Learning convex regularizers satisfying the variational source condition for inverse problems

Subhadip Mukherjee[1], Carola-Bibiane Schönlieb[1], and Martin Burger[2]
[1]Department of Applied Mathematics and Theoretical Physics, University of Cambridge, UK
[2]Department of Mathematics, University of Erlangen-Nuremberg, Germany
Emails: {sm2467, cbs31}@cam.ac.uk, martin.burger@fau.de

1. BACKGROUND ON INVERSE PROBLEMS

- Reconstruct image $x^* \in \mathbb{X}$ from noisy observation (data) $y^\delta = Ax^* + \text{noise} \in \mathbb{Y}$.
  - $A : \mathbb{X} \to \mathbb{Y}$ is the forward operator.
  - $\|\text{noise}\| \leqslant \delta$
- Inverse problems are ill-posed, i.e., A is either non-invertible or poorly conditioned.
- **Variational regularization**:

$$\boxed{x_\lambda \in \underset{x \in \mathbb{X}}{\arg\min}\, \frac{1}{2}\|y^\delta - Ax\|_2^2 + \lambda\, \psi_\theta(x)} \quad (1)$$

  - $\{\psi_\theta\}_{\theta \in \Theta}$ is a convex regularizer.
- **ψ_θ-minimizing solution**:

$$\boxed{x^\dagger \in \underset{x \in \mathbb{X}}{\arg\min}\, \psi_\theta(x) \text{ subject to } Ax = y^0} \quad (2)$$

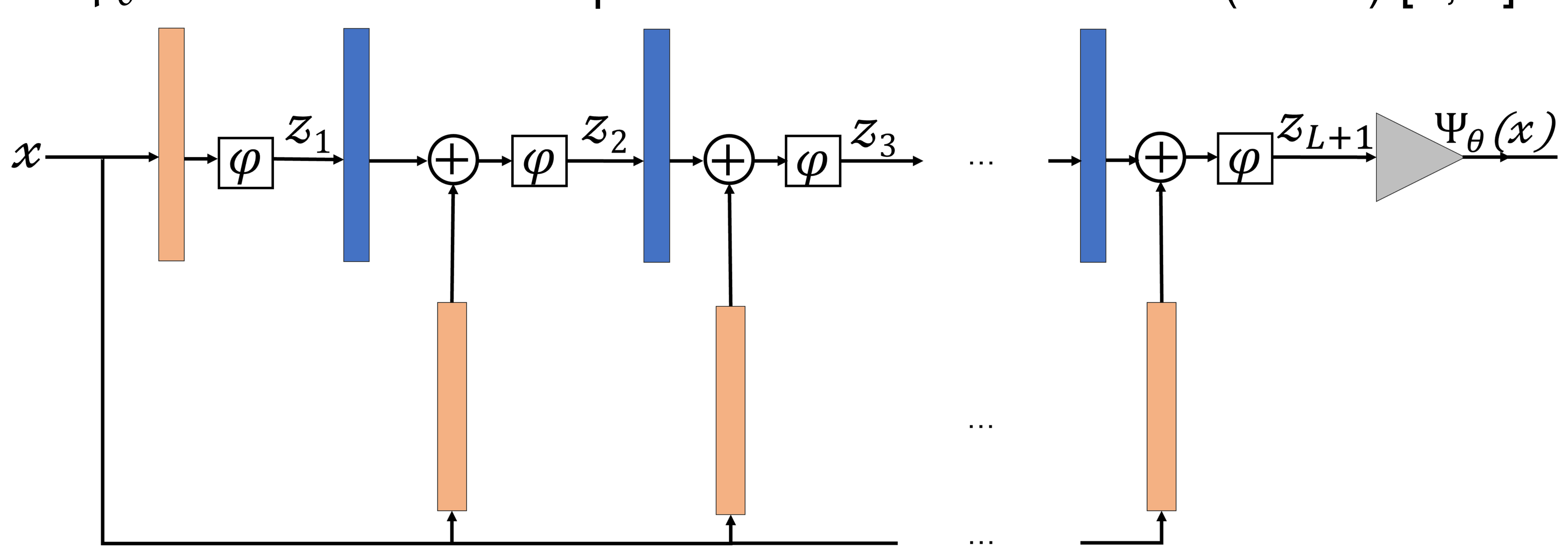
- **Variational source condition**: is satisfied if there exists some $w^\dagger \in \mathbb{Y}$ such that $A^* w^\dagger \in \partial\, \psi_\theta(x^\dagger)$.
- **Bregman distance**: $D_{\psi_\theta}(x_1, x_2) := \left\{\psi_\theta(x_1) - \psi_\theta(x_2) - \langle u, x_1 - x_2\rangle \middle| u \in \partial\psi_\theta(x_2)\right\}$
- **Convergence rate** [1]: If the source condition holds, then for each minimizer x_λ of (1), there exists $d \in D_{\psi_\theta}(x_\lambda, x^\dagger)$ such that

