CONTENTS

A RELATED WORK

Private Composition. In the blackbox composition setting, one can do no better than the $O(K\epsilon)$ privacy analysis for pure differential privacy Dwork et al. (2014). For approximate differential privacy, previous work has found optimal constants for advanced composition by reducing to the binary case of hypothesis testing with randomized response; and optimal tradeoffs between ϵ, δ for black box composition are given in Kairouz et al. (2015), where there could be a modest improvement 20%.

Thus, for specific applications, some work has turned to white-box composition analysis for improved utility analysis. Abadi et al. (2016) applied a technique called moment accountant for private SGD to reduce the $\log(1/\delta)$ dependence in the ϵ term and linear dependence on k in the δ term. For general private stochastic convex optimization, one can avoid the linear dependence on k in ϵ by using iterative application of contractive maps Feldman et al. (2018). For the specific case of model ensembles, Papernot et al. (2018) uses student model learning to privately aggregate a ensemble of teacher models trained on disjoint datasets and shows a data-dependent privacy bound that vanishes as the probability of disagreement goes to 0. Their method provides no utility analysis but they empirically observed less privacy loss when there is greater ensemble agreement.

When g is the maximization function, some previous work shows that an approximately maximum value can be outputted with high probability while incurring $O(\epsilon)$ privacy loss, independently of K. They proposed a random stopping mechanism for $m = 1$ that draws samples uniformly at random from $M_i(\mathcal{D})$ at each iteration. In any given iteration, the sampling halts with probability γ and the final output is computed based on the samples collected until that time. This leads to a final privacy cost of only 3ϵ for the maximization function g, which can be improved to 2ϵ (Papernot & Steinke,

2022). In addition to the aforementioned works, composing top-k and exponential mechanisms also enjoy slightly improved composition analysis via a bounded-range analysis Durfee & Rogers (2019); Dong et al. (2020).

Bypassing the Global Sensitivity. To ensure differential privacy, it is usually assumed the query function g has bounded global sensitivity — that is, the output of g does not change much on *any* adjacent input datasets differing in one entry. The noise added to the output is then proportional to the global sensitivity of g. If the sensitivity is large, the output utility will thus be terrible due to a large amount of noises added. However, the worst case global sensitivity can be rare in practice, and this observation has inspired a line of works on designing private algorithms with data-dependent sensitivity bound to reduce the amount of noises added.

Instead of using the maximum global sensitivity of g on any dataset, the classical Propose-Test-Release framework of Dwork Dwork & Lei (2009) uses a local sensitivity value for robust queries that is tested privately and if the sensitivity value is too large, the mechanism is halted before the query release. The halting mechanism incurs some failure probability but deals with the worst-case sensitivity situations, while allowing for lower noise injection in most average-case cases.

One popular way to estimate average-case sensitivity is to use the Subsample-and-Aggregate framework by introducing the notion of *perturbation stability*, also known as *local sensitivity* of a function g on a dataset $\mathcal{D}$ Thakurta & Smith (2013); Dwork et al. (2014), which represents the minimum number of entries in $\mathcal{D}$ needs to be changed to change $g(\mathcal{D})$. One related concept is *smooth sensitivity*, a measure of variability of g in the neighborhood of each dataset instance. To apply the framework under *smooth sensitivity*, one needs to privately estimate a function's local sensitivity L_s and adapt noise injection to be order of $O(\frac{L_s}{\epsilon})$, where L_s can often be as small as $O(e^{-n})$, where $n = |\mathcal{D}|$, the total dataset size Nissim et al. (2007). Generally, the private computation of the smooth sensitivity of a blackbox function is nontrivial but is aided by the Subsample and Aggregate approach for certain functions.

These techniques hinge on the observation that a function with higher stability on $\mathcal{D}$ requires less noise to ensure worst case privacy. Such techniques are also applied to answer multiple online functions/queries in model-agnostic learning Bassily et al. (2018). However, we highlight two key differences in our setting with a weaker stability assumption. First, in order to estimate the *perturbation stability* of g on $\mathcal{D}$, one needs to downsample or split $\mathcal{D}$ into multiple blocks Thakurta & Smith (2013); Dwork et al. (2014); Bassily et al. (2018), $\hat{\mathcal{D}}_1, \ldots, \hat{\mathcal{D}}_B$, and estimate the *perturbation stability* based on the mode of $g(\hat{\mathcal{D}}_1), \ldots, g(\hat{\mathcal{D}}_B)$. This essentially reduces the amount of change in the output of g due to a single entry in $\mathcal{D}$, with high probability and replaces the hard-to-estimate *perturbation stability* of g with an easy-to-compute *perturbation stability* of the mode. Such a notion of stability has also been successfully applied, along with the sparse vector technique, for model-agnostic private learning to handle exponentially number of queries to a model Bassily et al. (2018). Note that in these cases, since a private stochastic test is applied, one cannot achieve pure differential privacy Dwork et al. (2014). In practice, e.g. federated learning, however, one does not have direct access to $\mathcal{D}$, and thus it is impractical to draw samples from or to split $\mathcal{D}$. Second, to ensure good utility, one relies on a key assumption, i.e. the *subsampling stability* of g, which requires $g(\hat{\mathcal{D}}) = g(\mathcal{D})$ with high probability over the draw of subsamples $\hat{\mathcal{D}}$.

Learning the Optimal Noise Distribution. Most of these works only claim improved utility and there is no optimality guarantee. There have been limited works that attempt to derive or learn the best noise distribution. For deep neural networks inference, Mireshghallah et al. (2020) attempts to learn the best noise distribution to maximizing utility subject to an entropy Lagragian, but no formal privacy guarantees were derived. For queries with bounded sensitivity, Geng & Viswanath (2015) demonstrate that the optimal noise distribution is in fact a staircase distribution that approaches the Laplacian distribution as $\epsilon \to 0$.

B Preliminaries: Differential Privacy

Definition B.1 (Differential Privacy (DP) Dwork et al. (2014))**.** *A randomized mechanism $\mathcal{M} : \mathcal{D} \to \mathcal{R}$ with a domain $\mathcal{D}$ and range $\mathcal{R}$ satisfies (ϵ, δ)-differential privacy for $\epsilon, \delta \geq 0$ if for any two* **adjacent datasets** *$\mathcal{D}, \mathcal{D}'$ and for any subset of outputs $S \subseteq \mathcal{R}$ it holds that $\Pr[\mathcal{M}(\mathcal{D}) \in S] \leq$*

$e^\epsilon \Pr[\mathcal{M}(\mathcal{D}') \in S] + \delta$. *$\delta = 0$ is often called pure differential privacy; while $\delta > 0$ is often called approximate differential privacy.*

Theorem B.2 (Simple Composition Dwork et al. (2014)). *Let $\mathcal{M}_1 : \mathcal{D} \to \mathcal{R}_1$ be an ϵ_1-differentially privacy mechanism and $\mathcal{M}_2 : \mathcal{D} \to \mathcal{R}_2$ be an ϵ_2-differentially privacy mechanism, then their combination $\mathcal{M}_{1,2}(x) = (\mathcal{M}_1(x), \mathcal{M}_2(x))$ is $(\epsilon_1 + \epsilon_2)$-differentially private.*

Theorem B.3 (Advanced Composition Dwork et al. (2014)). *For all $\epsilon, \delta, \delta' \geq 0$, the class of (ϵ, δ)-differentially private mechanisms satisfies $(\epsilon', k\delta + \delta')$-differential privacy under k-fold adaptive composition for*

$$\epsilon' = \sqrt{2k \ln(1/\delta')}\epsilon + k\epsilon(e^\epsilon - 1) \tag{4}$$

C Details of Section 3: Private Majority Algorithms

C.1 Randomized Response with Constant γ

Recall the classical Randomized Response (RR) algorithm that provides a binary output algorithm with differential privacy guarantee proceeds as follows: With probability p_γ, one returns the true output of the algorithm; otherwise, one returns a random answer. In this section, we show the magnitude of the constant probability p_γ in RR to use RR to solve Problem 1.1 and to ensure RR is $(m\epsilon, \delta)$-differentially private. We can view the p_γ probability in RR as a constant $\gamma : \{0, 1, \ldots, K\} \to [0, 1]$ function such that $\gamma(l) = p_\gamma, \forall l \in [K]$.

Lemma C.1 (Randomized Response (constant) γ). *Consider Problem 1.1 with privacy allowance $m > 0$ and failure probability $\delta \geq 0$. Let p_γ be the probability of outputting the true majority based on K samples in Randomized Response (RR). Let the majority of K (ϵ, Δ) mechanisms be $(\tau\epsilon, \lambda)$-differentially private, reasoned by simple composition or advanced composition for some $0 < \tau \leq K, 0 \leq \lambda < 1$. If one sets*

$$p_\gamma = \frac{e^{m\epsilon} - 1 + 2\delta}{\frac{2(e^{\tau\epsilon} - e^{m\epsilon} + 2\lambda)}{e^{\tau\epsilon}+1}} + e^{m\epsilon} - 1 \tag{5}$$

then RR is $(m\epsilon, \delta)$-differentially private.

Proof. For convenience, let $x \in \{0, 1\}$ denote the output majority, and q_x, q'_x denote the probability the aggregated majority from K samples is x on dataset adjacent $\mathcal{D}$ and $\mathcal{D}'$ respectively. Recall each mechanism we aggregate is (ϵ, Δ)-differentially private. The output of the aggregated majority from K samples is $(\tau\epsilon, \lambda)$-differentially private, for some $\tau \leq K$. When $\Delta = 0$, $\tau = K$ and $\lambda = 0$ can be reasoned through simple composition. When $\Delta > 0$, $\tau \approx \sqrt{K}$ and $\lambda \approx K\Delta$ can be reasoned through advanced composition. And so simultaneously the following four constraints on q_x, q'_x apply:

$$q_x \leq e^{\tau\epsilon} q'_x + \lambda, \quad \text{and} \quad 1 - q'_x \leq e^{\tau\epsilon}(1 - q_x) + \lambda \tag{6}$$

$$q'_x \leq e^{\tau\epsilon} q_x + \lambda, \quad \text{and} \quad 1 - q_x \leq e^{\tau\epsilon}(1 - q'_x) + \lambda \tag{7}$$

To ensure RR is $(m\epsilon, \delta)$-differentially private, one needs γ such that for all possible q_x, q'_x,

$$\Pr[\text{RR}(\mathcal{D}) = x] \leq e^{m\epsilon} \Pr[\text{RR}(\mathcal{D}') = x] + \delta \tag{8}$$

$$\gamma \cdot q_x + \frac{1}{2}(1 - \gamma) \leq e^{m\epsilon}(\gamma \cdot q'_x + \frac{1}{2}(1 - \gamma)) + \delta \tag{9}$$

$$(q_x - e^{m\epsilon} q'_x + \frac{1}{2}e^{m\epsilon} - \frac{1}{2}) \cdot \gamma \leq \frac{1}{2}e^{m\epsilon} - \frac{1}{2} + \delta \tag{10}$$

To maximize the utility, one wants to maximize γ while conforming to the above privacy constraints. Hence, we solve solve the following Linear Programming (LP) problem:

$$\text{Objective:} \quad \max_{q_x, q'_x} f(q_x, q'_x) = q_x - e^{m\epsilon} q'_x + \frac{1}{2}e^{m\epsilon} - \frac{1}{2} \tag{11}$$

$$\text{Subject to:} \quad 0 \leq q_x \leq 1, 0 \leq q'_x \leq 1 \tag{12}$$

$$q_x \leq e^{\tau\epsilon} q'_x + \lambda, 1 - q'_x \leq e^{\tau\epsilon}(1 - q_x) + \lambda \tag{13}$$

$$q'_x \leq e^{\tau\epsilon} q_x + \lambda, 1 - q_x \leq e^{\tau\epsilon}(1 - q'_x) + \lambda \tag{14}$$

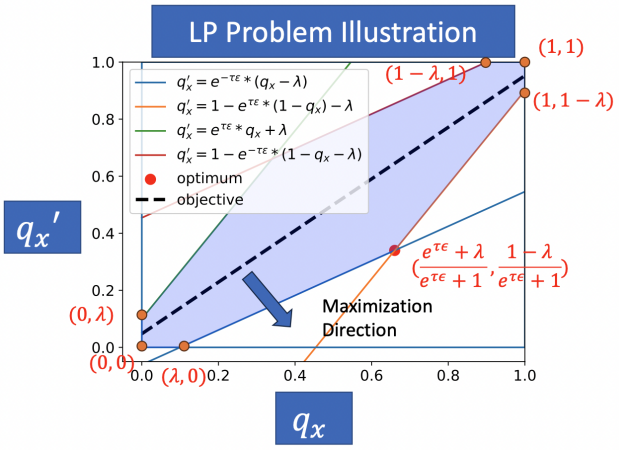


Figure 4: A visualization of the LP problem.

The optimum of any LP problem is at the corners of the feasible region. Here, the feasible region $\mathcal{F}$ is shown in Figure 4. This means $(q_x^*, q_x'^*) = \arg\max_{q_x, q'_x} f(q_x, q'_x) \in \{(0,0), (1,1), (0,\lambda), (\lambda,0), (1-\lambda,1), (1,1-\lambda), (\frac{1-\lambda}{e^{\tau\epsilon}+1}, \frac{e^{\tau\epsilon}+\lambda}{e^{\tau\epsilon}+1}), (\frac{e^{\tau\epsilon}+\lambda}{e^{\tau\epsilon}+1}, \frac{1-\lambda}{e^{\tau\epsilon}+1})\}$. The optimum of the above LP problem is at

$$q_x{}^* = \frac{e^{\tau\epsilon} + \lambda}{e^{\tau\epsilon} + 1}, \quad q'^*_x = \frac{1 - \lambda}{e^{\tau\epsilon} + 1} \tag{15}$$

We need to set γ according to the following upper bound to ensure privacy while maximizing γ to maximize utility,

$$\gamma \cdot \max_{q_x, q'_x} f(q_x, q'_x) \leq \frac{1}{2}(e^{m\epsilon} - 1) + \delta \tag{16}$$

Hence, we want γ that

$$\gamma \cdot \left(\frac{e^{\tau\epsilon} + \lambda}{e^{\tau\epsilon} + 1} - e^{m\epsilon} \frac{1 - \lambda}{e^{\tau\epsilon} + 1} + \frac{1}{2} e^{m\epsilon} - \frac{1}{2} \right) = \frac{1}{2}(e^{m\epsilon} - 1) + \delta \tag{17}$$

$$\gamma \cdot \left(\frac{e^{\tau\epsilon} - e^{m\epsilon} + 2\lambda}{e^{\tau\epsilon} + 1} + \frac{1}{2}(e^{m\epsilon} - 1) \right) = \frac{1}{2}(e^{m\epsilon} - 1) + \delta \tag{18}$$

$$\gamma = \frac{e^{m\epsilon} - 1 + 2\delta}{\frac{2(e^{\tau\epsilon} - e^{m\epsilon} + 2\lambda)}{e^{\tau\epsilon}+1} + e^{m\epsilon} - 1} \tag{19}$$

For small m, ϵ, K, using the approximation $e^y \approx 1 + y$,

$$\gamma \approx \frac{m\epsilon + 2\delta}{\frac{2(\tau\epsilon - m\epsilon + 2\lambda)}{\tau\epsilon+2} + m\epsilon} = \frac{m + 2\delta/\epsilon}{\frac{2(\tau - m + 2\lambda/\epsilon)}{\tau\epsilon+2} + m} \tag{20}$$

□

C.2 Proof of Lemma 3.1: the Subsampling γ Function

Lemma 3.1. *Consider Problem 1.1 and the sum of observed outcomes of the mechanisms,* $l = \sum_{i=1}^{K} S_i \in \{0, 1, \ldots, K\}$. *For* $m \in \mathbb{Z}^+$, $m \leq K$, *if one sets a success probability* $\gamma_{\textit{Subsampling}}$,

dependent on the value of l, by

$$\gamma_{\textit{Subsampling}}(l) = \begin{cases} \gamma_{\textit{Subsampling}}(K-l) = 1 - 2\sum_{j=\frac{m+1}{2}}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} & \textit{for odd } m \\ \gamma_{\textit{Subsampling}}(K-l) = 1 - 2\sum_{j=\frac{m}{2}+1}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} - \frac{\binom{l}{\frac{m}{2}}\binom{K-l}{\frac{m}{2}}}{\binom{K}{m}} & \textit{for even } m \end{cases} \tag{21}$$

then outputting the majority of m out of K subsampled mechanisms without replacement and DaRRM$_{\gamma_{Subsampling}}$ have the same output distribution.

Proof. Let algorithm SS denote outputting the majority based on m out of K subsampled mechanisms without replacement. Note the output of SS is the same as drawing one sample per mechanism $\mathcal{S} = \{S_i\}_{i=1}^K$, where $S_i \sim M_i(\mathcal{D})$, subsample m of the observed samples without replacement and outputs the majority based on the m subsamples. Let $\mathcal{L} = \sum_{i=1}^K S_i$ be the sum of observed outcomes from K mechanisms, and conditioned on $\mathcal{L}$, notice the output follows a hypergeometric distribution. Hence, the output probability of SS can be written as

$$\Pr[\text{SS}(\mathcal{D}) = 1] = \sum_{l=0}^{K} \Pr[\text{SS}(\mathcal{D}) = 1 \mid \mathcal{L} = l] \cdot \Pr[\mathcal{L} = l] \tag{22}$$

$$= \sum_{l=0}^{K} \Pr[\sum_{j=1}^{m} S_j \geq \frac{m}{2} \mid \mathcal{L} = l] \cdot \Pr[\mathcal{L} = l] \tag{23}$$

$$= \begin{cases} \sum_{l=0}^{K} (\sum_{j=\frac{m+1}{2}}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}}) \cdot \Pr[\mathcal{L} = l] & \text{if } m \text{ is odd} \\ \sum_{l=0}^{K} (\sum_{j=\frac{m}{2}+1}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} + \frac{1}{2}\frac{\binom{l}{\frac{m}{2}}\binom{K-l}{\frac{m}{2}}}{\binom{K}{m}}) \cdot \Pr[\mathcal{L} = l] & \text{if } m \text{ is even} \end{cases} \tag{24}$$

Recall $\gamma : \{0, 1, \ldots, K\} \to [0, 1]$ is the noise function in DaRRM. The output probability of DaRRM$_\gamma$ is:

$$\Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 1] = \sum_{l=0}^{K} \Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 1 \mid \mathcal{L} = l] \cdot \Pr[\mathcal{L} = l] \tag{25}$$

$$= \sum_{l=0}^{K} (\gamma(l) \cdot \mathbb{I}\{l \geq \frac{K+1}{2}\} + \frac{1}{2}(1 - \gamma(l))) \cdot \Pr[\mathcal{L} = l] \tag{26}$$

To let $\Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 1] = \Pr[\text{SS}(\mathcal{D}) = 1]$, if m is odd, for $l \leq \frac{K-1}{2}$,

$$\frac{1}{2}(1 - \gamma(l)) = \sum_{j=\frac{m+1}{2}}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} \quad \Rightarrow \gamma(l) = 1 - 2\sum_{j=\frac{m+1}{2}}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} \tag{27}$$

and for $l \geq \frac{K+1}{2}$,

$$\frac{1}{2} + \frac{1}{2}\gamma(l) = \sum_{j=\frac{m+1}{2}}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} \quad \Rightarrow \gamma(l) = 2\sum_{j=\frac{m+1}{2}}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} - 1 \tag{28}$$

