha\sqrt{d})^\alpha + d^\alpha\right), \tag{19}$$

$$\mathcal{A}(x, t) = \left(\frac{be^2}{\sqrt{d}}\left\|\Sigma^{-1/2}(x - m)\right\|^2 + 4e^2\sqrt{d}\right)\arcsin(e^{-bt}) - 10e^2\sqrt{d}\log\left(1 - e^{-2bt}\right), \tag{20}$$

and

$$\mathcal{K}(t) = \left(1 - e^{-2bt}\right)^{-5e^2\sqrt{d}} \cdot \exp\left\{2e^2\sqrt{d}\arcsin(e^{-bt})\right\} \tag{21}$$

Then, it holds that

$$e^{-\mathcal{A}(x,t)\mathcal{K}(t)} \left(\frac{\mathcal{T}_\infty g(x) e^{f(x)}}{G(x)}\right)^{\mathcal{K}(t)} \leqslant \frac{\mathcal{T}_t g(x) e^{f(x)}}{G(x)} \leqslant e^{\mathcal{A}(x,t)/\mathcal{K}(t)} \left(\frac{\mathcal{T}_\infty g(x) e^{f(x)}}{G(x)}\right)^{1/\mathcal{K}(t)}.$$

The proof of Lemma B.3 is postponed to Appendix B.3. The key ingredients of the proof are the Kolmogorov-Fokker-Planck equation, a Gronwall-type bound from Lemma D.1 and a sharp bound on the L_p-norm of a centered chi-squared random variable (see Lemma D.3).

B.1 Proof of Lemma B.1

Let us note that

$$\mathcal{T}_\infty e^{\psi(x)} \leqslant \mathcal{T}_\infty e^M \leqslant e^M.$$

This yields that

$$\psi(y) - \log \rho_T^\psi(y) = -\log \int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x)}{\mathcal{T}_t e^{\psi(x)}} \rho_0(x)\,\mathrm{d}x \leqslant M - \log \int\limits_{\mathbb{R}^d} \mathsf{q}(y \,|\, x)\rho_0(x)\,\mathrm{d}x.$$

The integral in the right-hand side is nothing but the marginal density of the base process X_t^0 at the moment $t = T$:

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} \mathsf{q}(y \,|\, x)\rho_0(x)\,\mathrm{d}x = \rho_T^0(y) \\
&\qquad = (2\pi)^{-d/2} \det\left(\Sigma_T + e^{-2bT} I_d\right)^{-1/2} \exp\left\{-\frac{1}{2}\left\|\Sigma_T^{-1/2}\big(y - m_T(0)\big)\right\|^2\right\}.
\end{aligned}$$

Hence, $\rho_T^\psi(y)$ satisfies the inequality

$$\begin{aligned}
\psi(y) - \log \rho_T^\psi(y) \leqslant M + \frac{d}{2}\log(2\pi) + \frac{1}{2}\log\det\left(\Sigma_T + e^{-2bT} I_d\right) \\
+ \frac{1}{2}\left\|\Sigma_T^{-1/2}\big(y - m_T(0)\big)\right\|^2.
\end{aligned}$$

According to the definition of Σ_T, we have

$$\Sigma_T = \frac{1 - e^{-2bT}}{2b}\Sigma \preceq \frac{1 - e^{-2bT}}{2b}\|\Sigma\| I_d,$$

and then

$$\det\left(\Sigma_T + e^{-2bT} I_d\right) \leqslant \det\left(\frac{\|\Sigma\|}{2b} I_d + e^{-2bT} I_d\right) = \left(\frac{\|\Sigma\|}{2b} + e^{-2bT}\right)^d.$$

Taking into account that

$$m_T(0) = (1 - e^{-bT})m$$

and using the Cauchy-Schwarz inequality

$$\frac{1}{2}\left\|\Sigma_T^{-1/2}\big(y - m_T(0)\big)\right\|^2 \leqslant \left\|\Sigma_T^{-1/2}\big(y - m\big)\right\|^2 + e^{-2bT}\left\|\Sigma_T^{-1/2} m\right\|^2,$$

we deduce the desired upper bound:

$$\begin{aligned}
\psi(y) - \log \rho_T^\psi(y) \leqslant M + \frac{d}{2}\log(2\pi) + \frac{d}{2}\log\left(\frac{\|\Sigma\|}{2b} + e^{-2bT}\right) \\
+ \frac{2b}{1 - e^{-2bT}}\left\|\Sigma^{-1/2}\big(y - m\big)\right\|^2 + \frac{2b e^{-2bT}}{1 - e^{-2bT}}\left\|\Sigma^{-1/2} m\right\|^2.
\end{aligned}$$

$\square$

B.2 Proof of Lemma B.2

Let us fix an arbitrary $y \in \mathbb{R}^d$ and consider

$$F(s) = \log \int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x)}{\mathcal{T}_t e^{f_s(x)}} \, \rho_0(x) \mathrm{d}x, \quad s \in [0,1],$$

where, for any $s \in [0,1]$, we introduced $f_s(x) = s f_1(x) + (1-s) f_0(x)$ for brevity. Then it is straightforward to observe that

$$\log \int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x)}{\mathcal{T}_T e^{f_1(x)}} \, \rho_0(x) \mathrm{d}x - \log \int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x)}{\mathcal{T}_T e^{f_0(x)}} \, \rho_0(x) \mathrm{d}x = F(1) - F(0).$$

Note that, due to the Lagrange mean value theorem, it is enough to show that

$$\begin{aligned} \left| \frac{\mathrm{d}F(s)}{\mathrm{d}s} \right| &\lesssim \left(\mathcal{T}_\infty |f_1 - f_0| \right)^{1/\mathcal{K}(T)} \left(d^2 + \left\| \Sigma^{-1/2} \big(y - m \big) \right\|^2 \right)^{1 - 1/\mathcal{K}(T)} \\ &\quad \cdot \exp\left\{ \mathcal{O}(d + (M + \log(A \vee B)) \sqrt{d} e^{-bT}) \right\} \\ &\quad \cdot \exp\left\{ 3 e^{-bT} \left\| \Sigma_T^{-1/2} (y - m) \right\|^2 + \mathcal{O}(e^{-bT}) \right\}. \end{aligned} \tag{22}$$

In the rest of the proof, we elaborate on the derivative $\mathrm{d}F(s)/\mathrm{d}s$ and derive the upper bound (22). Since the proof is quite long, we split it into several steps for reader's convenience.

Step 1: properties of the Ornstein-Uhlenbeck operator. Let us fix an arbitrary $s \in [0,1]$. Note that

$$\frac{\mathrm{d}F(s)}{\mathrm{d}s} = \left(\int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x) \mathcal{T}_T \left[\big(f_1(x) - f_0(x) \big) e^{f_s(x)} \right]}{\big(\mathcal{T}_T e^{f_s(x)} \big)^2} \, \rho_0(x) \mathrm{d}x \right) \Bigg/ \left(\int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x)}{\mathcal{T}_T e^{f_s(x)}} \, \rho_0(x) \mathrm{d}x \right).$$

Then the absolute value of $\mathrm{d}F(s)/\mathrm{d}s$ does not exceed

$$\left| \frac{\mathrm{d}F(s)}{\mathrm{d}s} \right| \leqslant \left(\int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x) \mathcal{T}_T \left[\big| f_1(x) - f_0(x) \big| e^{f_s(x)} \right]}{\big(\mathcal{T}_T e^{f_s(x)} \big)^2} \, \rho_0(x) \mathrm{d}x \right) \Bigg/ \left(\int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x)}{\mathcal{T}_T e^{f_s(x)}} \, \rho_0(x) \mathrm{d}x \right).$$

On this step, we focus our attention on $\mathcal{T}_T \left[\big| f_1(x) - f_0(x) \big| e^{f_s(x)} \right]$ and $\mathcal{T}_T e^{f_s(x)}$. Due to the conditions of the lemma, it holds that

$$f_s(x) = s f_1(x) + (1-s) f_0(x) \leqslant sM + (1-s)M = M$$

and, for any $x \in \mathbb{R}^d$,

$$|f_s(x)| \leqslant s |f_1(x)| + (1-s) |f_0(x)| \leqslant \max\left\{ A \|x\|^2 + B, M \right\} = A\|x\|^2 + B.$$

Here we took into account that $B \geqslant (M \vee 0)$. Then, according to Lemma B.3, it holds that

$$e^{-\mathcal{A}(x,T)\mathcal{K}(T)} \left(\frac{\mathcal{T}_\infty e^{f_s}}{e^M} \right)^{\mathcal{K}(T)} \leqslant \frac{\mathcal{T}_T e^{f_s(x)}}{e^M} \leqslant e^{\mathcal{A}(x,T)/\mathcal{K}(T)} \left(\frac{\mathcal{T}_\infty e^{f_s}}{e^M} \right)^{1/\mathcal{K}(T)} \tag{23}$$

and

$$\frac{\mathcal{T}_T \left[\big| f_1(x) - f_0(x) \big| e^{f_s(x)} \right]}{\mathcal{G}(x)} \leqslant e^{\mathcal{A}(x,T)/\mathcal{K}(T)} \left(\frac{\mathcal{T}_\infty \left[|f_1 - f_0| e^{f_s} \right]}{\mathcal{G}(x)} \right)^{1/\mathcal{K}(T)}, \tag{24}$$

where the functions $\mathcal{A}(x,t)$ and $\mathcal{K}(t)$ are defined in (20) and (21), respectively, and

$$\mathcal{G}(x) = B e^M + \frac{A e^M}{2} \left\| \Sigma^{-1/2} (x - m) \right\|^2 + \frac{A e^M}{8b} \left(400 d + d^2 \right).$$

The inequality (23) yields that

$$\int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x)}{\mathcal{T}_T e^{f_s(x)}} \, \rho_0(x) \mathrm{d}x \geqslant \frac{e^{M/\mathcal{K}(t) - M}}{\big(\mathcal{T}_\infty e^{f_s}\big)^{1/\mathcal{K}(T)}} \int\limits_{\mathbb{R}^d} e^{-\mathcal{A}(x,T)/\mathcal{K}(T)} \, \mathsf{q}(y \,|\, x) \rho_0(x) \mathrm{d}x$$

while (23) and (24) imply that

$$\begin{aligned}
&\left(\int\limits_{\mathbb{R}^d} \frac{\mathsf{q}(y \,|\, x) \mathcal{T}_T \left[\big|f_1(x) - f_0(x)\big| e^{f_s(x)} \right]}{\big(\mathcal{T}_T e^{f_s(x)}\big)^2} \, \rho_0(x) \mathrm{d}x \right) \\
&\leqslant \frac{e^{2M\mathcal{K}(T) - 2M} \left(\mathcal{T}_\infty |f_1 - f_0| e^{f_s}\right)^{1/\mathcal{K}(T)}}{\big(\mathcal{T}_\infty e^{f_s}\big)^{2\mathcal{K}(T)}} \int\limits_{\mathbb{R}^d} \frac{e^{2\mathcal{A}(x,T)\mathcal{K}(T) + \mathcal{A}(x,T)/\mathcal{K}(T)} \, \mathsf{q}(y \,|\, x)}{\big(\mathcal{G}(x)\big)^{1/\mathcal{K}(T) - 1}} \, \rho_0(x) \mathrm{d}x \\
&\leqslant \frac{e^{2M\mathcal{K}(T) + M/\mathcal{K}(T) - 2M} \left(\mathcal{T}_\infty |f_1 - f_0|\right)^{1/\mathcal{K}(T)}}{\big(\mathcal{T}_\infty e^{f_s}\big)^{2\mathcal{K}(T)}} \int\limits_{\mathbb{R}^d} \frac{e^{2\mathcal{A}(x,t)\mathcal{K}(T) + \mathcal{A}(x,T)/\mathcal{K}(T)} \, \mathsf{q}(y \,|\, x)}{\big(\mathcal{G}(x)\big)^{1/\mathcal{K}(T) - 1}} \, \rho_0(x) \mathrm{d}x.
\end{aligned}$$

In the last line we used the fact that $e^{f_s(x)} \leqslant e^M$. Taking these inequalities into account, we obtain that

$$\begin{aligned}
&\left| \frac{\mathrm{d}F(s)}{\mathrm{d}s} \right| \leqslant e^{2M\mathcal{K}(T) - M} \left(\mathcal{T}_\infty |f_1 - f_0|\right)^{1/\mathcal{K}(T)} \Big/ \left(\mathcal{T}_\infty e^{f_s}\right)^{2\mathcal{K}(T) - 1/\mathcal{K}(t)} \\
&\cdot \left(\int\limits_{\mathbb{R}^d} \frac{e^{2\mathcal{A}(x,T)\mathcal{K}(T) + \mathcal{A}(x,T)/\mathcal{K}(T)} \, \mathsf{q}(y \,|\, x)}{\big(\mathcal{G}(x)\big)^{1/\mathcal{K}(T) - 1}} \, \rho_0(x) \mathrm{d}x \right) \Bigg/ \left(\int\limits_{\mathbb{R}^d} e^{-\mathcal{A}(x,T)/\mathcal{K}(T)} \, \mathsf{q}(y \,|\, x) \rho_0(x) \mathrm{d}x \right).
\end{aligned} \tag{25}$$

Step 2: lower bound on $\mathcal{T}_\infty e^{f_s}$. A lower bound on $\mathcal{T}_\infty e^{f_s}$ easily follows from the normalization conditions $\mathcal{T}_\infty f_0 = \mathcal{T}_\infty f_1 = 0$. Indeed, according to Jensen's inequality, we have

$$\mathcal{T}_\infty e^{f_s} \geqslant e^{\mathcal{T}_\infty f_s} = 1.$$

Then

$$\left(\frac{e^M}{\mathcal{T}_\infty e^{f_s}} \right)^{2\mathcal{K}(T) - 1/\mathcal{K}(T)} \leqslant e^{2M\mathcal{K}(T) - M/\mathcal{K}(T)}$$

and (25) simplifies to

$$\begin{aligned}
&\left| \frac{\mathrm{d}F(s)}{\mathrm{d}s} \right| \leqslant e^{2M\mathcal{K}(T) - M} \left(\mathcal{T}_\infty |f_1 - f_0|\right)^{1/\mathcal{K}(T)} \\
&\cdot \left(\int\limits_{\mathbb{R}^d} \frac{e^{2\mathcal{A}(x,T)\mathcal{K}(T) + \mathcal{A}(x,T)/\mathcal{K}(T)} \, \mathsf{q}(y \,|\, x)}{\big(\mathcal{G}(x)\big)^{1/\mathcal{K}(T) - 1}} \, \rho_0(x) \mathrm{d}x \right) \Bigg/ \left(\int\limits_{\mathbb{R}^d} e^{-\mathcal{A}(x,T)/\mathcal{K}(T)} \, \mathsf{q}(y \,|\, x) \rho_0(x) \mathrm{d}x \right).
\end{aligned} \tag{26}$$

We have to bound the integrals ratio in the right-hand side of (26). This is the most technically involved part of the proof, so we do it in several steps. First, we focus our attention on the denominator.

Step 3: elaborating on the integrals ratio, part I (denominator). The goal of this step is to compute

$$\int\limits_{\mathbb{R}^d} e^{-\mathcal{A}(x,T)/\mathcal{K}(T)} \, \mathsf{q}(y \,|\, x) \rho_0(x) \mathrm{d}x.$$

The idea is to note that $\mathcal{A}(x,T)/\mathcal{K}(T) - \log\big(\mathsf{q}(y \,|\, x)\rho_0(x)\big)$ is a quadratic function with respect to x. Then the integral of interest can be reduced to an integral of a Gaussian density.

