Appendix

In this appendix, we report the proofs and the experimental results missing from the main body of the paper.

A MISSING PROOFS

Proof of Theorem 1. Let $\vec{x}$ be the vector containing the position of the agents and let $\vec{y}$ be the position of the facilities. We denote with k_j the capacity of the facility located at y_j for every $j \in [m]$. In what follows, we assume that the set of agents has an inner ordering that decides how to break ties.

Let us define $\mathcal{D}$ the set containing all the distances agents to facility, that is $\mathcal{D} = \{|x_i - y_j|\}_{i\in[n],j\in[m]}$.

Let $\vec{c} \in \mathbb{R}^m$ be the null vector, that is $c_j = 0$ for every $j \in [m]$. We now construct a Nash Equilibrium through the following iterative routine.

1. Let d be the minimum of the elements in $\mathcal{D}$. Up to a tie, there exist a couple $(i_1, j_1) \in [n]\times[m]$ such that $d = |x_{i_1} - y_{j_1}|$. We set $c_{j_1} = c_{j_1} + 1$, $s_{i_1} = j_1$, and remove all the elements of the form $|x_{i_1} - y_j|$ from $\mathcal{D}$. Then, if $c_{j_1} = k_{j_1}$, we remove from $\mathcal{D}$ all the elements of the form $|x_i - y_{j_1}|$.
2. We repeat the routine of point (1) until $\mathcal{D}$ becomes empty.
3. If $s_i = 0$ for some $i \in [n]$, we set them to be equal to 1.

Since $\mathcal{D}$ is discrete, the routine terminates in finite number of iterations and the output is a vector containing a set of agents' pure strategies.

We now show that the output of the routine $\vec{s}$ is a Nash Equilibrium by proving that no agent i can increase its payoff by deviating from playing s_i. Toward a contradiction, assume that an agent i can increase its payoff by playing s_i' rather than s_i. By definition of s_i, we have that if $|x_i - y_{s_i'}| < |x_i - y_{s_i}|$, then there are at least $k_{s_i'}$ agents that are closer to $y_{s_i'}$ or that have a higher priority order than agent i and play strategy s_i'. Thus the agent cannot gain a benefit from deviating from s_i, which proves that $\vec{s}$ is a pure Nash Equilibrium. □

Proof of Theorem 4. To complete the proof, we need to consider the case in which $i_1 < \lfloor \frac{k_1+1}{2} \rfloor$. First, we consider the case in which $i_2 < n - \lfloor \frac{k_2+1}{2} \rfloor$. By the same argument used to prove the case in which $i_1 \geq \lfloor \frac{k_1+1}{2} \rfloor$, we have that the worst case instance in this case is

$$x_i = \begin{cases} x_i = 0 & \text{if} \quad i = 1, \dots, i_1, \\ x_i = \lambda & \text{if} \quad i = i_1 + 1, \dots, i_2 - 1, \\ x_i = 1 & \text{otherwise.} \end{cases}$$

for some $\lambda \in [0, 1]$, since the SW of the mechanism is minimized when the i_1-th and i_2-th agents are at the extremes of the interval. For any value of λ, the SW of the mechanism is then

$$SW(\vec{x}) = i_1 + (n - i_2) + (1 - \lambda)(k_1 - i_1) + \lambda(k_2 - (n - i_2)).$$

Since $SW(\vec{x})$ is linear in λ, we have that the minimum is achieved at either $\lambda = 0$ or $\lambda = 1$. Thus the minimal SW achievable is

$$\min\{k_1 + (n - i_2), k_2 + i_1\}.$$

Since in both cases we have that the optimal SW is $k_1 + k_2$, we conclude the thesis for this specific case.