Similarly, if m is even, for $l \leq \frac{K-1}{2}$,

$$\frac{1}{2}(1 - \gamma(l)) = \sum_{j=\frac{m}{2}+1}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} + \frac{1}{2}\frac{\binom{l}{\frac{m}{2}}\binom{K-l}{\frac{m}{2}}}{\binom{K}{m}} \quad \Rightarrow \gamma(l) = 1 - 2\sum_{j=\frac{m}{2}+1}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} - \frac{\binom{l}{\frac{m}{2}}\binom{K-l}{\frac{m}{2}}}{\binom{K}{m}} \tag{29}$$

and for $l \geq \frac{K+1}{2}$,

$$\frac{1}{2} + \frac{1}{2}\gamma(l) = \sum_{j=\frac{m}{2}+1}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} + \frac{1}{2}\frac{\binom{l}{\frac{m}{2}}\binom{K-l}{\frac{m}{2}}}{\binom{K}{m}} \quad \Rightarrow \gamma(l) = 2\sum_{j=\frac{m}{2}+1}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} + \frac{\binom{l}{\frac{m}{2}}\binom{K-l}{\frac{m}{2}}}{\binom{K}{m}} - 1 \tag{30}$$

Note that this $\gamma(l)$ is symmetric around $\frac{K}{2}$, since for $l \leq \frac{K-1}{2}$ (and so $K-l \geq \frac{K+1}{2}$), if m is odd,

$$\gamma(K-l) = 2\sum_{j=\frac{m+1}{2}}^{m} \frac{\binom{K-l}{j}\binom{l}{m-j}}{\binom{K}{m}} - 1 = 2\Big(1 - \sum_{j=1}^{\frac{m-1}{2}} \frac{\binom{K-l}{j}\binom{l}{m-j}}{\binom{K}{m}}\Big) - 1 \tag{31}$$

$$= 1 - 2\sum_{j=1}^{\frac{m-1}{2}} \frac{\binom{K-l}{j}\binom{l}{m-j}}{\binom{K}{m}} = 1 - 2\sum_{j=\frac{m+1}{2}}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} \tag{32}$$

$$= \gamma(l) \tag{33}$$

Similarly, if m is even,

$$\gamma(K-l) = 2\sum_{j=\frac{m}{2}+1}^{m} \frac{\binom{K-l}{j}\binom{l}{m-j}}{\binom{K}{m}} + \frac{\binom{l}{\frac{m}{2}}\binom{K-l}{\frac{m}{2}}}{\binom{K}{m}} - 1 = 2\Big(1 - \sum_{j=1}^{\frac{m}{2}-1} \frac{\binom{K-l}{j}\binom{l}{m-j}}{\binom{K}{m}} - \frac{1}{2}\frac{\binom{l}{\frac{m}{2}}\binom{K-l}{\frac{m}{2}}}{\binom{K}{m}}\Big) - 1 \tag{34}$$

$$= 1 - 2\sum_{j=1}^{\frac{m}{2}-1} \frac{\binom{K-l}{j}\binom{l}{m-j}}{\binom{K}{m}} - \frac{\binom{l}{\frac{m}{2}}\binom{K-l}{\frac{m}{2}}}{\binom{K}{m}} = 1 - 2\sum_{j=\frac{m}{2}+1}^{m} \frac{\binom{l}{j}\binom{K-l}{m-j}}{\binom{K}{m}} - \frac{\binom{l}{\frac{m}{2}}\binom{K-l}{\frac{m}{2}}}{\binom{K}{m}} \tag{35}$$

$$= \gamma(l) \tag{36}$$

Therefore, setting γ as in Eq. 27 if m is odd, and as in Eq. 29 if m is even makes DaRRM$_\gamma$ have the same output distribution as SS. We hence call this γ function $\gamma_{Subsampling}$. □

C.3 Proof of Lemma 3.2: Generality of DaRRM

Lemma 3.2 (Generality of DaRRM). *Let $\mathcal{A}$ be any randomized algorithm to compute the majority function g on $\mathcal{S}$ such that for all $\mathcal{S}$, $\Pr[\mathcal{A}(\mathcal{S}) = g(\mathcal{S})] \geq 1/2$ (i.e. $\mathcal{A}$ is at least as good as a random guess). Then, there exists a a general γ: $\{0,1\}^{K+1} \to [0,1]$ such that if one sets p_γ by $\gamma(\mathcal{S})$ in DaRRM, the output distribution of DaRRM$_\gamma$ is the same as the output distribution of $\mathcal{A}$.*

Proof. For some $\mathcal{D}$ and conditioned on $\mathcal{S}$, we see that by definition $\Pr[\text{DaRRM}_\gamma(\mathcal{S}) = g(\mathcal{S})] = \gamma(\mathcal{S}) + (1/2)(1-\gamma(\mathcal{S}))$. We want to set γ such that $\Pr[\text{DaRRM}_\gamma(\mathcal{S}) = g(\mathcal{S})] = \Pr[\mathcal{A}(\mathcal{S}) = g(\mathcal{S})]$. Therefore, we set $\gamma(\mathcal{S}) = 2\Pr[\mathcal{A}(\mathcal{S}) = g(\mathcal{S})] - 1$.

Lastly, we need to justify that $\gamma \in [0,1]$. Clearly, $\gamma(\mathcal{S}) \leq 2 - 1 \leq 1$ since $\Pr[\mathcal{A}(\mathcal{S}) = g(\mathcal{S})] \leq 1$. Note that the non-negativity follows from assumption. □

C.4 Proof of Lemma 3.3: γ Privacy Condition

Lemma 3.3 (γ privacy condition and privacy cost objective). *Consider using DaRRM to solve Problem 1.1. Let $\alpha_l = \Pr[\mathcal{L}(\mathcal{D}) = l]$ and $\alpha'_l = \Pr[\mathcal{L}(\mathcal{D}') = l]$, for l in support $\{0,\dots,K\}$ and adjacent datasets $\mathcal{D}, \mathcal{D}'$. For $\gamma : \{0,1,\dots,K\} \to [0,1]$ such that $\gamma(l) = \gamma(K-l), \forall l$, DaRRM$_\gamma$ is $(m\epsilon, \delta)$-differentially private if and only if for all α_l, α'_l,*

$$f(p_1,\dots,p_K,p'_1,\dots,p'_K;\gamma) := \sum_{l=0}^{\frac{K-1}{2}} (e^{m\epsilon}\alpha'_l - \alpha_l)\cdot\gamma(l) + \sum_{l=\frac{K+1}{2}}^{K} (\alpha_l - e^{m\epsilon}\alpha'_l)\cdot\gamma(l) \leq e^{m\epsilon} - 1 + 2\delta \tag{37}$$

We call f the privacy cost objective.

Proof. By the definition of differential privacy,

DaRRM$_\gamma$ is $(m\epsilon, \delta)$-differentially private

$$\iff \Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 1] \leq e^{m\epsilon} \Pr[\text{DaRRM}_\gamma(\mathcal{D}') = 1] + \delta, \tag{38}$$

$$\text{and } \Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 0] \leq e^{m\epsilon} \Pr[\text{DaRRM}_\gamma(\mathcal{D}') = 0] + \delta, \quad \forall \text{ adjacent } \mathcal{D}, \mathcal{D}' \tag{39}$$

Consider random variables $\mathcal{L}(\mathcal{D}) = \sum_{i=1}^{K} S(\mathcal{D})$ and $\mathcal{L}(\mathcal{D}') = \sum_{i=1}^{K} S(\mathcal{D}')$, based on which one sets γ. Note, when the output is 1,

$$\Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 1] \leq e^{m\epsilon} \Pr[\text{DaRRM}_\gamma(\mathcal{D}') = 1] + \delta \tag{40}$$

$$\iff \sum_{l=0}^{K} \Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 1 \mid \mathcal{L}(\mathcal{D}) = 1] \cdot \Pr[\mathcal{L}(\mathcal{D}) = l] \tag{41}$$

$$\leq e^{m\epsilon} \Big(\sum_{l=0}^{K} \Pr[\text{DaRRM}_\gamma(\mathcal{D}') = 1 \mid \mathcal{L}(\mathcal{D}') = 1] \cdot \Pr[\mathcal{L}(\mathcal{D}') = l] \Big) + \delta$$

$$\iff \sum_{l=0}^{K} \Big(\gamma(l) \cdot \mathbb{I}\{l \geq \frac{K}{2}\} + \frac{1}{2}(1 - \gamma(l)) \Big) \cdot \Pr[\mathcal{L}(\mathcal{D}) = l] \tag{42}$$

$$\leq e^{m\epsilon} \Big(\sum_{l=0}^{K} \Big(\gamma(l) \cdot \mathbb{I}\{l \geq \frac{K}{2}\} + \frac{1}{2}(1 - \gamma(l)) \Big) \cdot \Pr[\mathcal{L}(\mathcal{D}') = l] \Big) + \delta$$

$$\iff \sum_{l=\frac{K+1}{2}}^{K} \Big(\gamma(l) + \frac{1}{2}(1 - \gamma(l)) \Big) \cdot \Pr[\mathcal{L}(\mathcal{D}) = l] + \sum_{l=0}^{\frac{K-1}{2}} \frac{1}{2}(1 - \gamma(l)) \cdot \Pr[\mathcal{L}(\mathcal{D}) = l] \tag{43}$$

$$\leq e^{m\epsilon} \Big(\sum_{l=\frac{K+1}{2}}^{K} \Big(\gamma(l) + \frac{1}{2}(1 - \gamma(l)) \Big) \cdot \Pr[\mathcal{L}(\mathcal{D}) = l] \Big) + e^{m\epsilon} \Big(\sum_{l=0}^{\frac{K-1}{2}} \frac{1}{2}(1 - \gamma(l)) \cdot \Pr[\mathcal{L}(\mathcal{D}') = l] \Big) + \delta$$

$$\iff \sum_{l=\frac{K+1}{2}}^{K} \frac{1}{2}\gamma(l)\alpha_l - \sum_{l=0}^{\frac{K-1}{2}} \frac{1}{2}\gamma(l)\alpha_l + \frac{1}{2} \tag{44}$$

$$\leq e^{m\epsilon} \sum_{l=\frac{K+1}{2}}^{K} \frac{1}{2}\gamma(l)\alpha'_l - e^{m\epsilon} \sum_{l=0}^{\frac{K-1}{2}} \frac{1}{2}\gamma(l)\alpha'_l + \frac{1}{2}e^{m\epsilon} + \delta$$

$$\iff \sum_{l=\frac{K+1}{2}}^{K} (\alpha_l - e^{m\epsilon}\alpha'_l)\gamma(l) - \sum_{l=0}^{\frac{K-1}{2}} (\alpha_l - e^{m\epsilon}\alpha'_l)\gamma(l) \leq e^{m\epsilon} - 1 + 2\delta \tag{45}$$

where $\alpha_l = \Pr[\mathcal{L}(\mathcal{D}) = l]$ and $\alpha'_l = \Pr[\mathcal{L}(\mathcal{D}') = l], \forall l \in \{0, 1, \ldots, K\}$.

Similarly, when the output is 0,

$$\Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 0] \leq e^{m\epsilon} \Pr[\text{DaRRM}_\gamma(\mathcal{D}') = 0] + \delta \tag{46}$$

$$\iff \sum_{l=0}^{K} \Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 0 \mid \mathcal{L}(\mathcal{D}) = 0] \cdot \Pr[\mathcal{L}(\mathcal{D}) = 0] \tag{47}$$

$$\leq e^{m\epsilon} \Big(\sum_{l=0}^{K} \Pr[\text{DaRRM}_\gamma(\mathcal{D}') = 0 \mid \mathcal{L}(\mathcal{D}') = 0] \cdot \Pr[\mathcal{L}(\mathcal{D}') = 0] \Big) + \delta$$

$$\iff \sum_{l=0}^{K} \Big(\gamma(l) \cdot \mathbb{I}\{l < \frac{K}{2}\} + \frac{1}{2}(1 - \gamma(l)) \Big) \cdot \Pr[\mathcal{L}(\mathcal{D}) = l] \tag{48}$$

$$\leq e^{m\epsilon} \Big(\sum_{l=0}^{K} \gamma(l) \cdot \mathbb{I}\{l < \frac{K}{2}\} + \frac{1}{2}(1 - \gamma(l)) \Big) \cdot \Pr[\mathcal{L}(\mathcal{D}') = l] + \delta$$

$$\iff \sum_{l=0}^{\frac{K-1}{2}} \Big(\gamma(l) + \frac{1}{2}(1-\gamma(l))\Big) \cdot \Pr[\mathcal{L}(\mathcal{D}) = l] + \sum_{l=\frac{K+1}{2}}^{K} \frac{1}{2}(1-\gamma(l)) \cdot \Pr[\mathcal{L}(\mathcal{D}) = l] \tag{49}$$

$$\leq e^{m\epsilon}\Big(\sum_{l=0}^{\frac{K-1}{2}} \Big(\gamma(l) + \frac{1}{2}(1-\gamma(l))\Big) \cdot \Pr[\mathcal{L}(\mathcal{D}') = l] + \sum_{l=\frac{K+1}{2}}^{K} \frac{1}{2}(1-\gamma(l)) \cdot \Pr[\mathcal{L}(\mathcal{D}') = l]\Big) + \delta$$

$$\iff \sum_{l=0}^{\frac{K-1}{2}} \frac{1}{2}\gamma(l)\alpha_l - \sum_{l=\frac{K+1}{2}}^{K} \frac{1}{2}\gamma(l)\alpha_l + \frac{1}{2} \tag{50}$$

$$\leq e^{m\epsilon} \sum_{l=0}^{\frac{K-1}{2}} \frac{1}{2}\gamma(l)\alpha'_l - e^{m\epsilon} \sum_{l=\frac{K+1}{2}}^{K} \frac{1}{2}\gamma(l)\alpha'_l + \frac{1}{2}e^{m\epsilon} + \delta$$

$$\iff \sum_{l=0}^{\frac{K-1}{2}} (\alpha_l - e^{m\epsilon}\alpha'_l)\gamma(l) - \sum_{l=\frac{K+1}{2}}^{K} (\alpha_l - e^{m\epsilon}\alpha'_l)\gamma(l) \leq e^{m\epsilon} - 1 + 2\delta \tag{51}$$

Therefore,

DaRRM_γ is $(m\epsilon, \delta)$-differentially private

$$\iff \sum_{l=\frac{K+1}{2}}^{K} (\alpha_l - e^{m\epsilon}\alpha'_l)\gamma(l) - \sum_{l=0}^{\frac{K-1}{2}} (\alpha_l - e^{m\epsilon}\alpha'_l)\gamma(l) \leq e^{m\epsilon} - 1 + 2\delta \tag{52}$$

$$\text{and} \sum_{l=0}^{\frac{K-1}{2}} (\alpha_l - e^{m\epsilon}\alpha'_l)\gamma(l) - \sum_{l=\frac{K+1}{2}}^{K} (\alpha_l - e^{m\epsilon}\alpha'_l)\gamma(l) \leq e^{m\epsilon} - 1 + 2\delta \tag{53}$$

where $\alpha_l = \Pr[\mathcal{L}(\mathcal{D}) = l]$ and $\alpha'_l = \Pr[\mathcal{L}(\mathcal{D}') = l], \forall l \in \{0, 1, \ldots, K\}$ and $\mathcal{D}, \mathcal{D}'$ are any adjacent datasets.

If γ is symmetric around $\frac{K}{2}$, i.e. $\gamma(l) = \gamma(K-l)$, we show as follows satisfying either one of Eq. 52 or Eq. 53 implies satisfying the other one. The intuition is that there is nothing special about outputting 0 or 1.

$$\sum_{l=\frac{K+1}{2}}^{K} (\alpha_l - e^{m\epsilon}\alpha'_l)\gamma(l) - \sum_{l=0}^{\frac{K-1}{2}} (\alpha_l - e^{m\epsilon}\alpha'_l)\gamma(l) \leq e^{m\epsilon} - 1 + 2\delta \tag{54}$$

$$\iff \sum_{l=0}^{\frac{K-1}{2}} (\alpha_{K-l} - e^{m\epsilon}\alpha'_{K-l}) \cdot \gamma(K-l) - \sum_{l=\frac{K-1}{2}}^{K} (\alpha_{K-l} - e^{m\epsilon}\alpha'_{K-l}) \cdot \gamma(K-l) \leq e^{m\epsilon} - 1 + 2\delta \tag{55}$$

$$\iff \sum_{l=0}^{\frac{K-1}{2}} (\alpha_{K-l} - e^{m\epsilon}\alpha'_{K-l}) \cdot \gamma(l) - \sum_{l=\frac{K-1}{2}}^{K} (\alpha_{K-l} - e^{m\epsilon}\alpha'_{K-l}) \cdot \gamma(l) \leq e^{m\epsilon} - 1 + 2\delta \tag{56}$$

Since $\gamma(l) = \gamma(K-l)$

$$\iff \sum_{l=0}^{\frac{K-1}{2}} (\alpha_l - e^{m\epsilon}\alpha'_l) \cdot \gamma(l) - \sum_{l=\frac{K-1}{2}}^{K} (\alpha_l - e^{m\epsilon}\alpha'_l) \cdot \gamma(l) \leq e^{m\epsilon} - 1 + 2\delta \tag{57}$$

Since the above holds for all possible α_l, α'_l, one can consider another distribution such that the pmf of $\mathcal{L}(\mathcal{D})$ is $\beta_l = \alpha_{K-l}$ and the pmf of $\mathcal{L}(\mathcal{D}')$ is $\beta'_l = \alpha'_{K-l}$. Then, we rename β_l as α_l and β'_l as α'_l.

The above implies Eq. 52 and Eq. 53 are equivalent. Therefore,

$$\text{DaRRM}_\gamma \text{ is } (m\epsilon, \delta)\text{-differentially private}$$
$$\iff \sum_{l=\frac{K+1}{2}}^{K} (\alpha_l - e^{m\epsilon}\alpha'_l)\gamma(l) - \sum_{l=0}^{\frac{K-1}{2}} (\alpha_l - e^{m\epsilon}\alpha'_l)\gamma(l) \leq e^{m\epsilon} - 1 + 2\delta \tag{58}$$

□

D PROVABLE PRIVACY AMPLIFICATION IN I.I.D. SETTING UNDER PURE DP

Recall in the i.i.d. mechanisms setting, $\Pr[M_i(\mathcal{D}) = 1] = p$ and $\Pr[M_i(\mathcal{D}') = 1] = p'$, for all mechanisms M_i. Under pure differential privacy setting, each mechanisms M_i is ϵ-differentially private, and we want the aggregated majority voting output by DaRRM$_\gamma$ with a certain choice of γ to be $m\epsilon$-differentially private for $m < K$.

For analysis, we restrict our search for a γ function with good utility to the class with a mild monotonicity assumption: $\gamma(l) \geq \gamma(l+1), \forall l \leq \frac{K-1}{2}$ and $\gamma(l) \leq \gamma(l+1), \forall l \geq \frac{K+1}{2}$. This matches our intuition that as $\mathcal{L}$, i.e., the number of mechanisms outputting 1, approaches 0 or K, there is a clearer majority and so not much noise is needed to ensure privacy, which implies a larger value of γ.

Worst case probabilities. We call $(p^*, p'^*) = \arg\max_{p,p'} f(p^*, p'^*; \gamma)$ the worst case probabilities since they incur the largest privacy loss, where f is the simplified privacy cost objective defined in Eq. 37 with $\delta = \Delta = 0$. If we can show $f(p^*, p'^*; \gamma) \leq e^\epsilon - 1$ for some γ, then DaRRM$_\gamma$ is is $m\epsilon$-differentially private by Lemma 3.3. To find the worst case probabilities, first note (p^*, p'^*) are close to each other and lie in a feasible region $\mathcal{F}$, due to each mechanism being ϵ-differentially private in our setting. The feasible region is illustrated in Figure 5, and the four boundaries of which, i.e. (a) $p' \leq e^\epsilon p$ (b) $p \leq e^\epsilon p'$ (c) $1 - p' \leq e^\epsilon(1-p)$, and (d) $1 - p \leq e^\epsilon(1-p')$, are derived from the definition of differential privacy.