According to the definitions of $\mathcal{A}(x,t)$ and $\mathcal{K}(t)$ (see (20) and (21)), it holds that

$$\frac{\mathcal{A}(x,T)}{\mathcal{K}(T)} = \frac{be^2 \arcsin(e^{-bT})}{\mathcal{K}(T)\sqrt{d}} \left\| \Sigma^{-1/2}(x - m) \right\|^2 + \frac{2\log\mathcal{K}(T)}{\mathcal{K}(T)}.$$

Let

$$\beta_T = \frac{e^2 \arcsin(e^{-bT})\big(1 - e^{-2bT}\big)}{\mathcal{K}(T)\sqrt{d}}. \tag{27}$$

With the introduced notation, we have

$$\frac{\mathcal{A}(x,T)}{\mathcal{K}(T)} = \frac{\beta_T}{2}\left\|\Sigma_T^{-1/2}(x-m)\right\|^2 + \frac{2\log\mathcal{K}(T)}{\mathcal{K}(T)},$$

and then

$$\begin{aligned}&\int\limits_{\mathbb{R}^d} e^{-\mathcal{A}(x,T)/\mathcal{K}(T)}\,\mathsf{q}(y\,|\,x)\rho_0(x)\mathrm{d}x\\ &= e^{-2\log\mathcal{K}(T)/\mathcal{K}(T)}\int\limits_{\mathbb{R}^d}\exp\left\{-\frac{\beta_T}{2}\left\|\Sigma_T^{-1/2}(x-m)\right\|^2\right\}\mathsf{q}(y\,|\,x)\rho_0(x)\mathrm{d}x.\end{aligned}$$

Let us elaborate on

$$\exp\left\{-\frac{\beta_T}{2}\left\|\Sigma_T^{-1/2}(x-m)\right\|^2\right\}\rho_0(x) = (2\pi)^{-d/2}\cdot\exp\left\{-\frac{\beta_T}{2}\left\|\Sigma_T^{-1/2}(x-m)\right\|^2 - \frac{\|x\|^2}{2}\right\}.$$

It holds that

$$\begin{aligned}\beta_T\left\|\Sigma_T^{-1/2}(x-m)\right\|^2 + \|x\|^2 &= x^\top\left(I_d + \beta_T\Sigma_t^{-1}\right)x - 2\beta_T x^\top\Sigma_T^{-1}m + \beta_T\left\|\Sigma_T^{-1/2}m\right\|^2\\ &= \left\|\left(I_d+\beta_T\Sigma_T^{-1}\right)^{1/2}\left(x - \beta_T(I_d+\beta_T\Sigma_T^{-1})^{-1}\Sigma_T^{-1}m\right)\right\|^2\\ &\quad + \beta_T\left\|\Sigma_T^{-1/2}m\right\|^2 - \beta_T^2\left\|\left(I_d+\beta_T\Sigma_T^{-1}\right)^{-1/2}\Sigma_T^{-1}m\right\|^2\\ &= \left\|\left(I_d+\beta_T\Sigma_T^{-1}\right)^{1/2}\left(x-\beta_T(\Sigma_T+\beta_T I_d)^{-1}m\right)\right\|^2\\ &\quad + \beta_T\left\|\Sigma_T^{-1/2}m\right\|^2 - \beta_T^2\left\|\left(\Sigma_T+\beta_T I_d\right)^{-1/2}\Sigma_T^{-1}m\right\|^2.\end{aligned}$$

Since

$$\Sigma_T^{-1} - \beta_T\Sigma_T^{-1}\left(I_d+\beta_T\Sigma_T^{-1}\right)^{-1}\Sigma_T^{-1} = \Sigma_T^{-1}\left(I_d+\beta_T\Sigma_T^{-1}\right)^{-1} = \left(\Sigma_T+\beta_T I_d\right)^{-1},$$

we obtain that

$$\beta_T\left\|\Sigma_T^{-1/2}m\right\|^2 - \beta_T^2\left\|\left(I_d+\beta_T\Sigma_T^{-1}\right)^{-1/2}\Sigma_T^{-1}m\right\|^2 = \beta_T\left\|\left(\Sigma_T+\beta_T I_d\right)^{-1/2}m\right\|^2$$

and, consequently,

$$\begin{aligned}\beta_T\left\|\Sigma_T^{-1/2}(x-m)\right\|^2 - \|x\|^2 &= \left\|\left(I_d+\beta_T\Sigma_T^{-1}\right)^{-1/2}\left(x-\beta_T(\Sigma_T+\beta_T I_d)^{-1}m\right)\right\|^2\\ &\quad + \beta_T\left\|\left(\Sigma_T+\beta_T I_d\right)^{-1/2}m\right\|^2.\end{aligned}$$

This means that

$$\begin{aligned}&\det\left(I_d+\beta_T\Sigma_T^{-1}\right)^{-1/2}\exp\left\{-\frac{\beta_T}{2}\left\|\Sigma_T^{-1/2}(x-m)\right\|^2 - \frac{\beta_T}{2}\left\|\left(\Sigma_T+\beta_T I_d\right)^{-1/2}m\right\|^2\right\}\rho_0(x)\\ &= (2\pi)^{-d/2}\det\left(I_d+\beta_T\Sigma_T^{-1}\right)^{-1/2}\\ &\quad\cdot\exp\left\{-\frac{1}{2}\left\|\left(I_d+\beta_T\Sigma_T^{-1}\right)^{1/2}\left(x-\beta_T(\Sigma_T+\beta_T I_d)^{-1}m\right)\right\|^2\right\}\end{aligned}$$

is the density of $\mathcal{N}\big(\beta_T(\Sigma_T+\beta_T I_d)^{-1}m, (I_d+\beta_T\Sigma_T^{-1})^{-1}\big)$. Then, due to Lemma D.2, it holds that

$$\begin{aligned}&\int\limits_{\mathbb{R}^d}\exp\left\{-\frac{\beta_T}{2}\left\|\Sigma_T^{-1/2}(x-m)\right\|^2\right\}\mathsf{q}(y\,|\,x)\rho_0(x)\mathrm{d}x\\ &= \frac{\det\left(I_d+\beta_T\Sigma_T^{-1}\right)^{1/2}}{(2\pi)^{d/2}\det\left(\Sigma_T + e^{-2bt}(I_d+\beta_T\Sigma_T^{-1})^{-1}\right)^{1/2}}\exp\left\{\frac{\beta_T}{2}\left\|\left(\Sigma_T+\beta_T I_d\right)^{-1/2}m\right\|^2\right\}\\ &\quad\cdot\exp\left\{-\frac{1}{2}\left\|\left(\Sigma_T+e^{-2bT}(I_d+\beta_T\Sigma_T^{-1})^{-1}\right)^{-1/2}\left(y-\mu_T(\beta_T)\right)\right\|^2\right\},\end{aligned}$$

where we introduced

$$\mu_T(\beta_T) = m_T\big(\beta_T(\Sigma_T + \beta_T I_d)^{-1} m\big), \tag{28}$$

and $m_T(\cdot)$ is defined in (4). Thus, we obtained that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} e^{-\mathcal{A}(x,T)/\mathcal{K}(T)}\, \mathsf{q}(y \,|\, x)\rho_0(x)\mathrm{d}x \\
&= \frac{e^{-2\log\mathcal{K}(T)/\mathcal{K}(T)} \det\big(I_d + \beta_T \Sigma_T^{-1}\big)^{1/2}}{(2\pi)^{d/2}\det\big(\Sigma_T + e^{-2bT}(I_d + \beta_T\Sigma_T^{-1})^{-1}\big)^{1/2}} \exp\left\{\frac{\beta_T}{2}\left\|(\Sigma_T + \beta_T I_d)^{-1/2}\, m\right\|^2\right\} \\
&\quad \cdot \exp\left\{-\frac{1}{2}\left\|\big(\Sigma_T + e^{-2bt}(I_d + \beta_T\Sigma_T^{-1})^{-1}\big)^{-1/2}\big(y - \mu_T(\beta_T)\big)\right\|^2\right\} \\
&\geqslant \frac{e^{-2\log\mathcal{K}(T)/\mathcal{K}(T)}}{(2\pi)^{d/2}\det\left(\Sigma_T + \beta_T I_d + e^{-2bT} I_d\right)^{1/2}} \\
&\quad \cdot \exp\left\{-\frac{1}{2}\left\|\big(\Sigma_T + e^{-2bT}(I_d + \beta_T\Sigma_T^{-1})^{-1}\big)^{-1/2}\big(y - \mu_T(\beta_T)\big)\right\|^2\right\}.
\end{aligned}$$

The expression in the right-hand side can be simplified even further, if one uses the inequalities

$$\big(\Sigma_T + e^{-2bT}(I_d + \beta_T\Sigma_T^{-1})^{-1}\big)^{-1} \preceq \Sigma_T^{-1} \quad \text{and} \quad \frac{\log u}{u} \leqslant e^{-1} \quad \text{for all } u > 0.$$

Then $\log\mathcal{K}(T)/\mathcal{K}(T) \leqslant e^{-1}$,

$$\left\|\big(\Sigma_T + e^{-2bT}(I_d + \beta_T\Sigma_T^{-1})^{-1}\big)^{-1/2}\big(y - \mu_T(\beta_T)\big)\right\|^2 \leqslant \left\|\Sigma_T^{-1/2}\big(y - \mu_T(\beta_T)\big)\right\|^2,$$

and it holds that

$$\int\limits_{\mathbb{R}^d} e^{-\mathcal{A}(x,T)/\mathcal{K}(T)}\, \mathsf{q}(y \,|\, x)\rho_0(x)\mathrm{d}x \geqslant \frac{e^{-2/e}}{(2\pi)^{d/2}\det\left(\Sigma_T + \beta_T I_d + e^{-2bT} I_d\right)^{1/2}} \tag{29}$$

$$\cdot \exp\left\{-\frac{1}{2}\left\|\Sigma_T^{-1/2}\big(y - \mu_T(\beta_T)\big)\right\|^2\right\}. \tag{30}$$

Step 4: elaborating on the integrals ratio, part II (intermediate). To bound the numerator

$$\int\limits_{\mathbb{R}^d} \big(\mathcal{G}(x)\big)^{1-1/\mathcal{K}(T)} e^{2\mathcal{A}(x,T)\mathcal{K}(T) + \mathcal{A}(x,T)/\mathcal{K}(T)}\, \mathsf{q}(y \,|\, x)\, \rho_0(x)\mathrm{d}x,$$

we rely on the same ideas as in the previous step. However, before we proceed, let us make some preparations. Namely, on this step we consider

$$2\mathcal{A}(x,T)\mathcal{K}(T) + \mathcal{A}(x,T)/\mathcal{K}(T).$$

Let us recall that, due to the definitions of $\mathcal{A}(x,t)$ and $\mathcal{K}(t)$ (see (20) and (21)), we have

$$\mathcal{A}(x,T) = \frac{be^2\arcsin(e^{-bT})}{\sqrt{d}}\left\|\Sigma^{-1/2}(x-m)\right\|^2 + 2\log\mathcal{K}(T).$$

Introducing

$$\alpha_T = \left(2\mathcal{K}(T) + \frac{1}{\mathcal{K}(T)}\right)\frac{e^2\arcsin(e^{-bT})(1-e^{-2bT})}{\sqrt{d}} \leqslant (2\mathcal{K}(T)+1)\,\frac{e^2\arcsin(e^{-bT})}{\sqrt{d}}. \tag{31}$$

we obtain that

$$2\mathcal{A}(x,T)\mathcal{K}(T) + \frac{\mathcal{A}(x,T)}{\mathcal{K}(T)} = \frac{\alpha_T}{2}\left\|\Sigma_T^{-1/2}(x-m)\right\|^2 + 4\mathcal{K}(T)\log\mathcal{K}(T) + \frac{2\log\mathcal{K}(T)}{\mathcal{K}(T)},$$

and then

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^d} \big(\mathcal{G}(x)\big)^{1-1/\mathcal{K}(T)} e^{2\mathcal{A}(x,T)\mathcal{K}(T)+\mathcal{A}(x,T)/\mathcal{K}(T)}\, \mathsf{q}(y \,|\, x)\, \rho_0(x) \mathrm{d}x \\
&= e^{2\log\mathcal{K}(T)(2\mathcal{K}(T)+1/\mathcal{K}(T))} \int\limits_{\mathbb{R}^d} \big(\mathcal{G}(x)\big)^{1-1/\mathcal{K}(T)} e^{\alpha_T \left\|\Sigma_T^{-1/2}(x-m)\right\|^2/2}\, \mathsf{q}(y \,|\, x)\, \rho_0(x) \mathrm{d}x \qquad (32) \\
&\leqslant e^{4\log\mathcal{K}(T)\mathcal{K}(T)+2/e} \int\limits_{\mathbb{R}^d} \big(\mathcal{G}(x)\big)^{1-1/\mathcal{K}(T)} e^{\alpha_T \left\|\Sigma_T^{-1/2}(x-m)\right\|^2/2}\, \mathsf{q}(y \,|\, x)\, \rho_0(x) \mathrm{d}x.
\end{aligned}
$$

In the last line, we used the inequality $\log u/u \leqslant 1/e$ for all $u > 0$. We are going to show that $\alpha_T \leqslant 40e^{-bT}$ under the conditions of the lemma. Let us first elaborate on $1 - 1/\mathcal{K}(T)$ for this purpose. We start with the inequalities

$$
u \leqslant \arcsin(u) \leqslant \frac{\pi u}{2} \quad \text{and} \quad u \leqslant -\log(1-u) \leqslant \frac{u}{1-u}
$$

holding for all $u \in (0, 1)$. They yield that

$$
e^{-bT} \leqslant \arcsin\left(e^{-bT}\right) \leqslant \frac{\pi e^{-bT}}{2} \quad \text{and} \quad e^{-2bT} \leqslant -\log\left(1 - e^{-2bT}\right) \leqslant \frac{e^{-2bT}}{1-e^{-2bT}}
$$

The right-hand side of the latter inequality can be simplified even further if one takes into account that $bT \geqslant 5 + \log d \geqslant 5$:

$$
-\log\left(1 - e^{-2bT}\right) \leqslant \frac{e^{-2bT}}{1-e^{-2bT}} \leqslant \frac{e^{-5-bT}}{1-e^{-10}}.
$$

This implies that

$$
\begin{aligned}
\log\mathcal{K}(T) &= 2e^2\sqrt{d}\arcsin\left(e^{-bT}\right) - 5e^2\sqrt{d}\log\left(1-e^{-2bT}\right) \\
&\leqslant e^2\sqrt{d}\, e^{-bT}\left(\pi + \frac{5e^{-5}}{1-e^{-10}}\right) \qquad (33) \\
&\leqslant 4e^2\sqrt{d}\, e^{-bT}.
\end{aligned}
$$

Since $bT \geqslant 5 + \log d$, it holds that

$$
4e^2\sqrt{d}\, e^{-bT} \leqslant 4e^{-3} < \frac{1}{5}.
$$

Then

$$
\alpha_T \leqslant (2\mathcal{K}(T)+1)\, \frac{e^2\arcsin(e^{-bT})}{\sqrt{d}} \leqslant \left(2e^5+1\right) \frac{\pi e^2 e^{-bT}}{2} \leqslant 40e^{-bT}.
$$

Let us note that the assumption $bT \geqslant \log(160b\|\Sigma^{-1}\|)$ ensures that

$$
\alpha_T \leqslant 40e^{-bT} \leqslant \frac{1}{4b\|\Sigma^{-1}\|} \leqslant \frac{1}{2\|\Sigma_T^{-1}\|}. \qquad (34)
$$

This fact will play an important role during the proof.

Step 5: elaborating on the integrals ratio, part III (numerator). We move to an upper bound on

$$
\int\limits_{\mathbb{R}^d} \big(\mathcal{G}(x)\big)^{1-1/\mathcal{K}(T)} e^{2\mathcal{A}(x,T)\mathcal{K}(T)+\mathcal{A}(x,T)/\mathcal{K}(T)}\, \mathsf{q}(y \,|\, x)\, \rho_0(x) \mathrm{d}x.
$$

As in the previous step, the idea is to represent the integral of interest as an expectation of a function of a Gaussian random vector and use Lemma D.2. Let us recall that (see (32))

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^d} \big(\mathcal{G}(x)\big)^{1-1/\mathcal{K}(T)} e^{2\mathcal{A}(x,T)\mathcal{K}(T)+\mathcal{A}(x,T)/\mathcal{K}(T)}\, \mathsf{q}(y \,|\, x)\, \rho_0(x) \mathrm{d}x \\
&\leqslant e^{4\log\mathcal{K}(T)\mathcal{K}(T)+2/e} \int\limits_{\mathbb{R}^d} \big(\mathcal{G}(x)\big)^{1-1/\mathcal{K}(T)} e^{\alpha_T \left\|\Sigma_T^{-1/2}(x-m)\right\|^2/2}\, \mathsf{q}(y \,|\, x)\, \rho_0(x) \mathrm{d}x.
\end{aligned}
$$

Similarly to the third step, we note that

$$
\begin{aligned}
-\alpha_T \left\| \Sigma_T^{-1/2}(x - m) \right\|^2 + \|x\|^2 &= x^\top \left(I_d - \alpha_T \Sigma_T^{-1} \right) x + 2\alpha_T x^\top \Sigma_T^{-1} m - \alpha_T \left\| \Sigma_T^{-1/2} m \right\|^2 \\
&= \left\| \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{1/2} \left(x + \alpha_T (\Sigma_T - \alpha_T I_d)^{-1} m \right) \right\|^2 \\
&\quad - \alpha_T \left\| \Sigma_T^{-1/2} m \right\|^2 + \alpha_T^2 \left\| \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1/2} \Sigma_T^{-1} m \right\|^2
\end{aligned}
$$

and

$$
-\alpha_T \left\| \Sigma_T^{-1/2} m \right\|^2 + \alpha_T^2 \left\| \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1/2} \Sigma_T^{-1} m \right\|^2 = -\alpha_T \left\| (\Sigma_T - \alpha_T I_d)^{-1/2} m \right\|^2 .
$$

We would like to emphasize that $\Sigma_T - \alpha_T I_d \succeq 0.5 \Sigma_T$ due to (34). Then

$$
\begin{aligned}
-\alpha_T \left\| \Sigma_T^{-1/2}(x - m) \right\|^2 - 2 \log \rho_0(x) &= \left\| \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{1/2} \left(x + \alpha_T (I_d - \alpha_T \Sigma_T^{-1})^{-1} \Sigma_T^{-1} m \right) \right\|^2 \\
&\quad - \alpha_T \left\| (\Sigma_T - \alpha_T I_d)^{-1/2} m \right\|^2 ,
\end{aligned}
$$

and we conclude that

$$
\det \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1/2} \exp \left\{ \frac{\alpha_T}{2} \left\| \Sigma_T^{-1/2}(x - m) \right\|^2 + \frac{\alpha_T}{2} \left\| (\Sigma_T - \alpha_T I_d)^{-1/2} m \right\|^2 \right\} \rho_0(x)
$$

is the density of the Gaussian distribution

$$
\mathcal{N}\left(-\alpha_T (\Sigma_T - \alpha_T I_d)^{-1} m, \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \right) .
$$

According to Lemma D.2, it holds that

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^d} \left(\mathcal{G}(x) \right)^{1 - 1/\mathcal{K}(T)} e^{2\mathcal{A}(x,T)\mathcal{K}(T) + \mathcal{A}(x,T)/\mathcal{K}(T)} \, \mathsf{q}(y \,|\, x) \, \rho_0(x) \mathrm{d}x \\
&= \frac{e^{4\mathcal{K}(T) \log \mathcal{K}(T) + 2/e} \det \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{1/2}}{(2\pi)^{d/2} \det \left(\Sigma_T + e^{-2bT} \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \right)^{1/2}} \exp \left\{ -\frac{\alpha_T}{2} \left\| (\Sigma_T - \alpha_T I_d)^{-1/2} m \right\|^2 \right\} \\
&\quad \cdot \exp \left\{ -\frac{1}{2} \left\| \left(\Sigma_T + e^{-2bt} \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \right)^{-1/2} \left(y - \mu_T(-\alpha_T) \right) \right\|^2 \right\} \mathbb{E}\big(\mathcal{G}(\xi) \big)^{1 - 1/\mathcal{K}(T)} \qquad (35) \\
&\leqslant \frac{e^{4\mathcal{K}(T) \log \mathcal{K}(T) + 2/e}}{(2\pi)^{d/2} \det \left(\Sigma_T - \alpha_T I_d + e^{-2bT} I_d \right)^{1/2}} \cdot \mathbb{E}\big(\mathcal{G}(\xi) \big)^{1 - 1/\mathcal{K}(T)} \\
&\quad \cdot \exp \left\{ -\frac{1}{2} \left\| \left(\Sigma_T + e^{-2bT} \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \right)^{-1/2} \left(y - \mu_T(-\alpha_T) \right) \right\|^2 \right\} ,
\end{aligned}
$$

where

$$
\mu_T(-\alpha_T) = m_T\big(-\alpha_T (\Sigma_T - \alpha_T I_d)^{-1} m \big) \qquad (36)
$$

and $\xi \sim \mathcal{N}(\breve{\mu}, \breve{\Omega})$ is a Gaussian random vector with mean

$$
\begin{aligned}
\breve{\mu} &= -\alpha_T (\Sigma_T - \alpha_T I_d)^{-1} m \\
&\quad + e^{-bT} \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \left(\Sigma_T + e^{-2bT} \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \right)^{-1} \left(y - \mu_T(-\alpha_T) \right) \qquad (37)
\end{aligned}
$$

and covariance

$$
\breve{\Omega} = \left(I_d - \alpha_T \Sigma_T^{-1} + e^{-2bT} \Sigma_T^{-1} \right)^{-1} . \qquad (38)
$$

