Principled Learning-to-Communicate in Cooperative MARL: An Information-Structure Perspective

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Abstract

Learning-to-communicate (LTC) in partially observable environments has gained increasing attention in deep multi-agent reinforcement learning, where the control and communication strategies are *jointly* learned. On the other hand, the impact of communication has been extensively studied in control theory, through the lens of *information structures* (ISs). In this paper, we seek to formalize and better understand LTC by bridging these two lines of work. To this end, we formalize LTC in decentralized partially observable Markov decision processes (Dec-POMDPs), and classify LTCs based on the ISs. We first show that non-classical LTCs are computationally intractable, and thus focus on quasi-classical (QC) LTCs. We then propose a series of conditions for QC LTCs, violating which can cause computational hardness in general. Further, we develop provable planning and learning algorithms for QC LTCs, and show that examples of QC LTCs satisfying the above conditions can be solved without computationally intractable oracles. Along the way, we also establish some relationship between (strictly) QC IS and the condition of having strategy-independent CIB beliefs (SI-CIB), as well as solving general Dec-POMDPs beyond those with SI-CIB, the only known condition that enables planning/learning in Dec-POMDPs without computationally intractable oracles, which may be of independent interest.

1 Introduction

Learning-to-communicate (LTC) has emerged and gained traction in the area of (deep) multi-agent reinforcement learning (MARL) (Foerster et al., 2016; Sukhbaatar et al., 2016; Jiang & Lu, 2018). Unlike classical MARL, which aims to learn a *control* strategy that minimizes the expected accumulated costs, LTC seeks to *jointly* minimize over both the *control* and the *communication* strategies of all the agents, as a way to mitigate the challenges due to the agents' *partial observability* of the environment. Despite the promising empirical successes, theoretical understandings of LTC remain largely underexplored.

On the other hand, in control theory, a rich literature has investigated the role of *communication* in decentralized/networked control (Tatikonda & Mitter, 2004; Nair et al., 2007; Xiao et al., 2005; Yüksel, 2013), inspiring us to examine LTCs from such a principled and rigorous perspective. Most of these studies, however, focused on linear systems, and did not explore the computational or sample complexity guarantees when the system knowledge is not (fully) known. A few recent studies (Sudhakara et al., 2021; Kartik et al., 2022) started to explore the settings with general discrete spaces, with special communication protocols and state transition dynamics.

More broadly, (the design of) communication strategy dictates the *information structure* (IS) of the control system, which characterizes *who knows what and when* (Witsenhausen, 1971). IS and its impact on the *optimization tractability*, especially for linear systems, have been extensively studied in decentralized control, see (Yüksel & Başar, 2023) for comprehensive overviews. In this work, we seek a more principled understanding of LTCs through the lens of information structures, with a focus on the computational and sample complexities of the problem.

Specifically, we formalize LTCs in the general framework of decentralized partially observable Markov decision processes (Dec-POMDPs) (Bernstein et al., 2002), as in the empirical works (Foerster et al., 2016; Sukhbaatar et al., 2016; Jiang & Lu, 2018). We detail our contributions as follows.

Contributions. (i) We formalize learning-to-communicate in Dec-POMDPs under the common-information-based framework (Nayyar et al., 2013b;a; Liu & Zhang, 2023), allowing *historical* information sharing. (ii) We classify LTCs through the lens of *information structure*, according to the ISs before additional information sharing. We then show that LTCs with *non-classical* (Mahajan et al., 2012) baseline IS is computationally intractable. (iii) Given the hardness, we thus focus on *quasi-classical* (QC) LTCs, and propose a series of conditions under which LTCs preserve the QC IS after sharing, while violating which can cause computational hardness in general. (iv) We propose both planning and learning algorithms for QC LTCs, by reformulating them as Dec-POMDPs with *strategy-independent (SI) common-information-based beliefs* (SI-CIB) (Nayyar et al., 2013a; Liu & Zhang, 2023), with quasi-polynomial time and sample complexities. Along the way, we also establish some relationship between *(strictly) quasi-classical* ((s)QC) ISs and the SI-CIB condition in the framework of (Nayyar et al., 2013a), as well as solving general Dec-POMDPs beyond those with SI-CIBs, the only known condition that enables planning/learning in Dec-POMDPs without computationally intractable, which may be of independent interest.

2 Preliminaries

2.1 Learning-to-Communicate

For $n > 1$ agents, a *Learning-to-Communicate* problem can be depicted by a tuple $\mathcal{L} = \langle H, \mathcal{S}, \{\mathcal{A}_{i,h}\}_{i\in[n],h\in[H]}, \{\mathcal{O}_{i,h}\}_{i\in[n],h\in[H]}, \{\mathcal{M}_{i,h}\}_{i\in[n],h\in[H]}, \mathbb{T}, \mathbb{O}, \mu_1, \{\mathcal{R}_h\}_{h\in[H]}, \{\mathcal{K}_h\}_{h\in[H]}\rangle$, where H denotes the length of each episode, and other components are introduced as follows.

Decision-making components We use $\mathcal{S}$ to denote the state space, and $\mathcal{A}_{i,h}$ to denote the *control action* space of agent i at timestep $h \in [H]$. We denote by $s_h \in \mathcal{S}$ the state and by $a_{i,h}$ the control action of agent i at timestep h. We use $a_h := (a_{1,h}, \cdots, a_{n,h}) \in \mathcal{A}_h := \prod_{i\in[n]} \mathcal{A}_{i,h}$ to denote the joint control action for all the n agents at timestep h. We denote by $\mathbb{T} = \{\mathbb{T}_h\}_{h\in[H]}$ the collection of state transition kernels, where $s_{h+1} \sim \mathbb{T}_h(\cdot\,|\,s_h, a_h) \in \Delta(\mathcal{S})$ at timestep h. We use $\mu_1 \in \Delta(\mathcal{S})$ to denote the initial state distribution. We denote by $\mathcal{O}_{i,h}$ the observation space and by $o_{i,h} \in \mathcal{O}_{i,h}$ the observation of agent i at timestep h. We use $o_h := (o_{1,h}, o_{2,h}, \cdots, o_{n,h}) \in \mathcal{O}_h := \mathcal{O}_{1,h} \times \mathcal{O}_{2,h} \times \cdots \mathcal{O}_{n,h}$ to denote the joint observation of all the n agents at timestep h. We use $\mathbb{O} = \{\mathbb{O}_h\}_{h\in[H]}$ to denote the collection of emission functions, where $o_h \sim \mathbb{O}_h(\cdot\,|\,s_h) \in \Delta(\mathcal{O}_h)$ at timestep h and state $s_h \in \mathcal{S}$. Also, we denote by $\mathbb{O}_{i,h}(\cdot\,|\,s_h)$ the emission for agent i, the marginal distribution of $o_{i,h}$ given $\mathbb{O}_h(\cdot\,|\,s_h)$ for all $s_h \in \mathcal{S}$. At each timestep h, agents will receive a common reward $r_h = \mathcal{R}_h(s_h, a_h)$, where $\mathcal{R}_h : \mathcal{S} \times \mathcal{A}_h \to [0, 1]$ denotes the reward function.

Communication components In addition to reward-driven decision-making, agents also need to decide and learn (what) to communicate with others. At timestep h, agents share part of their information $z_h \in \mathcal{Z}_h$ with other agents, where $\mathcal{Z}_h$ denotes the collection of all possible shared information at timestep h. Here we consider a general setting where the shared information z_h may contain two parts, the *baseline-sharing* part z_h^b that comes from some existing sharing protocol among agents, and the *additional-sharing* part $z_{i,h}^a$ for each agent i that comes from explicit communication *to be decided/learned*, with the joint additional-sharing information $z_h^a := \cup_{i=1}^n z_{i,h}^a$. This general setting covers those considered in most empirical works on LTC (Foerster et al., 2016; Sukhbaatar et al., 2016; Jiang & Lu, 2018), with a void baseline sharing part. We kept the baseline sharing since our focus is on the *finite-time* and *sample* tractability of LTC, for which a certain amount of information sharing is known to be necessary (Liu & Zhang, 2023). Note that $z_h = z_h^b \cup z_h^a$ and $z_h^b \cap z_h^a = \emptyset$. The shared information is part of the historical observations and (both *control* and *communication*) actions. We denote by $\mathcal{Z}_h^b$, $\mathcal{Z}_h^a$, and $\mathcal{Z}_{i,h}^a$ the collections of all z_h^b, z_h^a, and $z_{i,h}^a$.