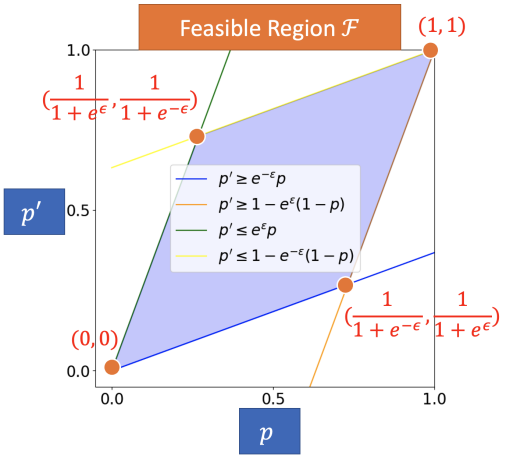


Figure 5: The feasible region $\mathcal{F}$ is plotted as the blue area. The four boundaries are implied by p, p' satisfying ϵ-differential privacy.

D.1 CHARACTERIZING WORST CASE PROBABILITIES

We first show a key lemma, later used in the proof of our main privacy amplification result, that allows us to further refine the search region for (p^*, p'^*) under γ: $\{0, 1, \ldots, K\} \to [0, 1]$ functions that are symmetric around $\frac{K}{2}$ and that satisfy the above mild monotonicity assumption. We call such γ functions *well-behaved*.

Lemma D.1 (Characteristics of worst case probabilities)**.** *Consider well-behaved γ functions such that $\gamma(\frac{K-1}{2}) > 0$ and $\gamma(\frac{K+1}{2}) > 0$, the worst case probabilities $(p^*, p'^*) = \arg\max_{p,p'} f(p, p'; \gamma)$ must satisfy exactly one of the following:*

$$p^* = e^\epsilon p'^*, \qquad \forall p \in [0, \frac{1}{e^{-\epsilon}+1}], p' \in [0, \frac{1}{1+e^\epsilon}] \tag{59}$$

$$1 - p'^* = e^\epsilon(1 - p^*), \qquad \forall p \in [\frac{1}{1+e^{-\epsilon}}, 1], p' \in [\frac{1}{1+e^\epsilon}, 1] \tag{60}$$

See the blue line and the orange line in Figure 5, respectively.

To show the above Lemma D.1, we show Lemma D.2 and Lemma D.3 as follows, each of which gives partial characteristics of the worst case probabilities. Lemma D.1 directly follows by combining the two lemmas.

Lemma D.2. *Consider a γ: $\{0, 1, \ldots, K\} \to [0, 1]$ function that is symmetric around $\frac{K}{2}$. If γ further satisfies: 1) $\gamma(l+1) \leq \gamma(l), \forall l \leq \frac{K}{2}$, 2) $\gamma(l+1) \geq \gamma(l), \forall l \geq \frac{K}{2}$, and 3) $\gamma(\frac{K-1}{2}) > 0, \gamma(\frac{K+1}{2}) > 0$, then the worst case probabilities $(p^*, p'^*) = \arg\max_{p,p'} f(p, p'; \gamma)$ must satisfy one of the following four equalities:*

$$p'^* = e^\epsilon p^*, \qquad \forall p \in [0, \frac{1}{1+e^\epsilon}], p' \in [0, \frac{1}{1+e^{-\epsilon}}] \tag{61}$$

$$p^* = e^\epsilon p'^*, \qquad \forall p \in [0, \frac{1}{e^{-\epsilon}+1}], p' \in [0, \frac{1}{1+e^\epsilon}] \tag{62}$$

$$1 - p^* = e^\epsilon(1 - p'^*), \qquad \forall p \in [\frac{1}{1+e^\epsilon}, 1], p' \in [\frac{1}{1+e^{-\epsilon}}, 1] \tag{63}$$

$$1 - p'^* = e^\epsilon (1 - p^*), \qquad \forall p \in [\frac{1}{1+e^{-\epsilon}}, 1], p' \in [\frac{1}{1+e^{\epsilon}}, 1] \tag{64}$$

Proof of Lemma D.2. Consider the privacy cost objective $f(p, p'; \gamma)$ as in Lemma 3.3, when the mechanisms are i.i.d. The gradients w.r.t. p and p' are

$$\begin{aligned}\nabla_p f(p, p'; \gamma) = &\sum_{l=0}^{\frac{K-1}{2}} -\binom{K}{l}\gamma(l) \cdot (lp^{l-1}(1-p)^{K-l} - p^l(K-l)(1-p)^{K-l-1}) \\ &+ \sum_{l=\frac{K+1}{2}}^{K} \binom{K}{l}\gamma(l) \cdot (lp^{l-1}(1-p)^{K-l} - p^l(K-l)(1-p)^{K-l-1})\end{aligned} \tag{65}$$

and

$$\begin{aligned}\nabla_{p'} f(p, p'; \gamma) = &\sum_{l=0}^{\frac{K-1}{2}} e^{m\epsilon}\binom{K}{l}\gamma(l) \cdot (lp'^{l-1}(1-p')^{K-l} - p'^l(K-l)(1-p')^{K-l-1}) \\ &+ \sum_{l=\frac{K+1}{2}}^{K} -e^{m\epsilon}\binom{K}{l}\gamma(l) \cdot (lp'^{l-1}(1-p')^{K-l} - p'^l(K-l)(1-p')^{K-l-1})\end{aligned} \tag{66}$$

We show $\forall p \in (0,1)$, $\nabla_p f(p, p'; \gamma) > 0$ and $\nabla_{p'} f(p, p'; \gamma) < 0$. This implies there is no local maximum inside $\mathcal{F}$, and so $p^*, p'^* = \arg\max_{p,p'} f(p, p'; \gamma)$ must be on one of the four boundaries of $\mathcal{F}$.

To show $\nabla_p f(p, p'; \gamma) > 0$ for $p \in (0,1)$, we first show for $p \in (0,1)$

$$\sum_{l=0}^{\frac{K-1}{2}} \gamma(l)\binom{K}{l} \cdot (p^l(K-l)(1-p)^{K-l-1} - lp^{l-1}(1-p)^{K-l}) > 0 \tag{67}$$

$$\Leftrightarrow \sum_{l=0}^{\frac{K-1}{2}} \gamma(l)\binom{K}{l} \cdot p^l(K-l)(1-p)^{K-l-1} > \sum_{l=0}^{\frac{K-1}{2}} \gamma(l)\binom{K}{l} \cdot lp^{l-1}(1-p)^{K-l} \tag{68}$$

$$\Leftrightarrow \sum_{l=0}^{\frac{K-1}{2}} \gamma(l)\binom{K-1}{l}\frac{K}{K-l} \cdot p^l(K-l)(1-p)^{K-l-1} \tag{69}$$

$$> \sum_{l=1}^{\frac{K-1}{2}} \gamma(l)\binom{K-1}{l-1}\frac{K}{l} \cdot lp^{l-1}(1-p)^{K-l}$$

$$\Leftrightarrow K\sum_{l=0}^{\frac{K-1}{2}} \gamma(l)\binom{K-1}{l}p^l(1-p)^{K-l-1} > K\sum_{l=1}^{\frac{K-1}{2}} \gamma(l)\binom{K-1}{l-1}p^{l-1}(1-p)^{K-l} \tag{70}$$

$$\Leftrightarrow \sum_{l=0}^{\frac{K-1}{2}} \gamma(l)\binom{K-1}{l}p^l(1-p)^{K-l-1} > \sum_{l=0}^{\frac{K-1}{2}-1} \gamma(l+1)\binom{K-1}{l}p^l(1-p)^{K-l-1} \tag{71}$$

Note that for $l \leq \frac{K-1}{2}$, $\gamma(l) \geq \gamma(l+1)$. Since $p \in (0,1)$, this implies for $l \in \{0, \ldots, \frac{K-1}{2} - 1\}$,

$$\gamma(l)\binom{K-1}{l}p^l(1-p)^{K-l-1} \geq \gamma(l+1)\binom{K-1}{l}p^l(1-p)^{K-l-1} \tag{72}$$

Furthermore, note the L.H.S. of Eq. 71 has one additional term $\gamma(\frac{K-1}{2})\binom{K-1}{\frac{K-1}{2}}p^{\frac{K-1}{2}}(1-p)^{\frac{K-1}{2}}$. Since $\gamma(\frac{K-1}{2}) > 0$ and $p \in (0,1)$,

$$\gamma(\frac{K-1}{2})\binom{K-1}{\frac{K-1}{2}}p^{\frac{K-1}{2}}(1-p)^{\frac{K-1}{2}} > 0 \tag{73}$$

Therefore, combining Eq. 72 and Eq. 73, we conclude Eq. 71 holds.

Next, we show for $p \in (0,1)$,

$$\sum_{l=\frac{K+1}{2}}^{K} \binom{K}{l} \gamma(l) \cdot (l p^{l-1}(1-p)^{K-l} - p^l (K-l)(1-p)^{K-l-1}) > 0 \tag{74}$$

$$\Leftrightarrow \sum_{l=\frac{K+1}{2}}^{K} \binom{K}{l} \gamma(l) \cdot l p^{l-1}(1-p)^{K-l} > \sum_{l=\frac{K+1}{2}}^{K} \binom{K}{l} p^l (K-l)(1-p)^{K-l-1} \tag{75}$$

$$\Leftrightarrow \sum_{l=\frac{K+1}{2}}^{K} \gamma(l) \binom{K-1}{l-1} \frac{K}{l} \cdot l p^{l-1}(1-p)^{K-l} \tag{76}$$

$$> \sum_{l=\frac{K+1}{2}}^{K-1} \gamma(l) \binom{K-1}{l} \frac{K}{K-l} \cdot p^l (K-l)(1-p)^{K-l-1}$$

$$\Leftrightarrow K \sum_{l=\frac{K+1}{2}}^{K} \gamma(l) \binom{K-1}{l-1} \cdot p^{l-1}(1-p)^{K-l} \tag{77}$$

$$> K \sum_{l=\frac{K+1}{2}}^{K-1} \gamma(l) \binom{K-1}{l} \cdot p^l (1-p)^{K-l-1}$$

$$\Leftrightarrow \sum_{l=\frac{K+1}{2}}^{K} \gamma(l) \binom{K-1}{l-1} \cdot p^{l-1}(1-p)^{K-l} > \sum_{l=\frac{K+1}{2}+1}^{K} \gamma(l-1) \binom{K-1}{l-1} \cdot p^{l-1}(1-p)^{K-l} \tag{78}$$

Note that for $l \geq \frac{K+1}{2}+1$, $\gamma(l) \geq \gamma(l-1)$. Since $p \in (0,1)$, this implies for $l \in \{\frac{K+1}{2}+1, \ldots, K\}$,

$$\gamma(l) \binom{K-1}{l-1} p^{l-1}(1-p)^{K-l} \geq \gamma(l-1) \binom{K-1}{l-1} p^{l-1}(1-p)^{K-l} \tag{79}$$

Furthermore, note the L.H.S. of Eq. 78 has one additional term $\gamma(\frac{K+1}{2}) \binom{K-1}{\frac{K-1}{2}} p^{\frac{K-1}{2}} (1-p)^{\frac{K-1}{2}}$. Since $\gamma(\frac{K+1}{2}) > 0$ and $p \in (0,1)$,

$$\gamma(\frac{K+1}{2}) \binom{K-1}{\frac{K-1}{2}} p^{\frac{K-1}{2}} (1-p)^{\frac{K-1}{2}} > 0 \tag{80}$$

Therefore, combining Eq. 79 and Eq. 80, we conclude Eq. 78 holds. Hence, combining Eq. 67 and Eq. 74, we have for $p \in (0,1)$, if γ satisfies the three conditions,

$$\nabla_p f(p, p'; \gamma) > 0 \tag{81}$$

Similarly, one can show for $p \in (0,1)$, if γ satisfies the three conditions,

$$\nabla_{p'} f(p, p'; \gamma) < 0 \tag{82}$$

This implies there is no local minima or local maxima inside the feasible region $\mathcal{F}$. Hence, the worst case probability $(p^*, p'^*) = \arg\max_{p,p'} f(p, p'; \gamma)$ is on one of the four boundaries of $\mathcal{F}$, that is, (p^*, p'^*) satisfy exactly one of the following:

$$p'^* = e^\epsilon p^*, \qquad \forall p \in [0, \frac{1}{1+e^\epsilon}], p' \in [0, \frac{1}{1+e^{-\epsilon}}]$$

$$p^* = e^\epsilon p'^*, \qquad \forall p \in [0, \frac{1}{e^{-\epsilon}+1}], p' \in [0, \frac{1}{1+e^\epsilon}]$$

$$1-p^* = e^\epsilon (1-p'^*), \qquad \forall p \in [\frac{1}{1+e^\epsilon}, 1], p' \in [\frac{1}{1+e^{-\epsilon}}, 1]$$

$$1-p'^* = e^\epsilon (1-p^*), \qquad \forall p \in [\frac{1}{1+e^{-\epsilon}}, 1], p' \in [\frac{1}{1+e^\epsilon}, 1]$$

□

Lemma D.3. *Consider a γ: $\{0,1,\ldots,K\} \to [0,1]$ function that is symmetric around $\frac{K}{2}$. If γ satisfies: $\gamma(l) \geq \gamma(l+1), \forall l \leq \frac{K}{2}$ and $\gamma(l+1) \geq \gamma(l), \forall l \geq \frac{K}{2}$, then the privacy cost objective $f(p,p';\gamma)$ is maximized when $p \geq p'$.*

Proof of Lemma D.3. WLOG, consider the output of DaRRM to be 1. By Eq. 41, the privacy cost objective $f(p,p';\gamma)$, defined in Lemma 3.3 when $\delta = 0$, when the mechanisms are i.i.d., is equivalent to

$$f(p,p';\gamma) = \frac{\Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 1]}{\Pr[\text{DaRRM}_\gamma(\mathcal{D}') = 1]} - 1 \tag{83}$$

Hence, $f(p,p';\gamma)$ is maximized when $\Pr[\mathcal{A}(\mathcal{D}) = 1] \geq \Pr[\mathcal{A}(\mathcal{D}') = 1]$. Note that

$$\Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 1] = \frac{1}{2}\sum_{l=\frac{K+1}{2}}^{K} \gamma(l)\binom{K}{l}p^l(1-p)^{K-l} - \frac{1}{2}\sum_{l=0}^{\frac{K-1}{2}} \gamma(l)\binom{K}{l}p^l(1-p)^{K-l-1} + \frac{1}{2} \tag{84}$$

Define $g(l) = \begin{cases} -\frac{1}{2}\gamma(l) & \forall l \leq \frac{K}{2} \\ \frac{1}{2}\gamma(l) & \forall l \geq \frac{K}{2} \end{cases}$. Since $\gamma(l) \geq \gamma(l+1), \forall l \leq \frac{K}{2}$ and $\gamma(l+1) \geq \gamma(l), \forall l \geq \frac{K}{2}$, there is $g(l+1) \geq g(l), \forall l \in \{0,\ldots,K\}$. And replacing $\gamma(l)$ with $g(l)$,

$$\Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 1] = \sum_{l=0}^{K} g(l)\binom{K}{l}p^l(1-p)^{K-l} \tag{85}$$

The gradient of the above probability w.r.t. p is

$$\nabla_p \Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 1] \tag{86}$$

$$= \sum_{l=0}^{K} g(l)\binom{K}{l}\left(lp^{l-1}(1-p)^{K-l} - (K-l)p^l(1-p)^{K-l-1}\right) \tag{87}$$

$$= \sum_{l=1}^{K} g(l)\binom{K-1}{l-1}\frac{K}{l}lp^{l-1}(1-p)^{K-l} - \sum_{l=0}^{K-1}\binom{K-1}{l}\frac{K}{K-l}(K-l)p^l(1-p)^{K-l-1} \tag{88}$$

$$= K\sum_{l=1}^{K}\binom{K-1}{l-1}p^{l-1}(1-p)^{K-l} - K\sum_{l=0}^{K-1}\binom{K-1}{l}p^l(1-p)^{K-l-1} \tag{89}$$

$$= K\sum_{l=0}^{K-1} g(l+1)\binom{K-1}{l}p^l(1-p)^{K-l-1} - K\sum_{l=0}^{K-1} g(l)\binom{K-1}{l}p^l(1-p)^{K-l-1} \tag{90}$$

$$= K\sum_{l=0}^{K-1}\Big(g(l+1) - g(l)\Big)\binom{K-1}{l}p^l(1-p)^{K-l-1} \tag{91}$$

Since $g(l+1) \geq g(l)$ and the binomial probability is always ≥ 0, $\nabla_p \Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 1] \geq 0$. This implies whenever $p \geq p'$, $\Pr[\text{DaRRM}_\gamma(\mathcal{D}) = 1] \geq \Pr[\text{DaRRM}_\gamma(\mathcal{D}') = 1]$. Hence, the privacy cost objective $f(p,p')$ is maximized when $p \geq p'$. □

D.2 Proof of Main Results on Privacy Amplification (Theorem 4.1)

Roadmap. To show Theorem 4.1, we show two parts separately: in section D.2.1, we show if the privacy allowance $m \geq \frac{K+1}{2}$, one can set $\gamma = 1$ (see Lemma D.4), and in section D.2.2 we show if $m \leq \frac{K-1}{2}$, one can set γ to be the one such that DaRRM$_\gamma$ has the same output distribution as outputting the majority based on $2m-1$ subsampled mechanisms and DaRRM$_\gamma$ still satisfies the privacy guarantee. We call this γ function $\gamma_{\text{Double Subsampling}}$ (see Lemma D.8).

Showing Lemma D.4 is relatively straightforward (see Section D.2.1). To show Lemma D.8, we first introduce a class of *well-behaved* γ functions called the *"symmetric-form"* family, and derive two clean sufficient conditions for γ functions from the *"symmetric-form"* family such that DaRRM$_\gamma$

is $m\epsilon$-differentially private. After that, we show setting $\gamma_{\text{Double Subsampling}}$ as in Lemma D.8 satisfies the two conditions, and hence, DaRRM$_{\gamma_{\text{Double Subsampling}}}$ is $m\epsilon$-differentially private. Details are in Section D.2.2.

Finally, Theorem 4.1 follows directly from combining Lemma D.4 and Lemma D.8.

D.2.1 Privacy Amplification under Large Privacy Allowance

We show the following lemma by showing that if one sets $\gamma(l) = 1, \forall l \in \{0, 1, \ldots, K\}$, then $m \geq \frac{K+1}{2}$ suffices to ensure the worst case probabilities $(p_i^*, p_i) = \arg\max_{p_i, p_i'} f(p, p'; \gamma)$ satisfy Eq. 37 in Lemma 3.3, and hence if $m \geq \frac{K+1}{2}$, DaRRM$_{\gamma=1}$ is $m\epsilon$-differentially private.