Step 6: bounding $\mathbb{E}\big(\mathcal{G}(\xi) \big)^{1 - 1/\mathcal{K}(T)}$. Using (33), we easily derive

$$
1 - \frac{1}{\mathcal{K}(T)} = 1 - e^{-\log \mathcal{K}(T)} \leqslant \log \mathcal{K}(T) \leqslant 4e^2 \sqrt{d} \, e^{-bT} .
$$

Let us recall that

$$\begin{aligned}\mathcal{G}(x) &= Be^M + \frac{Ae^M}{2}\left\|\Sigma^{-1/2}(x-m)\right\|^2 + \frac{Ae^M}{8b}\left(400d+d^2\right)\\ &= Be^M + \frac{Ae^M(1-e^{-2bt})}{4b}\left\|\Sigma_t^{-1/2}(x-m)\right\|^2 + \frac{Ae^M}{8b}\left(400d+d^2\right).\end{aligned}$$

Since $1-1/\mathcal{K}(T) \leqslant 4e^2\sqrt{d}\,e^{-bT} < 1$, it holds that

$$\begin{aligned}\mathbb{E}\big(\mathcal{G}(\xi)\big)^{1-1/\mathcal{K}(T)} &\leqslant e^{M-M/\mathcal{K}(T)}\left(B+\frac{A}{8b}\left(400d+d^2\right)\right)^{1-1/\mathcal{K}(T)}\\ &\qquad + \left(\frac{Ae^M(1-e^{-2bT})}{4b}\mathbb{E}\left\|\Sigma_T^{-1/2}(\xi-m)\right\|^2\right)^{1-1/\mathcal{K}(T)} \qquad (39)\\ &\lesssim e^{\mathcal{O}((M+\log B)\sqrt{d}e^{-bT})} + \left(\frac{Ae^M}{b}\right)^{\mathcal{O}(\sqrt{d}e^{-bT})}\left(d^2+\mathbb{E}\left\|\Sigma_T^{-1/2}(\xi-m)\right\|^2\right)^{1-1/\mathcal{K}(T)}.\end{aligned}$$

One can compute the expectation in the right-hand side explicitly:

$$\mathbb{E}\left\|\Sigma_T^{-1/2}(\xi-m)\right\|^2 = \mathrm{Tr}\left(\Sigma_T^{-1}\breve{\Omega}\right) + \left\|\Sigma_T^{-1/2}(\breve{\mu}-m)\right\|^2$$

Due to the definitions of $\mu_T(-\alpha_T)$ and $m_T(\cdot)$ (see (36) and (4)), we have

$$\mu_T(-\alpha_T) - m = e^{-bT}\big(-\alpha_T(\Sigma_T-\alpha_T I_d)^{-1}m - m\big).$$

This yields that

$$\begin{aligned}\breve{\mu}-m &= \left(1-e^{-bT}\right)\left(-\alpha_T(\Sigma_T-\alpha_T I_d)^{-1}m-m\right)\\ &\qquad + e^{-bT}\left(I_d-\alpha_T\Sigma_T^{-1}\right)^{-1}\left(\Sigma_T+e^{-2bT}\left(I_d-\alpha_T\Sigma_T^{-1}\right)^{-1}\right)^{-1}\big(y-m\big)\\ &= -\left(1-e^{-bT}\right)\left(I_d-\alpha_T\Sigma_T^{-1}\right)^{-1}m + e^{-bT}\left(\Sigma_T-\alpha_T I_d+e^{-2bT}I_d\right)^{-1}\big(y-m\big).\end{aligned}$$

The conditions of the lemma and (34) ensure that

$$\alpha_T \leqslant \frac{1}{2\|\Sigma_T^{-1}\|} \quad \text{and} \quad e^{-2bT} \leqslant \frac{1}{\|\Sigma_T^{-1}\|}.$$

Then, due to the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left\|\Sigma_T^{-1/2}(\breve{\mu}-m)\right\|^2 &\leqslant 2\left\|\left(1-e^{-bT}\right)\left(I_d-\alpha_T\Sigma_T^{-1}\right)^{-1}\Sigma_T^{-1/2}m\right\|^2\\ &\qquad + 2e^{-2bT}\left\|\left(\Sigma_T-\alpha_T I_d+e^{-2bT}I_d\right)^{-1}\Sigma_T^{-1/2}\big(y-m\big)\right\|^2\\ &\leqslant 8\left\|\Sigma_T^{-1/2}m\right\|^2 + 8e^{-2bT}\left\|\Sigma_T^{-3/2}\big(y-m\big)\right\|^2 \qquad (40)\\ &\leqslant 8\left\|\Sigma_T^{-1/2}m\right\|^2 + 8\left\|\Sigma_T^{-1/2}\big(y-m\big)\right\|^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{Tr}\left(\Sigma_T^{-1}\breve{\Omega}\right) &= \mathrm{Tr}\left(\Sigma_T^{-1}\left(I_d-\alpha_T\Sigma_T^{-1}+e^{-2bT}\Sigma_T^{-1}\right)^{-1}\right)\\ &= \mathrm{Tr}\left(\left(\Sigma_T-\alpha_T I_d+e^{-2bT}I_d\right)^{-1}\right) \qquad (41)\\ &\leqslant 2\mathrm{Tr}(\Sigma_T^{-1}).\end{aligned}$$

Substituting (40) and (41) into (39), we obtain that

$$\begin{aligned}&\mathbb{E}\big(\mathcal{G}(\xi)\big)^{1-1/\mathcal{K}(T)} \qquad (42)\\ &\lesssim e^{\mathcal{O}((M+\log B)\sqrt{d}e^{-bT})} + \left(\frac{Ae^M}{b}\right)^{\mathcal{O}(\sqrt{d}e^{-bT})}\left(d^2+\mathbb{E}\left\|\Sigma_T^{-1/2}(\xi-m)\right\|^2\right)^{1-1/\mathcal{K}(T)}\\ &\lesssim e^{\mathcal{O}((M+\log B)\sqrt{d}e^{-bT})} + \left(\frac{Ae^M}{b}\right)^{\mathcal{O}(\sqrt{d}e^{-bT})}\left(d^2+\left\|\Sigma_T^{-1/2}\big(y-m\big)\right\|^2\right)^{1-1/\mathcal{K}(T)}\\ &\lesssim e^{\mathcal{O}((M+\log B)\sqrt{d}e^{-bT})} + \left(\frac{Ae^M}{b}\right)^{\mathcal{O}(\sqrt{d}e^{-bT})}\left(d^2+b\left\|\Sigma^{-1/2}\big(y-m\big)\right\|^2\right)^{1-1/\mathcal{K}(T)}.\end{aligned}$$

Step 7: bounding the integrals ratio. Summing up (29) and (35), we obtain that

$$
\begin{aligned}
&\left(\int\limits_{\mathbb{R}^d} \frac{e^{2\mathcal{A}(x,T)\mathcal{K}(T) + \mathcal{A}(x,T)/\mathcal{K}(T)}\, \mathsf{q}(y \,|\, x)}{\big(\mathcal{G}(x)\big)^{1/\mathcal{K}(T)-1}}\, \rho_0(x) \mathrm{d}x \right) \Bigg/ \left(\int\limits_{\mathbb{R}^d} e^{-\mathcal{A}(x,T)/\mathcal{K}(T)}\, \mathsf{q}(y \,|\, x) \rho_0(x) \mathrm{d}x \right) \\
&\leqslant \frac{e^{4\log \mathcal{K}(T)\mathcal{K}(T) + 4/e} \det\left(\Sigma_T + \beta_T I_d + e^{-2bT} I_d\right)^{1/2}}{\det\left(\Sigma_T - \alpha_T I_d + e^{-2bT} I_d\right)^{1/2}}\, \mathbb{E}\big(\mathcal{G}(\xi)\big)^{1-1/\mathcal{K}(T)} \\
&\quad \cdot \exp\left\{ -\frac{1}{2} \left\| \left(\Sigma_T + e^{-2bT} \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \right)^{-1/2} \big(y - \mu_T(-\alpha_T) \big) \right\|^2 \right\} \\
&\quad \cdot \exp\left\{ \frac{1}{2} \left\| \Sigma_T^{-1/2} \big(y - \mu_T(\beta_T) \big) \right\|^2 \right\}.
\end{aligned}
$$

Due to the definitions of α_T and β_T (see (31) and (27) respectively) and (34), we have

$$
\beta_T \leqslant \alpha_T \leqslant 40 e^{-bT} \leqslant \frac{1}{2\|\Sigma_T^{-1}\|}. \tag{43}
$$

This implies that

$$
\frac{\det\left(\Sigma_T + \beta_T I_d + e^{-2bT} I_d\right)^{1/2}}{\det\left(\Sigma_T - \alpha_T I_d + e^{-2bT} I_d\right)^{1/2}} \leqslant \frac{\det\left(1.5\Sigma_T + e^{-2bT} I_d\right)^{1/2}}{\det\left(0.5\Sigma_T + e^{-2bT} I_d\right)^{1/2}} \leqslant 3^d.
$$

Taking into account that $\mathcal{K}(T) \log \mathcal{K}(T) = \mathcal{O}(\sqrt{d} e^{-bT})$, we deduce that

$$
\begin{aligned}
&\left(\int\limits_{\mathbb{R}^d} \frac{e^{2\mathcal{A}(x,T)\mathcal{K}(T) + \mathcal{A}(x,T)/\mathcal{K}(T)}\, \mathsf{q}(y \,|\, x)}{\big(\mathcal{G}(x)\big)^{1/\mathcal{K}(T)-1}}\, \rho_0(x) \mathrm{d}x \right) \Bigg/ \left(\int\limits_{\mathbb{R}^d} e^{-\mathcal{A}(x,T)/\mathcal{K}(T)}\, \mathsf{q}(y \,|\, x) \rho_0(x) \mathrm{d}x \right) \\
&\leqslant e^{\mathcal{O}(d)}\, \mathbb{E}\big(\mathcal{G}(\xi)\big)^{1-1/\mathcal{K}(T)} \\
&\quad \cdot \exp\left\{ -\frac{1}{2} \left\| \left(\Sigma_T + e^{-2bT} \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \right)^{-1/2} \big(y - \mu_T(-\alpha_T) \big) \right\|^2 \right\} \\
&\quad \cdot \exp\left\{ \frac{1}{2} \left\| \Sigma_T^{-1/2} \big(y - \mu_T(\beta_T) \big) \right\|^2 \right\}.
\end{aligned} \tag{44}
$$

Let us consider the difference

$$
\frac{1}{2} \left\| \Sigma_T^{-1/2} \big(y - \mu_T(\beta_T) \big) \right\|^2 - \frac{1}{2} \left\| \left(\Sigma_T + e^{-2bT} \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \right)^{-1/2} \big(y - \mu_T(-\alpha_T) \big) \right\|^2.
$$

It holds that

$$
\begin{aligned}
&\frac{1}{2} \left\| \Sigma_T^{-1/2} \big(y - \mu_T(\beta_T) \big) \right\|^2 - \frac{1}{2} \left\| \left(\Sigma_T + e^{-2bT} \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \right)^{-1/2} \big(y - \mu_T(-\alpha_T) \big) \right\|^2 \\
&= \frac{1}{2} (y - m)^\top \Sigma_T^{-1} (y - m) + (y - m)^\top \Sigma_T^{-1} \big(m - \mu_T(\beta_T) \big) + \frac{1}{2} \left\| \Sigma_T^{-1/2} \big(m - \mu_T(\beta_T) \big) \right\|^2 \\
&\quad - \frac{1}{2} (y - m)^\top \left(\Sigma_T + e^{-2bT} \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \right)^{-1} (y - m) \\
&\quad - (y - m)^\top \left(\Sigma_T + e^{-2bT} \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \right)^{-1} \big(m - \mu_T(-\alpha_T) \big) \\
&\quad - \frac{1}{2} \left\| \left(\Sigma_T + e^{-2bT} \left(I_d - \alpha_T \Sigma_T^{-1} \right)^{-1} \right)^{-1/2} \big(m - \mu_T(-\alpha_T) \big) \right\|^2.
\end{aligned}
$$

We can simplify the expression in the right-hand side noting that

$$
\begin{aligned}
&\Sigma_T^{-1} - \left(\Sigma_T + e^{-2bT}\left(I_d - \alpha_T \Sigma_T^{-1}\right)^{-1}\right)^{-1} \\
&= \Sigma_T^{-1}\left(I_d - \left(I_d + e^{-2bT}\left(\Sigma_T - \alpha_T I_d\right)^{-1}\right)^{-1}\right) \\
&= e^{-2bT}\Sigma_T^{-1}\left(\Sigma_T - \alpha_T I_d\right)^{-1}\left(I_d + e^{-2bT}\left(\Sigma_T - \alpha_T I_d\right)^{-1}\right)^{-1} \\
&= e^{-2bT}\Sigma_T^{-1}\left(\Sigma_T - \alpha_T I_d + e^{-2bT}\right)^{-1} \preceq 2e^{-2bT}\,\Sigma_T^{-2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&(y-m)^\top \Sigma_T^{-1}(y-m) - (y-m)^\top\left(\Sigma_T + e^{-2bT}\left(I_d - \alpha_T\Sigma_T^{-1}\right)^{-1}\right)^{-1}(y-m) \\
&\leqslant 2e^{-2bT}(y-m)^\top \Sigma_T^{-2}(y-m)
\end{aligned}
$$

and

$$
\begin{aligned}
&(y-m)^\top\left[\left(\Sigma_T + e^{-2bT}\left(I_d - \alpha_T\Sigma_T^{-1}\right)^{-1}\right)^{-1} - \Sigma_T^{-1}\right]\left(m - \mu_T(-\alpha_T)\right) \\
&e^{-2bT}(y-m)^\top\Sigma_T^{-1}\left(\Sigma_T - \alpha_T I_d + e^{-2bT}\right)^{-1}\left(m - \mu_T(-\alpha_T)\right) \\
&\leqslant e^{-2bT}\left\|\Sigma_T^{-1}(y-m)\right\|\,\left\|\left(\Sigma_T - \alpha_T I_d + e^{-2bT}\right)^{-1}\left(m - \mu_T(-\alpha_T)\right)\right\| \\
&\leqslant 2e^{-2bT}\left\|\Sigma_T^{-1}(y-m)\right\|\,\left\|\Sigma_T^{-1}\big(m - \mu_T(-\alpha_T)\big)\right\|.
\end{aligned}
$$

Thus, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\left\|\Sigma_T^{-1/2}\big(y - \mu_T(\beta_T)\big)\right\|^2 - \frac{1}{2}\left\|\left(\Sigma_T + e^{-2bT}\left(I_d - \alpha_T\Sigma_T^{-1}\right)^{-1}\right)^{-1/2}\left(y - \mu_T(-\alpha_T)\right)\right\|^2 \\
&\leqslant e^{-2bT}\left\|\Sigma_T^{-1}(y-m)\right\|^2 + (y-m)^\top\Sigma_T^{-1}\big(\mu_T(-\alpha_T) - \mu_T(\beta_T)\big) + \frac{1}{2}\left\|\Sigma_T^{-1/2}\big(m - \mu_T(\beta_T)\big)\right\|^2 \\
&\quad + 2e^{-2bT}\left\|\Sigma_T^{-1}(y-m)\right\|\left\|\Sigma_T^{-1}\big(m - \mu_T(-\alpha_T)\big)\right\|.
\end{aligned}
$$

Due to the Cauchy-Schwarz inequality, the right-hand side does not exceed

$$
\begin{aligned}
&2e^{-2bT}\left\|\Sigma_T^{-1}(y-m)\right\|^2 + (y-m)^\top\Sigma_T^{-1}\big(\mu_T(-\alpha_T) - \mu_T(\beta_T)\big) \\
&\quad + \frac{1}{2}\left\|\Sigma_T^{-1/2}\big(m - \mu_T(\beta_T)\big)\right\|^2 + e^{-2bT}\left\|\Sigma_T^{-1}\big(m - \mu_T(-\alpha_T)\big)\right\|^2.
\end{aligned}
$$

Since

$$
\begin{aligned}
\Sigma_T^{-1}\big(\mu_T(\beta_T) - \mu_T(-\alpha_T)\big) &= e^{-bT}\,\Sigma_T^{-1}\left(\beta_T(\Sigma_T + \beta_T I_d)^{-1} - \alpha_T(\Sigma_T - \alpha_T I_d)^{-1}\right)m \\
&= (\alpha_T + \beta_T)(\Sigma_T + \beta_T I_d)^{-1}(\Sigma_T - \alpha_T I_d)^{-1}m,
\end{aligned}
$$

we have

$$
\begin{aligned}
&(y-m)^\top\Sigma_T^{-1}\big(\mu_T(-\alpha_T) - \mu_T(\beta_T)\big) \\
&= e^{-bT}(\alpha_T + \beta_T)(y-m)^\top(\Sigma_T + \beta_T I_d)^{-1}(\Sigma_T - \alpha_T I_d)^{-1}m \\
&\leqslant e^{-bT}(\alpha_T + \beta_T)\left\|(\Sigma_T + \beta_T I_d)^{-1}(y-m)\right\|\,\left\|(\Sigma_T - \alpha_T I_d)^{-1}m\right\| \\
&\leqslant 2e^{-bT}(\alpha_T + \beta_T)\left\|\Sigma_T^{-1}(y-m)\right\|\,\left\|\Sigma_T^{-1}m\right\|.
\end{aligned}
$$

The inequality (43) yields that

$$
\begin{aligned}
&2e^{-bT}(\alpha_T + \beta_T)\left\|\Sigma_T^{-1}(y-m)\right\|\,\left\|\Sigma_T^{-1}m\right\| \\
&\leqslant 2e^{-bT}\left\|\Sigma_T^{-1/2}(y-m)\right\|\,\left\|\Sigma_T^{-1}m\right\| \\
&\leqslant e^{-bT}\left\|\Sigma_T^{-1/2}(y-m)\right\|^2 + e^{-bT}\left\|\Sigma_T^{-1}m\right\|^2.
\end{aligned}
$$

Hence, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\left\|\Sigma_T^{-1/2}\big(y-\mu_T(\beta_T)\big)\right\|^2-\frac{1}{2}\left\|\left(\Sigma_T+e^{-2bT}\left(I_d-\alpha_T\Sigma_T^{-1}\right)^{-1}\right)^{-1/2}\big(y-\mu_T(-\alpha_T)\big)\right\|^2\\
&\leqslant 2e^{-2bT}\left\|\Sigma_T^{-1}(y-m)\right\|^2+e^{-bT}\left\|\Sigma_T^{-1/2}(y-m)\right\|^2+e^{-bT}\left\|\Sigma_T^{-1}m\right\|^2\\
&\quad+\frac{1}{2}\left\|\Sigma_T^{-1/2}\big(m-\mu_T(\beta_T)\big)\right\|^2+e^{-2bT}\left\|\Sigma_T^{-1}\big(m-\mu_T(-\alpha_T)\big)\right\|^2\\
&\leqslant 3e^{-bT}\left\|\Sigma_T^{-1/2}(y-m)\right\|^2+\mathcal{O}(e^{-bT}).
\end{aligned}
\tag{45}
$$

In the last line, we used the inequality $e^{-bT}\|\Sigma^{-1}\|\leqslant 1$.