Lastly, we consider the case in which $n - i_2 \geq \lfloor \frac{k_2+1}{2} \rfloor$. In this case, the worst case instance places the first i_1 agents on the extreme left side, while places y_2 in between two clusters of agents. Therefore we consider the following instance

$$x_i = \begin{cases} x_i = 0 & \text{if} \quad i = 1, \dots, i_1, \\ x_i = \lambda & \text{if} \quad i = i_1 + 1, \dots, i_2 - 1, \\ x_{i_2} = \frac{\lambda+1}{2} & \\ x_i = 1 & \text{otherwise.} \end{cases}$$

The SW induced by the mechanism is then

$$SW(\vec{x}) = i_1 + 1 + (1 - \lambda)(k_1 - i_1) + \frac{1 + \lambda}{2}(k_2 - 1).$$

Again, since the SW is linear in λ, we have that the minimium is attained at either $\lambda = 0$ or $\lambda = 1$. Then the minimum SW achievable by the mechanism is

$$\min\Big\{k_1 + \frac{(k_2 + 1)}{2}, k_2 + i_1\Big\}.$$

To conclude notice that in both cases, the SW attained by the optimal solution is $k_1 + k_2$. □

Proof of Theorem 5. When $\Delta \geq \lceil \frac{k_1+k_2}{2} \rceil$, the indexes $i_1 = \lceil \frac{k_1}{2} \rceil$ and $i_2 = n - \lfloor \frac{k_2}{2} \rfloor$ are well defined. Owing to Theorem 3 and by definition of Δ, we have that $\mathcal{PM}_{\vec{v}}$ is ES. Finally, from Theorem 4, we infer that $ar(\mathcal{PM}_{\vec{v}}) = \frac{k_1+k_2}{\frac{k_1+1}{2}+k_2}$, which is the smallest approximation ratio achievable by an ES percentile mechanism.

To conclude the proof, we need to show that the points (ii) and (iii) hold. We do that by carefully tuning i_1 and i_2. For the sake of simplicity, we consider i_1 and i_2 to be rationals, to retrieve the real integer indexes, it suffices to take the floor or the ceil functions of the quantities we retrieve.

Let us consider the case (ii), that is $k_1 - k_2 \leq \Delta \leq \lfloor \frac{k_1+k_2}{2} \rfloor + 1$. Owing to Theorem 4, we retrieve the best values i_1 and i_2 by maximizing the quantity

$$\min\{k_1 + (n - i_2), i_1 + k_2\}.$$

Thus, we look for i_1 and i_2 such that

$$k_1 + (n - i_2) = i_1 + k_2,$$

subject to the constraint

$$n - i_2 + i_1 = \Delta,$$

since, owing to Theorem 2, $k_1 + k_2$ agents must lay between x_{i_1} and x_{i_2}. By a simple computation, we have that

$$n - i_2 = \frac{k_2 - k_1 + \Delta}{2},$$

thus $i_1 = \frac{\Delta-(k_2-k_1)}{2} = k_1 - k_2 + \frac{\Delta-(k_2-k_1)}{2}$ and $i_2 = n - \frac{k_2-k_1+\Delta}{2}$, which concludes the proof of case (ii).

Lastly, we consider case (iii). In this case, we have that $\Delta < k_1 - k_2$, thus we have

$$k_2 + i_1 - k_1 - (n - i_2) = i_2 - n + i_1 + k_2 - k_1 \leq \Delta + k_2 - k_1 < 0,$$

since $i_2 - n + i_1 < n - i_2 + i_1 \leq \Delta$. Thus the minimum SW attainable by the mechanism is $i_1 + k_2$, therefore, to maximize the minimum achievable SW, we need to set $i_1 = \Delta$ and $i_2 = n$, which concludes the proof. □

Proof of Theorem 6. The proof follows by the same argument used to prove Theorem 3. Indeed, by condition (4) for every $j \in [m]$ we have that at least $k_j + k_{j+1}$ agents are located between y_j and y_{j+1}, thus the Social Welfare generated by the facilities at y_j and y_{j+1} does not depend on the specific Nash equilibrium. To conclude the proof, it suffices to apply this argument to each couple of facilities (y_j, y_{j+1}).

□

Proof of Theorem 7. To conclude the proof, we need to consider the case in which either i_1 or $n-i_m$ are lower than $\lfloor\frac{k+1}{2}\rfloor$.