Lemma D.4 (Privacy amplification under large privacy allowance $m \geq \frac{K+1}{2}$)**.** *Consider using DaRRM to solve Problem 1.1 with $p_i = p$, $p_i' = p'$, $\forall i \in [K]$ and $\delta = \Delta = 0$. If the privacy allowance is $m \geq \frac{K+1}{2}$, one can set $\gamma(l) = 1, \forall l \in \{0, \ldots, K\}$ in DaRRM$_\gamma$ and DaRRM$_\gamma$ is $m\epsilon$ differentially private.*

Proof of Lemma D.4. Consider $\gamma(l) = 1, \forall l \in \{0, 1, \ldots, K\}$. Since $\gamma(l) \geq \gamma(l+1), \forall l \leq \frac{K-1}{2}$, $\gamma(l+1) \geq \gamma(l), \forall l \geq \frac{K+1}{2}$ and $\gamma(\frac{K-1}{2}) = \gamma(\frac{K+1}{2}) = 1 > 0$, by Lemma D.1, the worst case probabilities $(p^*, p'^*) = \arg\max_{p,p'} f(p, p')$ are on one of the two boundaries of $\mathcal{F}$, that is, they satisfy either $p = e^\epsilon p'$, $\forall p \in [0, \frac{1}{1+e^\epsilon}], p' \in [0, \frac{1}{1+e^\epsilon}]$ or $1 - p' = e^\epsilon(1-p)$, $\forall p \in [\frac{1}{1+e^{-\epsilon}}, 1], p' \in [\frac{1}{1+e^\epsilon}, 1]$. We now find the local maximums on the boundary $p = e^\epsilon p'$ and $1 - p' = e^\epsilon(1-p)$ separately and then find the global maximum $(p^*, p'^*) = \arg\max_{p,p'} f(p, p'; \gamma)$.

Part I: Finding the local worse case probabilities on the boundary $p = e^\epsilon p'$.

The privacy cost objective $f(p, p'; \gamma)$ on the boundary $p = e^\epsilon p'$, $\forall p \in [0, \frac{1}{e^{-\epsilon}+1}], p' \in [0, \frac{1}{1+e^\epsilon}]$, can be written as the following by substituting p with p' (and omitting γ for convenience):

$$
\begin{aligned}
f(p') = &\sum_{l=0}^{\frac{K-1}{2}} (e^{m\epsilon} \binom{K}{l} p'^l (1-p')^{K-l} - \binom{K}{l} (e^\epsilon p')^l (1 - e^\epsilon p')^{K-l}) \cdot \gamma(l) \\
&+ \sum_{l=\frac{K+1}{2}}^{K} (\binom{K}{l} (e^\epsilon p')^l (1 - e^\epsilon p')^{K-l} - e^{m\epsilon} \binom{K}{l} p'^l (1-p')^{K-l}) \cdot \gamma(l)
\end{aligned} \tag{92}
$$

And the gradient w.r.t. p' is

$$
\begin{aligned}
\nabla_{p'} f(p') = &\sum_{l=0}^{\frac{K-1}{2}} \Big(e^{m\epsilon} \binom{K}{l} (l p'^{l-1} (1-p')^{K-l} - p'^l (K-l)(1-p')^{K-l-1}) \\
&- e^\epsilon \binom{K}{l} (l (e^\epsilon p')^{l-1} (1 - e^\epsilon p')^{K-l} - e^{\epsilon l} p'^l (K-l)(1 - e^\epsilon p')^{K-l-1}) \Big) \cdot \gamma(l) \\
&+ \sum_{l=\frac{K+1}{2}}^{K} \Big(e^\epsilon \binom{K}{l} (l (e^\epsilon p')^{l-1} (1 - e^\epsilon p')^{K-l} - e^{\epsilon l} p'^l (K-l)(1 - e^\epsilon p')^{K-l-1}) \\
&- e^{m\epsilon} \binom{K}{l} (l p'^{l-1} (1-p')^{K-l} - p'^l (K-l)(1-p')^{K-l-1}) \Big) \cdot \gamma(l)
\end{aligned} \tag{93}
$$

$$
\begin{aligned}
&\nabla_{p'} f(p') \\
&= -K \sum_{l=0}^{\frac{K-1}{2}} e^{m\epsilon} \binom{K-1}{l} p'^l (1-p')^{K-l-1} \gamma(l) + K \sum_{l=\frac{K+1}{2}}^{K-1} e^{m\epsilon} \binom{K-1}{l} p'^l (1-p')^{K-l-1} \gamma(l) \\
&+ K \sum_{l=0}^{\frac{K-1}{2}} e^\epsilon \binom{K-1}{l} (\epsilon p')^\epsilon (1 - e^\epsilon p')^{K-l-1} \gamma(l) - K \sum_{l=\frac{K+1}{2}}^{K-1} e^\epsilon \binom{K-1}{l} (e^\epsilon p')^l (1 - e^\epsilon p')^{K-l-1} \gamma(l)
\end{aligned} \tag{94}
$$

$$+K\sum_{l=0}^{\frac{K-1}{2}-1}e^{m\epsilon}\binom{K-1}{l}p'^{l}(1-p')^{K-l-1}\gamma(l+1)-K\sum_{l=\frac{K-1}{2}}^{K-1}e^{m\epsilon}\binom{K-1}{l}p'^{l}(1-p')^{K-l-1}\gamma(l+1)$$
$$-K\sum_{l=0}^{\frac{K-1}{2}-1}e^{\epsilon}\binom{K-1}{l}(e^{\epsilon}p')^{l}(1-e^{\epsilon}p')^{K-l-1}\gamma(l+1)+K\sum_{l=\frac{K-1}{2}}^{K-1}e^{\epsilon}\binom{K-1}{l}(e^{\epsilon}p')^{l}(1-e^{\epsilon}p')^{K-l-1}\gamma(l+1)$$

That is,

$$\frac{\nabla_{p'}f(p')}{K} \tag{95}$$
$$=e^{m\epsilon}\sum_{l=0}^{\frac{K-1}{2}-1}\binom{K-1}{l}p'^{l}(1-p')^{K-l-1}\Big(\gamma(l+1)-\gamma(l)\Big)-2e^{m\epsilon}\binom{K-1}{\frac{K-1}{2}}p'^{\frac{K-1}{2}}(1-p')^{\frac{K-1}{2}}\gamma(\frac{K-1}{2})$$
$$+e^{m\epsilon}\sum_{l=\frac{K+1}{2}}^{K-1}\binom{K-1}{l}p'^{l}(1-p')^{K-l-1}\Big(\gamma(l)-\gamma(l+1)\Big)$$
$$+e^{\epsilon}\sum_{l=0}^{\frac{K-1}{2}-1}\binom{K-1}{l}(e^{\epsilon}p')^{l}(1-e^{\epsilon}p')^{K-l-1}\Big(\gamma(l)-\gamma(l+1)\Big)+2e^{\epsilon}\binom{K-1}{\frac{K-1}{2}}(e^{\epsilon}p')^{\frac{K-1}{2}}(1-e^{\epsilon}p')^{\frac{K-1}{2}}\gamma(\frac{K-1}{2})$$
$$+e^{\epsilon}\sum_{l=\frac{K+1}{2}}^{K-1}\binom{K-1}{l}(e^{\epsilon}p')^{l}(1-e^{\epsilon}p')^{K-l-1}\Big(\gamma(l+1)-\gamma(l)\Big)$$

When $\gamma(l)=1$, the above gradient is then

$$\frac{\nabla_{p'}f(p')}{K}=-2e^{m\epsilon}\binom{K-1}{\frac{K-1}{2}}p'^{\frac{K-1}{2}}(1-p')^{\frac{K-1}{2}}+2e^{\epsilon}\binom{K-1}{\frac{K-1}{2}}(e^{\epsilon}p')^{\frac{K-1}{2}}(1-e^{\epsilon}p')^{\frac{K-1}{2}} \tag{96}$$

If $p'=0$, then $p=0$, and the original privacy cost objective is $f(0,0';\gamma=1)=e^{m\epsilon}-1$, which satisfies Eq. 37 in Lemma 3.3 (i.e. DaRRM$_{\gamma=1}$ is $m\epsilon$-differentially private at $(p,p')=(0,0)$).

For $p'>0$, if $\nabla_{p'}f(p')\le 0$, then we know $f(p')\le f(0)$, i.e. the worst case probabilities on the boundary $p=e^{\epsilon}p'$ is $(p,p')=(0,0)$. To ensure $\nabla_{p'}f(p')\le 0$,

$$\nabla_{p'}f(p')\le 0 \tag{97}$$
$$\Leftrightarrow -2e^{m\epsilon}\binom{K-1}{\frac{K-1}{2}}p'^{\frac{K-1}{2}}(1-p')^{\frac{K-1}{2}}\le -2e^{\epsilon}\binom{K-1}{\frac{K-1}{2}}(e^{\epsilon}p')^{\frac{K-1}{2}}(1-e^{\epsilon}p')^{\frac{K-1}{2}} \tag{98}$$
$$\Leftrightarrow e^{m\epsilon}\binom{K-1}{\frac{K-1}{2}}p'^{\frac{K-1}{2}}(1-p')^{\frac{K-1}{2}}\ge e^{\epsilon}\binom{K-1}{\frac{K-1}{2}}(e^{\epsilon}p')^{\frac{K-1}{2}}(1-e^{\epsilon}p')^{\frac{K-1}{2}} \tag{99}$$

Let

$$\mathcal{R}:=\frac{\text{L.H.S.}}{\text{R.H.S.}}=\frac{e^{m\epsilon}\binom{K-1}{\frac{K-1}{2}}p'^{\frac{K-1}{2}}(1-p')^{\frac{K-1}{2}}}{e^{\epsilon}\binom{K-1}{\frac{K-1}{2}}(e^{\epsilon}p')^{\frac{K-1}{2}}(1-e^{\epsilon}p')^{\frac{K-1}{2}}}=\frac{e^{m\epsilon}}{e^{\frac{K+1}{2}\epsilon}}\cdot\Big(\frac{1-p'}{1-e^{\epsilon}p'}\Big)^{\frac{K-1}{2}} \tag{100}$$

and $\mathcal{R}\ge 1\Leftrightarrow\nabla_{p'}f(p')\le 0$. Since $\frac{1-p'}{1-e^{\epsilon}p'}\ge 1$, $\mathcal{R}\ge e^{(m-\frac{K+1}{2})\cdot\epsilon}$.

Hence, to make sure $\mathcal{R}\ge 1$ (and so $\nabla_{p'}f(p')\le 0$), $m\ge\frac{K+1}{2}$ suffices.

Part II: Finding the local worst case probabilities on the boundary $1-p'=e^{\epsilon}(1-p)$.

Now consider the maximum point on the other boundary $1-p'=e^{\epsilon}(1-p)$ for $p'\in[\frac{1}{1+e^{\epsilon}},1]$ and $p\in[\frac{1}{1+e^{-\epsilon}},1]$. Following the privacy cost objective $f(p,p';\gamma)$ and let $q=1-p$ and $p'=1-q'$, the objective is

$$f(q,q';\gamma)=\sum_{l=0}^{\frac{K-1}{2}}\left(e^{m\epsilon}\binom{K}{l}(1-q')^{l}q'^{K-l}-\binom{K}{l}(1-q)^{l}q^{K-l}\right)\cdot\gamma(l) \tag{101}$$

$$+\sum_{l=\frac{K+1}{2}}^{K}\Big(\binom{K}{l}(1-q)^l q^{K-l}-e^{m\epsilon}\binom{K}{l}(1-q')^l q'^{K-l}\Big)\cdot\gamma(l)$$

Substituting $1-p'=e^{\epsilon}(1-p)\Leftrightarrow q'=e^{\epsilon}q$ (and omitting γ for convenience), the privacy cost objective can be written as

$$\begin{aligned}f(q)=&\sum_{l=0}^{\frac{K-1}{2}}\Big(e^{m\epsilon}\binom{K}{l}(1-e^{\epsilon}q)^l(e^{\epsilon}q)^{K-l}-\binom{K}{l}(1-q)^l q^{K-l}\Big)\cdot\gamma(l)\\&+\sum_{l=\frac{K+1}{2}}^{K}\Big(\binom{K}{l}(1-q)^l q^{K-l}-e^{m\epsilon}\binom{K}{l}(1-e^{\epsilon}q)^l(e^{\epsilon}q)^{K-l}\Big)\cdot\gamma(l)\end{aligned}\tag{102}$$

And the gradient w.r.t. q is

$$\begin{aligned}\nabla_q f(q)=&\sum_{l=0}^{\frac{K-1}{2}}\Big(e^{m\epsilon}\binom{K}{l}\Big((-e^{\epsilon})l(1-e^{\epsilon}q)^{l-1}(e^{\epsilon}q)^{K-l}+e^{\epsilon}(K-l)(1-e^{\epsilon}q)^l(e^{\epsilon}q)^{K-l-1}\Big)\\&-\binom{K}{l}\Big(-l(1-q)^{l-1}q^{K-l}+(K-l)(1-q)^l q^{K-l-1}\Big)\Big)\cdot\gamma(l)\\&+\sum_{l=\frac{K+1}{2}}^{K}\Big(\binom{K}{l}\Big(-l(1-q)^{l-1}q^{K-l}+(K-l)(1-q)^l q^{K-l-1}\Big)\\&-e^{m\epsilon}\binom{K}{l}\Big((-e^{\epsilon})l(1-e^{\epsilon}q)^{l-1}(e^{\epsilon}q)^{K-l}+e^{\epsilon}(K-l)(1-e^{\epsilon}q)^l(e^{\epsilon}q)^{K-l-1}\Big)\Big)\cdot\gamma(l)\end{aligned}\tag{103}$$

$$\begin{aligned}\nabla_q f(q)=&-\sum_{l=1}^{\frac{K-1}{2}}e^{(m+1)\epsilon}\binom{K-1}{l-1}\frac{K}{l}l(1-e^{\epsilon}q)^{l-1}(e^{\epsilon}q)^{K-l}\cdot\gamma(l)\\&+\sum_{l=0}^{\frac{K-1}{2}}e^{(m+1)\epsilon}\binom{K-1}{l}\frac{K}{K-l}(K-l)(1-e^{\epsilon}q)^l(e^{\epsilon}q)^{K-l-1}\cdot\gamma(l)\\&+\sum_{l=1}^{\frac{K-1}{2}}\binom{K-1}{l-1}\frac{K}{l}l(1-q)^{l-1}q^{K-l}\cdot\gamma(l)-\sum_{l=0}^{\frac{K-1}{2}}\binom{K-1}{l}\frac{K}{K-l}(K-l)(1-q)^l q^{K-l-1}\cdot\gamma(l)\\&-\sum_{l=\frac{K+1}{2}}^{K}\binom{K-1}{l-1}\frac{K}{l}l(1-q)^{l-1}q^{K-l}\cdot\gamma(l)+\sum_{l=\frac{K+1}{2}}^{K-1}\binom{K-1}{l}\frac{K}{K-l}(K-l)(1-q)^l q^{K-l-1}\cdot\gamma(l)\\&+\sum_{l=\frac{K+1}{2}}^{K}e^{(m+1)\epsilon}\binom{K-1}{l-1}\frac{K}{l}l(1-e^{\epsilon}q)^{l-1}(e^{\epsilon}q)^{K-l}\cdot\gamma(l)\\&-\sum_{l=\frac{K+1}{2}}^{K-1}e^{(m+1)\epsilon}\binom{K-1}{l}\frac{K}{K-l}(K-l)(1-e^{\epsilon}q)^l(e^{\epsilon}q)^{K-l-1}\cdot\gamma(l)\end{aligned}\tag{104}$$

$$\begin{aligned}\nabla_q f(q)=&-K\sum_{l=1}^{\frac{K-1}{2}}e^{(m+1)\epsilon}\binom{K-1}{l-1}(1-e^{\epsilon}q)^{l-1}(e^{\epsilon}q)^{K-l}\cdot\gamma(l)\\&+K\sum_{l=0}^{\frac{K-1}{2}}e^{(m+1)\epsilon}\binom{K-1}{l}(1-e^{\epsilon}q)^l(e^{\epsilon}q)^{K-l-1}\cdot\gamma(l)\end{aligned}\tag{105}$$

$$
\begin{aligned}
&+ K \sum_{l=1}^{\frac{K-1}{2}} \binom{K-1}{l-1}(1-q)^{l-1}q^{K-l}\cdot\gamma(l) - K\sum_{l=0}^{\frac{K-1}{2}}\binom{K-1}{l}(1-q)^{l}q^{K-l-1}\cdot\gamma(l)\\
&- K \sum_{l=\frac{K+1}{2}}^{K} \binom{K-1}{l-1}(1-q)^{l-1}q^{K-l}\cdot\gamma(l) + K\sum_{l=\frac{K+1}{2}}^{K-1}\binom{K-1}{l}(1-q)^{l}q^{K-l-1}\cdot\gamma(l)\\
&+ K \sum_{l=\frac{K+1}{2}}^{K} e^{(m+1)\epsilon}\binom{K-1}{l-1}(1-e^{\epsilon}q)^{l-1}(e^{\epsilon}q)^{K-l}\cdot\gamma(l)\\
&- K\sum_{l=\frac{K+1}{2}}^{K-1} e^{(m+1)\epsilon}\binom{K-1}{l}(1-e^{\epsilon}q)^{l}(e^{\epsilon}q)^{K-l-1}\cdot\gamma(l)
\end{aligned}
$$

The above is

$$
\begin{aligned}
&\frac{\nabla_q f(q)}{K}\\
&= -\sum_{l=0}^{\frac{K-1}{2}-1} e^{(m+1)\epsilon}\binom{K-1}{l}(1-e^{\epsilon}q)^{l}(e^{\epsilon}q)^{K-l-1}\cdot\gamma(l+1) && (106)\\
&+ \sum_{l=0}^{\frac{K-1}{2}} e^{(m+1)\epsilon}\binom{K-1}{l}(1-e^{\epsilon}q)^{l}(e^{\epsilon}q)^{K-l-1}\cdot\gamma(l)\\
&+ \sum_{l=0}^{\frac{K-1}{2}-1}\binom{K-1}{l}(1-q)^{l}q^{K-l-1}\cdot\gamma(l+1) - \sum_{l=0}^{\frac{K-1}{2}}\binom{K-1}{l}(1-q)^{l}q^{K-l-1}\cdot\gamma(l)\\
&- \sum_{l=\frac{K-1}{2}}^{K-1}\binom{K-1}{l}(1-q)^{l}q^{K-l-1}\cdot\gamma(l+1) + \sum_{l=\frac{K+1}{2}}^{K-1}\binom{K-1}{l}(1-q)^{l}q^{K-l-1}\cdot\gamma(l)\\
&+ \sum_{l=\frac{K-1}{2}}^{K-1} e^{(m+1)\epsilon}\binom{K-1}{l}(1-e^{\epsilon}q)^{l}(e^{\epsilon}q)^{K-l-1}\cdot\gamma(l+1)\\
&- \sum_{l=\frac{K+1}{2}}^{K-1} e^{(m+1)\epsilon}\binom{K-1}{l}(1-e^{\epsilon}q)^{l}(e^{\epsilon}q)^{K-l-1}\cdot\gamma(l)\\
&= \sum_{l=0}^{\frac{K-1}{2}-1} e^{(m+1)\epsilon}\binom{K-1}{l}(1-e^{\epsilon}q)^{l}(e^{\epsilon}q)^{K-l-1}\cdot\Big(\gamma(l)-\gamma(l+1)\Big) && (107)\\
&+ \sum_{l=\frac{K+1}{2}}^{K-1}\binom{K-1}{l}(1-e^{\epsilon}q)^{l}(e^{\epsilon}q)^{K-l-1}\cdot\Big(\gamma(l+1)-\gamma(l)\Big)\\
&+ 2e^{(m+1)\epsilon}\binom{K-1}{\frac{K-1}{2}}(1-e^{\epsilon}q)^{\frac{K-1}{2}}(e^{\epsilon}q)^{\frac{K-1}{2}}\cdot\gamma(\frac{K-1}{2}) && \text{Since } \gamma(\frac{K+1}{2})=\gamma(\frac{K-1}{2})\\
&+ \sum_{l=0}^{\frac{K-1}{2}-1}\binom{K-1}{l}(1-q)^{l}q^{K-l-1}\cdot\Big(\gamma(l+1)-\gamma(l)\Big)\\
&+ \sum_{l=\frac{K+1}{2}}^{K-1}(1-q)^{l}q^{K-l-1}\cdot\Big(\gamma(l)-\gamma(l+1)\Big)\\
&- 2\binom{K-1}{\frac{K-1}{2}}(1-q)^{\frac{K-1}{2}}q^{\frac{K-1}{2}}\cdot\gamma(\frac{K-1}{2}) && \text{Since } \gamma(\frac{K-1}{2})=\gamma(\frac{K+1}{2})
\end{aligned}
$$

When $\gamma(l) = 1$, then the above gradient is then

$$\frac{\nabla_q f(q)}{K} = 2e^{(m+1)\epsilon}\binom{K-1}{\frac{K-1}{2}}(1-e^\epsilon q)^{\frac{K-1}{2}}(e^\epsilon q)^{\frac{K-1}{2}} - 2\binom{K-1}{\frac{K-1}{2}}(1-q)^{\frac{K-1}{2}}q^{\frac{K-1}{2}} \tag{108}$$

Since $p \in [\frac{1}{1+e^{-\epsilon}}, 1]$, and $q = 1-p \in [0, \frac{1}{1+e^\epsilon}]$, $(1-e^\epsilon q)(e^\epsilon q) \geq (1-q)q$. Furthermore, since $e^{(m+1)\epsilon} \geq 1$, $\nabla_q f(q) \geq 0$. Hence, $f(q)$ achieves the maximum at $q = \frac{1}{1+e^\epsilon}$ — that is, at $p = 1-\frac{1}{1+e^\epsilon} = \frac{1}{1+e^{-\epsilon}}$. Since $1-p' = e^\epsilon(1-p)$, $f(q)$ achieves the maximum at $p' = 1-e^\epsilon(1-p) = 1-e^\epsilon(1-\frac{1}{1+e^{-\epsilon}}) = \frac{1}{1+e^\epsilon}$. Notice that $p, p' = (\frac{1}{1+e^{-\epsilon}}, \frac{1}{1+e^\epsilon}) = \arg\min_{p,p'} f(p, p'; \gamma)$ on the other boundary $p = e^\epsilon p', \forall p \in [0, \frac{1}{e^{-\epsilon}+1}]$. Hence, the global worst case probabilities (p^*, p'^*) are not on the boundary $1-p' = e^\epsilon(1-p)$.