Step 8: final bound. The inequalities (42), (44), and (45), we deduce that

$$
\begin{aligned}
&\left(\int\limits_{\mathbb{R}^d}\frac{e^{2\mathcal{A}(x,T)\mathcal{K}(T)+\mathcal{A}(x,T)/\mathcal{K}(T)}\,\mathsf{q}(y\,|\,x)}{\big(\mathcal{G}(x)\big)^{1/\mathcal{K}(T)-1}}\,\rho_0(x)\mathrm{d}x\right)\Bigg/\left(\int\limits_{\mathbb{R}^d}e^{-\mathcal{A}(x,T)/\mathcal{K}(T)}\,\mathsf{q}(y\,|\,x)\rho_0(x)\mathrm{d}x\right)\\
&\lesssim e^{\mathcal{O}(d+M\sqrt{d}e^{-bT})}(A\vee B)^{\mathcal{O}(\sqrt{d}e^{-bT})}\left(d^2+\left\|\Sigma^{-1/2}\big(y-m\big)\right\|^2\right)^{1-1/\mathcal{K}(T)}\\
&\quad\cdot\exp\left\{3e^{-bT}\left\|\Sigma_T^{-1/2}(y-m)\right\|^2+\mathcal{O}(e^{-bT})\right\}.
\end{aligned}
$$

This, together with (26) yields that

$$
\begin{aligned}
\left|\frac{\mathrm{d}F(s)}{\mathrm{d}s}\right|&\lesssim e^{2M\mathcal{K}(t)-M}\left(\mathcal{T}_\infty|f_1-f_0|\right)^{1/\mathcal{K}(T)}\\
&\quad\cdot e^{\mathcal{O}(d+(M+\log(A\vee B))\sqrt{d}e^{-bT})}\left(d^2+\left\|\Sigma^{-1/2}\big(y-m\big)\right\|^2\right)^{1-1/\mathcal{K}(T)}\\
&\quad\cdot\exp\left\{3e^{-bT}\left\|\Sigma_T^{-1/2}(y-m)\right\|^2+\mathcal{O}(e^{-bT})\right\}.
\end{aligned}
$$

Finally, taking into account that $\mathcal{K}(T)=1+\mathcal{O}(\sqrt{d}e^{-bT})$, we conclude that

$$
\begin{aligned}
\left|\frac{\mathrm{d}F(s)}{\mathrm{d}s}\right|&\lesssim\left(\mathcal{T}_\infty|f_1-f_0|\right)^{1/\mathcal{K}(T)}\left(d^2+\left\|\Sigma^{-1/2}\big(y-m\big)\right\|^2\right)^{1-1/\mathcal{K}(T)}\\
&\quad\cdot\exp\left\{\mathcal{O}(d+(M+\log(A\vee B))\sqrt{d}e^{-bT})\right\}\\
&\quad\cdot\exp\left\{3e^{-bT}\left\|\Sigma_T^{-1/2}(y-m)\right\|^2+\mathcal{O}(e^{-bT})\right\},
\end{aligned}
$$

and the claim follows.

□

B.3 Proof of Lemma B.3

The proof of the lemma is quite cumbersome. For this reason, we split it into several steps for reader's convenience.

Step 1: Kolmogorov-Fokker-Planck equation. Let us note that $\mathcal{T}_t\big(g(x)e^{f(x)}\big)$ satisfies the Kolmogorov-Fokker-Planck equation, that is,

$$
\frac{\partial\mathcal{T}_t\big(g(x)e^{f(x)}\big)}{\partial t}=-b(x-m)^\top\nabla\mathcal{T}_t\big(g(x)e^{f(x)}\big)+\frac{1}{2}\mathrm{Tr}\left(\Sigma\nabla^2\mathcal{T}_t\big(g(x)e^{f(x)}\big)\right).
\tag{46}
$$

To simplify the expressions in the right-hand side, we represent both

$$
(x-m)^\top\nabla\mathcal{T}_t\big(g(x)e^{f(x)}\big)\quad\text{and}\quad\mathrm{Tr}\left(\Sigma\nabla^2\mathcal{T}_t\big(g(x)e^{f(x)}\big)\right)
$$

as expectations of functions of a Gaussian random vector. Indeed, direct computation yields that

$$\begin{aligned}\nabla \mathcal{T}_t\big(g(x)e^{f(x)}\big) &= \frac{e^{-bt}}{(2\pi)^{d/2}\sqrt{\det(\Sigma_t)}} \int\limits_{\mathbb{R}^d} \Sigma_t^{-1}\big(y - m_t(x)\big) g(y) \\ &\qquad\qquad\qquad\qquad \cdot \exp\left\{ f(y) - \frac{1}{2}\left\|\Sigma_t^{-1/2}\big(y - m_t(x)\big)\right\|^2 \right\} \mathrm{d}y \\ &= e^{-bt}\,\mathbb{E}_{\eta\sim\mathcal{N}(m_t(x),\Sigma_t)}\left[\Sigma_t^{-1}\big(\eta - m_t(x)\big) g(\eta) e^{f(\eta)}\right].\end{aligned}$$

Similarly, for the Hessian of $\mathcal{T}_t\big(g(x)e^{f(x)}\big)$ it holds that

$$\begin{aligned}\nabla^2 \mathcal{T}_t\big(g(x)e^{f(x)}\big) &= \frac{e^{-2bt}}{(2\pi)^{d/2}\sqrt{\det(\Sigma_t)}} \int\limits_{\mathbb{R}^d} \left(\Sigma_t^{-1}\big(y - m_t(x)\big)\big(y - m_t(x)\big)^\top \Sigma_t^{-1} - \Sigma_t^{-1}\right) g(y) \\ &\qquad\qquad\qquad\qquad \cdot \exp\left\{ f(y) - \frac{1}{2}\left\|\Sigma_t^{-1/2}\big(y - m_t(x)\big)\right\|^2 \right\} \mathrm{d}y.\end{aligned}$$

Taking into account the relation $\Sigma_t = (1 - e^{-2bt})\Sigma/(2b)$, it is straightforward to observe that $\mathrm{Tr}\big(\Sigma\nabla^2 \mathcal{Z}_t(x)\big)$ equals to

$$\frac{2be^{-2bt}}{1 - e^{-2bt}} \mathbb{E}_{\eta\sim\mathcal{N}(m_t(x),\Sigma_t)}\left[\left(\left\|\Sigma_t^{-1/2}\big(\eta - m_t(x)\big)\right\|^2 - d\right) g(\eta) e^{f(\eta)}\right].$$

Then the Kolmogorov-Fokker-Planck equation (46) and the triangle inequality imply that

$$\begin{aligned}\left|\frac{\partial \mathcal{T}_t\big(g(x)e^{f(x)}\big)}{\partial t}\right| &\leqslant be^{-bt}\left|\mathbb{E}_{\eta\sim\mathcal{N}(m_t(x),\Sigma_t)}\left[(x-m)^\top \Sigma_t^{-1}\big(\eta - m_t(x)\big) g(\eta) e^{f(\eta)}\right]\right| \\ &\quad + \frac{be^{-2bt}}{1 - e^{-2bt}}\left|\mathbb{E}_{\eta\sim\mathcal{N}(m_t(x),\Sigma_t)}\left[\left(\left\|\Sigma_t^{-1/2}\big(\eta - m_t(x)\big)\right\|^2 - d\right) g(\eta) e^{f(\eta)}\right]\right|.\end{aligned} \tag{47}$$

Step 2: Hölder's inequality. Let us fix arbitrary $x \in \mathbb{R}^d$ and $t > 0$ and let $p \geqslant 2$ be a parameter to be specified later. Applying Hölder's inequality, we obtain that

$$\begin{aligned}&\left|\mathbb{E}_{\eta\sim\mathcal{N}(m_t(x),\Sigma_t)}\left[(x-m)^\top \Sigma_t^{-1}\big(\eta - m_t(x)\big) g(\eta) e^{f(\eta)}\right]\right| \\ &= \left|\mathbb{E}_{\eta\sim\mathcal{N}(m_t(x),\Sigma_t)}\left[(x-m)^\top \Sigma_t^{-1}\big(\eta - m_t(x)\big)\left(g(\eta)e^{f(\eta)}\right)^{\frac{2}{p}} \cdot \left(g(\eta)e^{f(\eta)}\right)^{1-\frac{2}{p}}\right]\right| \\ &\leqslant \left(\mathbb{E}\left|(x-m)^\top \Sigma_t^{-1}\big(\eta - m_t(x)\big)\right|^p\right)^{\frac{1}{p}} \left(\mathbb{E} g^2(\eta) e^{2f(\eta)}\right)^{\frac{1}{p}} \left(\mathbb{E} g(\eta) e^{f(\eta)}\right)^{1-\frac{2}{p}}.\end{aligned}$$

The expression in the right-hand side is nothing but

$$\left(\mathbb{E}\left|(x-m)^\top \Sigma_t^{-1}\big(\eta - m_t(x)\big)\right|^p\right)^{\frac{1}{p}} \left(\mathcal{T}_t g^2(x) e^{2f(x)}\right)^{\frac{1}{p}} \left(\mathcal{T}_t g(x) e^{f(x)}\right)^{1-\frac{2}{p}}.$$

Thus, we showed that

$$\begin{aligned}&\left|\mathbb{E}_{\eta\sim\mathcal{N}(m_t(x),\Sigma_t)}\left[(x-m)^\top \Sigma_t^{-1}\big(\eta - m_t(x)\big) g(\eta) e^{f(\eta)}\right]\right| \\ &\leqslant \left(\mathbb{E}_{\eta\sim\mathcal{N}(m_t(x),\Sigma_t)}\left|(x-m)^\top \Sigma_t^{-1}\big(\eta - m_t(x)\big)\right|^p\right)^{\frac{1}{p}} \left(\mathcal{T}_t g^2(x) e^{2f(x)}\right)^{\frac{1}{p}} \left(\mathcal{T}_t g(x) e^{f(x)}\right)^{1-\frac{2}{p}} \\ &= \left(\mathbb{E}_{\xi\sim\mathcal{N}(0,I_d)}\left|(x-m)^\top \Sigma_t^{-1/2}\xi\right|^p\right)^{\frac{1}{p}} \left(\mathcal{T}_t g^2(x) e^{2f(x)}\right)^{\frac{1}{p}} \left(\mathcal{T}_t g(x) e^{f(x)}\right)^{1-\frac{2}{p}}.\end{aligned} \tag{48}$$

An upper bound on the absolute value of

$$\mathbb{E}_{\eta\sim\mathcal{N}(m_t(x),\Sigma_t)}\left[\left(\left\|\Sigma_t^{-1/2}\big(\eta - m_t(x)\big)\right\|^2 - d\right) g(\eta) e^{f(\eta)}\right]$$

can be derived in a similar way. With the same $p \geqslant 2$, it holds that

$$\begin{aligned}
&\left|\mathbb{E}_{\eta\sim\mathcal{N}(m_t(x),\Sigma_t)}\left[\left(\left\|\Sigma_t^{-1/2}\big(\eta - m_t(x)\big)\right\|^2 - d\right) g(\eta) e^{f(\eta)}\right]\right| \\
&\leqslant \left(\mathbb{E}_{\eta\sim\mathcal{N}(m_t(x),\Sigma_t)}\left|\left\|\Sigma_t^{-1/2}\big(\eta - m_t(x)\big)\right\|^2 - d\right|^p\right)^{\frac{1}{p}} \left(\mathcal{T}_t g^2(x) e^{2f(x)}\right)^{\frac{1}{p}} \left(\mathbb{E} g(x) e^{f(x)}\right)^{1-\frac{2}{p}} \\
&= \left(\mathbb{E}_{\xi\sim\mathcal{N}(0,I_d)}\left|\|\xi\|^2 - d\right|^p\right)^{\frac{1}{p}} \left(\mathcal{T}_t g^2(x) e^{2f(x)}\right)^{\frac{1}{p}} \left(\mathcal{T}_t g(x) e^{f(x)}\right)^{1-\frac{2}{p}}. \qquad (49)
\end{aligned}$$

Summing up the inequalities (47), (48), and (49), we conclude that

$$\begin{aligned}
\left|\frac{\partial \mathcal{T}_t\big(g(x)e^{f(x)}\big)}{\partial t}\right| &\leqslant b e^{-bt}\left(\mathbb{E}_{\xi\sim\mathcal{N}(0,I_d)}\left|(x-m)^\top \Sigma_t^{-1/2}\xi\right|^p\right)^{\frac{1}{p}} \left(\mathcal{T}_t g^2(x) e^{2f(x)}\right)^{\frac{1}{p}} \left(\mathcal{T}_t g(x) e^{f(x)}\right)^{1-\frac{2}{p}} \\
&\quad + \frac{b e^{-2bt}}{1-e^{-2bt}}\left(\mathbb{E}_{\xi\sim\mathcal{N}(0,I_d)}\left|\|\xi\|^2 - d\right|^p\right)^{\frac{1}{p}} \left(\mathcal{T}_t g^2(x) e^{2f(x)}\right)^{\frac{1}{p}} \left(\mathcal{T}_t g(x) e^{f(x)}\right)^{1-\frac{2}{p}}. \qquad (50)
\end{aligned}$$

Step 3: properties of Gaussian random vectors. Let us elaborate on $\mathbb{E}_{\xi\sim\mathcal{N}(0,I_d)}\left|\|\xi\|^2 - d\right|^p$ and $\mathbb{E}_{\xi\sim\mathcal{N}(0,I_d)}\left|(x-m)^\top \Sigma_t^{-1/2}\xi\right|^p$. Due to Lemma D.3, we have

$$\left(\mathbb{E}\left|\|\xi\|^2 - d\right|^p\right)^{1/p} \leqslant 10p\sqrt{d} \quad \text{for all } p \geqslant 1. \qquad (51)$$

An upper bound on $\mathbb{E}_{\xi\sim\mathcal{N}(0,I_d)}\left|(x-m)^\top \Sigma_t^{-1/2}\xi\right|^p$ follows from the properties of sub-Gaussian random variables. Note that $(x-m)^\top \Sigma_t^{-1/2}\xi \sim \mathcal{N}\big(0, \|\Sigma_t^{-1/2}(x-m)\|^2\big)$. Then it holds that

$$\mathbb{E}_{\xi\sim\mathcal{N}(0,I_d)}\left|(x-m)^\top \Sigma_t^{-1/2}\xi\right|^p = \left\|\Sigma_t^{-1/2}(x-m)\right\|^p \mathbb{E}_{\mathfrak{z}\sim\mathcal{N}(0,1)}|\mathfrak{z}|^p.$$

It is known that

$$\mathbb{P}_{\mathfrak{z}\sim\mathcal{N}(0,1)}\left(|\mathfrak{z}| \geqslant u\right) \leqslant 2e^{-u^2/2} \quad \text{for all } u > 0.$$

Then, according to [Vershynin, 2018, Proposition 2.5.2][3],

$$\left(\mathbb{E}_{\mathfrak{z}\sim\mathcal{N}(0,1)}|\mathfrak{z}|^p\right)^{1/p} \leqslant 2\sqrt{p}. \qquad (52)$$

The bounds (50), (51), and (52) yield that

$$\begin{aligned}
\left|\frac{\partial \mathcal{T}_t\big(g(x)e^{f(x)}\big)}{\partial t}\right| &\leqslant \left(2be^{-bt}\left\|\Sigma_t^{-1/2}(x-m)\right\|\sqrt{p} + \frac{10bpe^{-2bt}\sqrt{d}}{1-e^{-2bt}}\right) \\
&\quad \cdot \left(\mathcal{T}_t g^2(x) e^{2f(x)}\right)^{\frac{1}{p}} \left(\mathcal{T}_t g(x) e^{f(x)}\right)^{1-\frac{2}{p}}. \qquad (53)
\end{aligned}$$

Step 4: upper bound on $\mathcal{T}_t g^2(x) e^{2f(x)}$. Our next goal is to show that $\mathcal{T}_t g^2(x) e^{2f(x)} \leqslant G(x)$ uniformly over $t > 0$, where $G(x)$ is defined in (19). Using the condition (18), we observe that

$$\begin{aligned}
&\mathcal{T}_t g^2(x) e^{2f(x)} \leqslant e^{2M}\mathcal{T}_t g^2(x) \\
&= \frac{e^{2M}}{(2\pi)^{d/2}\sqrt{\det(\Sigma_t)}} \int_{\mathbb{R}^d} g^2(y) \exp\left\{-\frac{1}{2}\left\|\Sigma_t^{-1/2}\big(y - m_t(x)\big)\right\|^2\right\} \mathrm{d}y \\
&\leqslant \frac{e^{2M}}{(2\pi)^{d/2}\sqrt{\det(\Sigma_t)}} \int_{\mathbb{R}^d} \left(A\left\|\Sigma^{-1/2}(y-m)\right\|^\alpha + B\right)^2 \exp\left\{-\frac{1}{2}\left\|\Sigma_t^{-1/2}\big(y - m_t(x)\big)\right\|^2\right\} \mathrm{d}y \\
&= e^{2M}\,\mathbb{E}_{\eta\sim\mathcal{N}(m_t(x),\Sigma_t)}\left(A\left\|\Sigma^{-1/2}(\eta - m)\right\|^\alpha + B\right)^2.
\end{aligned}$$

[3] In the proof of the implication $1 \Rightarrow 2$ of Proposition 2.5.2, Vershynin shows that $K_2 = 2K_1$ (see p. 24).