At timestep h, the *common information* among all the agents is thus defined as the union of all the *shared information* so far: $c_{h^-} = \cup_{t=1}^{h-1} z_t \cup z_h^b$, and $c_{h^+} = \cup_{t=1}^{h} z_t$, where c_{h^-} and c_{h^+} denote the (accumulated) common information *before* and *after* additional sharing, respectively. Hence, the *private information* of agent i at time h *before* and *after* additional sharing is defined accordingly as $p_{i,h^-} = \{o_{i,1}, a_{i,1}, \cdots, a_{i,h-1}, o_{i,h}\} \backslash c_{h^-}, p_{i,h^+} = \{o_{i,1}, a_{i,1} \cdots, a_{i,h-1}, o_{i,h}\} \backslash c_{h^+}$, respectively. We denote by $p_{h^-} := (p_{1,h^-}, \cdots, p_{n,h^-})$ the joint private information *before* additional sharing, by $p_{h^+} := (p_{1,h^+}, \cdots, p_{n,h^+})$ the joint private information *after* additional sharing, at timestep h. We then denote by $\tau_{i,h^-} = p_{i,h^-} \cup c_{h^-}, \tau_{i,h^+} = p_{i,h^+} \cup c_{h^+}$ the *information available* to agent i at timestep h, before and after additional sharing, respectively, with $\tau_{h^-} = p_{h^-} \cup c_{h^-}, \tau_{h^+} = p_{h^+} \cup c_{h^+}$ denoting the associated joint information. We use $\mathcal{C}_{h^-}, \mathcal{C}_{h^+}, \mathcal{P}_{i,h^-}, \mathcal{P}_{i,h^+}, \mathcal{P}_{h^-}$, $\mathcal{P}_{h^+}, \mathcal{T}_{i,h^-}, \mathcal{T}_{i,h^+}, \mathcal{T}_{h^-}, \mathcal{T}_{h^+}$ to denote, respectively, the corresponding collections of all possible $c_{h^-}, c_{h^+}, p_{i,h^-}, p_{i,h^+}, p_{h^-}, p_{h^+}, \tau_{i,h^-}, \tau_{i,h^+}, \tau_{h^-}, \tau_{h^+}$.

We use $m_{i,h}$ to denote the *communication action* of agent i at timestep h, and it will determine what information $z_{i,h}^a$ she will share, through the way specified later. We denote by $\mathcal{M}_{i,h}$ the space of $m_{i,h}$, and by $m_h := (m_{1,h}, \cdots, m_{n,h}) \in \mathcal{M}_h := \mathcal{M}_{1,h} \times \cdots \mathcal{M}_{n,h}$ the joint communication action of all the agents. $\mathcal{K}_h : \mathcal{Z}_h^a \to [0, 1]$ denotes the *communication cost* function, and $\kappa_h = \mathcal{K}_h(z_h^a)$ denotes the incurred communication cost at timestep h, due to additional sharing.

System evolution The system's evolution alternates between the communication and control steps.

Communication step: At each timestep h, each agent i observes $o_{i,h}$ and may share part of her private information via baseline sharing, receives the baseline sharing of information from others, and forms p_{i,h^-} and c_{h^-}. Then, each agent i chooses her communication action, which determines the additional sharing of information, receives the additional-sharing of information from others, forms p_{i,h^+} and c_{h^+}, and incurs some communication cost κ_h. Formally, the evolution of the information is formalized as follows, which, unless otherwise noted, will be assumed throughout the paper.

Assumption 2.1 (*Information evolution*)**.** For each $h \in [H]$,

(a) (Baseline sharing). $z_{h+1}^b = \chi_{h+1}(p_{h^+}, a_h, o_{h+1})$ for some fixed transformation χ_{h+1};

(b) (Additional sharing). For each agent $i \in [n]$, $z_{i,h}^a = \phi_{i,h}(p_{i,h^-}, m_{i,h})$ for some function $\phi_{i,h}$, given communication action $m_{i,h}$, and $m_{i,h} \in z_{i,h}^a$; and the joint sharing $z_h^a := \cup_{i \in [n]} z_{i,h}^a$ is thus generated by $z_h^a = \phi_h(p_{h^-}, m_h)$, for some function ϕ_h;

(c) (Private information before sharing). For each agent $i \in [n]$, $p_{i,(h+1)^-} = \xi_{i,h+1}(p_{i,h^+}, a_{i,h}, o_{i,h+1})$ for some fixed transformation $\xi_{i,h+1}$, and the joint private information thus evolves as $p_{(h+1)^-} = \xi_{h+1}(p_{h^+}, a_h, o_{h+1})$ for some fixed transformation ξ_{h+1};

(d) (Private information after sharing). For each agent $i \in [n]$, $p_{i,h^+} = p_{i,h^-} \backslash z_{i,h}^a$;

(e) (Full memory). For each agent $i \in [n]$, $\tau_{i,h^-} \subseteq \tau_{i,h^+} \subseteq \tau_{i,(h+1)^-}$, and $o_{i,h} \in \tau_{i,h^-}$.

Note that as *fixed transformations* (e.g., χ_h and $\xi_{i,h}$ above), they are not affected by the *realized values* of the random variables, but dictate some *pre-defined* transformation of the input random variables. See (Nayyar et al., 2013b;a) and §B in (Liu & Zhang, 2023) for common examples of baseline sharing that admit such fixed transformations when there is no additional sharing, and examples in §A on how they are extended in the LTC setting. It should not be confused with some general *function* (e.g., $\phi_{i,h}$ above), which may depend on the *realized values* of the input random variables. (a) and (c) on baseline sharing follow from those in (Nayyar et al., 2013a; Liu & Zhang, 2023); (b) and (d) on additional sharing dictate how the communication action affects the sharing based on private information. For example, a common choice of $(\mathcal{M}_{i,h}, \phi_{i,h})$ is that $\mathcal{M}_{i,h} = \{0, 1\}^{|p_{i,h^-}|}$, for any $p_{i,h^-} \in \mathcal{P}_{i,h^-}$ and $m_{i,h} \in \mathcal{M}_{i,h}$, $\phi_{i,h}(p_{i,h^-}, m_{i,h})$ consists of the k-th element ($k \in [|p_{i,h^-}|]$) of p_{i,h^-} if and only if the k-th element of $m_{i,h}$ is 1. As $m_{i,h}$ (depicting what to share) will be known given $z_{i,h}^a$ (what has been shared), $m_{i,h}$ is thus also modeled as being shared, i.e., $m_{i,h} \in z_{i,h}^a$. This is also consistent with the models in (Sudhakara et al., 2021; Kartik et al., 2022) on control/communication joint optimization. (e) means that the agent has full memory of the information she has in the past and at present. We emphasize that this is closely related, but different from the common notion of *perfect recall* (Kuhn, 1953), where the agent has to recall all her own *past actions*. Condition (e), in contrast, relaxes the memorization of the actions, but includes the instantaneous observation $o_{i,h}$. This condition is satisfied by the models and examples in (Mahajan et al., 2012; Nayyar et al., 2013b;a; Liu & Zhang, 2023). See also §A for more examples that satisfy this assumption.

Decision-making step: After the communication, each agent i chooses her control action $a_{i,h}$, receives a reward r_h, and the joint action a_h drives the state to $s_{h+1} \sim \mathbb{T}_h(\cdot \mid s_h, a_h)$.

Strategies and solution concept At timestep h, each agent i has two strategies, a *control* strategy and a *communication* strategy. We define a control strategy as $g^a_{i,h} : \mathcal{T}_{i,h^+} \to \mathcal{A}_{i,h}$ and a communication strategy as $g^m_{i,h} : \mathcal{T}_{i,h^-} \to \mathcal{M}_{i,h}$. We denote by $g^a_h = (g^a_{1,h}, \cdots, g^a_{n,h})$ the joint control strategy and by $g^m_h = (g^m_{1,h}, \cdots, g^m_{n,h})$ the joint communication strategy. We denote by $\mathcal{G}^a_{i,h}, \mathcal{G}^m_{i,h}, \mathcal{G}^a_h, \mathcal{G}^m_h$ the corresponding spaces of $g^a_{i,h}, g^m_{i,h}, g^a_h, g^m_h$, respectively.

The objective of the agents in the LTC problem is to maximize the expected accumulated sum of the reward and the negative communication cost from timestep $h = 1$ to H: $J_{\mathcal{L}}(g^a_{1:H}, g^m_{1:H}) := \mathbb{E}_{\mathcal{L}}\left[\sum_{h=1}^H (r_h - \kappa_h) \,\middle|\, g^a_{1:H}, g^m_{1:H}\right]$, where the expectation $\mathbb{E}_{\mathcal{L}}$ is taken over all the randomness in the system evolution, given the strategies $(g^a_{1:H}, g^m_{1:H})$. With this objective, for any $\epsilon \geq 0$, we can define the solution concept of *ϵ-team optimum* for $\mathcal{L}$ as follows.

Definition 2.2 (ϵ-team optimum)**.** We call a joint strategy $(g^a_{1:H}, g^m_{1:H})$ an ϵ-team optimal strategy of the LTC $\mathcal{L}$ if $\max_{\widetilde{g}^a_{1:H} \in \mathcal{G}^a_{1:H}, \widetilde{g}^m_{1:H} \in \mathcal{G}^m_{1:H}} J_{\mathcal{L}}(\widetilde{g}^a_{1:H}, \widetilde{g}^m_{1:H}) - J_{\mathcal{L}}(g^a_{1:H}, g^m_{1:H}) \leq \epsilon$.