Since the other case is symmetric, we restrict our analysis to the case in which $i_1 \le n-i_2$. Again, since $i_1, n-i_m \le \lfloor\frac{k+1}{2}\rfloor$, we have that the worst case instance places the first i_1 agents at 0 and the last $n-i_m+1$ at 1. Since every facility has the same capacity, we have that the worst case instance has the following form

$$x_i = \begin{cases} 0 & \text{if} \quad i = 1,\dots,i_1, \\ \delta_1 & \text{if} \quad i = i_1+1,\dots,i_2-1, \\ \delta_1 + \frac{1-\delta_1-\delta_2}{2(m-2)} & \text{if} \quad i = i_2, \\ \delta_1 + 2\frac{1-\delta_1-\delta_2}{2(m-2)} & \text{if} \quad i = i_2+1,\dots,i_3-1, \\ \delta_1 + 3\frac{1-\delta_1-\delta_2}{2(m-2)} & \text{if} \quad i = i_3, \\ \delta_1 + 4\frac{1-\delta_1-\delta_2}{2(m-2)} & \text{if} \quad i = i_3+1,\dots,i_4-1, \\ \dots & \\ 1-\delta_2 & \text{if} \quad i = i_{m-1}+1,\dots,i_m-1, \\ 1 & \text{otherwise} \end{cases}$$

where $\delta_1, \delta_2 \ge 0$ and such that $\delta_1+\delta_2 \le 1$. The SW of the mechanism on this instance is

$$SW(\vec{x}) = i_1 + (n-i_2) + m - 2 + (k-i_1)(1-\delta_1) + \sum_{i=2}^{m-2}\left((k-1)\Big(\frac{m-3+\delta_1+\delta_2}{m-2}\Big)\right) + (k-(n-i_m))(1-\delta_2).$$

Again, this quantity is linear in δ_1 and δ_2, thus it is minimized when $\delta_1, \delta_2 \in \{0,1\}$ By plugging in the possible combinations, we infer that the minimum is achieved when $\delta_1 = 1$ and $\delta_2 = 0$ since $i_1 \le n-i_m$. ☐

Proof of Theorem 8. Owing to Theorem 7, the approximation ratio is lower when $\min\{i_1, n-i_m\}$ is maximized, thus when $i_1 = n-i_2$. Thus the best mechanism places the first and last facility at x_ℓ and $x_{n-\ell}$, where ℓ is a suitable integer. Since $i_1 + n - i_2 = n - 2k(m-1)+1$, we complete the first half of the proof.

Notice that, if i_1 or i_2 is less than $\lfloor\frac{k+1}{2}\rfloor$, then we have that

$$\min\{i_1, i_2\} \le \lfloor\frac{k+1}{2}\rfloor.$$

Therefore,

$$\Big(m-\frac{1}{2}\Big)k + \frac{1}{2} - (m-1)k - \min\{i_1,i_2\} \ge \frac{k}{2} + \frac{1}{2} - \lfloor\frac{k+1}{2}\rfloor \ge 0,$$

thus the approximation ratio of the mechanism is smaller when $i_1, i_2 \ge \lfloor\frac{k+1}{2}\rfloor$. Moreover, in this case, the approximation ratio does not depend on the specific $\vec{v}$, thus any ES percentile mechanism whose $\vec{v}$ is such that $i_1, i_2 \ge \lfloor\frac{k+1}{2}\rfloor$ achieves the minimum approximation ratio. Notice that, by definition, the vector $\vec{v}$ where $v_j = \frac{\alpha+(2k-1)(j-1)}{n}$ for $j \in [m]$ where $\alpha = \lfloor\frac{(n-2k(m-1)+1)}{2}\rfloor$ is such that $i_1, i_2 \ge \lfloor\frac{k+1}{2}\rfloor$. Moreover, owing to Theorem 2, it is also ES, hence it achieves the minimal approximation ratio.