Taking into account the relations

$$m_t(x) - m = e^{-bt}(x - m) \quad \text{and} \quad \Sigma_t = \frac{1 - e^{-2bt}}{2b}\Sigma,$$

we obtain that

$$\begin{aligned}
\mathcal{T}_t g^2(x) e^{2f(x)} &\leqslant e^{2M}\, \mathbb{E}_{\eta \sim \mathcal{N}(m_t(x), \Sigma_t)} \left(A \left\| \Sigma^{-1/2}(\eta - m) \right\|^\alpha + B \right)^2 \\
&= e^{2M}\, \mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \left(A \left\| \Sigma^{-1/2}\big(m_t(x) - m\big) + \Sigma^{-1/2}\Sigma_t^{1/2}\xi \right\|^\alpha + B \right)^2 \\
&= e^{2M}\, \mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \left(A \left\| e^{-bt}\, \Sigma^{-1/2}(x - m) + \sqrt{\frac{1 - e^{-2bt}}{2b}}\, \xi \right\|^\alpha + B \right)^2.
\end{aligned}$$

Due to the triangle inequality, it holds that

$$\begin{aligned}
\sqrt{\mathcal{T}_t g^2(x) e^{2f(x)}} &\leqslant e^M \sqrt{\mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \left(A \left\| e^{-bt}\, \Sigma^{-1/2}(x - m) + \sqrt{\frac{1 - e^{-2bt}}{2b}}\, \xi \right\|^\alpha + B \right)^2} \\
&\leqslant B e^M + A e^M \sqrt{\mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \left\| e^{-bt}\, \Sigma^{-1/2}(x - m) + \sqrt{\frac{1 - e^{-2bt}}{2b}}\, \xi \right\|^{2\alpha}}.
\end{aligned}$$

The inequality $(u + v)^\alpha \leqslant 2^{\alpha - 1} u^\alpha + 2^{\alpha - 1} v^\alpha$ holding for all non-negative u and v ensures that

$$\begin{aligned}
&\sqrt{\mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \left\| e^{-bt}\, \Sigma^{-1/2}(x - m) + \sqrt{\frac{1 - e^{-2bt}}{2b}}\, \xi \right\|^{2\alpha}} \\
&= \left(\left[\mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \left\| e^{-bt}\, \Sigma^{-1/2}(x - m) + \sqrt{\frac{1 - e^{-2bt}}{2b}}\, \xi \right\|^{2\alpha} \right]^{\frac{1}{2\alpha}} \right)^\alpha \\
&\leqslant \left(e^{-bt} \left\| \Sigma^{-1/2}(x - m) \right\| + \sqrt{\frac{1 - e^{-2bt}}{2b}}\ \left[\mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \|\xi\|^{2\alpha} \right]^{\frac{1}{2\alpha}} \right)^\alpha \\
&\leqslant 2^{\alpha - 1} e^{-\alpha bt} \left\| \Sigma^{-1/2}(x - m) \right\|^\alpha + 2^{\alpha - 1} \left(\frac{1 - e^{-2bt}}{2b} \right)^{\alpha/2} \sqrt{\mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \|\xi\|^{2\alpha}}.
\end{aligned}$$

Hence, we showed that

$$\begin{aligned}
\sqrt{\mathcal{T}_t g^2(x) e^{2f(x)}} &\leqslant B e^M + 2^{\alpha - 1} A e^{M - \alpha bt} \left\| \Sigma^{-1/2}(x - m) \right\|^\alpha \\
&\quad + 2^{\alpha - 1} A e^M \left(\frac{1 - e^{-2bt}}{2b} \right)^{\alpha/2} \sqrt{\mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \|\xi\|^{2\alpha}} \\
&\leqslant B e^M + 2^{\alpha - 1} A e^M \left(\left\| \Sigma^{-1/2}(x - m) \right\|^\alpha + (2b)^{-\alpha/2} \sqrt{\mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \|\xi\|^{2\alpha}} \right).
\end{aligned}$$

Finally, applying the inequality $(u + v)^\alpha \leqslant 2^{\alpha - 1} u^\alpha + 2^{\alpha - 1} v^\alpha$ once again and using Lemma D.3, we obtain that

$$\begin{aligned}
\mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \|\xi\|^{2\alpha} &\leqslant \mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \left(\left| \|\xi\|^2 - d \right| + d \right)^\alpha \\
&\leqslant 2^{\alpha - 1} \mathbb{E}_{\xi \sim \mathcal{N}(0, I_d)} \left| \|\xi\|^2 - d \right|^\alpha + 2^{\alpha - 1} d^\alpha \\
&\leqslant 2^{\alpha - 1} (10\alpha\sqrt{d})^\alpha + 2^{\alpha - 1} d^\alpha.
\end{aligned}$$

This yields the bound

$$\begin{aligned}
\sqrt{\mathcal{T}_t g^2(x) e^{2f(x)}} &\leqslant B e^M + 2^{\alpha - 1} A e^M \left\| \Sigma^{-1/2}(x - m) \right\|^\alpha \\
&\quad + 4^{\alpha - 1} A e^M (2b)^{-\alpha/2} \left((10\alpha\sqrt{d})^\alpha + d^\alpha \right) \qquad (54) \\
&= G(x),
\end{aligned}$$

which holds uniformly over $t > 0$.

Step 5: choice of p. The bounds (53), (54) and the equality $\Sigma_t = (1 - e^{-2bt})\Sigma/(2b)$ imply that

$$\begin{aligned}\left|\frac{\partial \mathcal{T}_t\big(g(x)e^{f(x)}\big)/\partial t}{\mathcal{T}_t\big(g(x)e^{f(x)}\big)}\right| &\leqslant \left(2be^{-bt}\left\|\Sigma_t^{-1/2}(x-m)\right\|\sqrt{p} + \frac{10bpe^{-2bt}\sqrt{d}}{1-e^{-2bt}}\right) \\ &\qquad \cdot \left(\mathcal{T}_t g^2(x)e^{2f(x)}\right)^{\frac{1}{p}}\left(\mathcal{T}_t g(x)e^{f(x)}\right)^{1-\frac{2}{p}} \\ &\leqslant \left(\frac{(2b)^{3/2}e^{-bt}\sqrt{p}}{\sqrt{1-e^{-2bt}}}\left\|\Sigma^{-1/2}(x-m)\right\| + \frac{10pbe^{-2bt}\sqrt{d}}{1-e^{-2bt}}\right)\left(\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}}\right)^{\frac{2}{p}}\end{aligned}$$

In the last line we used $\Sigma_t = (1 - e^{-2bt})\Sigma/(2b)$. Applying Young's inequality

$$(2b)^{3/2}\sqrt{p}\left\|\Sigma^{-1/2}(x-m)\right\| \leqslant \frac{1}{2}\left(\frac{2b^2\left\|\Sigma^{-1/2}(x-m)\right\|^2}{\sqrt{d}} + 4bp\sqrt{d}\right),$$

we obtain that

$$\begin{aligned}\left|\frac{\partial \mathcal{T}_t\big(g(x)e^{f(x)}\big)/\partial t}{\mathcal{T}_t\big(g(x)e^{f(x)}\big)}\right| &\leqslant \frac{b^2e^{-bt}}{\sqrt{1-e^{-2bt}}\cdot\sqrt{d}}\left\|\Sigma^{-1/2}(x-m)\right\|^2\left(\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}}\right)^{\frac{2}{p}} \\ &\qquad + 2bp\sqrt{d}\left(\frac{e^{-bt}}{\sqrt{1-e^{-2bt}}} + \frac{5e^{-2bt}}{1-e^{-2bt}}\right)\left(\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}}\right)^{\frac{2}{p}}.\end{aligned}$$

Let us choose $p = 2 \vee \log\big(G(x)/\mathcal{T}_t\big(g(x)e^{f(x)}\big)\big)$. Then it is easy to observe that

$$\left(\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}}\right)^{\frac{2}{p}} \leqslant e^2,$$

and, therefore,

$$\begin{aligned}\left|\frac{\partial \mathcal{T}_t\big(g(x)e^{f(x)}\big)/\partial t}{\mathcal{T}_t\big(g(x)e^{f(x)}\big)}\right| &\leqslant \frac{b^2e^{2-bt}}{\sqrt{1-e^{-2bt}}\cdot\sqrt{d}}\left\|\Sigma^{-1/2}(x-m)\right\|^2 \\ &\qquad + 2be^2\sqrt{d}\left(\frac{e^{-bt}}{\sqrt{1-e^{-2bt}}} + \frac{5e^{-2bt}}{1-e^{-2bt}}\right)\left(2 \vee \log\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}}\right).\end{aligned}$$

We would like to recall that (see (54))

$$\sqrt{\mathcal{T}_t g^2(x)e^{2f(x)}} \leqslant G(x).$$

This means that

$$\log\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}} \geqslant \log\frac{G(x)}{\sqrt{\mathcal{T}_t g^2(x)e^{2f(x)}}} \geqslant 0, \tag{55}$$

and, as a consequence, we have

$$2 \vee \log\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}} \leqslant 2 + \log\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}}.$$

Thus, we obtain that

$$\begin{aligned}\left|\frac{\partial \mathcal{T}_t\big(g(x)e^{f(x)}\big)/\partial t}{\mathcal{T}_t\big(g(x)e^{f(x)}\big)}\right| &\leqslant \frac{b^2e^{2-bt}}{\sqrt{1-e^{-2bt}}\cdot\sqrt{d}}\left\|\Sigma^{-1/2}(x-m)\right\|^2 \\ &\qquad + 2be^2\sqrt{d}\left(\frac{e^{-bt}}{\sqrt{1-e^{-2bt}}} + \frac{5e^{-2bt}}{1-e^{-2bt}}\right)\left(2 + \log\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}}\right).\end{aligned} \tag{56}$$

Step 6: properties of ODEs. Let us note that, due to (56), we have

$$\begin{aligned}\left|\frac{\partial}{\partial t}\log\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}}\right| &= \left|\frac{\partial \mathcal{T}_t\big(g(x)e^{f(x)}\big)/\partial t}{\mathcal{T}_t\big(g(x)e^{f(x)}\big)}\right| \\ &\leqslant \frac{b^2 e^{2-bt}}{\sqrt{1-e^{-2bt}}\cdot\sqrt{d}}\left\|\Sigma^{-1/2}(x-m)\right\|^2 \\ &\quad + 2be^2\sqrt{d}\left(\frac{e^{-bt}}{\sqrt{1-e^{-2bt}}}+\frac{5e^{-2bt}}{1-e^{-2bt}}\right)\left(2+\log\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}}\right).\end{aligned} \tag{57}$$

In other words the partial derivative of $\log(G(x)/\mathcal{T}_t g(x)e^{f(x)})$ is bounded by its value. On this step, we use properties of ordinary differential equations to convert the inequality (57) into an upper bound on the absolute value of

$$\log\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}} - \log\frac{G(x)}{\mathcal{T}_\infty g(x)e^{f(x)}}.$$

For this purpose, let us fix an arbitrary $x \in \mathbb{R}^d$ and apply Lemma D.1 with

$$a(t) = \frac{b^2 e^{2-bt}}{\sqrt{1-e^{-2bt}}\cdot\sqrt{d}}\left\|\Sigma^{-1/2}(x-m)\right\|^2 + 4be^2\sqrt{d}\left(\frac{e^{-bt}}{\sqrt{1-e^{-2bt}}}+\frac{5e^{-2bt}}{1-e^{-2bt}}\right)$$

and

$$\varkappa(t) = 2be^2\sqrt{d}\left(\frac{e^{-bt}}{\sqrt{1-e^{-2bt}}}+\frac{5e^{-2bt}}{1-e^{-2bt}}\right).$$

Note that the function

$$\log\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}}$$

is always non-negative due to (55). Since

$$\int\limits_t^{+\infty}\frac{be^{-bs}\,\mathrm{d}s}{\sqrt{1-e^{-2bs}}} = \arcsin\left(e^{-bt}\right) \quad \text{and} \quad \int\limits_t^{+\infty}\frac{2be^{-2bs}\,\mathrm{d}s}{1-e^{-2bs}} = -\log\left(1-e^{-2bt}\right),$$

it holds that

$$\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau = 2e^2\sqrt{d}\arcsin(e^{-bt}) - 5e^2\sqrt{d}\log\left(1-e^{-2bt}\right) = \log\mathcal{K}(t)$$

and

$$\begin{aligned}\int\limits_t^{+\infty} a(s)\,\mathrm{d}s &= \left(\frac{be^2}{\sqrt{d}}\left\|\Sigma^{-1/2}(x-m)\right\|^2 + 4e^2\sqrt{d}\right)\arcsin(e^{-bt}) \\ &\quad - 10e^2\sqrt{d}\log\left(1-e^{-2bt}\right) \\ &= \mathcal{A}(x,t),\end{aligned}$$

where the functions $\mathcal{K}(t)$ and $\mathcal{A}(x,t)$ are defined in (21) and (20), respectively. Then, according to Lemma D.1, it holds that

$$\frac{1}{\mathcal{K}(t)}\left(\log\frac{G(x)}{\mathcal{T}_\infty g(x)e^{f(x)}} - \mathcal{A}(x,t)\right) \leqslant \log\frac{G(x)}{\mathcal{T}_t g(x)e^{f(x)}} \leqslant \mathcal{K}(t)\left(\log\frac{G(x)}{\mathcal{T}_\infty g(x)e^{f(x)}} + \mathcal{A}(x,t)\right),$$

and we finally obtain that

$$e^{-\mathcal{A}(x,t)\mathcal{K}(t)}\left(\frac{\mathcal{T}_\infty g(x)e^{f(x)}}{G(x)}\right)^{\mathcal{K}(t)} \leqslant \frac{\mathcal{T}_t g(x)e^{f(x)}}{G(x)} \leqslant e^{\mathcal{A}(x,t)/\mathcal{K}(t)}\left(\frac{\mathcal{T}_\infty g(x)e^{f(x)}}{G(x)}\right)^{1/\mathcal{K}(t)}.$$

C Properties of subquadratic log-densities

In this section, we present some properties of probability densities $\mathsf{p}(x)$ such that $\log \mathsf{p}(x) = \mathcal{O}(\|x\|^2)$. In view of Assumption 3 and Lemma B.1, such densities naturally arise in the proof of Theorem 1. We start with the following preliminary bound, which helps us to show that the class of log-potentials Ψ satisfies a Bernstein-type condition. This is one of key moments in the proof of our main result, allowing us to derive rates of convergence possibly faster than $\mathcal{O}(n^{-1/2})$.

Lemma C.1. *For any two probability densities* p *and* q *on* $\mathbb{R}^d$ *such that*

$$\int\limits_{\mathbb{R}^d} \log^2\left(\frac{\mathsf{q}(x)}{\mathsf{p}(x)}\right) \mathsf{p}(x)\,\mathrm{d}x < +\infty$$

and any $\omega \in (0,1)$*, it holds that*

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \log^2\left(\frac{\mathsf{p}(x)}{\mathsf{q}(x)}\right) \mathsf{p}(x)\,\mathrm{d}x &\leqslant 2\log(1/\omega)\,\mathsf{KL}(\mathsf{p},\mathsf{q}) \\ &\quad + 2\int\limits_{\mathbb{R}^d} \log^2\left(\frac{(1-\omega)\mathsf{q}(x)+\omega\mathsf{p}(x)}{\mathsf{q}(x)}\right)\mathsf{p}(x)\,\mathrm{d}x. \qquad (58)\end{aligned}$$

The proof of Lemma C.1 is postponed to Appendix C.1. The next lemma ensures that, under Assumption 2, the second term in the right-hand side of (58) grows polynomially with ω.

Lemma C.2. *Let* p *and* q *be arbitrary probability densities on* $\mathbb{R}^d$*. Assume that* p *is a probability density of a centered sub-Gaussian distribution on* $\mathbb{R}^d$ *with a variance proxy* v^2*, that is,*

$$\mathbb{E}_{\xi\sim\mathsf{p}}\, e^{u^\top \xi} \leqslant e^{\mathsf{v}^2\|u\|^2/2} \quad \textit{for all } u \in \mathbb{R}^d.$$

Let $\mathsf{p} \ll \mathsf{q}$ *and suppose that there are constants* $A \geqslant 0$ *and* $B \in \mathbb{R}$ *such that*

$$\log\frac{\mathsf{p}(x)}{\mathsf{q}(x)} \leqslant A\|x\|^2 + B \qquad \textit{for all } x \textit{ from the support of } \mathsf{p}(x).$$

Then, for any $\omega \in (0, 1/2]$*, it holds that*

$$\begin{aligned}&\int\limits_{\mathbb{R}^d} \log^2\left(\frac{(1-\omega)\mathsf{q}(x)+\omega\mathsf{p}(x)}{\mathsf{q}(x)}\right)\mathsf{p}(x)\,\mathrm{d}x \\ &\leqslant 4\omega^2 + e^B\omega + 6^d\left(0.5\log(1/\omega) - 0.5B + 16A\sigma^2\right) e^{B/(16A\mathsf{v}^2)}\omega^{1/(16A\mathsf{v}^2)}.\end{aligned}$$

The proof of Lemma C.2 is moved to Appendix C.2. Lemma C.1 and Lemma C.2 imply that, if one takes $\omega = n^{-16A\mathsf{v}^2} \wedge n^{-1}$, then

$$\int\limits_{\mathbb{R}^d} \log^2\left(\frac{\mathsf{p}(x)}{\mathsf{q}(x)}\right)\mathsf{p}(x)\,\mathrm{d}x \lesssim \log(n)\,\mathsf{KL}(\mathsf{p},\mathsf{q}) + \mathcal{O}(1/n).$$

In other words, under the conditions of these lemmata, the variance of $\log\big(\mathsf{p}(\xi)/\mathsf{q}(\xi)\big)$, where $\xi\sim\mathsf{p}$, is controlled by its expectation. Finally, we would like to present a result on an upper bound on the ψ_1-norm of $\log\big(\mathsf{p}(\xi)/\mathsf{q}(\xi)\big)$, $\xi\sim\mathsf{p}$. An upper bound on the Orlicz norm is necessary for understanding behaviour of distribution tails.