Part III: Global maximum point (p, p')

The above implies the global maximum points (aka. global worst case probabilities) $(p^*, p'^*) = \arg\max_{p,p'} f(p, p') = (0, 0)$ if $m \geq \frac{K+1}{2}$, and $f(0,0) = e^{m\epsilon} - 1$.

Therefore, by Lemma 3.3, if $m \geq \frac{K+1}{2}$, setting $\gamma(l) = 1, \forall l \in \{0, \ldots, K\}$ ensures DaRRM$_\gamma$ is $m\epsilon$-differentially private. □

D.2.2 Privacy Amplification under Small Privacy Allowance

Roadmap. To show the privacy amplification under a small privacy allowance $m \leq \frac{K-1}{2}$ in Lemma D.8, we first observe that the γ function corresponding to natural subsampling as shown in Lemma 3.1 falls into a special family of γ functions, which we call the "symmetric form family", that are a combination of two functions of a specific form on support $\{0, \ldots, \frac{K}{2}\}$ and $\{\frac{K}{2}, \ldots, K\}$ and are symmetric around $\frac{K}{2}$ — that is, $\gamma(l) = \begin{cases} 1-2h(l) & l \leq \frac{K}{2} \\ 2h(l)-1 & l \geq \frac{K}{2} \end{cases}$ and $h(l) + h(K-1) = 1$, where $h(l)$ is monotonically increasing on the support. It is not hard to see these functions are *well-behaved*, and so we can apply Lemma D.1 in such cases to limit the region to search for the worst case probabilities. For a γ function that falls under this "symmetric form family", we show two clean sufficient conditions for DaRRM$_\gamma$ to be $m\epsilon$-differentially private in terms of the expectation of the γ function applied to some Binomial random variables, as in Lemma D.5.

To show the privacy amplification results under a small privacy allowance m, we further need two building blocks on recurrence relationships in expectation of Binomial random variables and Hypergeometric random variables in Lemma D.6 and Lemma D.7.

Finally, based on Lemma D.6 and Lemma D.7, we show in Lemma D.8 that $\gamma_{\text{Double Subsampling}}$, i.e., the γ function that enables DaRRM to have the same distribution as outputting the majority of $2m-1$ subsampled mechanisms, belongs to the "symmetric form family", and satisfies the sufficient conditions as stated in Lemma D.5. Hence DaRRM$_{\gamma_{\text{Double Subsampling}}}$ is $m\epsilon$-differentially private.

Lemma D.5 (Privacy conditions of "symmetric form family"). *Consider a $\gamma : \{0, 1, \ldots, K\} \to [0, 1]$ function that is of the form*

$$\gamma(l) = \begin{cases} 1-2h(l) & l \in \{0, 1, \ldots, \frac{K-1}{2}\} \\ 2h(l)-1 & l \in \{\frac{K+1}{2}, \ldots, K\} \end{cases} \tag{109}$$

where $h(l)$ is a monotonically increasing function on $l \in \{0, \ldots, K\}$ and $h(l) + h(K-l) = 1$. Let random variables $X \sim$ Binom$(K-1, p)$ and $Y \sim$ Binom$(K-1, e^\epsilon p)$. Let random variables $\hat{X} \sim$ Binom$(K-1, 1-e^\epsilon(1-p))$ and $\hat{Y} \sim$ Binom$(K-1, p)$. If this γ function further satisfies the following two conditions:

$$e^{m\epsilon}\mathbb{E}_X[h(X+1) - h(X)] \geq e^\epsilon \mathbb{E}_Y[h(Y+1) - h(Y)], \quad \forall p \in [0, \frac{1}{1+e^\epsilon}] \tag{110}$$

$$e^{(m+1)\epsilon}\mathbb{E}_{\hat{X}}[h(\hat{X}+1) - h(\hat{X})] \geq \mathbb{E}_{\hat{Y}}[h(\hat{Y}+1) - h(\hat{Y})], \quad \forall p \in [\frac{1}{1+e^{-\epsilon}}, 1] \tag{111}$$

then Algorithm DaRRM$_\gamma$ is $m\epsilon$-differentially private.

Proof of Lemma D.5. Since $h(l+1) \geq h(l)$ on $l \in \{0, \dots, K\}$, there is $\gamma(l) \geq \gamma(l+1), \forall l \leq \frac{K}{2}$ and $\gamma(l+1) \geq \gamma(l), \forall l \geq \frac{K}{2}$. Furthermore, since $h(l) + h(K-l) = 1$, $\gamma(\frac{K-1}{2}) = 1 - 2h(\frac{K-1}{2}) = 1 - 2(1 - h(\frac{K+1}{2})) = 2h(\frac{K+1}{2}) - 1$. And so by Lemma D.1, the worst case probabilities $(p^*, p'^*) = \arg\max_{p,p'} f(p, p'; \gamma)$ satisfy one of the two following: $p = e^\epsilon p'$, $\forall p \in [0, \frac{1}{1+e^{-\epsilon}}], p' \in [0, \frac{1}{1+e^\epsilon}]$, or $1 - p' = e^\epsilon(1-p)$, $\forall p \in [\frac{1}{1+e^{-\epsilon}}, 1], p' \in [\frac{1}{1+e^\epsilon}, 1]$.

On the boundary $p = e^\epsilon p'$, where $p' \in [0, \frac{1}{1+e^\epsilon}]$, the privacy cost objective can be re-written as

$$f(p, p') = f(p') = \sum_{l=0}^{\frac{K-1}{2}} (e^{m\epsilon} \binom{K}{l} p'^l (1-p')^{K-l} - \binom{K}{l} (e^\epsilon p')^l (1 - e^\epsilon p')^{K-l}) \cdot \gamma(l) \tag{112}$$
$$+ \sum_{l=\frac{K+1}{2}}^{K} (\binom{K}{l} (e^\epsilon p')^l (1 - e^\epsilon p')^{K-l} - e^{m\epsilon} \binom{K}{l} p'^l (1-p')^{K-l}) \cdot \gamma(l)$$

as in Eq. 92, and as in Eq. 95, the gradient w.r.t. p' is

$$\frac{\nabla_{p'} f(p')}{K} = e^{m\epsilon} \sum_{l=0}^{\frac{K-1}{2}-1} \binom{K-1}{l} p'^l (1-p')^{K-l-1} \Big(\gamma(l+1) - \gamma(l)\Big) - 2e^{m\epsilon} \binom{K-1}{\frac{K-1}{2}} p'^{\frac{K-1}{2}} (1-p')^{\frac{K-1}{2}} \gamma(\frac{K-1}{2}) \tag{113}$$
$$+ e^{m\epsilon} \sum_{l=\frac{K+1}{2}}^{K-1} \binom{K-1}{l} p'^l (1-p')^{K-l-1} \Big(\gamma(l) - \gamma(l+1)\Big)$$
$$+ e^\epsilon \sum_{l=0}^{\frac{K-1}{2}-1} \binom{K-1}{l} (e^\epsilon p')^l (1 - e^\epsilon p')^{K-l-1} \Big(\gamma(l) - \gamma(l+1)\Big) + 2e^\epsilon \binom{K-1}{\frac{K-1}{2}} (e^\epsilon p')^{\frac{K-1}{2}} (1 - e^\epsilon p')^{\frac{K-1}{2}} \gamma(\frac{K-1}{2})$$
$$+ e^\epsilon \sum_{l=\frac{K+1}{2}}^{K-1} \binom{K-1}{l} (e^\epsilon p')^l (1 - e^\epsilon p')^{K-l-1} \Big(\gamma(l+1) - \gamma(l)\Big)$$

With this family of γ function,

1. When $l \leq \frac{K}{2}$, $\gamma(l) - \gamma(l+1) = (1 - 2h(l)) - (1 - 2h(l+1)) = 2h(l+1) - 2h(l)$
2. When $l \geq \frac{K}{2}$, $\gamma(l+1) - \gamma(l) = (2h(l+1) - 1) - (2h(l) - 1) = 2h(l+1) - 2h(l)$
3. Since $\gamma(\frac{K-1}{2}) = \gamma(\frac{K+1}{2})$,
$$2\gamma(\frac{K-1}{2}) = \Big(\gamma(\frac{K-1}{2}) + \gamma(\frac{K+1}{2})\Big) \tag{114}$$
$$= \Big(1 - 2h(\frac{K-1}{2}) + 2h(\frac{K+1}{2}) - 1\Big) \tag{115}$$
$$= 2h(\frac{K+1}{2}) - 2h(\frac{K-1}{2}) \tag{116}$$

and so the gradient is equivalent to

$$\frac{\nabla_{p'} f(p')}{K} = -e^{m\epsilon} \sum_{l=0}^{K-1} \binom{K-1}{l} p'^l (1-p')^{K-l} \Big(2h(l+1) - 2h(l)\Big) \tag{117}$$
$$+ e^\epsilon \sum_{l=0}^{K-1} \binom{K-1}{l} (e^\epsilon p')^l (1 - e^\epsilon p')^{K-l-1} \Big(2h(l+1) - 2h(l)\Big)$$
$$= -2e^{m\epsilon} \mathbb{E}_X[h(X+1) - h(X)] + 2e^\epsilon \mathbb{E}_Y[h(Y+1) - h(Y)] \tag{118}$$

where $X \sim \text{Binom}(K-1, p')$ and $Y \sim \text{Binom}(K-1, e^\epsilon p')$. Hence,

$$\nabla_{p'} f(p') \leq 0 \Leftrightarrow e^\epsilon \mathbb{E}_Y[h(Y+1) - h(Y)] \leq e^{m\epsilon} \mathbb{E}_X[h(X+1) - h(X)] \tag{119}$$

On the boundary $1-p' = e^{\epsilon}(1-p)$, where $p \in [\frac{1}{1+e^{-\epsilon}}, 1]$. Let $q = 1-p$ and $q' = 1-p'$ for $q \in [0, \frac{1}{1+e^{\epsilon}}]$, the privacy cost objective can be re-written as

$$f(q) = \sum_{l=0}^{\frac{K-1}{2}} \left(e^{m\epsilon} \binom{K}{l} (1-e^{\epsilon}q)^l (e^{\epsilon}q)^{K-l} - \binom{K}{l} (1-q)^l q^{K-l} \right) \cdot \gamma(l) \tag{120}$$
$$+ \sum_{l=\frac{K+1}{2}}^{K} \left(\binom{K}{l} (1-q)^l q^{K-l} - e^{m\epsilon} \binom{K}{l} (1-e^{\epsilon}q)^l (e^{\epsilon}q)^{K-l} \right) \cdot \gamma(l)$$

as in Eq. 102, and as in Eq. 103, the gradient w.r.t. q is

$$\frac{\nabla_q f(q)}{K} = \sum_{l=0}^{\frac{K-1}{2}-1} e^{(m+1)\epsilon} \binom{K-1}{l} (1-e^{\epsilon}q)^l (e^{\epsilon}q)^{K-l-1} \cdot \Big(\gamma(l) - \gamma(l+1) \Big) \tag{121}$$
$$+ \sum_{l=\frac{K+1}{2}}^{K-1} \binom{K-1}{l} (1-e^{\epsilon}q)^l (e^{\epsilon}q)^{K-l-1} \cdot \Big(\gamma(l+1) - \gamma(l) \Big)$$
$$+ 2e^{(m+1)\epsilon} \binom{K-1}{\frac{K-1}{2}} (1-e^{\epsilon}q)^{\frac{K-1}{2}} (e^{\epsilon}q)^{\frac{K-1}{2}} \cdot \gamma(\frac{K-1}{2})$$
$$+ \sum_{l=0}^{\frac{K-1}{2}-1} \binom{K-1}{l} (1-q)^l q^{K-l-1} \cdot \Big(\gamma(l+1) - \gamma(l) \Big)$$
$$+ \sum_{l=\frac{K+1}{2}}^{K-1} (1-q)^l q^{K-l-1} \cdot \Big(\gamma(l) - \gamma(l+1) \Big)$$
$$- 2 \binom{K-1}{\frac{K-1}{2}} (1-q)^{\frac{K-1}{2}} q^{\frac{K-1}{2}} \cdot \gamma(\frac{K-1}{2})$$

With this family of γ function, the gradient above is equivalent to

$$\frac{\nabla_q f(q)}{K} = e^{(m+1)\epsilon} \sum_{l=0}^{K-1} \binom{K-1}{l} (1-e^{\epsilon}q)^l (e^{\epsilon}q)^{K-l-1} \cdot \Big(2h(l+1) - 2h(l) \Big) \tag{122}$$
$$- \sum_{l=0}^{K} \binom{K-1}{l} (1-q)^l q^{K-l-1} \cdot \Big(2h(l+1) - 2h(l) \Big)$$
$$= 2e^{(m+1)\epsilon} \mathbb{E}_{\hat{X}}[h(\hat{X}+1) - h(\hat{X})] - 2\mathbb{E}_{\hat{Y}}[h(\hat{Y}+1) - h(\hat{Y})] \tag{123}$$

where $\hat{X} \sim \text{Binom}(K-1, 1-e^{\epsilon}(1-p))$ and $\hat{Y} \sim \text{Binom}(K-1, p)$.

$$\nabla_q f(q) \geq 0 \Leftrightarrow e^{(m+1)\epsilon} \mathbb{E}_{\hat{X}}[h(\hat{X}+1) - h(\hat{X})] \geq \mathbb{E}_{\hat{Y}}[h(\hat{Y}+1) - h(\hat{Y})] \tag{124}$$

Recall $q \in [0, \frac{1}{1+e^{\epsilon}}]$. The above implies the maximum on this boundary is at point $q = \frac{1}{1+e^{\epsilon}}$ — that is, at point $(p, p') = (\frac{1}{1+e^{-\epsilon}}, \frac{1}{1+e^{\epsilon}})$. Notice this is the minimum on the first boundary $p = e^{\epsilon}p'$. Hence, the global maximum of the cost objective is at $(p, p') = (0, 0)$, and since the maximum $f(0,0) = e^{m\epsilon} - 1 \leq e^{m\epsilon} - 1$, this further implies the algorithm is $m\epsilon$ differentially private. □

Lemma D.6 (Binomial Expectation Recurrence Relationship (Theorem 2.1 of Zhang et al. (2019))). *Let $X_{(K-1)} \sim Binom(K-1, p)$ and $X_{(K)} \sim Binom(K, p)$. Let $g(x)$ be a function with $-\infty < \mathbb{E}[g(X_{(K-1)})] < \infty$ and $-\infty < g(-1) < \infty$, then*

$$Kp\mathbb{E}_{X_{(K-1)}}[g(X_{(K-1)})] = \mathbb{E}_{X_{(K)}}[X_{(K)} g(X_{(K)} - 1)] \tag{125}$$

Lemma D.7. *Given $i, m, K \in \mathbb{Z}$, $K \geq 1$, $0 \leq i \leq m \leq K$, let $X_{(K)} \sim Binom(K, p)$ for some $p \in [0, 1]$, there is*

$$\frac{1}{\binom{K}{m}} \mathbb{E}_{X_{(K)}} \left[\binom{X}{i} \binom{K-X}{m-i} \right] = \binom{m}{i} p^i (1-p)^{m-i} \tag{126}$$

Proof of Lemma D.7. We show the above statement by induction on K and m.

Base Case: $K = 1$.

1. If $m = 0$, then $i = 0$. $\frac{1}{\binom{1}{0}}\mathbb{E}_{X_{(1)}}[\binom{X}{0}\binom{1-X}{0}] = \mathbb{E}_{X_{(1)}}[1] = 1$, and $\binom{0}{0}p^0(1-p)^0 = 1$.
2. If $m = 1$,
   (a) $i = 0$, $\frac{1}{\binom{1}{1}}\mathbb{E}_{X_{(1)}}[\binom{X}{0}\binom{1-X}{1}] = \mathbb{E}_{X_{(1)}}[1-X] = 1-p$, and $\binom{1}{0}p^0(1-p)^1 = 1-p$
   (b) $i = 1$, $\frac{1}{\binom{1}{1}}\mathbb{E}_{X_{(1)}}[\binom{X}{1}\binom{1-X}{0}] = \mathbb{E}_{X_{(1)}}[X] = p$, and $\binom{1}{1}p^1(1-p)^0 = p$.

Hence, the statement holds for the base case.