$$\boxed{d \leqslant \lambda \frac{\|w^\dagger\|^2}{2} + \frac{\delta^2}{2\lambda}}. \quad (3)$$

Therefore, choosing $\lambda \propto \delta$ leads to an $\mathcal{O}(\delta)$ convergence rate of the variational reconstruction x_λ to $x^\dagger$.

2. PARAMETRIZING THE REGULARIZER

- ψ_θ is taken to be an input-convex neural network (ICNN) [2, 3].



- ψ_θ is constructed recursively by taking non-negative sums of convex functions (starting from affine), and then applying a (point-wise) monotonically-increasing convex activation.
- The filter weights in the orange layers need to be $\geqslant 0$, whereas the blue layers can have any real-valued filters.
- The activation functions are taken to be leaky-ReLU with negative slope 0.2 (convex and monotone).

3. LEARNING THE REGULARIZER

- **Main idea**: If A is invertible, source condition dictates that for any solution x of (1), the following holds:

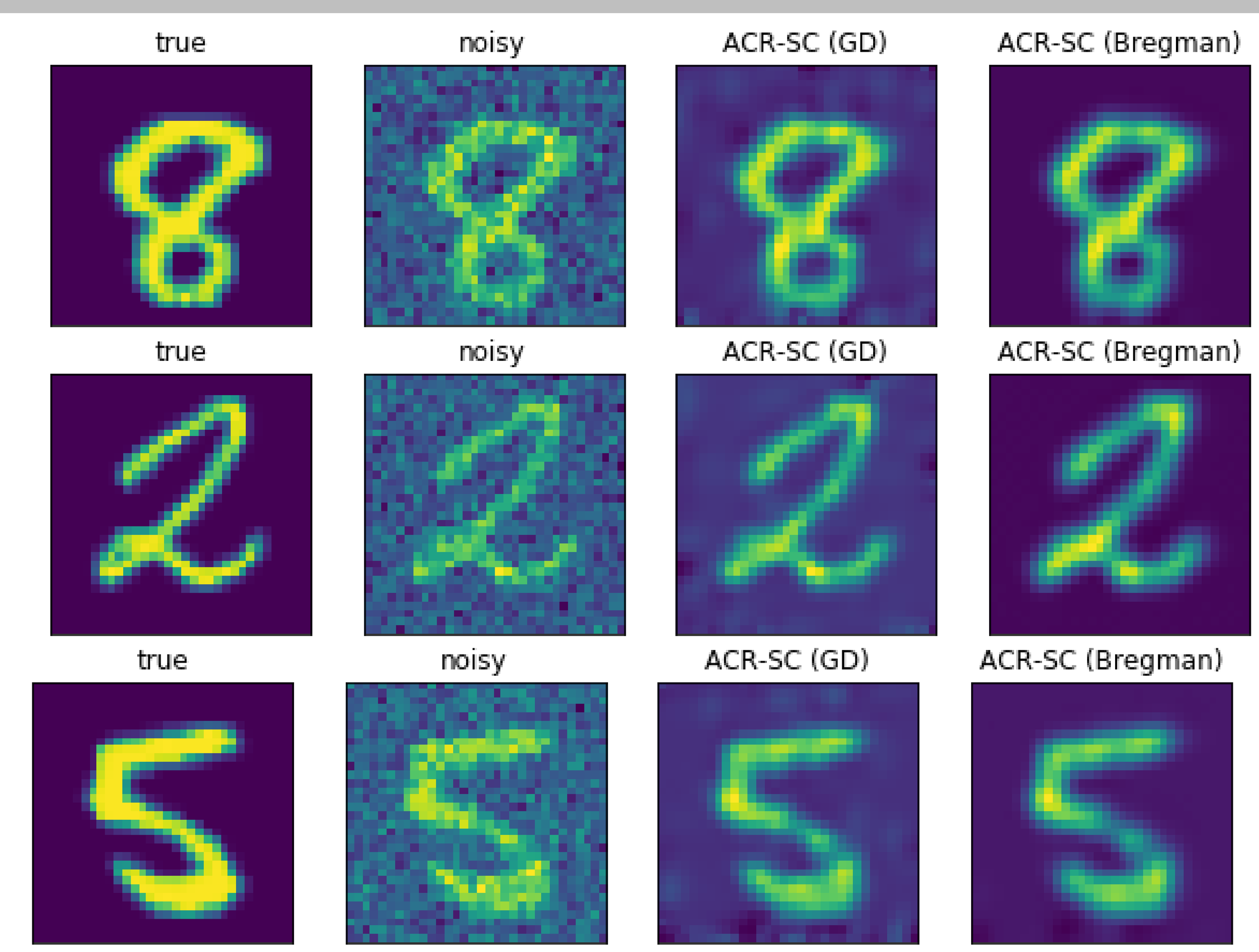
$$\boxed{\ell_{\text{sc}}(x;\theta) = \|(A^*)^{-1}\nabla_x \psi_\theta(x)\| < \infty}$$