Lastly, notice that

$$\frac{mk}{(m-\frac{1}{2})k+\frac{1}{2}} \le \frac{mk}{(m-\frac{1}{2})k} = \frac{(m-\frac{1}{2})k+\frac{k}{2}}{(m-\frac{1}{2})k} = 1 + \frac{1}{2m-1},$$

which concludes the proof. ☐

Proof of Theorem 9. It follows directly from Theorem 4. Indeed, it suffices to prove that even if we have m facilities to locate, the optimal SW we can obtain by locating m facilities with capacity l is the same as locating two facilities with capacity $\lceil\frac{m}{2}\rceil k$ and $\lfloor\frac{m}{2}\rfloor k$. Since the worst case instance of any $\mathcal{PM}_{\vec{v}}$ with $\vec{v} \in [0,1]^2$ places i_1 agents 0 and the others at 1, the optimal SW remains mk even though we locate m facilities separately. ☐

Proof of Theorem 10. By definition of $\vec{v} = (0.5, 0.5, \dots, 0.5)$ and $\mathcal{PM}_{\vec{v}}$, we have that for every input $\vec{x} \in [0,1]^n$ the facility is placed at $\lfloor\frac{n+1}{2}\rfloor$. The number of agents on the left of y_1 and the number of agents on the right of y_1 is the same, hence the SW of the mechanism is minimizes when $x_i = 0$ when $i < \lfloor \frac{n+1}{2} \rfloor$, $x_{\lfloor \frac{n+1}{2} \rfloor} = \frac{1}{2}$, and $x_i = 1$ otherwise. The SW of the mechanism is $\frac{mk+1}{2}$.

If $n \leq (m+1)k$, the optimal SW on the instance is $(m-1)k + \frac{n-(m-1)k}{2} + \frac{1}{2}$. Indeed, we can locate $m-1$ facilities at either 0 or 1 that only accommodate the agents at 0 and 1. The total combined utility of the agents accommodated by these $m-1$ facilities is $(m-1)k$. Since the agents are divided evenly among 0 and 1, the maximum utility attainable by the last facility is at most $\frac{n-(m-1)k}{2} + \frac{1}{2}$. Therefore the total utility of the optimal SW is $(m-1)k + \frac{n-(m-1)k}{2} + \frac{1}{2}$.

If $n > (m+1)k$, the optimal SW on this instance is mk, and it is attained when $\lfloor \frac{m}{2} \rfloor$ facilities are placed at 0 and the others at 1. To conclude the thesis it suffices to take the ratio of the optimal SW and the SW of the mechanism. $\square$

B ADDITIONAL EXPERIMENTAL RESULTS

In this section, we report the experimental results missing from the main body of the paper.

In Table 4, we report all our results for the case in which the facilities have balanced capacity, that is $k_1 = k_2$.

In Table 4, we report all our results for the case in which the facilities have unbalanced capacity, that is $k_1 > k_2$.

In Table 6, we report all our experiments non identical for different values of Λ.

We observe no major changes across all the different cases we considered.

