The Human Brain as a Combinatorial Complex

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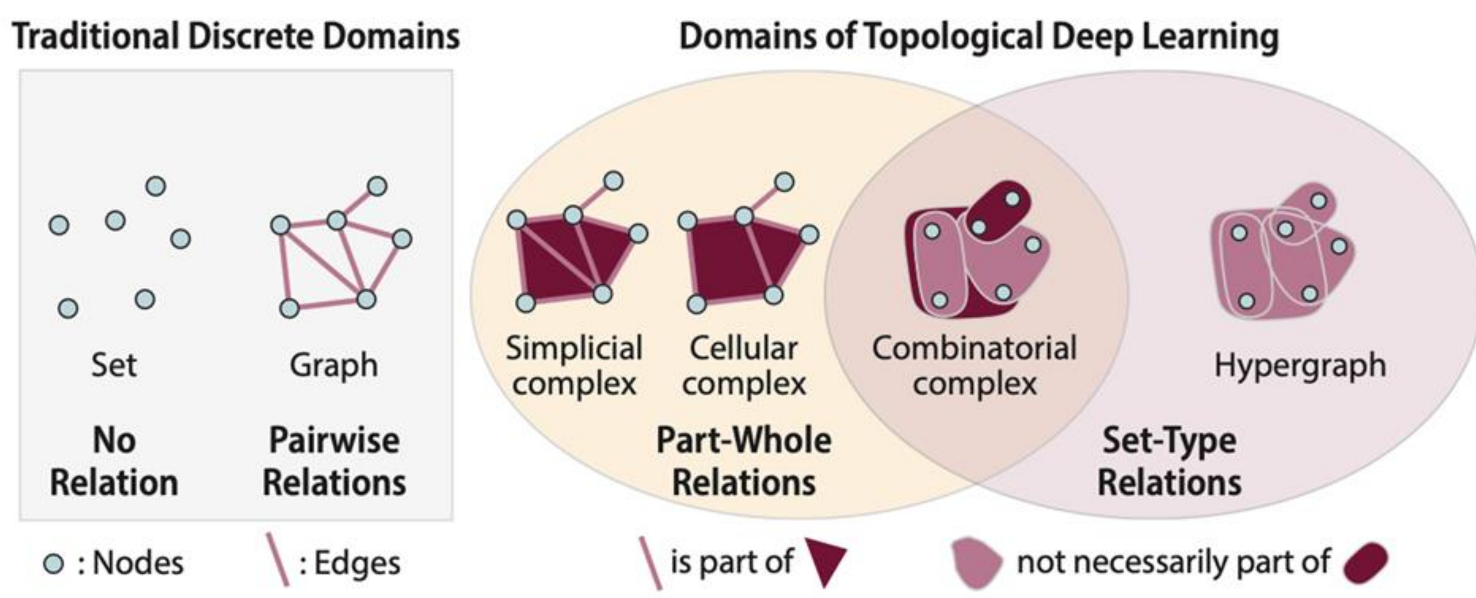
NEURAL INFORMATION PROCESSING SYSTEMS

Detecting Higher-Order Structure via Information Theory

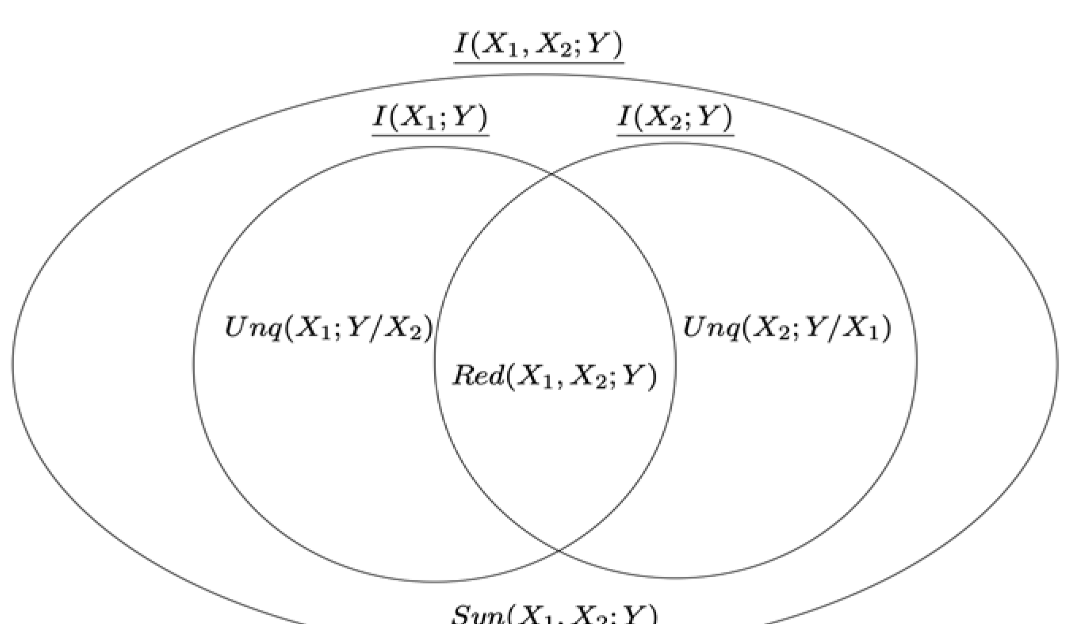
Brain networks are typically represented as graphs where nodes are regions and edges encode pairwise statistical relationships. This framework and its topological liftings (clique expansion, simplicial complexes) are **fundamentally limited to pairwise structure.** We construct combinatorial complexes directly from multivariate information measures, detecting statistical dependencies at each rank independently. This data-driven approach captures **irreducible multivariate structure in complex systems** including the brain, providing natural representations for topological deep learning (**TDL**) architectures.