2.2 Information Structures of LTC

In decentralized stochastic control, the notion of information structure (Witsenhausen, 1975; Mahajan et al., 2012) captures *who knows what and when* as the system evolves. In LTC, as the additional sharing via communication will also affect the IS and is *not* determined *beforehand*, when we discuss the *IS of an LTC problem*, we will refer to that of the problem *with only baseline sharing*. In particular, an LTC $\mathcal{L}$ without additional sharing is essentially a Dec-POMDP (with potential baseline information sharing), as defined in §E for completeness. We call a Dec-POMDP *induced* by $\mathcal{L}$ as the problem without additional sharing, (as defined in F.3).

(Strictly) quasi-classical ISs are important subclasses of ISs, which were first introduced for decentralized stochastic control (Witsenhausen, 1975; Mahajan & Yüksel, 2010; Yüksel & Başar, 2023) (see the instantiation for Dec-POMDPs in §F.2). An IS that is not QC is *non-classical* (Mahajan et al., 2012; Yüksel & Başar, 2023). We extend such a categorization to LTC problems as follows.

Definition 2.3 ((Strictly) quasi-classical LTC)**.** We call an LTC $\mathcal{L}$ *(strictly) quasi-classical* if the Dec-POMDP induced by $\mathcal{L}$ (cf. Definition F.3) is *(strictly) quasi-classical.* Namely, each agent in the intrinsic model of $\overline{\mathcal{D}}_{\mathcal{L}}$ knows the information (and the actions) of the agents who influence her, either directly or indirectly.

Note that intrinsic model (defined in F.3) is oftentimes used for discussing information structure, where each agent only *acts once* throughout the problem evolution, and the same agent in the state-space model at different timesteps is now treated as *different agents*.

3 Structural Assumptions and Hardness

It is known that computing an (approximate) team-optimum in Dec-POMDPs, which are LTCs *without* information-sharing, is `NEXP-hard` (Bernstein et al., 2002). The hardness cannot be fully circumvented even when agents are allowed to share information: even if agents share all the information, the LTC problem becomes a Partially Observable Markov Decision Process (POMDP), which is known to be `PSPACE-hard` (Papadimitriou & Tsitsiklis, 1987; Lusena et al., 2001). Hence, additional assumptions are necessary to make LTCs computationally tractable. We introduce several such assumptions and their justifications below, whose proofs can be found in §B.

Recently, (Golowich et al., 2023) showed that *observable* POMDPs, a class of POMDPs with relatively *informative* observations, allow *quasi-polynomial time* algorithms to solve. Such a condition was then generalized to the *joint* emission function of Dec-POMDPs in (Liu & Zhang, 2023) . As solving LTCs is at least as hard as solving the Dec-POMDPs considered in (Liu & Zhang, 2023) , we first also make such an observability assumption, to avoid computationally intractable oracles.

Assumption 3.1 (γ-observability (Golowich et al., 2023))**.** There exists a $\gamma > 0$ such that $\forall h \in [H]$, the emission $\mathbb{O}_h$ satisfies that $\forall b_1, b_2 \in \Delta(\mathcal{S})$, $\left\|\mathbb{O}_h^\top b_1 - \mathbb{O}_h^\top b_2\right\|_1 \geq \gamma \left\|b_1 - b_2\right\|_1$.

However, Assumption 3.1 is not enough when it comes to LTC, if the baseline sharing IS is not favorable, in particular, *non-classical* (Mahajan et al., 2012). The hardness persists even under a few additional assumptions to be introduced later (as shown in Lemma B.3).

Hence, we will focus on the *quasi-classical* LTCs hereafter. Indeed, QC is also known to be critical for efficiently solving *continuous-space* and *linear* decentralized control (Ho et al., 1972; Lamperski & Lessard, 2015). However, in our discrete setting, even QC LTCs may not be computationally tractable: the additional sharing may *break* the QC IS, and introduce computational hardness. We formalize this intuition with the following discussions on when *QC may break*, and computational hardness results to justify the associated assumptions.

Firstly, QC may break by additional sharing, if an agent influences others (only) via such sharing, while others cannot fully access the information used for determining the *communication action*. Indeed, the general communication-strategy space in §2.1 allows the dependence on agents' *private information*, making this case possible. We show that this causes computational hardness in general.

To avoid this hardness, we thus focus on communication strategies that only condition on the *common information*. Intuitively, this assumption is not unreasonable, as it means that *which historical information to share* is determined by *what has been shared* (in the common information). Note that, this does not lose the generality in the sense that the private information p_{i,h^-} *can still* be shared. It only means that the communication action is not determined based on p_{i,h^-}, and the additional sharing is still dictated by $z_{i,h}^a = \phi_{i,h}(p_{i,h^-}, m_{i,h})$ (cf. Assumption 2.1), depending on p_{i,h^-}.

Assumption 3.2 (Common-information-based communication strategy)**.** The communication strategies take *common information* as input, with the following form:

$$\forall i \in [n], h \in [H], \quad g_{i,h}^m : \mathcal{C}_{h^-} \to \mathcal{M}_{i,h}. \tag{3.1}$$

Secondly, QC may break by additional sharing if it makes an agent *influence* others(' available information) by *sharing* her *control* actions, while these other agents were *not influenced* by the agent in the baseline sharing, and thus did not have to access the available information that the agent decided her control actions upon. We make the following two assumptions to avoid the related pessimistic cases, followed by the hardness results when they are missing. The common idea behind the hardness results in both Lemmas B.5 and B.6 exactly follows from this insight.

Specifically, in some special cases, the action of some agents may not influence the state transition. Such actions are thus *useless* in terms of decision-making, when there is *no* information sharing. However, if they were deemed *non-influential*, but shared via additional sharing, then QC may break for the LTC problem. We thus make the following assumption, followed by a justification result.

Assumption 3.3 (Control-useless action is not used)**.** For each $i \in [n], h \in [H]$, if agent i's action $a_{i,h}$ does not influence the state s_{h+1}, namely, $\forall s_h \in \mathcal{S}, a_h \in \mathcal{A}_h, a'_{i,h} \in \mathcal{A}_{i,h}, a'_{i,h} \neq a_{i,h}, \mathbb{T}_h(\cdot \mid s_h, a_h) = \mathbb{T}_h(\cdot \mid s_h, (a'_{i,h}, a_{-i,h}))$. Then, $\forall h' > h, a_{i,h} \notin \tau_{h'^-}$ and $a_{i,h} \notin \tau_{h'^+}$.

Note that other than the justification above based on computational hardness, Assumption 3.3 has been *implicitly* made in the IS examples in the literature when there are *uncontrolled* state dynamics, see e.g., (Nayyar et al., 2013a; Liu & Zhang, 2023). Moreover, we emphasize that for common cases where actions *do* affect the state transition, this assumption becomes not necessary.

Other than *not influencing* state transition, an action may also be non-influential if the emission functions of other agents are *degenerate*: they cannot *sense* the influence from previous agents' actions. We thus make the following assumption on the emissions, followed by a justification result.

Assumption 3.4 (Other agents' emissions are non-degenerate)**.** For $\forall h \in [H], i \in [n]$, $\mathbb{O}_{-i,h}$ satisfies $\forall b_1, b_2 \in \Delta(\mathcal{S}), b_1 \neq b_2,\ \mathbb{O}_{-i,h}^{\top} b_1 \neq \mathbb{O}_{-i,h}^{\top} b_2$.

Finally, for both the baseline and additional sharing protocols, we follow the convention in the series of works on partial history/information sharing (Nayyar et al., 2013b;a; Liu & Zhang, 2023; Sudhakara et al., 2021; Kartik et al., 2022) that, if an agent shares, she will share the information with *all other* agents. We make it more formally as follows.

Assumption 3.5. $\forall i_1, i_2 \in [n], h_1, h_2 \in [H], i_1 \neq i_2, h_1 < h_2$, if $\sigma(o_{i_1,h_1}) \subseteq \sigma(\tau_{i_2,h_2^-})$, then $\sigma(o_{i_1,h_1}) \subseteq \sigma(c_{h_2^-})$, and if $\sigma(a_{i_1,h_1}) \subseteq \sigma(\tau_{i_2,h_2^-})$, then $\sigma(a_{i_1,h_1}) \subseteq \sigma(c_{h_2^-})$; if $\sigma(o_{i_1,h_1}) \subseteq \sigma(\tau_{i_2,h_2^+})$, then $\sigma(o_{i_1,h_1}) \subseteq \sigma(c_{h_2^+})$, and if $\sigma(a_{i_1,h_1}) \subseteq \sigma(\tau_{i_2,h_2^+})$, then $\sigma(a_{i_1,h_1}) \subseteq \sigma(c_{h_2^+})$.

As will be shown later (cf. Theorem 4.1), LTCs under Assumptions 3.2, 3.3, 3.4, and 3.5 can indeed *preserve* the QC/sQC information structure after additional sharing, making it possible for the overall LTC to be computationally tractable, as we will show next. Some more examples that satisfy these assumptions can also be found in §A.