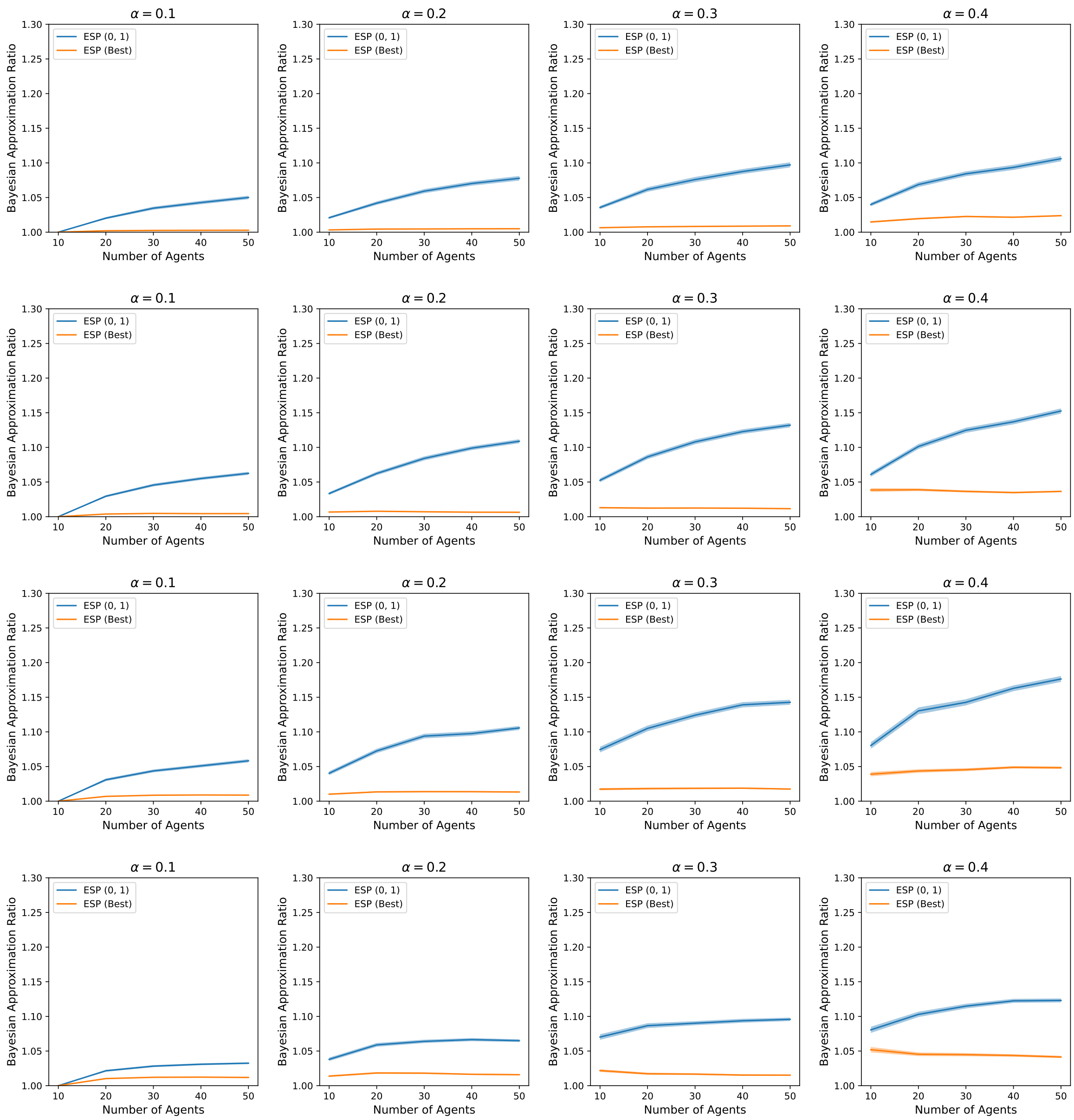

Figure 4: The Bayesian approximation ratio of $\mathcal{PM}_{best}$ and $\mathcal{PM}_{\vec{w}}$ in the balanced case, i.e. $k_1 = k_2$ for $n = 10, 20, \dots, 50$. Every column contains the results for different vector $\vec{k}$. The first and second row contains the results for the Beta distribution. In the first row, we consider an asymmetric Beta distribution, that is $\mathcal{B}(1, 9)$; in the second row a symmetric Beta, that is $\mathcal{B}(5, 5)$. The third row contains the results for the triangular distribution $\mathcal{T}$. The last row contains the results for the Uniform distribution $\mathcal{U}$.