Theoretical Foundation

Synergy occurs when $I(X_1,X_2;Y) > I(X_1;Y) + I(X_2;Y)$: the joint provides more information than the sum of individual sources. This represents irreducible multivariate dependencies—information that emerges only when variables are observed together.

Contextuality arises when locally consistent pairwise marginals $P(X_i, X_j)$ cannot extend to a global joint $P(X_1, \ldots, X_n)$. Graph-based methods assume this global consistency, but it fails for synergistic systems: XOR exhibits $I(X;Z) \approx 0$, $I(Y;Z) \approx 0$ yet $I(X,Y;Z) >> 0$—no graph captures this.



Information decomposition of $I(X_1,X_2;Y)$.

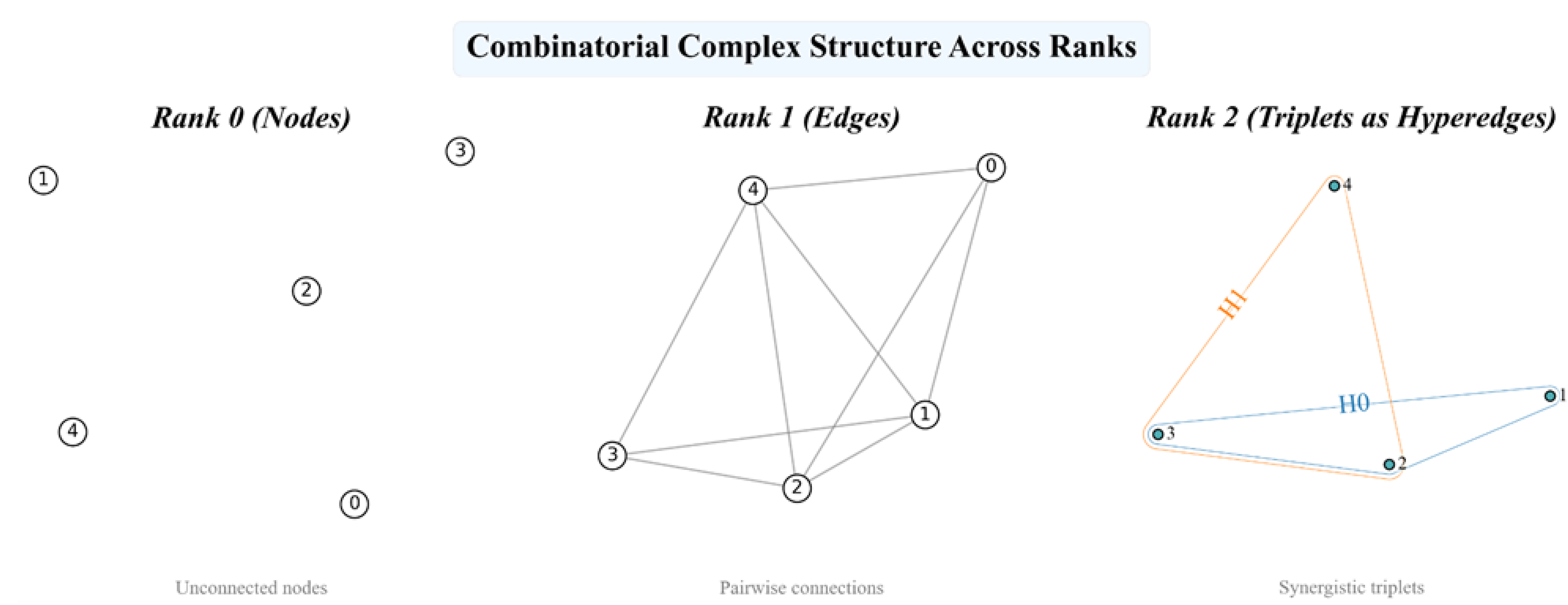
What is a Combinatorial Complex?