4 Solving LTC Problems Provably

We now study how to solve LTC provably, via either *planning* (with model knowledge) or *learning* (without model knowledge). Proofs of the results can be found in §C.

4.1 An Equivalent Dec-POMDP

Given any H-steps LTC $\mathcal{L}$, we can reformulate it as an $2H$-steps Dec-POMDP $\mathcal{D}_{\mathcal{L}}$ such that $\mathcal{L}$ and $\mathcal{D}_{\mathcal{L}}$ are equivalent. The elements in the odd timestep $2h-1$ of $\mathcal{D}_{\mathcal{L}}$ is constructed from elements of communication step (h^-) in $\mathcal{L}$, and the elements in the even timestep $2h$ of $\mathcal{D}_{\mathcal{L}}$ is constructed from decision-making step (h^+) in $\mathcal{L}$. We defer the formal reformulation in §C.1. The Dec-POMDP $\mathcal{D}_{\mathcal{L}}$ inherits the QC IS from $\mathcal{L}$, formally stated as follows.

Theorem 4.1 (Preserving (s)QC)**.** If $\mathcal{L}$ is (s)QC and satisfies Assumptions 3.2, 3.3, 3.4, and 3.5, then the reformulated Dec-POMDP $\mathcal{D}_{\mathcal{L}}$ is also (s)QC.

4.2 Strict Expansion of $\mathcal{D}_{\mathcal{L}}$

Despite being QC/sQC, it is not clear if one can solve $\mathcal{D}_{\mathcal{L}}$ without computationally intractable oracles. Note that, to the best of our knowledge, the only known finite-time computational complexity results for planning in such decentralized control models were in (Liu & Zhang, 2023), which were established under the *strategy independence* assumption (Nayyar et al., 2013a) on the common-information-based beliefs (Nayyar et al., 2013b;a). This SI assumption was shown critical for *computation* (Liu & Zhang, 2023) – it eliminates the need to *enumerate* the past strategies in dynamic programming, which would otherwise be prohibitively large. Thus, we need to connect QC/sQC to SI-CIB for tractable computation.

Interestingly, under certain conditions, one can connect QC with SI-CIB for the reformulated Dec-POMDP $\mathcal{D}_{\mathcal{L}}$. As the first step, we will *expand* the QC $\mathcal{D}_{\mathcal{L}}$ by adding the *actions* of the agents who influence the later agents in the intrinsic model of $\mathcal{D}_{\mathcal{L}}$ to the shared information. We denote the strictly expanded Dec-POMDP as $\mathcal{D}_{\mathcal{L}}^{\dagger}$. We replace the $\widetilde{\ }$ notation in $\mathcal{D}_{\mathcal{L}}$ by the $\breve{\ }$ notation in $\mathcal{D}_{\mathcal{L}}^{\dagger}$. The horizon, states, actions, observations, transitions, and reward functions remain the same, but the sets of information $\breve{p}_h, \breve{c}_h, \breve{\tau}_h, \breve{p}_{i,h}, \breve{\tau}_{i,h}$ are different: for any $h \in [\widetilde{H}], i \in [n]$

$$\breve{c}_h = \widetilde{c}_h \cup \{\widetilde{a}_{j,t} \,|\, j \in [n], t < h, \sigma(\widetilde{\tau}_{j,t}) \subseteq \sigma(\widetilde{c}_h)\},\ \breve{p}_{i,h} = \widetilde{p}_{i,h} \backslash \{\widetilde{a}_{i,t} \,|\, t < h, \sigma(\widetilde{\tau}_{i,t}) \subseteq \sigma(\widetilde{c}_h)\}. \tag{4.1}$$

It is not hard to verify that $\mathcal{D}_{\mathcal{L}}^{\dagger}$ is sQC (as shown in Lemma C.3). Also, as shown below, a benefit of obtaining an *sQC* $\mathcal{D}_{\mathcal{L}}^{\dagger}$ is that, it is *SI-CIB* (as shown in Theorem C.5), making it possible to be solved without computationally intractable oracles as in (Liu & Zhang, 2023). Furthermore, we can get the solution of $\mathcal{D}_{\mathcal{L}}$ by solving $\mathcal{D}_{\mathcal{L}}^{\dagger}$ (as shown in Theorem C.4).

4.3 Refinement of $\mathcal{D}_{\mathcal{L}}^{\dagger}$

Despite of being SI, $\mathcal{D}_{\mathcal{L}}^{\dagger}$ is still not eligible for applying the results in (Liu & Zhang, 2023): the information evolution rules of $\mathcal{D}_{\mathcal{L}}^{\dagger}$ break those in (Nayyar et al., 2013a; Liu & Zhang, 2023). To address this issue, we propose to further *refine* the $\mathcal{D}_{\mathcal{L}}^{\dagger}$ to obtain a Dec-POMDP $\mathcal{D}'_{\mathcal{L}}$, which satisfies the information evolution rules. We replace the $\breve{}$ notation in $\mathcal{D}_{\mathcal{L}}^{\dagger}$ by the $^-$ notation in $\mathcal{D}'_{\mathcal{L}}$. The elements in $\mathcal{D}'_{\mathcal{L}}$ remain the same as those in $\mathcal{D}_{\mathcal{L}}^{\dagger}$, except that the private information at odd steps is now refined as $\overline{p}_{i,2t-1} = p_{i,t^-} \backslash \breve{c}_{2t-1}$.

The new Dec-POMDP $\mathcal{D}'_{\mathcal{L}}$ is not equivalent to $\mathcal{D}_{\mathcal{L}}^{\dagger}$ in general, since it enlarges the strategy space at the odd timesteps. However, if we define new strategy spaces in $\mathcal{D}'_{\mathcal{L}}$ as $\overline{\mathcal{G}}_{i,2t-1} : \overline{\mathcal{C}}_{2t-1} \to \overline{\mathcal{A}}_{i,2t-1}, \overline{\mathcal{G}}_{i,2t} : \overline{\mathcal{T}}_{i,2t} \to \overline{\mathcal{A}}_{i,2t}$ for each $t \in [H], i \in [n]$, and thus define $\overline{\mathcal{G}}_h$ to be the associated joint space, then solving $\mathcal{D}_{\mathcal{L}}^{\dagger}$ is equivalent to finding a *best-in-class* team-optimal strategy of $\mathcal{D}'_{\mathcal{L}}$ within space $\overline{\mathcal{G}}_{1:\overline{H}}$, as shown below.

Theorem 4.2. Let $\mathcal{D}_{\mathcal{L}}^{\dagger}$ be an sQC Dec-POMDP generated from $\mathcal{L}$ after reformulation and strict expansion, and $\mathcal{D}'_{\mathcal{L}}$ be the refinement of $\mathcal{D}_{\mathcal{L}}^{\dagger}$ as above. Then, finding the optimal strategy in $\mathcal{D}_{\mathcal{L}}^{\dagger}$ is equivalent to finding the optimal strategy of $\mathcal{D}'_{\mathcal{L}}$ in the space $\overline{\mathcal{G}}_{1:\overline{H}}$, and $\mathcal{D}'_{\mathcal{L}}$ satisfies the information evolution rule. Furthermore, $\mathcal{D}'_{\mathcal{L}}$ has SI-CIB with respect to the strategy spaces $\overline{\mathcal{G}}_{1:\overline{H}}$, i.e., for any $h \in [\overline{H}], \overline{s}_h \in \overline{\mathcal{S}}, \overline{p}_h \in \overline{\mathcal{P}}_h, \overline{c}_h \in \overline{\mathcal{C}}_h, \overline{g}_{1:h-1}, \overline{g}'_{1:h-1} \in \overline{\mathcal{G}}_{1:h-1}$, it holds that

$$\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}_{1:h-1}) = \mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}'_{1:h-1}). \tag{4.2}$$

4.4 Planning in QC LTC with Quasi-polynomial Time

Now we focus on how to solve the SI-CIB Dec-POMDP $\mathcal{D}'_{\mathcal{L}}$ *computationally tractably*, which has been studied in (Liu & Zhang, 2023). Given any such a Dec-POMDP $\mathcal{D}'_{\mathcal{L}}$, (Liu & Zhang, 2023) proposed to construct an (ϵ_r, ϵ_z)-*expected*-approximate common information model $\mathcal{M}$ through *finite memory* (as defined in §C.6), when $\mathcal{D}'_{\mathcal{L}}$ is γ-observable. ϵ_r and ϵ_z here denote the approximation errors for rewards and transitions, respectively, for which we defer a detailed introduction to §C.6).