Lemma C.3. *Let* p *be a sub-Gaussian probability density on* $\mathbb{R}^d$ *with variance proxy* v^2*, that is,*

$$\mathbb{E}_{\xi\sim\mathsf{p}} e^{u^\top\xi} \leqslant e^{\mathsf{v}^2\|u\|^2/2} \quad \textit{for all } u\in\mathbb{R}^d.$$

Let q *be an arbitrary probability density such that*

$$\log\frac{\mathsf{p}(x)}{\mathsf{q}(x)} \leqslant A\|x\|^2 + B \quad \textit{for all } x \in \operatorname{supp}(\mathsf{p}),$$

where A *and* B *are some non-negative constants. Let* $\xi\sim\mathsf{p}$*. Then it holds that*

$$\left\|\log\frac{\mathsf{p}(\xi)}{\mathsf{q}(\xi)}\right\|_{\psi_1} \leqslant 1 \vee \big(2B + 2(d+2)A\mathsf{v}^2\big).$$

The proof of Lemma C.3 is deferred to Appendix C.3.

C.1 Proof of Lemma C.1

According to the Newton-Leibniz formula, for any $u < 1$ we have

$$\log^2(1-u) = \left(\int_0^1 \frac{u\,\mathrm{d}s}{1-su}\right)^2.$$

Using Young's inequality, we obtain that

$$\begin{aligned}
\log^2(1-u) &\leqslant 2\left(\int_0^{1-\omega} \frac{u\,\mathrm{d}s}{1-su}\right)^2 + 2\left(\int_{1-\omega}^1 \frac{u\,\mathrm{d}s}{1-su}\right)^2 \\
&= 2\left(\int_0^{1-\omega} \frac{u\,\mathrm{d}s}{1-su}\right)^2 + 2\big(\log(1-u) - \log\big(1-(1-\omega)u\big)\big)^2 \qquad (59)\\
&= 2\left(\int_0^{1-\omega} \frac{u\,\mathrm{d}s}{1-su}\right)^2 + 2\log^2\left(1 + \frac{\omega u}{1-u}\right).
\end{aligned}$$

The first term in the right-hand side does not exceed

$$\begin{aligned}
2\left(\int_0^{1-\omega} \frac{1}{\sqrt{1-s}} \cdot \frac{u\sqrt{1-s}}{1-su}\,\mathrm{d}s\right)^2 &\leqslant 2\int_0^{1-\omega} \frac{\mathrm{d}s}{1-s} \int_0^{1-\omega} \frac{u^2(1-s)\,\mathrm{d}s}{(1-su)^2} \\
&= 2\log(1/\omega) \int_0^{1-\omega} \frac{u^2(1-s)\,\mathrm{d}s}{(1-su)^2} \qquad (60)\\
&\leqslant 2\log(1/\omega) \int_0^1 \frac{u^2(1-s)\,\mathrm{d}s}{(1-su)^2}.
\end{aligned}$$

On the other hand, due to Taylor's expansion with an integral remainder term, it holds that

$$\log(1-u) = -u - \int_0^1 \frac{u^2(1-s)\,\mathrm{d}s}{(1-su)^2}.$$

In other words, the right-hand side in (60) is equal to

$$2\log(1/\omega)\big(-u - \log(1-u)\big).$$

This, together with (59), yields that

$$\log^2(1-u) \leqslant 2\log(1/\omega)\big(-u - \log(1-u)\big) + 2\log^2\left(1 + \frac{\omega u}{1-u}\right).$$

Substituting u in the expression above with $(\mathsf{p}(x) - \mathsf{q}(x))/\mathsf{p}(x)$, we observe that

$$\log^2\left(\frac{\mathsf{q}(x)}{\mathsf{p}(x)}\right) \leqslant 2\log(1/\omega)\left(\frac{\mathsf{q}(x) - \mathsf{p}(x)}{\mathsf{p}(x)} - \log\frac{\mathsf{q}(x)}{\mathsf{p}(x)}\right) + 2\log^2\left(\frac{(1-\omega)\mathsf{q}(x) + \omega\mathsf{p}(x)}{\mathsf{q}(x)}\right) \quad (61)$$

for all x such that $\mathsf{p}(x) > 0$. Let us note that the condition

$$\int_{\mathbb{R}^d} \log^2\left(\frac{\mathsf{q}(x)}{\mathsf{p}(x)}\right)\mathsf{p}(x)\,\mathrm{d}x < +\infty$$

implies that $\mathsf{p} \ll \mathsf{q}$. This means that $\mathsf{q}(x) > 0$ whenever x belongs to the support of $\mathsf{p}(x)$. Integrating (61), we finally obtain that

$$\int_{\mathbb{R}^d} \log^2\left(\frac{\mathsf{p}(x)}{\mathsf{q}(x)}\right)\mathsf{p}(x)\,\mathrm{d}x \leqslant 2\log(1/\omega)\,\mathsf{KL}(\mathsf{p},\mathsf{q}) + 2\int_{\mathbb{R}^d} \log^2\left(\frac{(1-\omega)\mathsf{q}(x) + \omega\mathsf{p}(x)}{\mathsf{q}(x)}\right)\mathsf{p}(x)\,\mathrm{d}x.$$

□

C.2 Proof of Lemma C.2

Due to the conditions of the lemma, for any x belonging to the support of p, it holds that

$$\begin{aligned}\log(1-\omega) \leqslant \log\left(\frac{(1-\omega)\mathsf{q}(x)+\omega\mathsf{p}(x)}{\mathsf{q}(x)}\right) \\ \leqslant \log\left(1+\omega e^{\log(\mathsf{p}(x)/\mathsf{q}(x))}\right) \\ \leqslant \log\left(1+\omega e^{A\|x\|^2+B}\right).\end{aligned}$$

Note that the expression in the left-hand side is at least -2ω, since $\omega \in (0, 1/2]$. This yields that

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \log^2\left(\frac{(1-\omega)\mathsf{q}(x)+\omega\mathsf{p}(x)}{\mathsf{q}(x)}\right)\mathsf{p}(x)\,\mathrm{d}x &\leqslant \int\limits_{\mathbb{R}^d} \max\left\{4\omega^2, \log^2\left(1+\omega e^{A\|x\|^2+B}\right)\right\}\mathsf{p}(x)\,\mathrm{d}x \\ &\leqslant \int\limits_{\mathbb{R}^d}\left(4\omega^2+\log^2\left(1+\omega e^{A\|x\|^2+B}\right)\right)\mathsf{p}(x)\,\mathrm{d}x \\ &= 4\omega^2 + \int\limits_{\mathbb{R}^d}\log^2\left(1+\omega e^{A\|x\|^2+B}\right)\mathsf{p}(x)\,\mathrm{d}x.\end{aligned}$$

Let us elaborate on the second term in the right-hand side. Let us introduce $\varepsilon = e^B\omega$ and split the integral into two parts:

$$\begin{aligned}&\int\limits_{\mathbb{R}^d}\log^2\left(1+\omega e^{A\|x\|^2+B}\right)\mathsf{p}(x)\,\mathrm{d}x = \int\limits_{\mathbb{R}^d}\log^2\left(1+\varepsilon e^{A\|x\|^2}\right)\mathsf{p}(x)\,\mathrm{d}x \\ &= \int\limits_{2A\|x\|^2\leqslant\log(1/\varepsilon)}\log^2\left(1+\varepsilon e^{A\|x\|^2}\right)\mathsf{p}(x)\,\mathrm{d}x + \int\limits_{2A\|x\|^2>\log(1/\varepsilon)}\log^2\left(1+\varepsilon e^{A\|x\|^2}\right)\mathsf{p}(x)\,\mathrm{d}x.\end{aligned}$$

On the set $\{x \in \mathbb{R}^d : 2A\|x\|^2 \leqslant \log(1/\varepsilon)\}$ we have $\varepsilon e^{A\|x\|^2} \leqslant \varepsilon^{1/2}$. This yields that

$$\int\limits_{2A\|x\|^2\leqslant\log(1/\varepsilon)}\log^2\left(1+\varepsilon e^{A\|x\|^2}\right)\mathsf{p}(x)\,\mathrm{d}x \leqslant \int\limits_{2A\|x\|^2\leqslant\log(1/\varepsilon)}\log^2\left(1+\sqrt{\varepsilon}\right)\mathsf{p}(x)\,\mathrm{d}x \leqslant \varepsilon.$$

It remains to bound

$$\int\limits_{2A\|x\|^2>\log(1/\varepsilon)}\log^2\left(1+\varepsilon e^{A\|x\|^2}\right)\mathsf{p}(x)\,\mathrm{d}x.$$

We use properties of sub-Gaussian distributions for this purpose. We start with the observation

$$\begin{aligned}\int\limits_{2A\|x\|^2>\log(1/\varepsilon)}\log^2\left(1+\varepsilon e^{A\|x\|^2}\right)\mathsf{p}(x)\,\mathrm{d}x &\leqslant \int\limits_{2A\|x\|^2>\log(1/\varepsilon)}\log^2\left(1+e^{A\|x\|^2}\right)\mathsf{p}(x)\,\mathrm{d}x \\ &\leqslant A\int\limits_{2A\|x\|^2>\log(1/\varepsilon)}\|x\|^2\,\mathsf{p}(x)\,\mathrm{d}x.\end{aligned}$$

Let us introduce a random vector $\xi \sim \mathsf{p}$. Then it is straightforward to observe that the integral

$$A\int\limits_{2A\|x\|^2>\log(1/\varepsilon)}\|x\|^2\,\mathsf{p}(x)\,\mathrm{d}x$$

is nothing but the expectation of the non-negative random variable $A\|\xi\|^2 \mathbb{1}\big[2A\|\xi\|^2 > \log(1/\varepsilon)\big]$. Then it holds that

$$A\,\mathbb{E}\|\xi\|^2 \mathbb{1}\big(2A\|\xi\|^2 > \log(1/\varepsilon)\big) = \frac{1}{2}\int\limits_0^{+\infty} \mathbb{P}\left(2A\|\xi\|^2 \mathbb{1}\big[2A\|\xi\|^2 > \log(1/\varepsilon)\big] \geqslant u\right) \mathrm{d}u$$

$$= \frac{1}{2}\int\limits_0^{\log(1/\varepsilon)} \mathbb{P}\left(2A\|\xi\|^2 > \log(1/\varepsilon)\right) \mathrm{d}u + \frac{1}{2}\int\limits_{\log(1/\varepsilon)}^{+\infty} \mathbb{P}\left(2A\|\xi\|^2 \geqslant u\right) \mathrm{d}u$$

$$= 0.5\log(1/\varepsilon)\,\mathbb{P}\left(2A\|\xi\|^2 > \log(1/\varepsilon)\right) + \frac{1}{2}\int\limits_{\log(1/\varepsilon)}^{+\infty} \mathbb{P}\left(2A\|\xi\|^2 \geqslant u\right) \mathrm{d}u.$$

Standard results on large deviation inequalities for the Euclidean norm of a sub-Gaussian random vector (see, for instance, the proof of Theorem 1.19 from Rigollet and Hütter [2023]) imply that

$$\mathbb{P}\left(A\|\xi\|^2 \geqslant u\right) = \mathbb{P}\left(\|\xi\| \geqslant \sqrt{\frac{u}{2A}}\right) \leqslant 6^d \exp\left\{-\frac{u}{16A\sigma^2}\right\} \quad \text{for any } u > 0.$$

This yields that

$$\begin{aligned}
&A\,\mathbb{E}\|\xi\|^2 \mathbb{1}\big(2A\|\xi\|^2 > \log(1/\varepsilon)\big) \\
&= 0.5\log(1/\varepsilon)\,\mathbb{P}\left(2A\|\xi\|^2 > \log(1/\varepsilon)\right) + \frac{1}{2}\int\limits_{\log(1/\varepsilon)}^{+\infty} \mathbb{P}\left(2A\|\xi\|^2 \geqslant u\right) \mathrm{d}u \\
&\leqslant 0.5\log(1/\varepsilon)\cdot 6^d \varepsilon^{1/(16A\sigma^2)} + \frac{6^d}{2}\int\limits_{\log(1/\varepsilon)}^{+\infty} \exp\left\{-\frac{u}{16A\sigma^2}\right\} \mathrm{d}u \\
&= 0.5\log(1/\varepsilon)\cdot 6^d \varepsilon^{1/(16A\sigma^2)} + 8\cdot 6^d A\sigma^2 \varepsilon^{1/(16A\sigma^2)}.
\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} \log^2\left(\frac{(1-\omega)\mathsf{q}(x) + \omega\mathsf{p}(x)}{\mathsf{q}(x)}\right) \mathsf{p}(x)\,\mathrm{d}x \\
&\leqslant 4\omega^2 + \varepsilon + 6^d\left(0.5\log(1/\varepsilon) + 16A\sigma^2\right)\varepsilon^{1/(16A\sigma^2)} \\
&= 4\omega^2 + e^B\omega + 6^d\left(0.5\log(1/\omega) - 0.5B + 16A\sigma^2\right) e^{B/(16A\sigma^2)}\omega^{1/(16A\sigma^2)}.
\end{aligned}$$

□

C.3 Proof of Lemma C.3

Let us introduce $u = 1 \vee (B + (d+2)A\mathrm{v}^2)$ and note that

$$\begin{aligned}
\mathbb{E}_{\xi\sim\mathsf{p}} \exp\left\{\frac{1}{2u}\left|\log\frac{\mathsf{p}(\xi)}{\mathsf{q}(\xi)}\right|\right\} &\leqslant \left(\mathbb{E}_{\xi\sim\mathsf{p}} \exp\left\{\frac{1}{u}\left|\log\frac{\mathsf{p}(\xi)}{\mathsf{q}(\xi)}\right|\right\}\right)^{1/2} \\
&\leqslant \left(\mathbb{E}_{\xi\sim\mathsf{p}} \exp\left\{\frac{1}{u}\log\frac{\mathsf{q}(\xi)}{\mathsf{p}(\xi)}\right\} + \mathbb{E}_{\xi\sim\mathsf{p}} \exp\left\{\frac{1}{u}\log\frac{\mathsf{p}(\xi)}{\mathsf{q}(\xi)}\right\}\right)^{1/2} \\
&\leqslant \left(1 + \mathbb{E}_{\xi\sim\mathsf{p}} \exp\left\{\frac{1}{u}\log\frac{\mathsf{p}(\xi)}{\mathsf{q}(\xi)}\right\}\right)^{1/2}.
\end{aligned}$$

In the last inequality we used the fact that $u \geqslant 1$ and then

$$\mathbb{E}_{\xi\sim\mathsf{p}} \exp\left\{\frac{1}{u}\log\frac{\mathsf{q}(\xi)}{\mathsf{p}(\xi)}\right\} \leqslant \mathbb{E}_{\xi\sim\mathsf{p}}\frac{\mathsf{q}(\xi)}{\mathsf{p}(\xi)} = 1.$$

Hence, it is enough to show that

$$\mathbb{E}_{\xi\sim\mathsf{p}} \exp\left\{\frac{1}{u}\log\frac{\mathsf{p}(\xi)}{\mathsf{q}(\xi)}\right\} \leqslant 3$$

to finish the proof of the lemma. For this purpose, we use the condition $\mathsf{p}(x)/\mathsf{q}(x) \leqslant A\|x\|^2 + B$ for all $x \in \operatorname{supp}(\mathsf{p})$, which yields that

$$\mathbb{E}_{\xi\sim\mathsf{p}} \exp\left\{\frac{1}{u}\log\frac{\mathsf{p}(\xi)}{\mathsf{q}(\xi)}\right\} \leqslant e^{B/u}\, \mathbb{E}_{\xi\sim\mathsf{p}} e^{A\|\xi\|^2/u}.$$

Let $\gamma \sim \mathcal{N}(0, I_d)$ be a Gaussian random vector in $\mathbb{R}^d$ which is independent of ξ. Then we can represent the exponential moment $\mathbb{E}_{\xi\sim\mathsf{p}} e^{A\|\xi\|^2/u}$ in the following form:

$$\mathbb{E}_{\xi\sim\mathsf{p}} e^{A\|\xi\|^2/u} = \mathbb{E}_{\xi\sim\mathsf{p}}\mathbb{E}_{\gamma\sim\mathcal{N}(0,I_d)} \exp\left\{\sqrt{\frac{2A}{u}}\,\xi^\top\gamma\right\}.$$

According to the conditions of the lemma, ξ is a sub-Gaussian random vector with variance proxy $\mathtt{v}^2$. This yields that

$$\begin{aligned}
\mathbb{E}_{\xi\sim\mathsf{p}} e^{A\|\xi\|^2/u} &= \mathbb{E}_{\xi\sim\mathsf{p}}\mathbb{E}_{\gamma\sim\mathcal{N}(0,I_d)} \exp\left\{\sqrt{\frac{2A}{u}}\,\xi^\top\gamma\right\} \\
&\leqslant \mathbb{E}_{\gamma\sim\mathcal{N}(0,I_d)} e^{A\mathtt{v}^2\|\gamma\|^2/u} \\
&= \left(1 - \frac{2A\mathtt{v}^2}{u}\right)^{-d/2}.
\end{aligned}$$

Since $u = B + (d+2)A\mathtt{v}^2 \geqslant (d+2)A\mathtt{v}^2$, we obtain that

$$-\frac{d}{2}\log\left(1 - \frac{2A\mathtt{v}^2}{u}\right) \leqslant \frac{d}{2}\cdot\frac{2A\mathtt{v}^2/u}{1 - 2/(d+2)} = \frac{(d+2)A\mathtt{v}^2}{u}.$$

Hence, it holds that

$$\mathbb{E}_{\xi\sim\mathsf{p}} \exp\left\{\frac{1}{u}\log\frac{\mathsf{p}(\xi)}{\mathsf{q}(\xi)}\right\} \leqslant e^{B/u}\, \mathbb{E}_{\xi\sim\mathsf{p}} e^{A\|\xi\|^2/u} \leqslant \exp\left\{\frac{B + (d+2)A\mathtt{v}^2}{u}\right\}.$$

Due to the definition, u is not less than $B + (d+2)A\mathtt{v}^2$. This yields that the expression in the right-hand side not exceed e. This implies that

$$\mathbb{E}_{\xi\sim\mathsf{p}} \exp\left\{\frac{1}{2u}\left|\log\frac{\mathsf{p}(\xi)}{\mathsf{q}(\xi)}\right|\right\} \leqslant \left(1 + \mathbb{E}_{\xi\sim\mathsf{p}} \exp\left\{\frac{1}{u}\log\frac{\mathsf{p}(\xi)}{\mathsf{q}(\xi)}\right\}\right)^{1/2} < \sqrt{1+3} = 2.$$

The proof is finished.