Induction Hypothesis: Suppose the statement holds for some $K \geq 1$ and $0 \leq i \leq m \leq K$. Consider $1 \leq i \leq m \leq K+1$,

$$\frac{1}{\binom{K+1}{m}}\mathbb{E}_{X_{(K+1)}}\left[\binom{X}{i}\binom{K+1-X}{m-i}\right] \tag{127}$$

$$= \frac{1}{\binom{K+1}{m}}\mathbb{E}_{X_{(K+1)}}[\frac{X!}{i!(X-i)!}\frac{(K+1-X)!}{(m-i)!(K+1-X-(m-i))!}] \tag{128}$$

$$= \frac{1}{\binom{K+1}{m}i!(m-i)!}\mathbb{E}_{X_{(K+1)}}[X\frac{(X-1)!}{((X-1)-(i-1))!}\frac{(K-(X-1))!}{(K-(X-1)-((m-1)-(i-1)))!}] \tag{129}$$

$$= \frac{1}{\binom{K+1}{m}i!(m-i)!}\mathbb{E}_{X_{(K)}}[\frac{X!}{(X-(i-1))!}\frac{(K-X)!}{(K-X-((m-1)-(i-1)))!}] \tag{130}$$

(By Lemma D.6)

$$= \frac{(i-1)!(m-i)!}{\binom{K+1}{m}i!(m-i)!}\mathbb{E}_{X_{(K)}}[\binom{X}{i-1}\binom{K-X}{(m-1)-(i-1)}] \tag{131}$$

$$= \frac{(i-1)!}{\binom{K+1}{m}i!}(K+1)p\binom{K}{m-1}\binom{m-1}{i-1}p^{i-1}(1-p)^{m-i} \tag{132}$$

(By Induction Hypothesis)

$$= \frac{m!(K+1-m)!}{(K+1)!i}\frac{K!}{(m-1)!(K-m+1)!}\frac{(m-1)!}{(i-1)!(m-i)!}(K+1)p^i(1-p)^{m-i} \tag{133}$$

$$= \frac{m!}{i!(m-i)!}p^i(1-p)^{m-i} = \binom{m}{i}p^i(1-p)^{m-i} \tag{134}$$

Now we consider the edge cases when $0 = i \leq m$.

If $i = 0$ and $m = 0$,

$$\frac{1}{\binom{K+1}{0}}\mathbb{E}_{X_{(K+1)}}[\binom{X}{0}\binom{K+1-X}{0}] = 1 \cdot \mathbb{E}_{X_{(K+1)}}[1] = 1 = \binom{0}{0}p^0(1-p)^0 \tag{135}$$

If $i = 0$ and $m > 0$,

$$\frac{1}{\binom{K+1}{m}}\mathbb{E}_{X_{(K+1)}}[\binom{K+1-X}{m}] \tag{136}$$

$$= \frac{1}{\binom{K+1}{m}}\sum_{x=0}^{K+1}\binom{K+1-x}{m}\binom{K+1}{x}p^x(1-p)^{K+1-x} \tag{137}$$

$$= \frac{1}{\binom{K+1}{m}}\sum_{x=0}^{K+1}\binom{K+1-x}{m}\left(\binom{K}{x} + \binom{K}{x-1}\mathbb{I}\{x \geq 1\}\right)p^x(1-p)^{K+1-x} \tag{138}$$

$$= \frac{1}{\binom{K+1}{m}} \sum_{x=0}^{K} \binom{K+1-x}{m} \binom{K}{x} p^x (1-p)^{K+1-x} + \frac{1}{\binom{K+1}{m}} \sum_{x=1}^{K+1} \binom{K+1-x}{m} \binom{K}{x-1} p^x (1-p)^{K+1-x} \tag{139}$$

(Since when $x = K+1$ and $m > 0$, $\binom{K+1-x}{m} = 0$)

$$= \frac{1}{\binom{K+1}{m}} \Big(\sum_{x=0}^{K} \binom{K-x}{m} \binom{K}{x} p^x (1-p)^{K+1-x} + \sum_{x=0}^{K} \binom{K-x}{m-1} \binom{K}{x} p^x (1-p)^{K+1-x} \Big) \tag{140}$$

$$+ \frac{1}{\binom{K+1}{m}} \sum_{x=0}^{K} \binom{K-x}{m} \binom{K}{x} p^{x+1} (1-p)^{K-x}$$

(Since $\binom{K+1-x}{m} = \binom{K-x}{m} + \binom{K-x}{m-1}$)

$$= \frac{1}{\binom{K+1}{m}} \Big((1-p)\mathbb{E}_{X_{(K)}}[\binom{K-X}{m}] + (1-p)\mathbb{E}_{X_{(k)}}[\binom{K-X}{m-1}] \Big) + \frac{1}{\binom{K+1}{m}} p \mathbb{E}_{X_{(K)}}[\binom{K-X}{m}] \tag{141}$$

$$= \frac{1}{\binom{K+1}{m}} \Big(\mathbb{E}_{X_{(K)}}[\binom{K-X}{m}] + (1-p)\mathbb{E}_{X_{(K)}}[\binom{K-X}{m-1}] \Big) \tag{142}$$

$$= \frac{1}{\binom{K+1}{m}} \Big(\binom{K}{m} (1-p)^m + (1-p) \binom{K}{m-1} (1-p)^{m-1} \Big) \tag{143}$$

(By Induction Hypothesis) (144)

$$= \frac{1}{\binom{K+1}{m}} \binom{K+1}{m} (1-p)^m \tag{145}$$

$$= (1-p)^m \tag{146}$$

□

Based on Lemma D.6 and Lemma D.7 and using the sufficient conditions in Lemma D.5, we are now ready to present the privacy amplification results under a small privacy allowance m as follows.

Lemma D.8 (Privacy amplification under small privacy allowance $m \leq \frac{K-1}{2}$)**.** *Consider using DaRRM to solve Problem 1.1 with $p_i = p$, $p_i' = p'$, $\forall i \in [K]$ and $\delta = \Delta = 0$. If the privacy allowance is $m \leq \frac{K-1}{2}$, one can set $\gamma(l) = \begin{cases} 1 - 2h(l) & \forall l \in \{0, 1, \ldots, \frac{K-1}{2}\} \\ 2h(l) - 1 & \forall l \in \{\frac{K+1}{2}, \ldots, K\} \end{cases}$, where $h : \{0, 1, \ldots, K\} \to [0, 1]$ and $h(l) = \sum_{i=m}^{2m-1} \frac{\binom{l}{i}\binom{K-l}{2m-1-i}}{\binom{K}{2m-1}}$, and Algorithm DaRRM$_\gamma$ is $m\epsilon$-differentially private.*

Proof of Lemma D.8. Let $X_{(K-1)} \sim \text{Binom}(K-1, p)$ and $Y_{(K-1)} \sim \text{Binom}(K-1, e^\epsilon p)$.

$$\mathbb{E}_{X_{(K-1)}}[h(X+1)] = \frac{1}{\binom{K}{2m-1}} \sum_{i=m}^{2m-1} \mathbb{E}_{X_{(K-1)}}[\binom{X+1}{i} \binom{K-X-1}{2m-1-i}] \tag{147}$$

$$= \frac{1}{\binom{K}{2m-1}} \sum_{i=m}^{2m-1} \mathbb{E}_{X_{(K-1)}}[\binom{X}{i} \binom{K-X-1}{2m-1-i} + \binom{X}{i-1} \binom{K-X-1}{2m-1-i}] \tag{148}$$

(Since $\binom{X+1}{i} = \binom{X}{i} + \binom{X}{i-1} \mathbb{I}\{i \geq 1\}$)

$$= \frac{1}{\binom{K}{2m-1}} \sum_{i=m}^{2m-1} \left(\mathbb{E}_{X_{(K-1)}}[\binom{X}{i}\binom{K-1-X}{2m-1-i}] + \mathbb{E}_{X_{(K-1)}}[\binom{X}{i-1}\binom{K-1-X}{(2m-2)-(i-1)}] \right) \tag{149}$$

$$= \frac{1}{\binom{K}{2m-1}} \sum_{i=m}^{2m-1} \Big(\binom{K-1}{2m-1}\binom{2m-1}{i} p^i (1-p)^{2m-1-i} \tag{150}$$

$$+ \binom{K-1}{2m-2}\binom{2m-2}{i-1} p^{i-1}(1-p)^{2m-1-i} \Big)$$

(By Lemma D.7)

$$\mathbb{E}_{X_{(K-1)}}[h(X)] = \frac{1}{\binom{K}{2m-1}} \sum_{i=m}^{2m-1} \mathbb{E}_{X_{(K-1)}}[\binom{X}{i}\binom{K-X}{2m-1-i}] \tag{151}$$

(Since $\binom{K-X}{2m-1-i} = \binom{K-1-X}{2m-1-i} + \binom{K-1-X}{2m-2-i}$)

$$= \frac{1}{\binom{K}{2m-1}} \sum_{i=m}^{2m-1} \left(\mathbb{E}_{X_{(K-1)}}[\binom{X}{i}\binom{K-1-X}{2m-1-i}] + \mathbb{E}_{X_{(K-1)}}[\binom{X}{i}\binom{K-1-X}{2m-2-i}] \mathbb{I}\{i \le 2m-2\} \right) \tag{152}$$

$$= \frac{1}{\binom{K}{2m-1}} \sum_{i=m}^{2m-1} \Big(\binom{K-1}{2m-1}\binom{2m-1}{i} p^i (1-p)^{2m-1-i} \tag{153}$$

$$+ \binom{K-1}{2m-2}\binom{2m-2}{i} p^i (1-p)^{2m-2-i} \mathbb{I}\{i \le 2m-2\} \Big)$$

(By Lemma D.7)

Hence,

$$\mathbb{E}_{X_{(K-1)}}[h(X+1) - h(X)] \tag{154}$$

$$= \frac{1}{\binom{K}{2m-1}} \Big(\sum_{i=m}^{2m-1} \binom{K-1}{2m-2}\binom{2m-2}{i-1} p^{i-1}(1-p)^{2m-1-i} - \sum_{i=m}^{2m-2} \binom{K-1}{2m-2}\binom{2m-2}{i} p^i (1-p)^{2m-2-i} \Big) \tag{155}$$

$$= \frac{1}{\binom{K}{2m-1}} \Big(\sum_{i=m-1}^{2m-2} \binom{K-1}{2m-2}\binom{2m-2}{i} p^i (1-p)^{2m-2-i} - \sum_{i=m}^{2m-2} \binom{K-1}{2m-2}\binom{2m-2}{i} p^i (1-p)^{2m-2-i} \Big) \tag{156}$$

$$= \frac{2m-1}{K}\binom{2m-2}{m-1} p^{m-1}(1-p)^{m-1} \tag{157}$$

Similarly,

$$\mathbb{E}_{Y_{(K-1)}}[h(Y+1) - h(Y)] = \frac{2m-1}{K}\binom{2m-2}{m-1} (e^\epsilon p)^{m-1}(1-e^\epsilon p)^{m-1} \tag{158}$$

Since $p(1-p) \ge e^{-\epsilon} e^\epsilon p(1-e^\epsilon p)$ for $p \in [0, \frac{1}{1+e^\epsilon}]$,

$$e^{(m-1)\epsilon}\mathbb{E}_{X_{(K-1)}}[h(X+1) - h(X)] = \frac{2m-1}{K}\binom{2m-2}{m-1} e^{(m-1)\epsilon} p^{m-1}(1-p)^{m-1} \tag{159}$$

$$\ge \frac{2m-1}{K}\binom{2m-2}{m-1} e^{(m-1)\epsilon} (e^{-\epsilon} e^\epsilon p(1-e^\epsilon p))^{m-1} \tag{160}$$

$$= \frac{2m-1}{K}\binom{2m-2}{m-1} (e^\epsilon p)^{m-1}(1-e^\epsilon p)^{m-1} \tag{161}$$

$$= \mathbb{E}_{Y_{(K-1)}}[h(Y+1) - h(Y)] \tag{162}$$

and so

$$e^{m\epsilon}\mathbb{E}_{X_{(K-1)}}[h(X+1) - h(X)] \geq e^{\epsilon}\mathbb{E}_{Y_{(K-1)}}[h(Y+1) - h(Y)] \tag{163}$$

The above shows $\gamma(l) = \begin{cases} 1 - 2h(l) & l \in \{0, 1, \ldots, \frac{K-1}{2}\} \\ 2h(l) - 1 & l \in \{\frac{K+1}{2}, \ldots, K\} \end{cases}$, where $h = \sum_{i=m}^{2m-1} \frac{\binom{l}{i}\binom{K-1}{2m-1-i}}{\binom{K}{2m-1}}$, satisfies the first condition in Eq. 110 of Lemma D.5. To ensure the Algorithm is $m\epsilon$ differentially private, we next show this γ also satisfies the second condition in Eq. 111 of Lemma D.5.

Let $\hat{X}_{(K-1)} \sim \text{Binom}(K-1, 1-e^{\epsilon}(1-p))$ and $\hat{Y}_{(K-1)} \sim \text{Binom}(K-1, p)$. By Eq. 149, we know

$$\mathbb{E}_{\hat{X}_{(K-1)}}[h(X+1)] = \frac{1}{\binom{K}{2m-1}} \sum_{i=m}^{2m-1} \left(\mathbb{E}_{\hat{X}_{(K-1)}}[\binom{\hat{X}}{i}\binom{K-1-\hat{X}}{2m-1-i}] + \mathbb{E}_{\hat{X}_{(K-1)}}[\binom{\hat{X}}{i-1}\binom{K-1-\hat{X}}{(2m-2)-(i-1)}] \right) \tag{164}$$

$$= \frac{1}{\binom{K}{2m-1}} \sum_{i=m}^{2m-1} \left(\binom{K-1}{2m-1}\binom{2m-1}{i}(1-e^{\epsilon}(1-p))^{i}(e^{\epsilon}(1-p))^{2m-1-i} \right. \tag{165}$$

$$\left. + \binom{K-1}{2m-2}\binom{2m-2}{i-1}(1-e^{\epsilon}(1-p))^{i-1}(e^{\epsilon}(1-p))^{2m-1-i} \right)$$

By Lemma D.7

and by Eq. 152, we know

$$\mathbb{E}_{\hat{X}_{(K-1)}}[h(\hat{X})] = \frac{1}{\binom{K}{2m-1}} \sum_{i=m}^{2m-1} \left(\mathbb{E}_{\hat{X}_{(K-1)}}[\binom{\hat{X}}{i}\binom{K-1-\hat{X}}{2m-1-i}] + \mathbb{E}_{\hat{X}_{(K-1)}}[\binom{\hat{X}}{i}\binom{K-1-\hat{X}}{2m-2-i}]\mathbb{I}\{i \leq 2m-2\} \right) \tag{166}$$

$$= \frac{1}{\binom{K}{2m-1}} \sum_{i=m}^{2m-1} \left(\binom{K-1}{2m-1}\binom{2m-1}{i}(1-e^{\epsilon}(1-p))^{i}(e^{\epsilon}(1-p))^{2m-1-i} \right. \tag{167}$$

$$\left. + \binom{K-1}{2m-2}\binom{2m-2}{i}(1-e^{\epsilon}(1-p))^{i}(e^{\epsilon}(1-p))^{2m-2-i}\mathbb{I}\{i \leq 2m-2\} \right)$$

By Lemma D.7

Hence,

$$\mathbb{E}_{\hat{X}_{(K-1)}}[h(\hat{X}+1) - h(\hat{X})] \tag{168}$$

$$= \frac{1}{\binom{K}{2m-1}} \left(\sum_{i=m}^{2m-1} \binom{K-1}{2m-2}\binom{2m-2}{i-1}(1-e^{\epsilon}(1-p))^{i-1}(e^{\epsilon}(1-p))^{2m-1-i} \right. \tag{169}$$

$$\left. - \sum_{i=m}^{2m-2} \binom{K-1}{2m-2}\binom{2m-2}{i}(1-e^{\epsilon}(1-p))^{i}(e^{\epsilon}(1-p))^{2m-2-i} \right)$$

$$= \frac{1}{\binom{K}{2m-1}} \left(\sum_{i=m-1}^{2m-2} \binom{K-1}{2m-2}\binom{2m-2}{i}(1-e^{\epsilon}(1-p))^{i}(e^{\epsilon}(1-p))^{2m-2-i} \right. \tag{170}$$

$$\left. - \sum_{i=m}^{2m-2} \binom{K-1}{2m-2}\binom{2m-2}{i}(1-e^{\epsilon}(1-p))^{i}(e^{\epsilon}(1-p))^{2m-2-i} \right)$$

$$= \frac{2m-1}{K}\binom{2m-2}{m-1}(1-e^{\epsilon}(1-p))^{m-1}(e^{\epsilon}(1-p))^{m-1} \tag{171}$$

Similarly,

$$\mathbb{E}_{\hat{Y}_{(K-1)}}[h(\hat{Y}+1) - h(\hat{Y})] = \frac{2m-1}{K}\binom{2m-2}{m-1}p^{m-1}(1-p)^{m-1} \tag{172}$$

Hence,

$$e^{(m+1)\epsilon}\mathbb{E}_{\hat{X}_{(K-1)}}[h(\hat{X}+1)-h(\hat{X})] = e^{(m+1)\epsilon}\frac{2m-1}{K}\binom{2m-2}{m-1}(1-e^{\epsilon}(1-p))^{m-1}(e^{\epsilon}(1-p))^{m-1} \tag{173}$$

$$\geq \frac{2m-1}{K}\binom{2m-2}{m-1}(1-e^{\epsilon}(1-p))^{m-1}e^{(m-1)\epsilon}(1-p)^{m-1} \tag{174}$$

$$= \frac{2m-1}{K}\binom{2m-2}{m-1}(e^{\epsilon}-e^{2\epsilon}(1-p))^{m-1}(1-p)^{m-1} \tag{175}$$

Note that

$$e^{\epsilon}-e^{2\epsilon}(1-p) = e^{\epsilon}-e^{2\epsilon}+e^{2\epsilon}p \geq p \tag{176}$$

$$\Leftrightarrow (e^{\epsilon}+1)(e^{\epsilon}-1)p \geq e^{\epsilon}(e^{\epsilon}-1) \tag{177}$$

$$\Leftrightarrow p \geq \frac{e^{\epsilon}}{e^{\epsilon}+1} = \frac{1}{1+e^{-\epsilon}} \tag{178}$$

and the second condition in Eq. 111 of Lemma D.5 is on $p \in [\frac{1}{1+e^{-\epsilon}}, 1]$.

Therefore, following Eq. 175,

$$e^{(m+1)\epsilon}\mathbb{E}_{\hat{X}_{(K-1)}}[h(\hat{X}+1)-h(\hat{X})] \geq \frac{2m-1}{K}\binom{2m-2}{m-1}p^{m-1}(1-p)^{m-1} \tag{179}$$

$$= \mathbb{E}_{\hat{Y}_{(K-1)}}[h(\hat{Y}+1)-h(\hat{Y})] \tag{180}$$

which means the second condition in Eq. 111 of Lemma D.5 is also satisfied.

Therefore, by Lemma D.5, DaRRM$_\gamma$ with this specific choice of γ is $m\epsilon$-differentially private. □

Now, Theorem 4.1 follows from combining Lemma D.4 and Lemma D.8.