Hence, we can leverage the approaches in (Liu & Zhang, 2023) to find the optimal strategy $\overline{g}^*_{1:\overline{H}}$ by finding an optimal prescriptions $\gamma^*_{1:\overline{H}}$ under each possible $\widehat{c}_{1:\overline{H}}$ with backward induction over the timesteps $h = \overline{H}, \cdots, 1$. Meanwhile, it is worth mentioning that at each step $h \in [\overline{H}]$, it requires maximizing the Q-value functions (as defined in §C.6) as follows

$$\left(\widehat{g}^*_{1,h}(\cdot \,|\, \widehat{c}_h, \cdot), \cdots, \widehat{g}^*_{n,h}(\cdot \,|\, \widehat{c}_h, \cdot)\right) \leftarrow \operatorname*{argmax}_{\gamma_h} Q_h^{*,\mathcal{M}}(\widehat{c}_h, \gamma_h). \tag{4.3}$$

Note that solving Eq. (4.3) is `NP-hard` in general (Tsitsiklis & Athans, 1985). Hence, the guarantee for the algorithms in (Liu & Zhang, 2023) also relies on the tractability of the *one-step* team-decision problem (Tsitsiklis & Athans, 1985). Note that this assumption is minimal for the computational tractability of finding a team-optimum in Dec-POMDPs/LTCs, since otherwise, even the $\overline{H} = 1$ case is intractable (Tsitsiklis & Athans, 1985). That said, the structural results so far still hold without this assumption, and the hardness results in §3 still hold even with this assumption.

Assumption 4.3 (One-step tractability)**.** Eq. (4.3) can be solved in polynomial time.

Assumption 4.3 is satisfied for several classes of Dec-POMDPs with information sharing (Liu & Zhang, 2023), which could result from structures of either the decision-making components of the model, or the information structures. We also include several such structural conditions in §G for completeness. With this assumption, we can obtain a planning algorithm with quasi-polynomial time complexity (cf. §C.7), and also shown in the Fig. 6 in §J.

4.5 LTC with Quasi-polynomial Time and Samples

Based on the previous results on planning, we are ready to solve the *learning* problem without model knowledge with both time and sample complexity guarantees. Now, one can only sample from $\mathcal{L}$, making it difficult to obtain an SI $\mathcal{D}'_{\mathcal{L}}$ from $\mathcal{L}$ as before. Fortunately, the *reformulation* step (§4.1) does not change the system dynamics, but only maps the information to different random variables; the *expansion* step (§4.2) only requires agents to share more actions with each other, without changing the input and output of the environment; the *refinement* step (§4.3) only recovers the private information the agents had in the original $\mathcal{L}$. Therefore, we can treat the samples from $\mathcal{L}$ as the samples from $\mathcal{D}'_{\mathcal{L}}$. This way, we can utilize similar algorithmic ideas in (Liu & Zhang, 2023) to develop the learning algorithm for LTC problems.

Specifically, we construct an (ϵ_r, ϵ_z)-*expected* approximate common information model that *depends on some given a strategy $\overline{g}^{1:\overline{H}}$ that generates the data for such a construction, which we denote by* $\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})$, and thus denote (ϵ_r, ϵ_z) as $(\epsilon_r(\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})), \epsilon_z(\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})))$. For such a model, one could *simulate* and *sample* by running the strategy $\overline{g}^{1:\overline{H}}$ in the true model $\mathcal{D}'_{\mathcal{L}}$. The choice of $\overline{g}^{1:\overline{H}}$ will be carefully specified to ensure $\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})$ to be a good approximation of $\mathcal{D}'_{\mathcal{L}}$. Then one can learn an empirical estimator $\widehat{\mathcal{M}}(\overline{g}^{1:\overline{H}})$ of $\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})$ by sampling under $\overline{g}^{1:\overline{H}}$ and solving the planning problem in $\widehat{\mathcal{M}}(\overline{g}^{1:\overline{H}})$. Meanwhile, the sample complexity analysis of such an algorithm will depend on the notion of *length* for the approximate common information, denoted as $\widehat{L}$. We defer the formal introduction for $\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})$, $\widehat{L}$, and corresponding algorithm to §C. Finally, we present our main results for learning in the LTC problem.

Theorem 4.4. Given any QC LTC problem $\mathcal{L}$ satisfying Assumptions 3.1, 3.2, 3.3, and 3.4, we can construct an SI-CIB Dec-POMDP problem $\mathcal{D}'_{\mathcal{L}}$ such that the following holds. Given a strategy $\overline{g}^{1:\overline{H}}$, $\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})$ satisfying Assumption 4.3, and $\widehat{L}$, where each $\overline{g}^h$ is a complete strategy with $\overline{g}^h_{h-\widehat{L}:h} = \text{Unif}(\overline{\mathcal{A}})$ for $h \in [\overline{H}]$, we define the statistical error for estimating $\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})$ as $\epsilon_{apx}(\overline{g}^{1:\overline{H}}, \widehat{L})$. Then, there exists an algorithm that can learn an ϵ-team-optimal strategy for $\mathcal{L}$ with probability at least $1-\delta_1$, using a sample complexity $N_0 = \texttt{poly}(\max_{h\in[\overline{H}]} |\mathcal{P}_h|, \max_{h\in[\overline{H}]} |\widehat{\mathcal{C}}_h|, H, \max_{h\in[\overline{H}]} |\mathcal{A}_h|, \max_{h\in[\overline{H}]} |\mathcal{O}_h|) \log(1/\delta_1)$, where $\epsilon := \texttt{poly}(\epsilon_{apx}, \epsilon_r(\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})), \epsilon_z(\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}}))$. Specifically, if $\mathcal{L}$ has the baseline sharing protocols as in §A, there exists an algorithm that learns an ϵ-team optimal strategy for $\mathcal{L}$ with both quasi-polynomial time and sample complexities.

5 Solving General QC Dec-POMDPs

In §4, we developed a pipeline for solving a special class of QC Dec-POMDPs generated by LTCs, without computationally intractable oracles. In fact, the pipeline can be extended to solving general QC Dec-POMDPs, which thus advances the results in (Liu & Zhang, 2023) that can only address *SI-CIB* Dec-POMDPs, a result of independent interest. Without much confusion given the context, we will adapt the notation of LTC to studying general Dec-POMDPs: we set $h^+ = h^- = h$ and void the additional sharing protocol. We extend the results to general QC Dec-POMDPs as follows.

Theorem 5.1. Consider a Dec-POMDP $\mathcal{D}$ that satisfies Assumptions 2.1 (e). If $\mathcal{D}$ is sQC and satisfies Assumptions 3.3, 3.4, and 3.5, then it has SI-CIB. Meanwhile, if $\mathcal{D}$ has SI-CIB and perfect recall, then it is sQC (up to null sets).

Perfect recall here (Kuhn, 1953) means that the agents will never forget their own past information and actions (as formally defined in §D). Note that Assumption 2.1 (e) is similar but different from perfect recall: it is implied by the latter with $o_{i,h} \in \tau_{i,h}$. Also, Assumptions 3.3, 3.4, and 3.5 were made for LTCs, and here we meant to impose them for Dec-POMDPs with $h^+ = h^- = h$. Finally, by sQC up to null sets, we meant that if agent (i_1, h_1) influences agent (i_2, h_2) in the intrinsic model of the Dec-POMDP, then under any strategy $\overline{g}_{1:\overline{H}}$, $\sigma(\overline{\tau}_{i_1,h_1}) \subseteq \sigma(\overline{\tau}_{i_2,h_2})$ except the null sets generated by $\overline{g}_{1:\overline{H}}$, where we add $^-$ for all the notation in the Dec-POMDP. Given Theorem 5.1 and the results in §4, we illustrate the relationship between LTCs and Dec-POMDPs with different assumptions and ISs in Fig. 1 in §H, which may be of independent interest.

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A Examples of QC LTC

In this section, we introduce 8 examples of QC LTC problems, and 4 of them are extended from the information structures of the baseline sharing protocol considered in the literature (Nayyar et al., 2013a; Liu & Zhang, 2023). It can be shown that LTC with any of these 8 examples as baseline sharing is QC.