$\square$

D Auxiliary results

This section contains auxiliary results used in the proofs of Lemma B.2 and Lemma B.3. The first one is a Gronwall-type inequality helping us to relate the operators $\mathcal{T}_t$ and $\mathcal{T}_\infty$ (see Lemma B.3).

Lemma D.1. *Let $\varphi : (0, +\infty) \to \mathbb{R}_+$ be a non-negative function such that there exists*

$$\lim_{t\to+\infty} \varphi(t) = \varphi(+\infty) \in \mathbb{R}.$$

Let the functions $a(t)$ and $\varkappa(t)$ take non-negative values on $(0, +\infty)$ and assume that the integrals

$$\int_t^{+\infty} a(s)\,\mathrm{d}s \quad \text{and} \quad \int_t^{+\infty} \varkappa(s)\,\mathrm{d}s$$

are finite for any $t > 0$. If

$$-\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} \leqslant a(t) + \varkappa(t)\varphi(t) \quad \textit{for all } t > 0,$$

then

$$\varphi(t) \leqslant \left(\varphi(+\infty) + \int\limits_t^{+\infty} a(s)\,\mathrm{d}s\right) \exp\left\{\int\limits_t^{+\infty} \varkappa(\tau)\,\mathrm{d}\tau\right\} \quad \textit{for any } t > 0.$$

Otherwise, if

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} \leqslant a(t) + \varkappa(t)\varphi(t) \quad \textit{for all } t > 0,$$

then it holds that

$$\varphi(t) \geqslant \left(\varphi(+\infty) - \int\limits_t^{+\infty} a(s)\,\mathrm{d}s\right) \exp\left\{-\int\limits_t^{+\infty} \varkappa(\tau)\,\mathrm{d}\tau\right\} \quad \textit{for any } t > 0.$$

The proof of Lemma D.1 is quite similar to the one of the original Gronwall lemma. Nevertheless, we provide all the derivations in Appendix D.1. We move to the next auxiliary result.

Lemma D.2. *Let $\rho_{\mu,\Omega}(x)$ stand for the density of $\mathcal{N}(\mu, \Omega)$ and let*

$$\mathsf{q}(y \,|\, x) = (2\pi)^{-d/2} \det(\Sigma_T)^{-1/2} \exp\left\{-\frac{1}{2}\left\|\Sigma_T^{-1/2}\big(y - m_T(x)\big)\right\|^2\right\}.$$

Then for any function $f : \mathbb{R}^d \to \mathbb{R}$ it holds that

$$\int\limits_{\mathbb{R}^d} f(x)\mathsf{q}(y \,|\, x)\rho_{\mu,\Omega}(x)\,\mathrm{d}x = \varphi(y)\,\mathbb{E}f(\xi),$$

where $\xi \sim \mathcal{N}(\breve{\mu}, \breve{\Omega})$ with

$$\breve{\mu} = \mu + e^{-bT}\Omega(\Sigma_T + e^{-2bT}\Omega)^{-1}\big(y - m_T(\mu)\big), \quad \breve{\Omega} = \left(\Omega^{-1} + e^{-2bT}\Sigma_T^{-1}\right)^{-1},$$

and $\varphi(y)$ is the density of $\mathcal{N}\big(m_T(\mu), \Sigma_T + e^{-2bT}\Omega\big)$:

$$\varphi(y) = (2\pi)^{-d/2} \det\left(\Sigma_T + e^{-2bT}\Omega\right)^{-1/2} \exp\left\{-\frac{1}{2}\left\|\left(\Sigma_T + e^{-2bT}\Omega\right)^{-1/2}\left(y - m_T(\mu)\right)\right\|^2\right\}.$$

The proof of Lemma D.2 is deferred to Appendix D.2. We use this lemma in the proof of our key technical result, Lemma B.2, which allows us to ensure that, under Assumptions 2 and 3 the log-density $\log \rho_T^{\psi}(y)$ is continuous with respect to the log-potential ψ. Finally, we present a sharp bound on L_p-norm of a centered chi-squared random variable.

Lemma D.3. *Let $\xi \sim \mathcal{N}(0, I_d)$ be a Gaussian vector in $\mathbb{R}^d$. Then, for any $p \geqslant 1$, it holds that*

$$\left(\mathbb{E}\left|\|\xi\|^2 - d\right|^p\right)^{1/p} \leqslant 10p\sqrt{d}.$$

The proof of Lemma D.3 is moved to Appendix D.3. Let us note that, unlike the L_p-norm of $\|\xi\|^2$, $\xi \sim \mathcal{N}(0, I_d)$, which is of order $\Omega(d)$, the L_p-norm of $\|\xi\|^2 - d$ is much smaller and grows as fast as $\mathcal{O}(\sqrt{d})$.

D.1 Proof of Lemma D.1

We perform the proof in two steps starting with the upper bound.

Step 1: upper bound. Let us introduce

$$\Phi(t) = \varphi(t) \exp\left\{-\int\limits_t^{+\infty} \varkappa(s)\,\mathrm{d}s\right\}, \quad t > 0.$$

Then it is easy to observe that for any $t > 0$ the derivative of ψ satisfies the inequality

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = \varkappa(t)\varphi(t)\exp\left\{-\int\limits_t^{+\infty}\varkappa(s)\,\mathrm{d}s\right\} + \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\exp\left\{-\int\limits_t^{+\infty}\varkappa(s)\,\mathrm{d}s\right\}$$
$$\geqslant -a(t)\exp\left\{-\int\limits_t^{+\infty}\varkappa(s)\,\mathrm{d}s\right\}.$$

Applying the Newton-Leibniz formula, we obtain that

$$\Phi(t) - \Phi(+\infty) = -\int\limits_t^{+\infty}\frac{\mathrm{d}\Phi(s)}{\mathrm{d}s}\,\mathrm{d}s \leqslant \int\limits_t^{+\infty} a(s)\exp\left\{-\int\limits_s^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s.$$

This yields that

$$\varphi(t)\exp\left\{-\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\} \leqslant \varphi(+\infty) + \int\limits_t^{+\infty} a(s)\exp\left\{\int\limits_s^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s.$$

Taking into account that

$$\exp\left\{\int\limits_t^{s}\varkappa(\tau)\,\mathrm{d}\tau\right\} \leqslant \exp\left\{\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\} \quad \text{for all } s \geqslant t,$$

we finally deduce

$$\varphi(t) \leqslant \varphi(+\infty)\exp\left\{\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\} + \int\limits_t^{+\infty} a(s)\exp\left\{\int\limits_t^{s}\varkappa(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s$$
$$\leqslant \varphi(+\infty)\exp\left\{\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\} + \int\limits_t^{+\infty} a(s)\exp\left\{\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s$$
$$= \left(\varphi(+\infty) + \int\limits_t^{+\infty} a(s)\,\mathrm{d}s\right)\exp\left\{\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\}.$$

Step 2: lower bound. The proof of the lower bound is quite similar. The only difference is that we have to replace $\Phi(t)$ by

$$\chi(t) = \varphi(t)\exp\left\{\int\limits_t^{+\infty}\varkappa(s)\,\mathrm{d}s\right\}, \quad t > 0.$$

Then the derivative of χ obeys the inequality

$$\frac{\mathrm{d}\chi(t)}{\mathrm{d}t} = -\varkappa(t)\varphi(t)\exp\left\{\int\limits_t^{+\infty}\varkappa(s)\,\mathrm{d}s\right\} + \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\exp\left\{\int\limits_t^{+\infty}\varkappa(s)\,\mathrm{d}s\right\}$$
$$\leqslant a(t)\exp\left\{\int\limits_t^{+\infty}\varkappa(s)\,\mathrm{d}s\right\}.$$

According to the Newton-Leibniz formula, it holds that

$$\chi(+\infty) - \chi(t) = \int\limits_t^{+\infty}\frac{\mathrm{d}\chi(s)}{\mathrm{d}s}\,\mathrm{d}s \leqslant \int\limits_t^{+\infty} a(s)\exp\left\{\int\limits_s^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s.$$

This implies that

$$\varphi(t)\exp\left\{\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\} \geqslant \varphi(+\infty) - \int\limits_t^{+\infty} a(s)\exp\left\{\int\limits_s^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s.$$

Since, for any $s \geqslant t$, it holds that

$$\exp\left\{\int\limits_t^{s}\varkappa(\tau)\,\mathrm{d}\tau\right\} \leqslant \exp\left\{\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\}$$

we obtain the desired bound:

$$\begin{aligned}
\varphi(t) &\geqslant \varphi(+\infty)\exp\left\{-\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\} - \int\limits_t^{+\infty} a(s)\exp\left\{-\int\limits_t^{s}\varkappa(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s \\
&\geqslant \varphi(+\infty)\exp\left\{-\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\} - \int\limits_t^{+\infty} a(s)\exp\left\{-\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s \\
&= \left(\varphi(+\infty) - \int\limits_t^{+\infty} a(s)\,\mathrm{d}s\right)\exp\left\{-\int\limits_t^{+\infty}\varkappa(\tau)\,\mathrm{d}\tau\right\}.
\end{aligned}$$

□

D.2 Proof of Lemma D.2

Let us note that $\mathsf{q}(y \mid x)$ is the conditional density of X_T^0 given $X_0^0 = x$, where X_t^0 is the Ornstein-Uhlenbeck process

$$\mathrm{d}X_t^0 = b(m - X_t)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}W_t, \quad X_0^0 \sim \rho_{\mu,\Omega}.$$

Due to the properties of the Ornstein-Uhlenbeck process, $X_T - e^{-bT}X_0$ is independent of X_0 and

$$X_T - e^{-bT}X_0 \sim \mathcal{N}\big((1 - e^{-bT})m, \Sigma_T\big).$$

Since $X_0 \sim \mathcal{N}(\mu, \Omega)$ by the condition of the lemma, X_0 and $X_T - e^{-bT}X_0$ have a joint distribution

$$\begin{pmatrix} X_0 \\ X_T - e^{-bT}X_0 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ (1 - e^{-bT})m \end{pmatrix}, \begin{pmatrix} \Omega & O_d \\ O_d & \Sigma_T \end{pmatrix}\right),$$

where O_d is a matrix of size $d \times d$ with zero entries. This yields that

$$\begin{pmatrix} X_0 \\ X_T \end{pmatrix} = \begin{pmatrix} I_d & O_d \\ e^{-bT}I_d & I_d \end{pmatrix}\begin{pmatrix} X_0 \\ X_T - e^{-bT}X_0 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ m_T(\mu) \end{pmatrix}, \begin{pmatrix} \Omega & e^{-bT}\Omega \\ e^{-bT}\Omega & \Sigma_T + e^{-2bT}\Omega \end{pmatrix}\right).$$

Then the integral

$$\int\limits_{\mathbb{R}^d} f(x)\frac{\mathsf{q}(y \,|\, x)\rho_{\mu,\Omega}(x)}{\varphi(y)}\,\mathrm{d}x$$

is nothing but the the conditional expectation of $f(X_0)$ given $X_T = y$. It is known that the conditional distribution of X_0 given $X_T = y$ is Gaussian with mean $\breve{\mu}$ and covariance $\breve{\Omega}$

$$\breve{\mu} = \mu + e^{-bT}\Omega(\Sigma_T + e^{-2bT}\Omega)^{-1}\big(y - m_T(\mu)\big), \quad \breve{\Omega} = \big(\Omega^{-1} + e^{-2bT}\Sigma_T^{-1}\big)^{-1},$$

Hence, we obtain that

$$\int\limits_{\mathbb{R}^d} f(x)\frac{\mathsf{q}(y \,|\, x)\rho_{\mu,\Omega}(x)}{\varphi(y)}\,\mathrm{d}x = \mathbb{E}_{\xi\sim\mathcal{N}(\breve{\mu},\breve{\Omega})}f(\xi).$$

□

D.3 Proof of Lemma D.3

First, let us fix $\lambda > 0$ and consider the exponential moment $\mathbb{E}e^{\lambda|\|\xi\|^2 - d|}$. Using explicit expressions for moment generating functions of the chi-squared distribution with d degrees of freedom, we obtain that

$$\mathbb{E}e^{\lambda|\|\xi\|^2 - d|} \leqslant \mathbb{E}e^{\lambda\|\xi\|^2 - \lambda d} + \mathbb{E}e^{\lambda d - \lambda\|\xi\|^2} = \frac{e^{-\lambda d}}{(1-2\lambda)^{d/2}} + \frac{e^{\lambda d}}{(1+2\lambda)^{d/2}}.$$

Let us note that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(-\frac{1}{2}\log(1-2x)\right) = \frac{2}{(1-2x)^2} \leqslant 8 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(-\frac{1}{2}\log(1+2x)\right) = \frac{2}{(1+2x)^2} \leqslant 2$$

for all $0 \leqslant x \leqslant 1/4$. This and Taylor's expansion with a Lagrange remainder term imply that

$$-\frac{1}{2}\log(1-2x) \leqslant x + 4x^2 \quad \text{and} \quad -\frac{1}{2}\log(1+2x) \leqslant x + x^2 \quad \text{for all } 0 \leqslant x \leqslant 1/4.$$

Thus, we obtain that

$$\mathbb{E}e^{\lambda|\|\xi\|^2 - d|} \leqslant \frac{e^{-\lambda d}}{(1-2\lambda)^{d/2}} + \frac{e^{\lambda d}}{(1+2\lambda)^{d/2}} \leqslant e^{4\lambda^2 d} + e^{\lambda^2 d} \quad \text{for all } 0 < \lambda \leqslant 1/4.$$

We apply this inequality to bound the L_p-norm of $\|\xi\|^2 - d$ using a standard technique. To be more precise, for any $0 < \lambda \leqslant 1/4$, it holds that

$$\begin{aligned}
\mathbb{E}\left|\|\xi\|^2 - d\right|^p &= \int\limits_0^{+\infty} \mathbb{P}\left(\left|\|\xi\|^2 - d\right|^p \geqslant u\right) \mathrm{d}u \\
&\leqslant \int\limits_0^{+\infty} e^{-\lambda u^{1/p}} \mathbb{E}e^{\lambda|\|\xi\|^2 - d|} \,\mathrm{d}u \\
&\leqslant \left(e^{4\lambda^2 d} + e^{\lambda^2 d}\right) \int\limits_0^{+\infty} e^{-\lambda u^{1/p}} \,\mathrm{d}u.
\end{aligned}$$

Let us take $\lambda = 1/(4\sqrt{d})$. Then, substituting $u^{1/p}/(4\sqrt{d})$ by v, we obtain that

$$\begin{aligned}
\mathbb{E}\left|\|\xi\|^2 - d\right|^p \leqslant \left(e^{1/4} + e^{1/16}\right) \int\limits_0^{+\infty} e^{-\lambda u^{1/p}} \,\mathrm{d}u \\
&= (4\sqrt{d})^p \left(e^{1/4} + e^{1/16}\right) \int\limits_0^{+\infty} p v^{p-1} e^{v} \,\mathrm{d}v \\
&= (4\sqrt{d})^p \left(e^{1/4} + e^{1/16}\right) \cdot p\Gamma(p) \\
&= (4\sqrt{d})^p \left(e^{1/4} + e^{1/16}\right) \Gamma(p+1).
\end{aligned}$$

Since $e^{1/4} + e^{1/16} \leqslant 5/2$ and $\Gamma(p+1) \leqslant p^p$ for all $p \geqslant 1$, the expression in the right-hand side does not exceed

$$(4\sqrt{d})^p \left(e^{1/4} + e^{1/16}\right) \Gamma(p+1) \leqslant \frac{5}{2} \cdot (4p\sqrt{d})^p \leqslant (10p\sqrt{d})^p \quad \text{for all } p \geqslant 1.$$

The proof is finished.