E Details of Optimizing γ in DaRRM

E.1 Deriving the Optimization Objective

For γ that is symmetric around $\frac{K}{2}$, we can write the objective as

$$\mathbb{E}_{p_1,p_2,\ldots,p_K\sim\mathcal{T}}[\mathcal{E}(\text{DaRRM}_\gamma)] \tag{181}$$
$$= \mathbb{E}_{p_1,p_2,\ldots,p_K\sim\mathcal{T}}[\mathcal{D}_{TV}(\text{DaRRM}_\gamma(\mathcal{S}) \parallel f(\mathcal{S}))] \tag{182}$$
$$= \mathbb{E}_{p_1,p_2,\ldots,p_K\sim\mathcal{T}}[|\Pr[\text{DaRRM}_\gamma(\mathcal{S})=1] - \Pr[f(\mathcal{S})=1]|] \tag{183}$$
$$= \mathbb{E}_{p_1,p_2,\ldots,p_K\sim\mathcal{T}}\left[\left|\sum_{l=\frac{K+1}{2}}^{K}\left(\alpha_l\cdot(\gamma(l)+\frac{1}{2}(1-\gamma(l)))-\alpha_l\right)+\sum_{l=0}^{\frac{K-1}{2}}\alpha_l\cdot\frac{1}{2}(1-\gamma(l))\right|\right] \tag{184}$$
$$= \mathbb{E}_{p_1,p_2,\ldots,p_K\sim\mathcal{T}}\left[\left|\sum_{l=0}^{\frac{K-1}{2}}\alpha_l(\frac{1}{2}\gamma(l)-\frac{1}{2})+\sum_{l=\frac{K+1}{2}}^{K}\alpha_l(\frac{1}{2}-\frac{1}{2}\gamma(l))\right|\right] \tag{185}$$

The above follows by conditioning on $\mathcal{L}=\{0,1,\ldots,K\}$, i.e. the sum of observed outcomes in $\mathcal{S}$

$$= \mathbb{E}_{p_1,p_2,\ldots,p_K\sim\mathcal{T}}\left[\left|\frac{1}{2}\sum_{l=\frac{K+1}{2}}^{K}(\alpha_l-\alpha_{K-l})\,(1-\gamma(l))\right|\right] \tag{186}$$

The above follows by symmetry of γ

Furthermore, notice the objective is symmetric around 0, and can be written as

$$\mathbb{E}_{p_1,p_2,\ldots,p_K\sim\mathcal{T}}\left[\frac{1}{2}\sum_{l=\frac{K+1}{2}}^{K}(\alpha_l-\alpha_{K-l})\,(1-\gamma(l))\right] \tag{187}$$
$$= \frac{1}{2}\mathbb{E}_{p_1,p_2,\ldots,p_K\sim\mathcal{T}}\left[\sum_{l=\frac{K+1}{2}}^{K}\left((\alpha_l-\alpha_{K-l})-(\alpha_l-\alpha_{K-l})\gamma(l)\right)\right] \tag{188}$$
$$= \frac{1}{2}\mathbb{E}_{p_1,p_2,\ldots,p_K\sim\mathcal{T}}\left[\sum_{l=\frac{K+1}{2}}^{K}(\alpha_l-\alpha_{K-l})\right]-\frac{1}{2}\mathbb{E}_{p_1,p_2,\ldots,p_K\sim\mathcal{T}}\left[\sum_{l=\frac{K+1}{2}}^{K}(\alpha_l-\alpha_{K-l})\gamma(l)\right] \tag{189}$$

and this is the same as optimizing

$$-\frac{1}{2}\mathbb{E}_{p_1,p_2,\ldots,p_K\sim\mathcal{T}}\left[\sum_{l=\frac{K+1}{2}}^{K}(\alpha_l-\alpha_{K-l})\gamma(l)\right]=-\frac{1}{2}\sum_{l=\frac{K+1}{2}}^{K}\mathbb{E}_{p_1,p_2,\ldots,p_K\sim\mathcal{T}}\left[(\alpha_l-\alpha_{K-l})\right]\gamma(l) \tag{190}$$

which is linear in γ.

E.2 Practical Approximation of the Objective

Since the optimization objective in Eq. 190 requires taking an expectation over $p_1,\ldots,p_K$, and this invovles integrating over K variables, which can be slow in practice, we propose the following approximation to efficiently compute the objective. We start with a simple idea to compute the objective, by sampling p_i's from $[0,1]$ and take an empirical average of the objective value over all subsampled sets of $p_1,\ldots,p_K$ as the approximation of the expectation in Section E.2.1. However, we found this approach is less numerically stable. We then propose the second approach to approximate the objective in Section E.2.2, which approximates the integration over p_i's instead of directly approximating the objective value. We use the second approximation approach in our experiments and empirically demonstrates its effectiveness. Note approximation the optimization objective has no affect on the privacy guarantee.

E.2.1 APPROXIMATION VIA DIRECT SAMPLING OF p_i'S

We start with a straightforward way of approximating the objective:

1. Step 1: Sample $p_1, p_2, \ldots, p_K \sim \mathcal{T}$
2. Step 2: Compute the sampled objective value $g = -\frac{1}{2}\sum_{l=\frac{K+1}{2}}^{K}(\alpha_l - \alpha_{K-l})\gamma(l))$ based on the sampled p_i's.
3. Repeat Step 1 and Step 2 for $T = 10000$ times. Let g_t denotes the objective value in t-th trial. Use $\frac{1}{T}\sum_{t=1}^{T} g_t$ as an unbiased estimation of the true objective.

However, we found this approximation is less numerically stable in the experiments and so we propose and adpot the second approach as follows.

E.2.2 APPROXIMATING THE INTEGRATION OVER p_i'S

Consider the following surrogate objective:

$$-\frac{1}{2}\sum_{l=\frac{K+1}{2}}^{K}\int_{0.5}^{1}\int_{0.5}^{1}\cdots\int_{0.5}^{1}(\alpha_l - \alpha_{K-l})dp_1dp_2\ldots dp_K \cdot \gamma(l) \tag{191}$$

where we approximate the integration instead of directly approximating the objective value. The approximation of the integration is based on the rectangular rule and that the Poison Binomial (PB) distribution is invariant to the order of its probability parameters.

First, we discretize the integration over p_i's: pick $\tau = 50$ points representing probabilities between $[0.5, 1)$ with equal distance θ. Denote this set of points as $\mathcal{W}$. We pick only $\tau = 50$ samples to ensure the distance between each sample, i.e., θ, is not too small; or this can cause numerical instability. For each $l \in \{\frac{K+1}{2}, \frac{K+1}{2}+1, \ldots, K\}$, we want to compute an approximated coefficient for $\gamma(l)$ as follows:

$$\int_{0.5}^{1}\int_{0.5}^{1}\cdots\int_{0.5}^{1}(\alpha_l - \alpha_{K-l})dp_1dp_2\ldots dp_K \approx \sum_{p_1\in\mathcal{W}}\sum_{p_2\in\mathcal{W}}\cdots\sum_{p_K\in\mathcal{W}}(\alpha_l - \alpha_{K-l}) \tag{192}$$

which approximates integration over a K-dimensional grid $\mathcal{W}^K$.

The idea is then to sample points from this K-dimensional grid $\mathcal{W}^K$ and compute an empirical mean of the integration based on the sample probabilities for $p_1, \ldots, p_K$ from $\mathcal{W}^K$ as the approximation of the integration in the objective.

Let $(s_1, s_2, \ldots, s_K)$ be randomly sampled probability values from $\mathcal{W}^K$ and we want to compute $(\alpha_l - \alpha_{K-l})$ for all l based on $(p_1, \ldots, p_K) = (s_1, \ldots, s_K)$. To apply the rectangular rule, since the grid of probabilities is K-dimensional, the weight of $(\alpha_l - \alpha_{K-l})$ in the approximate integration is θ^K. Furthermore, observe that α_l is the pmf at l from a Poison Binomial (PB) distribution in our case, and $\text{PB}(p_1, \ldots, p_K) \sim \text{PB}(\pi(p_1, \ldots, p_K))$, where π denotes a permutation of $p_1, \ldots, p_K$ and $\sim$ denotes "the same distribution". Hence, with a single probability sample $(s_1, \ldots, s_K)$, we can indeed compute $\alpha_l - \alpha_{K-l}$ for each l at $K!$ points from the grid $\mathcal{W}^K$, since they all have the same value. Therefore, we should set the weight of $\alpha_l - \alpha_{K-l}$ in the approximate integration as $w = \theta^K \cdot K!$. Furthermore, since the order of $(p_1, \ldots, p_K)$ does not affect the objective value, there is a total of (τ choose K with replacement) $= \binom{\tau+K-1}{K} := P$ different points in the grid $\mathcal{W}^K$.

In summary, our approximation of the integration proceeds as follows: let $w = \theta^K \cdot K!$ and $P = \binom{\tau+K-1}{K}$.

1. Step 1: Generate a set $\mathcal{W}$ with 50 values of equal distance between 0.5 and 1.
2. Step 2: Randomly sample $(s_1, s_2, \ldots, s_K) \sim \mathcal{W}^K$. Compute $w \cdot (\alpha_l - \alpha_{K-l})$ based on $(p_1, p_2, \ldots, p_K) = (s_1, s_2, \ldots, s_K)$.
3. Step 3: repeat Step 2 for $N = 10000$ times.
4. Step 4: Let $g_t = \sum_{l=\frac{K+1}{2}}^{K} w \cdot (\alpha_l - \alpha_{K-l})$ denotes the approximate integration value in t-th trial.
   Form an unbiased estimation of the integration as $\frac{P}{N}\sum_{t=1}^{N} g_t$.

E.3 Reducing # Constraints From ∞ to A Polynomial Set

Lemma 5.1. *Consider using DaRRM to solve Problem 1.1. Given an arbitrary γ, let the global worst case probabilities be $(p_1^*, \ldots, p_K^*, p_1'^*, \ldots, p_K'^*) = \arg\max_{\{(p_i, p_i')\}_{i=1}^K} f(p_1, \ldots, p_K, p_1', \ldots, p_K'; \gamma)$, where f is the privacy cost objective defined in Lemma 3.3. Each pair $(p_i^*, p_i'^*)$ satisfies $(p_i^*, p_i'^*) \in \{(0,0), (1,1), (0,\Delta), (\Delta, 0), (1-\Delta, 1), (1, 1-\Delta), (\frac{e^\epsilon+\Delta}{e^\epsilon+1}, \frac{1-\Delta}{e^\epsilon+1}), (\frac{1-\Delta}{e^\epsilon+1}, \frac{e^\epsilon+\Delta}{e^\epsilon+1})\}$, $\forall i \in [K]$. Furthermore, there exists a set $\mathcal{P}$ of size $O(K^7)$ such that $(p_1^*, \ldots, p_K^*, p_1'^*, \ldots, p_K'^*) = \arg\max_{\{(p_i, p_i')\}_{i=1}^K \in \mathcal{P}} f(p_1, \ldots, p_K, p_1', \ldots, p_K'; \gamma)$ if $\delta > 0$ and a set $\mathcal{P}$ of size $O(K^3)$ if $\delta = 0$.*

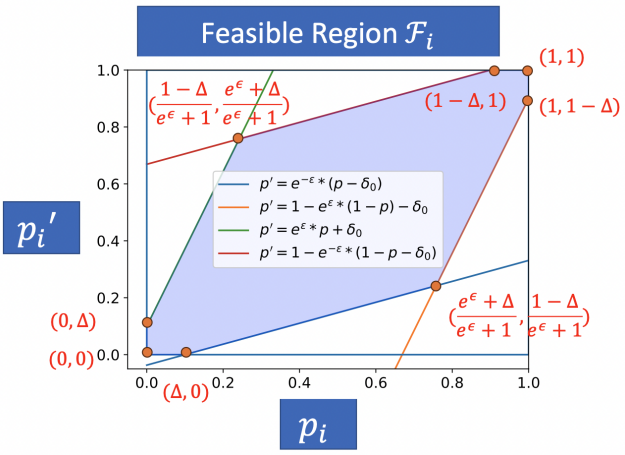


Figure 6: An illustration of the feasible region $\mathcal{F}_i$.

Proof. **Part I: Reducing # privacy constraints from ∞ to exponential.** Consider (p_i, p_i') for an arbitrary $i \in [K]$ and fixing $(p_j, p_j'), \forall j \neq i$. The privacy cost objective $f(p_1, \ldots, p_K, p_1', \ldots, p_K'; \gamma)$, as defined in Lemma 3.3, is then linear in (p_i, p_i'). To ensure DaRRM$_\gamma$ is differentially private with a target privacy loss $m\epsilon$, we need to consider the worst case probabilities $(p_i^*, p_i'^*) = \arg\max_{(p_i, p_i')} f$, given$(p_j, p_j'), \forall j \neq i$. Since mechanism M_i is (ϵ, Δ)-differentially private, by definition, the following constraints on (p_i, p_i') apply simultaneously,

$$p_i \leq e^\epsilon p_i' + \Delta, \quad p_i' \leq e^\epsilon p + \Delta$$
$$1 - p_i \leq e^\epsilon (1 - p_i') + \Delta, \quad 1 - p_i' \leq e^\epsilon (1 - p_i) + \Delta$$

This implies (p_i, p_i') lies in a feasible region $\mathcal{F}_i$ (see Figure 6). Notice the constraints on (p_i, p_i'), that is, the boundaries of $\mathcal{F}_i$, are linear in p_i and p_i', $\max_{(p_i, p_i')} f(p_1, \ldots, p_K, p_1', \ldots, p_K'; \gamma)$ is hence a Linear Programming (LP) problem in (p_i, p_i') for $i \in [K]$. Hence, the $(p_i^*, p_i'^*)$ has to be on one of the eight corners of $\mathcal{F}_i$ — that is $(p_i^*, p_i'^*) \in \{(0,0), (1,1), (0,\Delta), (\Delta,0), (1-\Delta, 1), (1, 1-\Delta), (\frac{e^\epsilon+\Delta}{e^\epsilon+1}, \frac{1-\Delta}{e^\epsilon+1}), (\frac{1-\Delta}{e^\epsilon+1}, \frac{e^\epsilon+\Delta}{e^\epsilon+1})\} := \mathcal{C}$. Therefore, the infinitely many privacy constraints are now reduced to only 8^K in optimizing for the best γ function in DaRRM.

Part II: Reducing # privacy constraints from exponential to polynomial. To further reduce the number of privacy constraints in optimization, recall by Lemma 3.3 we need γ such that

$$f(p_1, \ldots, p_K, p_1', \ldots, p_K'; \gamma) = \sum_{l=0}^{\frac{K-1}{2}} (e^{m\epsilon} \alpha_l' - \alpha_l) \cdot \gamma(l) + \sum_{l=\frac{K+1}{2}}^{K} (\alpha_l - e^{m\epsilon} \alpha_l') \cdot \gamma(l) \leq e^{m\epsilon} - 1 + 2\delta \tag{193}$$

where $\alpha_l = \Pr[\mathcal{L} = \sum_{i=1}^K M_i(\mathcal{D}) = l]$ and $\alpha_l' = \Pr[\mathcal{L}' = \sum_{i=1}^K M_i(\mathcal{D}') = l]$. Note $\mathcal{L}$ follows a Poisson Binomial (PB) distribution parameterized by $p_1, \ldots, p_K$, and $\mathcal{L}'$ follows a PB distribution

parameterized by $p'_1, \ldots, p'_K$. Observe that PB distribution[9] is invariant under the permutation of parameters. That is, $\text{PB}(p_1, \ldots, p_K)$ has the same distribution as $\text{PB}(\pi(p_1, \ldots, p_K))$, where π denotes permutation; and similarly, $\text{PB}(p'_1, \ldots, p'_K)$ has the same distribution as $\text{PB}(\pi(p'_1, \ldots, p'_K))$.

Consider a set $\mathcal{P}$ of privacy constraints as Eq. 193, where each constraint in $\mathcal{P}$ is constructed by setting $(p_1, p'_1), (p_2, p'_2), \ldots, (p_K, p'_K) = (v_1, v_2, \ldots, v_K)$, where $v_i \in \mathcal{C}, \forall i \in [K]$, such that constraints constructed by $(p_1, p'_1), (p_2, p'_2), \ldots, (p_K, p'_K) = \pi(v_1, v_2, \ldots, v_K)$ is not in $\mathcal{P}$ — that is, $\mathcal{P}$ has size (8 chooses K with replacement) = $\binom{K+8-1}{K} = O(K^7)$. Then, the global worst case probabilities $(p^*_1, \ldots, p^*_K, p'^*_1, \ldots, p'^*_K)$ must satisfy one of the constraints in $\mathcal{P}$, i.e. $(p^*_1, \ldots, p^*_K, p'^*_1, \ldots, p'^*_K) = \max_{\{(p_i, p'_i)\}_{i=1}^K \in \mathcal{P}} f(p_1, \ldots, p_K, p'_1, \ldots, p'_K; \gamma)$. This implies we only need $O(K^7)$ privacy constraints in optimizing for the best noise function γ in DaRRM.

Note when $\Delta = 0$, i.e., under pure differential privacy setting, the feasible region $\mathcal{F}_i$ has only 4 corners instead of 8, that is, $(p^*_i, p'^*_i) \in \mathcal{C} = \{(0,0), (1,1), (\frac{e^\epsilon}{e^\epsilon+1}, \frac{1}{e^\epsilon+1}), (\frac{1}{e^\epsilon+1}, \frac{e^\epsilon}{e^\epsilon+1})\}$. Hence, when $\Delta = 0$, $\mathcal{P}$ has size (4 choose K with replacement) = $\binom{K+4-1}{K} = O(K^3)$, implying we only need $O(K^3)$ privacy constrants in optimizing for the best noise function γ. □

F Full Experiment Results

F.1 Optimal γ in Simulations

F.1.1 Comparison Against Advanced Composition

Advanced composition indiates less privacy loss than simple composition when the number of compositions, m, is large, or when the failure probability δ is large. To enable meaningful comparison against advanced composition, we consider a larger K and a larger failure probability.

Consider $K = 35, \epsilon = 0.1, \Delta = 10^{-5}$. By advanced composition, if one outputs the majority of M subsampled mechanisms for some $M < K$, the majority output is $(\sqrt{2M\log(1/\delta')}\epsilon + M\epsilon(e^\epsilon - 1), M\Delta + \delta')$-differentially private for some $\delta' > 0$. We set this as the privacy guarantee of all majority ensembling algorithms. That is, if we want the majority output to be $(m\epsilon, \delta)$-differentially private, we set $m = \sqrt{2M\log(1/\delta')} + M(e^\epsilon - 1)$ and $\delta = M\Delta + \delta'$ accordingly. The parameters τ and λ for the constant γ in randomized response (see Lemma C.1) are set to be $\tau = \sqrt{2K\log(1/\delta')} + K(e^\epsilon - 1)$ and $\lambda = K\Delta + \delta'$.

In the experiments, we consider $M = \{10, 13, 15, 20\}$ and $\delta' = 0.1$.

All values of the parameters of the private ensembling algorithms are computed and listed in the table:

# Subsampled mechanisms	M	10	13	15	20
Privacy allowance	m	7.8378	9.1046	9.8888	11.7005
Parameter of constant γ	τ	16.3766	16.3767	16.3767	16.3767
Parameter of constant γ	λ	0.10035	0.10035	0.10035	0.10035
Overall privacy loss	$m\epsilon$	0.7837	0.9104	0.9889	1.1700
Overall failure probability	δ	0.10010	0.10013	0.10015	0.1002

Table 2: Parameters of all algorithms. Note all the private ensembling algorithms for comparison in the experiment is required to be $(m\epsilon, \delta)$-differentially private. $K = 35, \epsilon = 0.1, \delta = 10^{-5}$ and $\delta' = 0.1$.

[9] See, e.g. https://en.wikipedia.org/wiki/Poisson_binomial_distribution, for the pmf of Poisson Binomial (PB) distribution.