- **Example 1: One-step delayed information sharing:** At timestep $h \in [H]$, agents will share all the action-observation history in the private information until timestep $h-1$. Namely, for any $h \in [H], i \in [n], c_{h^-} = c_{(h-1)^+} \cup \{o_{h-1}, a_{h-1}\}$ and $p_{i,h^-} = \{o_{i,h}\}$.
- **Example 2: State controlled by one controller with asymmetric delayed information sharing:** The state dynamics and reward are controlled by only one agent (without loss of generality, agent 1), i.e., $\mathbb{T}_h(\cdot \mid s_h = S_h, a_{1,h} = A_{1,h}, a_{-1,h} = A_{1,h}) = \mathbb{T}_h(\cdot \mid s_h = S_h, a_{1,h} = A_{1,h}, a_{-1,h} = A'_{-1,h}), \mathcal{R}_h(\cdot \mid s_h = S_h, a_{1,h} = A_{1,h}, a_{-1,h} = A_{-1,h}) = \mathcal{R}_h(\cdot \mid s_h = S_h, a_{1,h} = A_{1,h}, a_{-1,h} = A'_{-1,h})$ for all $S_h \in \mathcal{S}, A_{1,h} \in \mathcal{A}_{1,h}, A_{-1,h} \in \mathcal{A}_{-1,h}, A'_{-1,h} \in \mathcal{A}_{-1,h}$. Agent 1 will share all of her information immediately, while others will share their information with a delay of $d \geq 1$ timesteps in the baseline sharing. Namely, for any $h \in [H], i \neq 1$, $c_{h^-} = c_{(h-1)^+} \cup \{a_{1,h-1}, o_{1,h}, o_{-1,h-d}\}, p_{1,h^-} = \emptyset, p_{i,h^-} = p_{i,(h-1)^+} \cup \{o_{i,h}\} \backslash \{o_{i,h-d}\}$.
- **Example 3: Information sharing with one-directional-one-step-delay:** For convenience, we assume there are 2 agents, and this example can be readily generalized to the multi-agent case. In this case, agent 1 will share the information immediately, while agent 2 will share information with one-step delay. Namely, $c_{1^-} = \{o_{1,1}\}, p_{1,1^-} = \emptyset, p_{2,1^-} = \{o_{2,1}\}$; for any $h \geq 2, i \in [n], c_{h^-} = c_{(h-1)^+} \cup \{o_{1,h}, o_{2,h-1}, a_h\}, p_{1,h^-} = \emptyset, p_{2,h^-} = \{o_{2,h}\}$.
- **Example 4: Uncontrolled state process:** The state transition does not depend on the action of agents, i.e., $\mathbb{T}_h(\cdot \mid s_h = S_h, a_h = A_h) = \mathbb{T}_h(\cdot \mid s_h = S_h, a_h = A'_h)$ for any $s_h \in \mathcal{S}, a'_h, a_h \in \mathcal{A}_h$. All agents will share their information with a delay of $d \geq 1$. For any $h \in [H], i \in [n]$, $c_{h^-} = c_{(h-1)^+} \cup \{o_{h-d}\}, p_{i,h^-} = p_{i,(h-1)^+} \cup \{o_{i,h}\} \backslash \{o_{i,h-d}\}$.
- **Example 5: One-step delayed observation sharing:** At timestep $h, h \in [H]$, each agent has access to observations of all agents until timestep $h-1$ and her present observation. Namely, for any $h \in [H], i \in [n], c_{h^-} = c_{(h-1)^+} \cup \{o_{h-1}\}$ and $p_{i,h^-} = \{o_{i,h}\}$.
- **Example 6: One-step delayed observation and two-step delayed control sharing:** At timestep $h, h \in [H]$, each agent will share the observations history until timestep $h-1$ and actions history until timestep $h-2$ from the private information. Namely, for any $h \in [H], i \in [n], c_{h^-} = c_{(h-1)^+} \cup \{o_{h-1}, a_{h-2}\}, p_{i,h^-} = \{o_{i,h}, a_{i,h-1}\}$.
- **Example 7: State controlled by one controller with asymmetric delayed observation sharing:** The state dynamics and reward are controlled by only one agent (i.e., system dynamics are the same as **Example 2**). Agent 1 will share all of her observations immediately, while others will share their observations with a delay of $d \geq 1$ timesteps in baseline sharing. Namely, for any $h \in [H], i \neq 1, c_{h^-} = c_{(h-1)^+} \cup \{o_{1,h}, o_{-1,h-d}\}, p_{1,h^-} = \emptyset, p_{i,h^-} = p_{i,(h-1)^+} \cup \{o_{i,h}\} \backslash \{o_{i,h-d}\}$.
- **Example 8: State controlled by one controller with asymmetric delayed observation and two-step delayed action sharing:** The state dynamics and reward are controlled by only one agent (i.e., system dynamics are the same as **Example 2**). At timestep $h, h \in [H]$, agent 1 will share all of her observations immediately and her actions history until timestep $h-2$, while others will share their observations with a delay of $d \geq 1$. Namely, for any $h \in [H], i \neq 1, c_{h^-} = c_{(h-1)^+} \cup \{o_{1,h}, a_{1,h-2}, o_{-1,h-d}\}, p_{1,h^-} = \{a_{1,h-1}\}, p_{i,h^-} = p_{i,(h-1)^+} \cup \{o_{i,h}\} \backslash \{o_{i,h-d}\}$.

In fact, the first 4 examples are all sQC LTC problems, while the other 4 examples are QC but not sQC problems, as shown in the following lemma.

Lemma A.1. Given an LTC problem $\mathcal{L}$. If the baseline sharing of $\mathcal{L}$ is one of the first 4 examples above, then $\mathcal{L}$ is sQC. If the baseline sharing of $\mathcal{L}$ is one of the last 4 examples above, then $\mathcal{L}$ is QC but not sQC.

Proof. Let $\overline{\mathcal{D}}_{\mathcal{L}}$ denote the Dec-POMDP induced by $\mathcal{L}$ (cf. F.3). We prove this lemma case by case. For convenience, we use $\dot{}$ in the notation for the elements in $\overline{\mathcal{D}}_{\mathcal{L}}$.