Definition (CC). *A combinatorial complex (CC) is a triple* $(S, \mathcal{X}, rk)$ *consisting of a set* S*, a subset* $\mathcal{X}$ *of* $\mathcal{P}(S) \setminus \{\emptyset\}$*, and a function* $rk : \mathcal{X} \to \mathbb{Z}_{\geq 0}$ *with:*

1. for all $s \in S$, $\{s\} \in \mathcal{X}$, and
2. the function rk is order-preserving: if $x, y \in \mathcal{X}$ satisfy $x \subseteq y$, then $\mathrm{rk}(x) \leq \mathrm{rk}(y)$.

Elements of S *are called* ***entities*** *or* ***vertices****, elements of* $\mathcal{X}$ *are called* ***relations*** *or* ***cells****, and rk is the* ***rank function*** *of the CC.*

A Data-Driven Combinatorial Complex



Toy combinatorial complex from NetSim neural time series. **Rank-0** cells are brain regions (nodes); **Rank-1** edges capture pairwise interactions (mutual information ≥ 0.02); **Rank-2** hyperedges denote synergistic triplets (S-information ≥ 0.45, O-information ≲ 0). Top triplets: (2,3,4) with S = 0.51, Ω = 0.06, and (1,2,3) with S = 0.48, Ω = 0.04.

Poster's Takeouts:

- **Graphs miss synergy**: Pairwise methods cannot capture irreducible multivariate dependencies
- **Information-theoretic CCs**: Σ and Ω detect synergy vs redundancy without assuming global consistency
- **Enables TDL on real data**: Natural input for combinatorial complex neural networks

Method Validation

We extended Dynamic Causal Modeling with XOR operations to generate ground truth synergy and test our method. XOR's synergy signature:

$$I(X;Z) \approx 0 \quad | \quad I(Y;Z) \approx 0 \quad | \quad I(X,Y;Z) \gg 0$$

Inputs separately provide no information; jointly they fully determine output.

Standard DCM

$$\frac{dz}{dt} = \sigma A z + C u$$

DCM_XOR

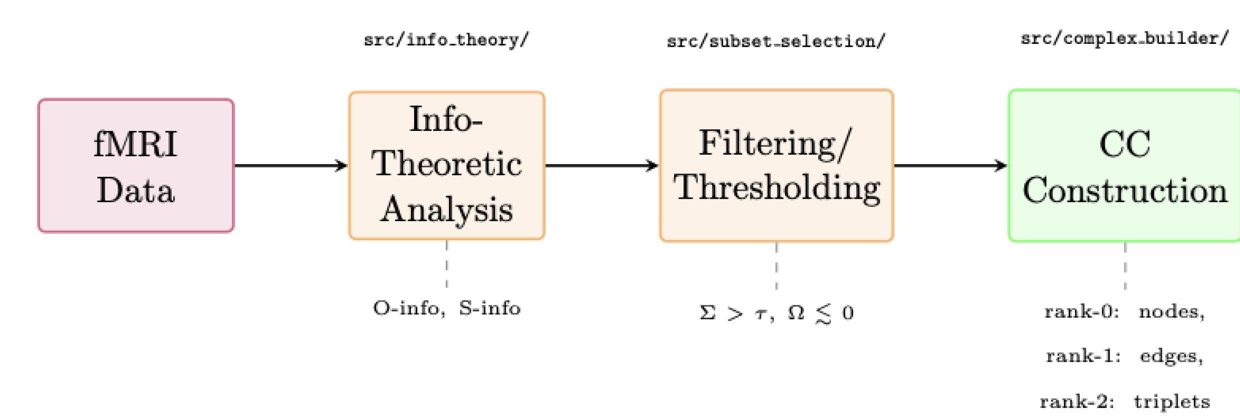
$$\frac{dz}{dt} = -z + F_{\text{XOR}}(z) + F_{\text{edge}}(z) + Cu$$

Experiment	Arch.	BOLD	Nodes	GT	Det.	TP/FP/FN	Sens.	Prec.
Exp 1: Linear DCM	Linear	No	5	0	0	0/0/0	—	—
Exp 2: XOR Connected	Conn.	No	10	3	42	2/40/1	66.7%	4.8%
Exp 3: XOR Isolated	Isol.	No	9	3	3	3/0/0	**100%**	**100%**
Exp 4: Linear + BOLD	Linear	Yes	5	0	5	0/5/0	—	—
Exp 5: XOR Conn. + BOLD	Conn.	Yes	10	3	98	3/95/0	100%	3.1%
Exp 6: XOR Isol. + BOLD	Isol.	Yes	9	3	22	3/19/0	**100%**	13.6%

Ground truth validation results across six DCM-based experiments.

Construction pipeline

S-information Σ = TC + DTC quantifies total statistical interdependence; **O-information** Ω = TC - DTC distinguishes synergy (Ω < 0) from redundancy (Ω > 0). We apply dual thresholding: Σ > τ selects statistically strong subsets, while Ω ≲ 0 ensures synergy-dominated structure.



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