$\square$

E On Schrödinger potentials in the Gaussian case

In conclusion, we would like to focus on a Gaussian setup. Given an initial distribution $\mathcal{N}(\mu_0, Q_0)$, a target distribution $\mathcal{N}(\mu_T, Q_T)$, and a reference process

$$\mathrm{d}X_t = b(m - X_t)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}W_t, \quad 0 \leqslant t \leqslant T,$$

we are going to derive explicit expressions for Schrödinger potentials ν_0 and ν_T and show that the log-density of ν_T with respect to the Lebesgue measure satisfies Assumption 3. A similar question was studied in [Bunne et al., 2023], where the authors obtained an explicit expression for a solution of a dynamic Schrödinger Bridge problem. However, the authors did not specify the potentials. In this section, we fill this gap. We would like to remind that, according to [Dai Pra, 1991, Theorem 2.2.], there exist unique (up to a multiplicative constant) ν_0 and ν_T such that the measure

$$\pi(\mathrm{d}x_0, \mathrm{d}x_T) = \mathsf{Q}(x_T, T \mid x_0, 0)\ \nu_0(\mathrm{d}x_0)\nu_T(\mathrm{d}x_T),$$

where $\mathsf{Q}(x_T, T \mid x_0, 0)$ is the transition density of the base process, has the marginals $\mathcal{N}(\mu_0, Q_0)$ and $\mathcal{N}(\mu_T, Q_T)$. Throughout this section,

$$\upsilon_0(x) = \frac{\mathrm{d}\nu_0}{\mathrm{d}x} \quad \text{and} \quad \upsilon_T(x) = \frac{\mathrm{d}\nu_T}{\mathrm{d}x}$$

stand for the Radon-Nikodym derivatives of ν_0 and ν_T, respectively. With this notation, we have

$$\pi(\mathrm{d}x_0, \mathrm{d}x_T) = \mathsf{Q}(x_T, T \mid x_0, 0)\ \upsilon_0(x_0)\upsilon_T(x_T)\ \mathrm{d}x_0\mathrm{d}x_T.$$

Let us introduce $Z_t = \Sigma^{-1/2}X_t$, $0 \leqslant t \leqslant T$, and let $\mathsf{P}(z_T, T \mid z_0, 0)$ stand for the transition density of a scaled reference process

$$\mathrm{d}Z_t = b(\Sigma^{-1/2}m - Z_t)\mathrm{d}t + \mathrm{d}W_t, \quad 0 \leqslant t \leqslant T.$$

Our idea is based on an observation that it is enough to find such ϱ_0 and ϱ_T that the scaled coupling

$$\varpi(\mathrm{d}z_0, \mathrm{d}z_T) = \mathsf{P}(z_T, T \mid z_0, 0)\ \varrho_0(z_0)\varrho_T(z_T)\ \mathrm{d}z_0\mathrm{d}z_T \tag{62}$$

has the marginals $\mathcal{N}(\Sigma^{-1/2}\mu_0, S_0)$ and $\mathcal{N}(\Sigma^{-1/2}\mu_T, S_T)$, where

$$S_0 = \Sigma^{-1/2}Q_0\Sigma^{-1/2} \quad \text{and} \quad S_T = \Sigma^{-1/2}Q_T\Sigma^{-1/2}. \tag{63}$$

Then, making an inverse substitution, it is easy to observe that

$$\upsilon_0(x_0) = \det(\Sigma)^{-1/2}\varrho_0(\Sigma^{-1/2}x_0) \quad \text{and} \quad \upsilon_T(x_T) = \det(\Sigma)^{-1/2}\varrho_T(\Sigma^{-1/2}x_T). \tag{64}$$

Similarly to [Bunne et al., 2023], our approach uses the fact that an entropic optimal transport plan between the Gaussian measures $\mathcal{N}(\Sigma^{-1/2}\mu_0, S_0)$ and $\mathcal{N}(\Sigma^{-1/2}\mu_T, S_T)$ has a form (see, for instance, [Janati et al., 2020, Theorem 1])

$$\mathcal{N}\left(\begin{pmatrix}\Sigma^{-1/2}\mu_0 \\ \Sigma^{-1/2}\mu_T\end{pmatrix}, \begin{pmatrix}S_0 & A_\sigma \\ A_\sigma^\top & S_T\end{pmatrix}\right) \tag{65}$$

with

$$D_\sigma = \left(4S_0^{1/2}S_TS_0^{1/2} + \sigma^4 I_d\right)^{1/2} \tag{66}$$

and

$$A_\sigma = \frac{1}{2}\left(S_0^{1/2}D_\sigma S_0^{-1/2} - \sigma^2 I_d\right). \tag{67}$$

Here $\sigma > 0$ is a penalization parameter in the entropic optimal transport problem (see [Bunne et al., 2023], eq. (1)). It turns out that the coupling ϖ from (62) is equal to (65) with an appropriate σ. We are ready to move to the main result of this section.

Proposition E.1. *Set*

$$\sigma^2 = \frac{1 - e^{-2bT}}{2b} \cdot e^{bT} \tag{68}$$

and let

$$\breve{S}_0 = S_0 - A_\sigma S_T^{-1}A_\sigma^\top, \quad \breve{S}_T = S_T - A_\sigma^\top S_0^{-1}A_\sigma, \tag{69}$$

where A_σ is defined in (67)*. With the notations introduced above, it holds that*

$$\log \varrho_0(z_0) = -\frac{1}{2}\left\| \breve{S}_0^{-1/2}(z_0 - \Sigma^{-1/2}\mu_0)\right\|^2 + \frac{be^{-2bT}\|z_0\|^2}{(1-e^{-2bT})} - \frac{2b\big((1-e^{-bT})m + \mu_T\big)^\top \Sigma^{-1/2} z_0}{e^{bT}(1-e^{-2bT})} + C_0$$

and

$$\log \varrho_T(z_T) = -\frac{1}{2}\left\| \breve{S}_T^{-1/2}(z_T - \Sigma^{-1/2}\mu_T)\right\|^2 + \frac{b\|z_T\|^2}{(1-e^{-2bT})} - \frac{2b\big((1-e^{-bT})m + e^{-bT}\mu_0\big)^\top \Sigma^{-1/2} z_T}{(1-e^{-2bT})} + C_T,$$

where C_0 and C_T are some constants.

The proof of Proposition E.1 is moved to Appendix E.1. In view of (64), it yields that there are some constants $\widetilde{C}_0$ and $\widetilde{C}_T$ such that

$$\log \upsilon_0(x_0) = -\frac{1}{2}\left\| \breve{S}_0^{-1/2}\Sigma^{-1/2}(x_0 - \mu_0)\right\|^2 + \frac{be^{-2bT}\|\Sigma^{-1/2}x_0\|^2}{(1-e^{-2bT})} - \frac{2b\big((1-e^{-bT})m + \mu_T\big)^\top \Sigma^{-1} x_0}{e^{bT}(1-e^{-2bT})} + \widetilde{C}_0$$

and

$$\log \upsilon_T(x_T) = -\frac{1}{2}\left\| \breve{S}_T^{-1/2}\Sigma^{-1/2}(x_T - \mu_T)\right\|^2 + \frac{b\|\Sigma^{-1/2}x_T\|^2}{(1-e^{-2bT})} - \frac{2b\big((1-e^{-bT})m + e^{-bT}\mu_0\big)^\top \Sigma^{-1} x_T}{(1-e^{-2bT})} + \widetilde{C}_T.$$

Let us note that $-\log \upsilon_T(x_T)$ grows as fast as $\mathcal{O}(\|x_T\|^2)$. Besides, if

$$S_T = \Sigma^{-1/2} Q_T \Sigma^{-1/2} \preceq \frac{1-e^{-2bT}}{2b}\, I_d,$$

then

$$\breve{S}_T = S_T - A_\sigma^\top S_0^{-1} A_\sigma \prec S_T \preceq \frac{1-e^{-2bT}}{2b}\, I_d,$$

and the potential $\varrho_T(z_T)$ (and, hence, $\upsilon_T(x_T)$ as well) is bounded. This gives a hint on how a learner should choose b and T.

E.1 Proof of Proposition E.1

Let $\sigma > 0$ be as defined in (68) and let the measure ϖ from (62) be equal to

$$\mathcal{N}\left(\begin{pmatrix} \Sigma^{-1/2}\mu_0 \\ \Sigma^{-1/2}\mu_T \end{pmatrix}, \begin{pmatrix} S_0 & A_\sigma \\ A_\sigma^\top & S_T \end{pmatrix} \right),$$

where S_0 and S_T are defined in (63) and A_σ is given by (67). Using the block-matrix inversion formula

$$\begin{pmatrix} S_0 & A_\sigma \\ A_\sigma^\top & S_T \end{pmatrix}^{-1} = \begin{pmatrix} \breve{S}_0^{-1} & -\breve{S}_0^{-1} A_\sigma S_T^{-1} \\ -S_T^{-1} A_\sigma^\top \breve{S}_0^{-1} & \breve{S}_T^{-1} \end{pmatrix}$$

with the Schur complements $\breve{S}_0$ and $\breve{S}_T$ defined in (69), we obtain that the log-density of ϖ with respect to the Lebesgue measure on $\mathbb{R}^{2d}$ satisfies

$$
\begin{aligned}
\log \frac{\varpi(\mathrm{d}z_0, \mathrm{d}z_T)}{\mathrm{d}z_0 \mathrm{d}z_T} &= -\frac{1}{2} \begin{pmatrix} z_0 - \Sigma^{-1/2}\mu_0 \\ z_T - \Sigma^{-1/2}\mu_T \end{pmatrix}^\top \begin{pmatrix} S_0 & A_\sigma \\ A_\sigma^\top & S_T \end{pmatrix}^{-1} \begin{pmatrix} z_0 - \Sigma^{-1/2}\mu_0 \\ z_T - \Sigma^{-1/2}\mu_T \end{pmatrix} + C \\
&= -\frac{1}{2} \begin{pmatrix} z_0 - \Sigma^{-1/2}\mu_0 \\ z_T - \Sigma^{-1/2}\mu_T \end{pmatrix}^\top \begin{pmatrix} \breve{S}_0^{-1} & -\breve{S}_0^{-1} A_\sigma S_T^{-1} \\ -S_T^{-1} A_\sigma^\top \breve{S}_0^{-1} & \breve{S}_T^{-1} \end{pmatrix} \begin{pmatrix} z_0 - \Sigma^{-1/2}\mu_0 \\ z_T - \Sigma^{-1/2}\mu_T \end{pmatrix} + C \\
&= -\frac{1}{2} \left\| \breve{S}_0^{-1/2}(z_0 - \Sigma^{-1/2}\mu_0) \right\|^2 - \frac{1}{2} \left\| \breve{S}_T^{-1/2}(z_T - \Sigma^{-1/2}\mu_T) \right\|^2 \\
&\quad + (z_0 - \Sigma^{-1/2}\mu_0)^\top \breve{S}_0^{-1} A_\sigma S_T^{-1} (z_T - \Sigma^{-1/2}\mu_T) + C,
\end{aligned} \tag{70}
$$

where C is a normalizing constant. We are going to show that $\breve{S}_0 S_T = \sigma^2 A_\sigma$. According to the definition of $\breve{S}_0$ (see (69)), it holds that

$$
\begin{aligned}
\breve{S}_0 S_T &= (S_0 - A_\sigma S_T^{-1} A_\sigma^\top) S_T \\
&= S_0 S_T - A_\sigma S_T^{-1} A_\sigma^\top S_T \\
&= S_0 S_T - \frac{1}{4} \left(S_0^{1/2} D_\sigma S_0^{-1/2} - \sigma^2 I_d \right) S_T^{-1} \left(S_0^{-1/2} D_\sigma S_0^{1/2} - \sigma^2 I_d \right) S_T \\
&= S_0 S_T - \frac{1}{4} S_0^{1/2} D_\sigma S_0^{-1/2} S_T^{-1} S_0^{-1/2} D_\sigma S_0^{1/2} S_T \\
&\quad + \frac{\sigma^2}{4} S_0^{1/2} D_\sigma S_0^{-1/2} + \frac{\sigma^2}{4} S_T^{-1} S_0^{-1/2} D_\sigma S_0^{1/2} S_T - \frac{\sigma^4}{4} I_d.
\end{aligned} \tag{71}
$$

Let us elaborate on the second and the fourth terms in the right-hand side of (71). Let us note that, due to the definition of D_σ (see (66)), it commutes with $S_0^{1/2} S_T S_0^{1/2}$, because these symmetric matrices share the same eigenvectors. This yields that

$$
\begin{aligned}
S_0^{1/2} D_\sigma S_0^{-1/2} S_T^{-1} S_0^{-1/2} D_\sigma S_0^{1/2} S_T &= S_0^{1/2} D_\sigma^2 S_0^{-1/2} \\
&= S_0^{1/2} (4 S_0^{1/2} S_T S_0^{1/2} + \sigma^4 I_d) S_0^{-1/2} \\
&= 4 S_0 S_T + \sigma^4 I_d.
\end{aligned} \tag{72}
$$

Similarly, it holds that

$$
S_T^{-1} S_0^{-1/2} D_\sigma S_0^{1/2} S_T = S_T^{-1} S_0^{-1/2} D_\sigma S_0^{1/2} S_T S_0^{1/2} S_0^{-1/2} = S_0^{1/2} D_\sigma S_0^{-1/2}. \tag{73}
$$

Summing up (71), (72), and (73), we obtain that

$$
\begin{aligned}
\breve{S}_0 S_T &= S_0 S_T - \frac{1}{4} \left(4 S_0 S_T + \sigma^4 I_d \right) + \frac{\sigma^2}{4} S_0^{1/2} D_\sigma S_0^{-1/2} + \frac{\sigma^2}{4} S_0^{1/2} D_\sigma S_0^{-1/2} - \frac{\sigma^4}{4} I_d \\
&= \frac{\sigma^2}{2} \left(S_0^{1/2} D_\sigma S_0^{-1/2} - \sigma^2 I_d \right) = \sigma^2 A_\sigma,
\end{aligned}
$$

as we announced. This and (70) yield that

$$
\begin{aligned}
\log \frac{\varpi(\mathrm{d}z_0, \mathrm{d}z_T)}{\mathrm{d}z_0 \mathrm{d}z_T} &= -\frac{1}{2} \left\| \breve{S}_0^{-1/2}(z_0 - \Sigma^{-1/2}\mu_0) \right\|^2 - \frac{1}{2} \left\| \breve{S}_T^{-1/2}(z_T - \Sigma^{-1/2}\mu_T) \right\|^2 \\
&\quad + (z_0 - \Sigma^{-1/2}\mu_0)^\top \breve{S}_0^{-1} A_\sigma S_T^{-1} (z_T - \Sigma^{-1/2}\mu_T) + C \\
&= -\frac{1}{2} \left\| \breve{S}_0^{-1/2}(z_0 - \Sigma^{-1/2}\mu_0) \right\|^2 - \frac{1}{2} \left\| \breve{S}_T^{-1/2}(z_T - \Sigma^{-1/2}\mu_T) \right\|^2 \\
&\quad + \frac{1}{\sigma^2} (z_0 - \Sigma^{-1/2}\mu_0)^\top (z_T - \Sigma^{-1/2}\mu_T) + C.
\end{aligned} \tag{74}
$$

On the other hand, in view of (62), we have

$$
\log \frac{\varpi(\mathrm{d}z_0, \mathrm{d}z_T)}{\mathrm{d}z_0 \mathrm{d}z_T} = \log \mathsf{P}(z_T, T \mid z_0, 0) + \log \varrho_0(z_0) + \log \varrho_T(z_T),
$$

where the transition density $\mathsf{P}(z_T, T \mid z_0, 0)$ of the scaled reference process is given by

$$\begin{aligned}\mathsf{P}(z_T, T \mid z_0, 0) &= (2\pi)^{-d/2} \cdot \left(\frac{1-e^{-2bT}}{2b}\right)^{-d/2} \\ &\qquad \cdot \exp\left\{-\frac{1}{2} \cdot \frac{2b}{1-e^{-2bT}} \cdot \left\| z_T - (1-e^{-bT})\Sigma^{-1/2} m - e^{-bT} z_0 \right\|^2\right\} \\ &= \left(\frac{\pi(1-e^{-2bT})}{b}\right)^{-d/2} \exp\left\{-\frac{e^{bT}}{2\sigma^2} \left\| z_T - (1-e^{-bT})\Sigma^{-1/2} m - e^{-bT} z_0 \right\|^2\right\}.\end{aligned}$$

This equality, combined with (74), implies that

$$\begin{aligned}&\log \varrho_0(z_0) + \log \varrho_T(z_T) \\ &= \log \frac{\varpi(\mathrm{d}z_0, \mathrm{d}z_T)}{\mathrm{d}z_0 \mathrm{d}z_T} - \log \mathsf{P}(z_T, T \mid z_0, 0) \\ &= -\frac{1}{2}\left\| \breve{S}_0^{-1/2}(z_0 - \Sigma^{-1/2}\mu_0)\right\|^2 - \frac{1}{2}\left\| \breve{S}_T^{-1/2}(z_T - \Sigma^{-1/2}\mu_T)\right\|^2 \\ &\quad + \frac{e^{bT}\|z_T\|^2}{2\sigma^2} + \frac{e^{-bT}\|z_0\|^2}{2\sigma^2} + \frac{e^{bT}(1-e^{-bT})^2\|m\|^2}{2\sigma^2} \\ &\quad - \frac{\big((1-e^{-bT})m + \mu_T\big)^\top \Sigma^{-1/2} z_0}{\sigma^2} - \frac{\big(e^{bT}(1-e^{-bT})m + \mu_0\big)^\top \Sigma^{-1/2} z_T}{\sigma^2} \\ &\quad + \frac{\mu_0^\top \Sigma^{-1} \mu_T}{\sigma^2} + \frac{d}{2}\log\left(\frac{\pi(1-e^{-2bT})}{b}\right) + C.\end{aligned}$$

Hence, there exist constants C_0 and C_T such that

$$\begin{aligned}\log \varrho_0(z_0) &= -\frac{1}{2}\left\| \breve{S}_0^{-1/2}(z_0 - \Sigma^{-1/2}\mu_0)\right\|^2 + \frac{e^{-bT}\|z_0\|^2}{2\sigma^2} \\ &\qquad - \frac{\big((1-e^{-bT})m + \mu_T\big)^\top \Sigma^{-1/2} z_0}{\sigma^2} + C_0 \\ &= -\frac{1}{2}\left\| \breve{S}_0^{-1/2}(z_0 - \Sigma^{-1/2}\mu_0)\right\|^2 + \frac{b e^{-2bT}\|z_0\|^2}{(1-e^{-2bT})} \\ &\qquad - \frac{2b\big((1-e^{-bT})m + \mu_T\big)^\top \Sigma^{-1/2} z_0}{e^{bT}(1-e^{-2bT})} + C_0\end{aligned}$$

and

$$\begin{aligned}\log \varrho_T(z_T) &= -\frac{1}{2}\left\| \breve{S}_T^{-1/2}(z_T - \Sigma^{-1/2}\mu_T)\right\|^2 + \frac{e^{bT}\|z_T\|^2}{2\sigma^2} \\ &\qquad - \frac{\big(e^{bT}(1-e^{-bT})m + \mu_0\big)^\top \Sigma^{-1/2} z_T}{\sigma^2} + C_T \\ &= -\frac{1}{2}\left\| \breve{S}_T^{-1/2}(z_T - \Sigma^{-1/2}\mu_T)\right\|^2 + \frac{b\|z_T\|^2}{(1-e^{-2bT})} \\ &\qquad - \frac{2b\big((1-e^{-bT})m + e^{-bT}\mu_0\big)^\top \Sigma^{-1/2} z_T}{(1-e^{-2bT})} + C_T.\end{aligned}$$

Here we used the fact that we chose σ^2 according to (68).

$\square$