- **Example 1:** The information in $\overline{\mathcal{D}}_{\mathcal{L}}$ evolves as $\forall h \in [H], i \in [n], \dot{c}_h = \{\dot{o}_{1:h-1}, \dot{a}_{1:h-1}\}$ and $\dot{p}_{i,h} = \{\dot{o}_{i,h}\}$. Then, for any $i_1, i_2 \in [n], h_1, h_2 \in [H], h_1 < h_2$, $\dot{\tau}_{i_1,h_1} = \{\dot{o}_{1:h_1-1}, \dot{a}_{1:h_1-1}, \dot{o}_{i_1,h_1}\} \subseteq \dot{c}_{h_1+1} \subseteq \dot{c}_{h_2} \subseteq \dot{\tau}_{i_2,h_2}$, and $\dot{a}_{i_1,h_1} \subseteq \dot{c}_{h_1+1} \subseteq \dot{c}_{h_2} \subseteq \dot{\tau}_{i_2,h_2}$. Therefore, we have $\sigma(\dot{\tau}_{i_1,h_1}) \subseteq \sigma(\dot{\tau}_{i_2,h_2})$, and thus $\mathcal{L}$ is sQC.
- **Example 2:** The information in $\overline{\mathcal{D}}_{\mathcal{L}}$ evolves as $\forall h \in [H], i \neq 1, \dot{c}_h = \{\dot{a}_{1,1:h-1}, \dot{o}_{1,1:h-1}, \dot{o}_{-1,1:h-d}\}, \dot{p}_{1,h} = \emptyset, \dot{p}_{i,h} = \{o_{i,h-d+1:h}\}$. Then, for any $i_1, i_2 \in [n], h_1, h_2 \in [H], h_1 < h_2$. If $i_1 \neq 1$, then agent (i_1, h_1) will not influence agent (i_2, h_2). If $i_1 = 1$, then $\dot{\tau}_{i_1,h_1} = \{\dot{o}_{1,1:h_1}, \dot{a}_{1,1:h_1-1}, \dot{o}_{-1,1:h_1-d}\} \subseteq \dot{c}_{h_1+1} \subseteq \dot{c}_{h_2} \subseteq \dot{\tau}_{i_2,h_2}$, and $\dot{a}_{i_1,h_1} \subseteq \dot{c}_{h_1+1} \subseteq \dot{c}_{h_2} \subseteq \dot{\tau}_{i_2,h_2}$. Therefore, we have $\sigma(\dot{\tau}_{i_1,h_1}) \subseteq \sigma(\dot{\tau}_{i_2,h_2})$ if agent (i_1, h_1) influences agent (i_2, h_2), and thus $\mathcal{L}$ is sQC.
- **Example 3:** The information in $\overline{\mathcal{D}}_{\mathcal{L}}$ evolves as $\forall h \in [H], \dot{c}_h = \{\dot{o}_{1:h-1}, \dot{a}_{1:h-1}, \dot{o}_{1,h}\}$ and $\dot{p}_{1,h} = \emptyset, \dot{p}_{2,h} = \{\dot{o}_{i,h}\}$. Then, for any $i_1, i_2 \in [n], h_1, h_2 \in [H], h_1 < h_2$, $\dot{a}_{i_1,h_1} \subseteq \dot{c}_{h_1+1} \subseteq \dot{c}_{h_2} \subseteq \dot{\tau}_{i_2,h_2}$. If $i_1 = 1$, then $\dot{\tau}_{i_1,h_1} = \{\dot{o}_{1:h_1-1}, \dot{a}_{1:h_1-1}, \dot{o}_{1,h_1}\} \subseteq \dot{c}_{h_1+1} \subseteq \dot{c}_{h_2} \subseteq \dot{\tau}_{i_2,h_2}$. If $i_1 = 2$, then $\dot{\tau}_{i_1,h_1} = \{\dot{o}_{1:h_1}, \dot{a}_{1:h_1-1}\} \subseteq \dot{c}_{h_1+1} \subseteq \dot{c}_{h_2} \subseteq \dot{\tau}_{i_2,h_2}$. Therefore, we have $\sigma(\dot{\tau}_{i_1,h_1}) \subseteq \sigma(\dot{\tau}_{i_2,h_2})$, and thus $\mathcal{L}$ is sQC.
- **Example 4:** Since in $\overline{\mathcal{D}}_{\mathcal{L}}$, for any $i_1, i_2 \in [n], h_1, h_2 \in [H]$, agent (i_1, h_1) does not influence agent (i_2, h_2), then $\mathcal{L}$ is sQC.
- **Example 5:** The information in $\overline{\mathcal{D}}_{\mathcal{L}}$ evolves as $\forall h \in [H], i \in [n], \dot{c}_h = \{\dot{o}_{1:h-1}\}$ and $\dot{p}_{i,h} = \{\dot{o}_{i,h}\}$. Then, for any $i_1, i_2 \in [n], h_1, h_2 \in [H], h_1 < h_2$, $\dot{\tau}_{i_1,h_1} = \{\dot{o}_{1:h_1-1}, \dot{o}_{i_1,h_1}\} \subseteq \dot{c}_{h_1+1} \subseteq \dot{c}_{h_2} \subseteq \dot{\tau}_{i_2,h_2}$. However, agent $(1,1)$ may influence agent $(1,2)$ but $\sigma(\dot{a}_{1,1}) \nsubseteq \sigma(\dot{\tau}_{1,2})$. Hence, $\mathcal{L}$ is QC but not sQC.
- **Example 6:** The information in $\overline{\mathcal{D}}_{\mathcal{L}}$ evolves as $\forall h \in [H], i \in [n], \dot{c}_h = \{\dot{o}_{1:h-1}, \dot{a}_{1:h-2}\}$ and $\dot{p}_{i,h} = \{\dot{o}_{i,h}, \dot{a}_{i,h-1}\}$. Then, for any $i_1, i_2 \in [n], h_1, h_2 \in [H], h_1 < h_2$, $\dot{\tau}_{i_1,h_1} = \{\dot{o}_{1:h_1-1}, \dot{a}_{1:h_1-2}, \dot{o}_{i_1,h_1}, \dot{a}_{i_1,h_1-1}\} \subseteq \dot{c}_{h_1+1} \subseteq \dot{c}_{h_2} \subseteq \dot{\tau}_{i_2,h_2}$, and $\dot{a}_{i_1,h_1} \subseteq \dot{c}_{h_1+1} \subseteq \dot{c}_{h_2} \subseteq \dot{\tau}_{i_2,h_2}$. However, agent $(1,1)$ may influence agent $(2,2)$ but $\sigma(\dot{a}_{1,1}) \nsubseteq \sigma(\dot{\tau}_{2,2})$. Hence, $\mathcal{L}$ is QC but not sQC.
- **Example 7:** The information in $\overline{\mathcal{D}}_{\mathcal{L}}$ evolves as $\forall h \in [H], i \neq 1, \dot{c}_h = \{\dot{o}_{1,1:h-1}, \dot{o}_{-1,1:h-d}\}, \dot{p}_{1,h} = \emptyset, \dot{p}_{i,h} = \{o_{i,h-d+1:h}\}$. Then, for any $i_1, i_2 \in [n], h_1, h_2 \in [H], h_1 < h_2$. If $i_1 \neq 1$, then agent (i_1, h_1) will not influence agent (i_2, h_2). If $i_1 = 1$, then $\dot{\tau}_{i_1,h_1} = \{\dot{o}_{1,1:h_1}, \dot{o}_{-1,1:h_1-d}\} \subseteq \dot{c}_{h_1+1} \subseteq \dot{c}_{h_2} \subseteq \dot{\tau}_{i_2,h_2}$. Therefore, we have $\sigma(\dot{\tau}_{i_1,h_1}) \subseteq \sigma(\dot{\tau}_{i_2,h_2})$ if agent (i_1, h_1) influences agent (i_2, h_2). However, agent $(1,1)$ may influence agent $(1,2)$ but $\sigma(\dot{a}_{1,1}) \nsubseteq \sigma(\dot{\tau}_{1,2})$. Hence, $\mathcal{L}$ is QC but not sQC.
- **Example 8:** The information in $\overline{\mathcal{D}}_{\mathcal{L}}$ evolves as $\forall h \in [H], i \neq 1, \dot{c}_h = \{\dot{o}_{1,1:h-1}, \dot{a}_{1,1:h-2}, \dot{o}_{-1,1:h-d}\}, \dot{p}_{1,h} = \{\dot{a}_{1,h-1}\}, \dot{p}_{i,h} = \{o_{i,h-d+1:h}\}$. Then, for any $i_1, i_2 \in [n], h_1, h_2 \in [H], h_1 < h_2$. If $i_1 \neq 1$, then agent (i_1, h_1) will not influence agent (i_2, h_2). If $i_1 = 1$, then $\dot{\tau}_{i_1,h_1} = \{\dot{o}_{1,1:h_1}, \dot{a}_{1,h_1-1}, \dot{o}_{-1,1:h_1-d}\} \subseteq \dot{c}_{h_1+1} \subseteq \dot{c}_{h_2} \subseteq \dot{\tau}_{i_2,h_2}$. Therefore, we have $\sigma(\dot{\tau}_{i_1,h_1}) \subseteq \sigma(\dot{\tau}_{i_2,h_2})$ if agent (i_1, h_1) influences agent (i_2, h_2). However, agent $(1,1)$ may influence agent $(2,2)$ but $\sigma(\dot{a}_{1,1}) \nsubseteq \sigma(\dot{\tau}_{2,2})$. Hence, $\mathcal{L}$ is QC but not sQC.

This completes the proof. $\square$

B Deferred Details of §3

Remark B.1. In the following proofs, for clarity, we use $O, A, M, C, P, \mathscr{T}$ to denote the realizations of random variables o, a, m, c, p, τ with the same subscripts.

As a preliminary, we first have the following lemma.

Lemma B.2. Given any QC LTC $\mathcal{L}$, its induced Dec-POMDP $\overline{\mathcal{D}}_{\mathcal{L}}$ and any $i_1, i_2 \in [n], h_1, h_2 \in [H]$. If agent (i_1, h_1) influences agent (i_2, h_2) in the intrinsic model of $\overline{\mathcal{D}}_{\mathcal{L}}$, then for the random variables $\tau_{i_1,h_1^-}, \tau_{i_2,h_2^-}$ in $\mathcal{L}$, we have $\sigma(\tau_{i_1,h_1^-}) \subseteq \sigma(\tau_{i_2,h_2^-})$. Moreover, if $\mathcal{L}$ is sQC, then for random variables $a_{i_1,h_1}, \tau_{i_2,h_2^-}$ in $\mathcal{L}$, we have $\sigma(a_{i_1,h_1}) \subseteq \sigma(\tau_{i_2,h_2^-})$.

Proof. We denote by $\dot{\tau}_{i_1,h_1}, \dot{\tau}_{i_2,h_2}$ the information of agent $(i_1, h_1), (i_2, h_2)$ in the problem $\overline{\mathcal{D}}_{\mathcal{L}}$. From the definition of $\overline{\mathcal{D}}_{\mathcal{L}}$ being QC, if agent (i_1, h_1) influences agent (i_2, h_2), then $\sigma(\dot{\tau}_{i_1,h_1}) \subseteq \sigma(\dot{\tau}_{i_2,h_2})$. Since for any $h \in [H], i \in [n], \dot{\tau}_{i,h}$ is the information of agent (i, h) without additional sharing, then we know that $\tau_{i,h^-} \backslash \dot{\tau}_{i,h} \subseteq \cup_{t=1}^{h-1} z_t^a, \tau_{i,h^+} \backslash \dot{\tau}_{i,h} \subseteq \cup_{t=1}^{h} z_t^a$. Therefore, we know that $\sigma(\tau_{i_1,h_1^-} \backslash \dot{\tau}_{i_1,h_1}) \subseteq \sigma(\cup_{t=1}^{h-1} z_t^a) \subseteq \sigma(c_{h_1^-}) \subseteq \sigma(c_{h_2^-}) \subseteq \sigma(\tau_{i_2,h_2^-})$. Also, we know $\sigma(\dot{\tau}_{i_1,h_1}) \subseteq \sigma(\dot{\tau}_{i_2,h_2}) \subseteq \sigma(\tau_{i_2,h_2^-})$. Thus, we can conclude that $\sigma(\tau_{i_1,h_1^-}) \subseteq \sigma(\tau_{i_2,h_2^-})$. Moreover, if $\mathcal{L}$ is sQC, then from the the definition of $\overline{\mathcal{D}}_{\mathcal{L}}$ being sQC and agent (i_1, h_1) influences agent (i_2, h_2) in $\overline{\mathcal{D}}_{\mathcal{L}}$, it holds that $\sigma(a_{i_1,h_1}) \subseteq \sigma(\dot{\tau}_{i_2,h_2}) \subseteq \sigma(\tau_{i_2,h_2^-})$. $\square$

B.1 Hardness results

Lemma B.3 (Non-classical LTCs are hard)**.** For non-classical LTCs under Assumption 3.1, 3.2, 3.3, 3.4, and 4.3, finding an $\frac{\epsilon}{H}$-team optimum is `PSPACE-hard`.