- The smaller the quantity $\ell_{\text{sc}}(x;\theta)$ is, the more suitable x would be as a variational solution.
  - Encourages the ground-truth images to the solution of the resulting variational problem.
- For non-invertible A, replace the inverse with the Moore-Penrose pseudo-inverse.
- **Training loss**:

$$\boxed{L(\theta) = \frac{1}{n}\sum_{i=1}^{n} \psi_\theta(x_i) - \frac{1}{n}\sum_{i=1}^{n} \psi_\theta(z_i) + \lambda_{\text{gp}} L_{\text{gp}}(\theta) + \lambda_{\text{sc}} L_{\text{sc}}(\theta)} \quad (4)$$

  - x_i: ensemble of clean images
  - $z_i = A^\dagger y_i^\delta$: ensemble of noisy images
  - L_{gp}: soft gradient penalty to enforce 1-Lipschitz bound on ψ_θ
- Can solve the variational problem with the learned regularizer via sub-gradient or Bregman iterations [4].

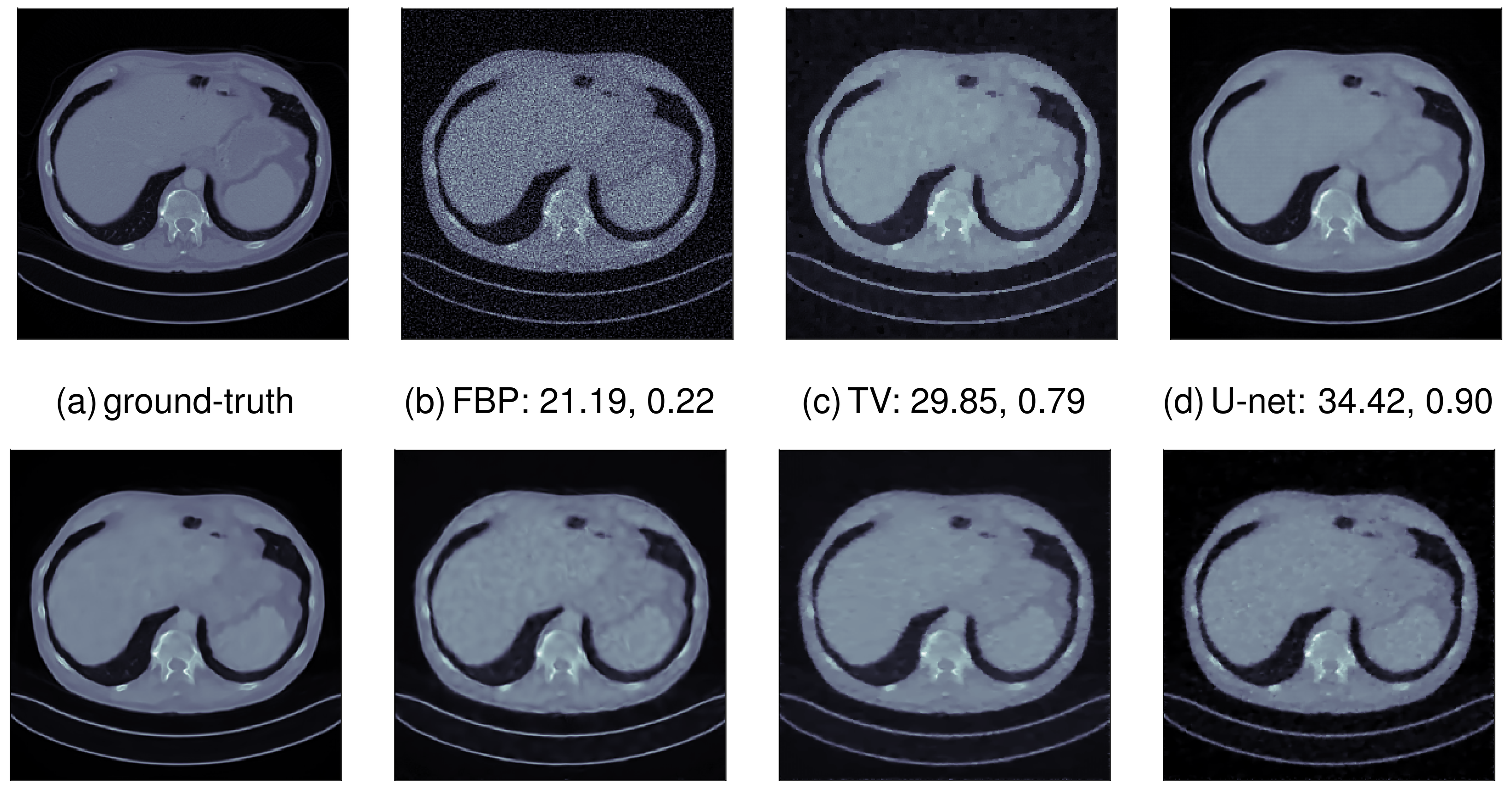
4. NUMERICAL EXAMPLES: DENOISING



- $\lambda_{\text{sc}} = 2.0$, $\lambda_{\text{gp}} = 10.0$
- The average PSNR and SSIM over 100 randomly chosen test images:
  1. noisy: 13.93 ± 0.13 dB, 0.51 ± 0.08
  2. ACR-SC (GD): 22.72 ± 0.64, 0.77 ± 0.04
  3. ACR-SC (Bregman): 20.29 ± 0.88, 0.86 ± 0.03
- Took $\lambda = 25$ for the Bregman technique and $\lambda = 5$ for vanilla gradient-descent.
- Bregman iterations perform better in terms of recovering the contrast while yielding effective denoising.