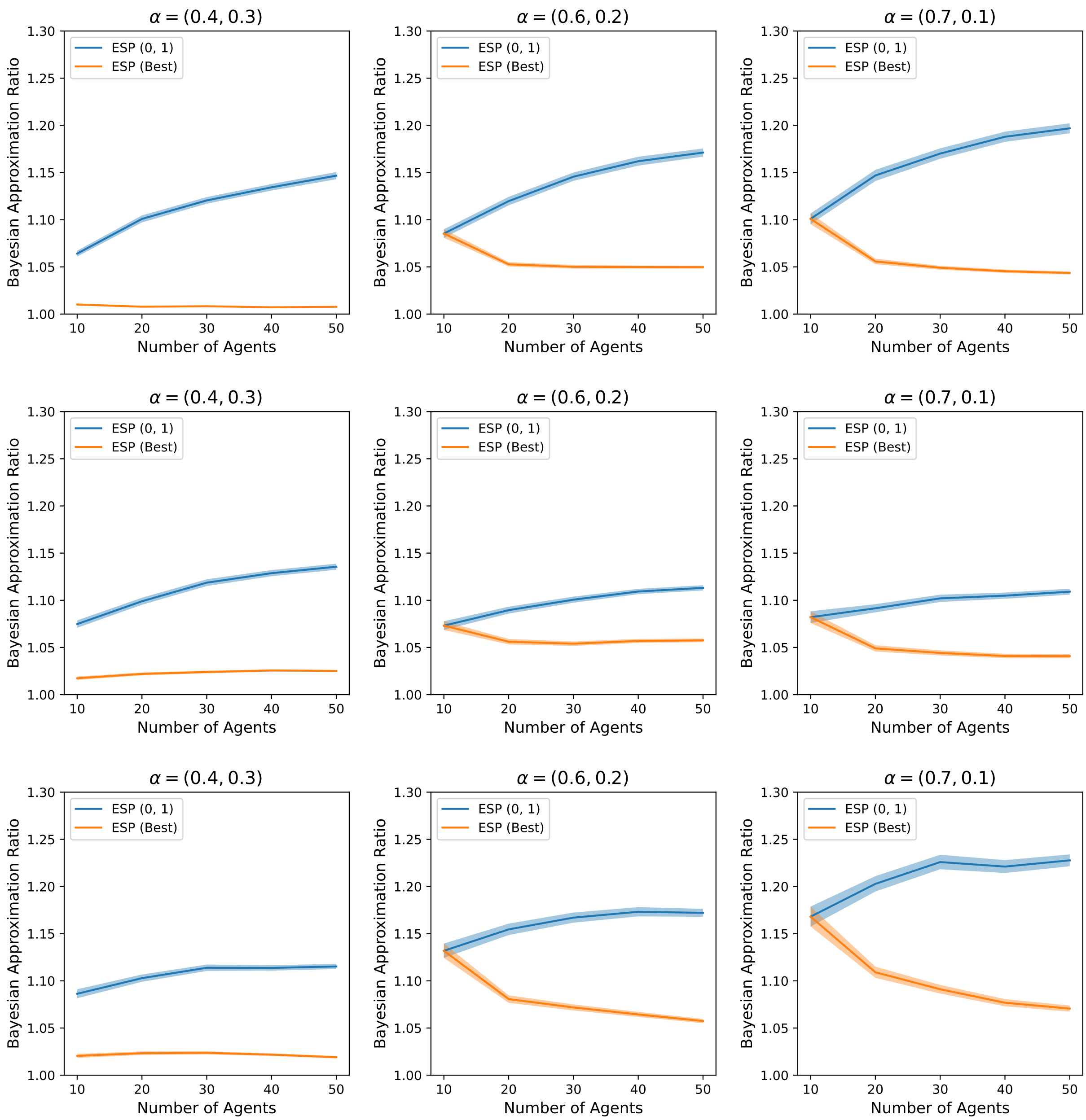

Figure 5: The Bayesian approximation ratio of $\mathcal{PM}_{best}$ and $\mathcal{PM}_{\vec{w}}$ when the agents are distributed according to $\mathcal{T}$ and the facilities are unbalanced, i.e. $k_1 = \alpha_1 n \neq k_2 = \alpha_2 n$ for $n = 10, 20, \dots, 50$. Every column contains the results for different vector $\vec{k}$. The first row contains the results for a symmetric Beta distribution, that is $\mathcal{B}(5,5)$. The second row contains the results for the triangular distribution $\mathcal{T}$. The last row contains the results for the Uniform distribution $\mathcal{U}$.