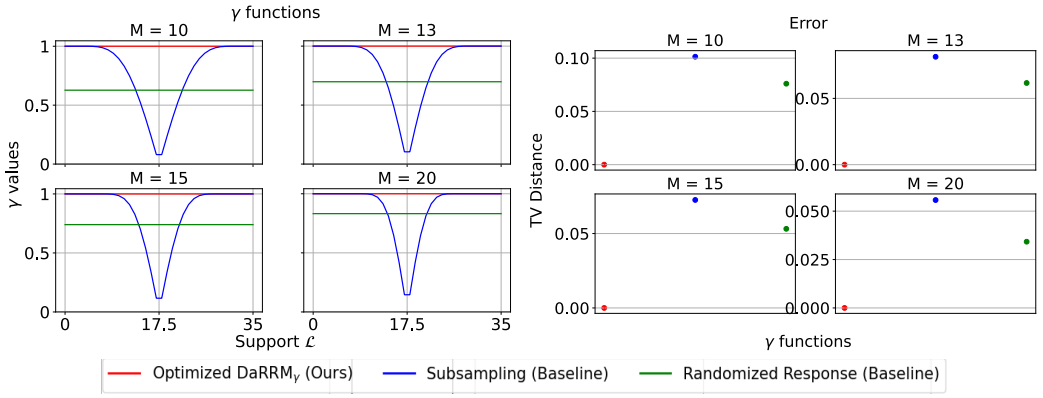


Figure 7: Plots of γ functions corresponding to optimized, subsampling (whose privacy guarantee is reasoned through advanced composition), the data-independent γ in randomized response and the error in TV distance of the majority ensembling output of DaRRM with different γ functions, when $K = 35$, the number of subsamples $M \in \{10, 20, 30, 40\}$, $\Delta = 10^{-5}$, and $\delta' = 0.1$.

F.1.2 Comparison Under Pure DP Settings

Consider the pure differential privacy setting, where $\Delta = \delta = 0$. Note in such pure differential privacy case, simple composition is tight. The subsampling baseline here outputs the majority of m out of K subsampeld mechansisms (without replacement). The majority output of different ensembling algorithms for comparison is required to be $m\epsilon$-differentially private. Furthermore, since the number of constraints in our optimization framework is $O(K^3)$ under pure differential privacy (see Lemma 5.1), we can optimize DaRRM$_\gamma$ for aggregating a larger number K of mechanisms. In this section, we present the simulation results for $K \in \{11, 101\}$ and compare the utility of three majority ensembling algorithms: optimized DaRRM, subsampling and randomizied response, under the same privacy loss.

Setting 1. $K = 11, m \in \{1, 3, 5, 7\}$.

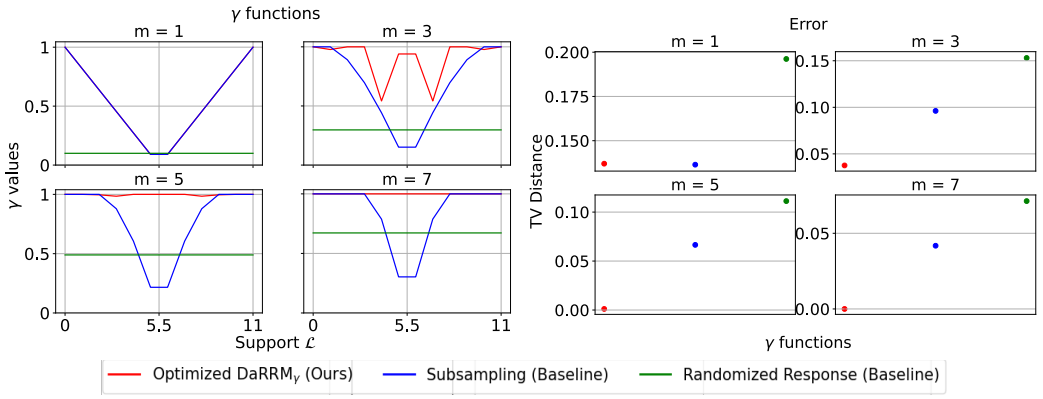


Figure 8: Plots of γ functions corresponding to optimized, subsampling, the data-independent γ in randomized response and the error in TV distance of the majority ensembling output of DaRRM with different γ functions, when $K = 11, m \in \{1, 3, 5, 7\}$, $\Delta = \delta = 0$.

Setting 2. $K = 101, m \in \{10, 20, 30, 40\}$.

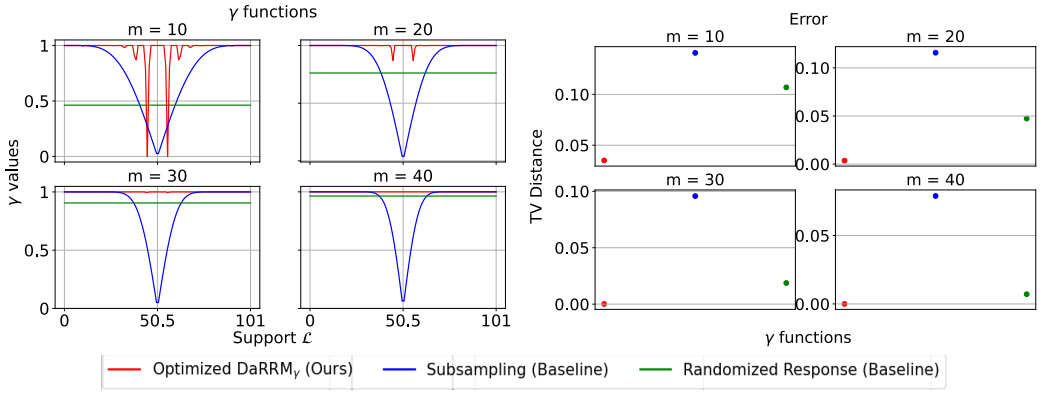


Figure 9: Plots of γ functions corresponding to optimized, subsampling, the data-independent γ in randomized response and the error in TV distance of the majority ensembling output of DaRRM with different γ functions, when $K = 101, m \in \{10, 20, 30, 40\}$, $\Delta = \delta = 0$.

F.1.3 Comparison Under Different Prior Distributions of p_i's

Recall $p_i = \Pr[M_i(\mathcal{D}) = 1], \forall i \in [K]$. We stress that our optimization procedure applies to any prior distribution of p_i's. Let $\mathcal{U}$ denote the distribution Uniform$([0, 1])$. We present results when $p_i \sim \mathcal{U}$, $\forall i \in [K]$, in the previous sections to show the performance of optimized DaRRM$_\gamma$ in the most general case when we do not have any prior knowledge of the mechanisms M_i's output, i.e., p_i. It is possible to consider a different prior distribution $\mathcal{T}$ of p_i's. If the true distribution of p_i's is closer to $\mathcal{T}$ than $\mathcal{U}$, than we get improved utility with the same privacy guarantee by optimizing γ under $\mathcal{T}$ than under $\mathcal{U}$; otherwise, if the true distribution of p_i is very different from $\mathcal{T}$, we suffer utility loss.

To illustrate this point, consider the following experiment setting. Suppose our prior belief is that each mechanism M_i has a clear tendency towards voting 0 or 1, i.e., p_i is far from 0.5. Let the new distribution $\mathcal{T}$ be Uniform$([0, 0.3] \cup [0.7, 1])$.

To optimize γ under $\mathcal{T}$, we change the approximate objective in Eq. 191, which optimizes γ assuming $p_i \sim \mathcal{U}$, to be the following, which optimizes γ assuming $p_i \sim \mathcal{T}, \forall i \in [K]$,

$$-\frac{1}{2} \sum_{l=\frac{K+1}{2}}^{K} \int_{0.7}^{1} \int_{0.7}^{1} \cdots \int_{0.7}^{1} (\alpha_l - \alpha_{K-l}) dp_1 dp_2 \ldots dp_K \cdot \gamma(l) \tag{194}$$

Setting. $K = 11, m \in \{3, 5\}, \delta = \Delta = 0$.

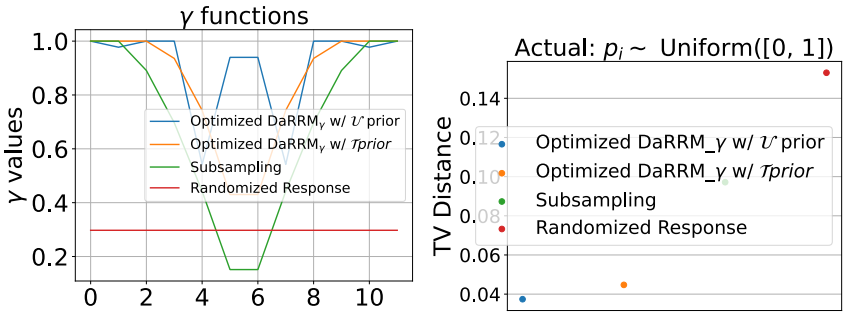
We compute the error of the optimized DaRRM$_\gamma$ in three different settings with three different actual p_i distributions:

1. "Actual: Uniform$([0, 1])$", which means we take $p_i \sim \mathcal{U}, \forall i \in [K]$ when computing the error
2. "Actual: Uniform$([0, 0.1])$", which means we take $p_i \sim$ Uniform$([0, 0.1]), \forall i \in [K]$ when computing the error

   This setting implies the mechanisms have a clear majority (of 0)
3. "Actual: $p_i = 0.5$", which means we take $p_i = 0.5, \forall i \in [K]$ when computing the error

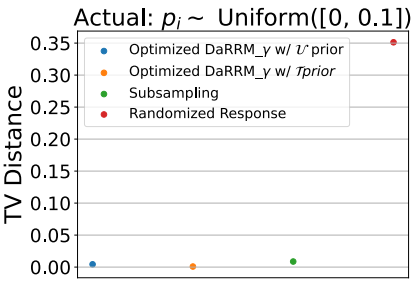
   This setting implies the mechanisms have no clear majority

Note since $\mathcal{T}$ is closer to the distribution in the second setting, we would expect DaRRM$_\gamma$ has a lower error when γ is optimized under $\mathcal{T}$ than under $\mathcal{U}$ in this setting. Also, since $\mathcal{T}$ is very different from the distribution in the third setting, we would expect DaRRM$_\gamma$ has a lower error when γ is optimized under $\mathcal{U}$ than under $\mathcal{T}$.

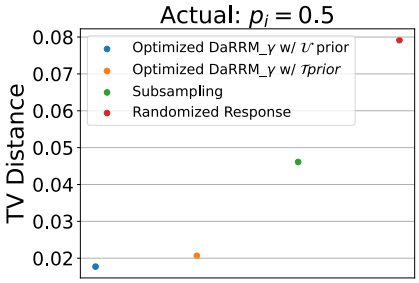


(a) γ function ($m = 3$)

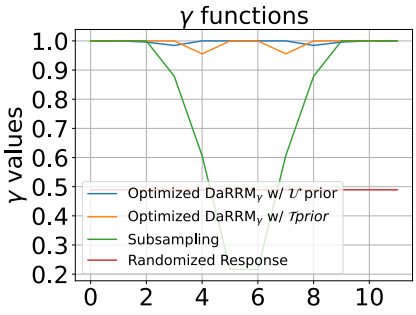
(b) Uniform p_i's ($m = 3$)



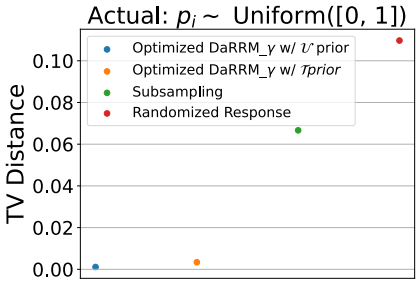
(c) Clear majority ($m = 3$)



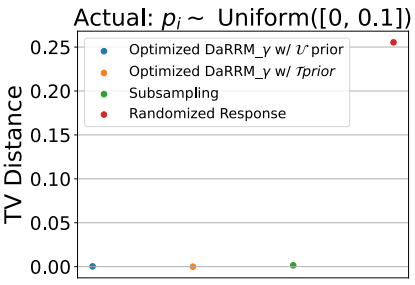
(d) No clear majority ($m = 3$)



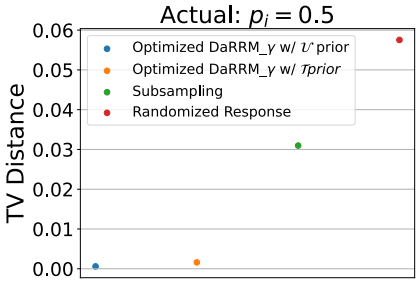
(e) γ function ($m = 5$)



(f) Uniform p_i's ($m = 5$)



(g) Clear majority ($m = 5$)



(h) No clear majority ($m = 5$)

Figure 10: Comparison of the error of DaRRM$_\gamma$ with optimized γ under two different prior distributions of p_i, i.e., $\mathcal{U}$ and $\mathcal{T}$, in the setting where $m \in \{3, 5\}, K = 11$. Observe that if the prior distribution of p_i we use when optimizing γ is closer to the actual distribution, we have additional utility gain (i.e., decreased error); otherwise, we suffer utility loss (i.e., increased error), compared to optimize γ under the uniform distribution $\mathcal{U}$ of p_i over $[0, 1]$. Furthermore, regardless of the choice of the prior distribution of p_i, optimized DaRRM$_\gamma$ achieves a lower error compared to the two baselines: Subsampling and Randomized Response.

F.2 PRIVATE DISTRIBUED SIGN-SGD

Notation. $w(t)$ denotes the parameter of the model at the t-th communication round.

Additional Experiment Details. In our experiments, each client computes its gradient based on the entire local dataset at each communication around. Also for simplicity, all clients are participated in training at each round.

Algorithm 2 β-Stochastic Sign SGD (Algorithm 2 of Xiang & Su (2023b)) without client subsampling

Input: K clients, T communication rounds, batch size n, hyperparameters B, β,
Output: $w(T)$
Initialization: $w(0) \leftarrow \mu$ for each $i \in [K]$
for communication round $t = 1, 2, \ldots, T$ **do**
  Client:
  for Client $i \in [K]$ **do**
    Each client $i \in [K]$ computes n stochastic gradients $\mathbf{g}_i^1(t), \ldots, \mathbf{g}_i^{(n)}(t)$
    for coordinate $j = 1, 2, \ldots, d$ **do**
      $\hat{g}_{i,j}(t) \leftarrow 1$ with probability $\frac{B+\beta+\text{clip}\{\frac{1}{n}\sum_{i=1}^{n} \mathbf{g}_{m,j}^{(i)}(t)\}}{2B+2\beta}$; $\hat{g}(t)_{i,j} \leftarrow -1$ otherwise
    end for
    Report $\hat{\mathbf{g}}_i(t)$ to the server.
  end for
  Server:
  On receiving one-bit encoded gradients $\hat{\mathbf{g}}_1(t), \ldots, \hat{\mathbf{g}}_K(t)$ from the clients, compute $\tilde{\mathbf{g}}(t) \leftarrow$ **Aggregate**$(\{\hat{\mathbf{g}}_i(t)\}_{i=1}^K)$
  Send $\tilde{\mathbf{g}}(t)$ to all clients
  Upon receiving $\tilde{\mathbf{g}}(t)$: $w(t+1) \leftarrow w(t) - \eta\tilde{\mathbf{g}}(t)$
end for

Coordinate wise pure DP guarantee:

Theorem F.1 (Theorem 4 of Xiang & Su (2023b)). *0-StoSign is not differentially private. When $\beta > 0$, β-StoSign is coordinate-wise $\log(\frac{2B+\beta}{\beta})$-differentially private. That is, β-StoSign is $d \cdot \log(\frac{2B+\beta}{\beta})$-differentially private.*

F.3 PRIVATE SEMI-SUPERVISED KNOLWEDGE TRANSFER

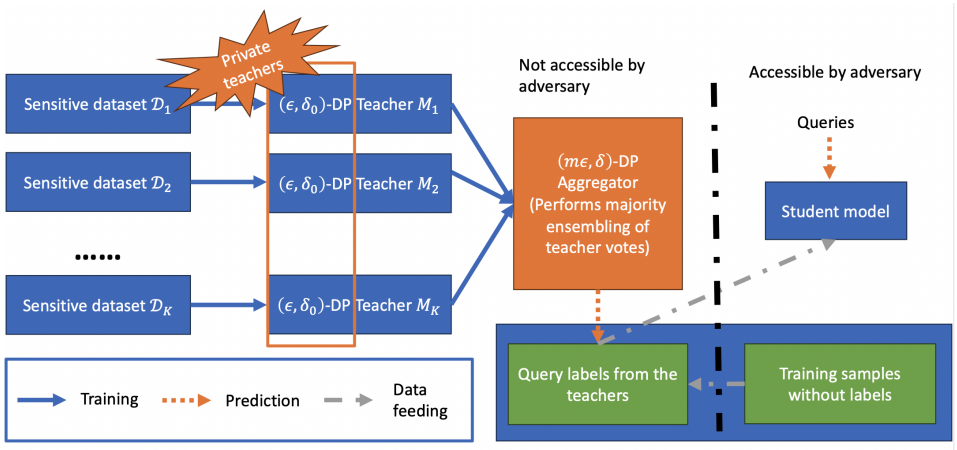


Figure 11: Semi supervised knowledge transfer setting. This figure is adapted from Figure 1 of PATE Papernot et al. (2017). Unlike PATE, we consider an untrustworthy aggregator and aggregate private teachers through private majority ensembling.

More Details About the Baseline GNMax Papernot et al. (2018)

The GNMax aggregation mechanism proceeds as follows (Section 4.1 of Papernot et al. (2018)): on input x,

$$M_\sigma(x) = \arg\max_i \{n_i(x) + \mathcal{N}(0, \sigma^2)\} \tag{195}$$

where $n_i(x)$ is # teachers who vote for class i.

Note GNMax works perfectly in aggregating non-private teachers, in our setting, it does not exploit the fact that each teacher is (ϵ, Δ)-differentially private. Hence, GNMax can add more noise than necessary to ensure the final aggregated output is $(m\epsilon, \delta)$-differentially private, especially when ϵ is small.

The privacy analysis Papernot et al. (2018) mainly focuses on computing the overall privacy loss of multiple private majority ensembling queries, while our analysis focuses on the privacy loss of a single-step aggregation from "prviate" teachers. Note the privacy composition analysis in Papernot et al. (2018) also applies to our setting to reason about the privacy loss through multiple queries.

How to set σ in GNMax?

Section 4.1 of Papernot et al. (2018) states the GNMax mechanism is $(\lambda, \lambda/\sigma^2)$-Renyi differentially private (RDP), for all $\lambda \geq 1$.

Although there is a data-dependent bound for GNMax that is tighter than the above mentioned RDP bound in Section 4.1 and in Appendix A of Papernot et al. (2018), according to Corollary 11 of Papernot et al. (2018), this analysis applies to majority voting when the number of output classes is ≥ 3, which does not directly apply to our binary-output case. Hence, we use the data-independent RDP bound for GNMax.

The following theorem shows the relationship between RDP and differential privacy (DP):

Theorem F.2 (RDP to DP (Theorem 5 of Papernot et al. (2018))). *If a mechanism M guarantees (λ, ϵ)-RDP, then M guarantees $(\epsilon + \frac{\log 1/\delta}{\lambda - 1}, \delta)$-differential privacy for $\delta \in (0, 1)$.*

Therefore, GNMax with parameter σ^2 guarantees $(\frac{\lambda}{\sigma^2} + \frac{\log 1/\delta}{\lambda - 1}, \delta)$-differential privacy, $\forall \lambda \geq 1$. Now, if we want the aggregated output to be $(m\epsilon, \delta)$-differentially private, the σ^2 in GNMax can be set as follows: 1) Since the above holds for all $\lambda \geq 1$, we first pick a proper λ that does not cause numerical instability and that ensures $\sigma^2 > 0$ by setting $\lambda = \frac{\log 1/\delta}{\epsilon} + 5$. 2) Now set $\sigma^2 = \lambda/(\epsilon - \frac{\log 1/\delta}{\lambda - 1})$ by the above theorem.