5. CT RECONSTRUCTION EXPERIMENTS

- **Experiments on Mayo-clinic low-dose CT data (2016)**:
  1. Extracted 2D slices of size 512×512 from 3D scans.
  2. Trained on 9 patients (2250 slices), evaluated on one (128 slices).
  3. Parallel-beam projection, 200 angles, 400 lines/angle, Gaussian noise with $\sigma_e = 2.0$ (25 dB of signal-to-noise ratio in the data space).
  4. No. of learnable parameters in ψ_θ: 590928.
  5. $\lambda_{\text{sc}} = 2.0$ and $\lambda_{\text{gp}} = 5.0$, trained for 10 epochs, batch-size was four. Adam optimizer with $\eta, \beta_1, \beta_2 = 10^{-5}, 0.90, 0.99$.



(a) ground-truth (b) FBP: 21.19, 0.22 (c) TV: 29.85, 0.79 (d) U-net: 34.42, 0.90

(e) LPD: 35.76, 0.92 (f) AR: 33.52, 0.86 (g) ACR: 31.24, 0.86 (h) ACR-SC: 30.93, 0.85

- ACR-SC is only marginally inferior to ACR, while it still outperforms classical model-based methods such as total-variation (TV).

6. SUMMARY

- Developed a novel training loss for learning a data-driven convex regularizer satisfying the source condition.
- Unsupervised learning, no paired data needed.
- Possible to derive convergence rate estimates for the resulting variational problem.
- Theoretical grounding for Bregman iterations.
- Enforcing the source condition does not lead to any significant drop in reconstruction quality.

REFERENCES

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