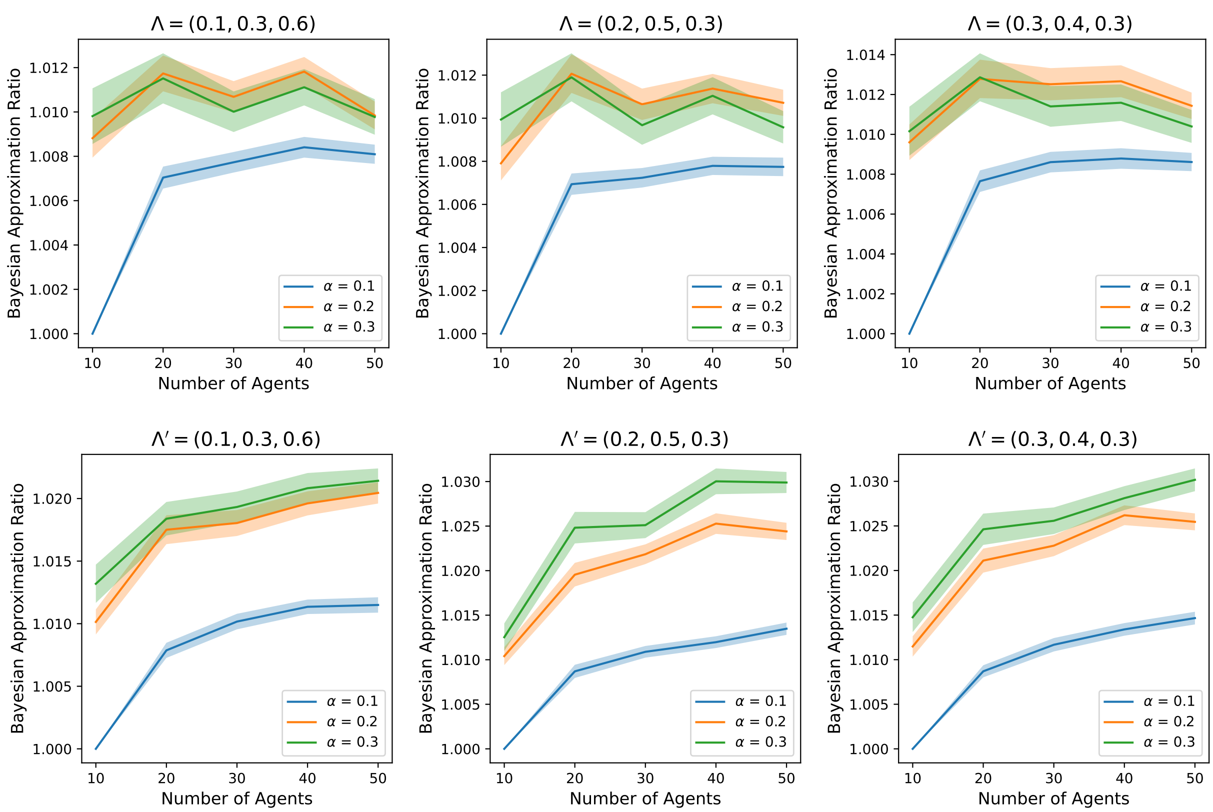

Figure 6: The Bayesian approximation ratio of $\mathcal{PM}_{best}$ for a population non i.d.. The capacities of the facilities are balanced, i.e. $k_1 = k_2 = \alpha n$ with $\alpha = 0.1, 0.2, 0.3$, and for $n = 10, 20, \dots, 50$. In the first raw, the Beta distribution is symmetric, in particular $\mathcal{B}(5, 5)$, in the second raw the Beta distribution is asymmetric, in particular $\mathcal{B}(1, 9)$. Every column contains the results for different Λ.