Lemma B.4 (QC LTCs with full-history-dependent communication strategies are hard)**.** For QC LTCs under Assumption 3.1, together with Assumptions 3.3, 3.4, and 4.3, computing a team-optimum in the general space of $(\mathcal{G}_{1:H}^a, \mathcal{G}_{1:H}^m)$ with $\mathcal{G}_{i,h}^m := \{g_{i,h}^m : \mathcal{T}_{i,h^-} \to \mathcal{M}_{i,h}\}$ is `NP-hard`.

Lemma B.5 (QC LTCs without Assumption 3.3 are hard)**.** For QC LTCs under Assumptions 3.1, 3.2, 3.4 and 4.3, finding a team-optimum is still `NP-hard`.

Lemma B.6 (QC LTCs without Assumption 3.4 are hard)**.** For QC LTCs under Assumption 3.1, 3.2, 3.3, and 4.3, finding an ϵ/H-team optimum is still `PSPACE-hard`.

B.2 Proof of Lemma B.3

Proof. We first have the following proposition on the hardness of solving POMDPs.

Proposition B.7. There exists an $\epsilon > 0$, such that computing an ϵ-additive optimal strategy in POMDPs is `PSPACE-hard`.

One can adapt the proof of (Lusena et al., 2001, Theorem 4.11), which proved the `PSPACE-hardness` of computing an ϵ-*relative* optimal strategy in POMDPs, to obtain such a result for an ϵ-*additive* one. In particular, any ϵ-additive optimal strategy in the POMDP constructed in the proof of Theorem 4.11 therein is also an ϵ-relative optimal strategy.

Now we proceed with the proof of Lemma B.3 based on the Proposition B.7. Given any POMDP $\mathcal{P} = (\mathcal{S}^{\mathcal{P}}, \mathcal{A}^{\mathcal{P}}, \mathcal{O}^{\mathcal{P}}, \{\mathbb{O}_h^{\mathcal{P}}\}_{h\in[H^{\mathcal{P}}]}, \{\mathbb{T}_h^{\mathcal{P}}\}_{h\in[H^{\mathcal{P}}]}, \{\mathcal{R}_h^{\mathcal{P}}\}_{h\in[H^{\mathcal{P}}]}, \mu_1^{\mathcal{P}})$, we can construct an LTC $\mathcal{L}$ as follows:

- Number of agents: $n = 3$; length of episode: $H = 2H^{\mathcal{P}}$.
- Underlying state space: $\mathcal{S} = \mathcal{S}^{\mathcal{P}} \times [2]$. For any $s \in \mathcal{S}$, we can split $s = (s^1, s^2)$, where $s^1 \in \mathcal{S}^{\mathcal{P}}, s^2 \in [2]$. Intial state distribution: $\forall s \in \mathcal{S}, \mu_1(s) = \mu_1^{\mathcal{P}}(s^1)/2$.
- Control action space: For any $h \in [H], \mathcal{A}_{1,h} = \mathcal{A}^{\mathcal{P}}, \mathcal{A}_{2,h} = [2], \mathcal{A}_{3,h} = \{\emptyset\}$.

- Transition functions: For any $h \in [H-1], s_h, s_{h+1} \in \mathcal{S}, a_h \in \mathcal{A}_h$, if $h = 2t-1$ with $t \in [H^{\mathcal{P}}], \mathbb{T}_h(s_{h+1} \mid s_h, a_h) = \mathbb{T}_t^{\mathcal{P}}(s_{h+1}^1 \mid s_h^1, a_{1,h}) \mathbb{1}[s_{h+1}^2 = s_h^2]$; if $h = 2t$ with $t \in [H^{\mathcal{P}} - 1], \mathbb{T}_h(s_{h+1} \mid s_h, a_h) = \mathbb{1}[s_{h+1}^1 = s_h^1, s_{h+1}^2 = a_{2,h}]$.
- Observation space: For any $h \in [H]$, if $h = 2t-1$ with $t \in [H^{\mathcal{P}}], \mathcal{O}_{1,h} = \mathcal{O}_t^{\mathcal{P}}, \mathcal{O}_{2,h} = \mathcal{O}_{3,h} = \mathcal{S}$; if $h = 2t$ with $t \in [H^{\mathcal{P}}], \mathcal{O}_{1,h} = [2], \mathcal{O}_{2,h} = \mathcal{O}_{3,h} = \mathcal{S}$.
- Emission matrix: For any $h \in [H]$, if $h = 2t-1$ with $t \in [H^{\mathcal{P}}], \forall o_h \in \mathcal{O}_h, s_h \in \mathcal{S}, \mathbb{O}_h(o_h \mid s_h) = \mathbb{O}_h^{\mathcal{P}}(o_{1,h} \mid s_h^1) \mathbb{1}[o_{2,h} = o_{3,h} = s_h]$; if $h = 2t$ with $t \in [H^{\mathcal{P}}], \forall o_h \in \mathcal{O}_h, s_h \in \mathcal{S}, \mathbb{O}_h(o_h \mid s_h) = \mathbb{1}[o_{1,h} = s_h^1, o_{2,h} = o_{3,h} = s_h]$.
- The baseline sharing: null.
- The communication action space: For any $h \in [H], \mathcal{M}_{1,h} = \mathcal{M}_{2,h} = \{0,1\}^{2h-1}, \mathcal{M}_{3,h} = \{0,1\}^h$. For any $i \in [2], p_{i,h^-} \in \mathcal{P}_{i,h^-}, \phi_{i,h}(p_{i,h^-}, m_{i,h}) = \{o_{i,k} \mid k \leq h, (2k-1)\text{-th digit of } p_{i,h^-} \text{ is 1 and } o_{i,k} \in p_{i,h^-}\} \cup \{a_{i,k} \mid k \leq h, 2k\text{-th digit of } p_{i,h^-} \text{ is 1 and } a_{i,k} \in p_{i,h^-}\} \cup \{m_{i,h}\}$. For agent $3, p_{3,h^-} \in \mathcal{P}_{3,h^-}, \phi_{3,h}(p_{3,h^-}, m_{3,h}) = \{o_{3,k} \mid k \leq h, k\text{-th digit of } p_{3,h^-} \text{ is 1 and } o_{3,k} \in p_{3,h^-}\} \cup \{m_{3,h}\}$.
- Reward function: For any $h \in [H], i \in [3], s_h \in \mathcal{S}, a_h \in \mathcal{A}_h$, if $h = 2t-1$ with $t \in [H^{\mathcal{P}}], \mathcal{R}_h(s_h, a_h) = \mathcal{R}_t^{\mathcal{P}}(s_h^1, a_{1,h})/H$; if $h = 2t$ with $t \in [H^{\mathcal{P}}], \mathcal{R}_h(s_h, a_h) = \mathbb{1}[a_{2,h} = 1]$.
- Communication cost function: For any $h \in [H], z_h^a \in \mathcal{Z}_h^a, \mathcal{K}_h(z_h^a) = \mathbb{1}[z_h^a \neq \{m_h\}]$. It means that the communication cost is 1 unless there is no additional sharing.
- We restrict the communication strategy only to use c_h as input. And for any $t \in [H-1]$, we remove $a_{3,t}$ in τ_h for any $h > t$.

We first verify that such a construction satisfies Assumptions 3.1, 3.2, 3.3, 3.4, and 4.3.

- $\mathcal{L}$ satisfies Assumption 3.1, 3.4 because both agent 2 and agent 3 have individual γ-observability. That is, for any $b_1, b_2 \in \Delta(\mathcal{S}), i = 2, 3$, we have

$$
\begin{aligned}
||\mathbb{O}_{i,h}^{\top}(b_1 - b_2)||_1 &= \sum_{o_{i,h} \in \mathcal{O}_h} \Big| \sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h)) \mathbb{O}_{i,h}(o_{i,h} \mid s_h) \Big| \\
&= \sum_{o_{i,h} \in \mathcal{O}_h} \Big| \sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h)) \mathbb{1}[o_{i,h} = s_h] \Big| \\
&= \sum_{o_{i,h} \in \mathcal{O}_h} |b_1(o_{i,h}) - b_2(o_{i,h})| = ||b_1 - b_2||_1.
\end{aligned}
$$

- $\mathcal{L}$ satisfies Assumption 3.2 because we restrict communication strategy can only use c_h as input.
- $\mathcal{L}$ satisfies Assumption 3.3 since only $a_{3,t}, t \in [H-1]$ do not influence underlying state, and we remove $a_{3,t}$ from τ_h for any $h > t$.
- $\mathcal{L}$ satisfies Assumption 4.3 since it satisfies the **Turn-based structures** condition in §G, with $ct(2t-1) = 1, ct(2t) = 2$ for any $t \in [H^{\mathcal{P}}]$.

In this LTC problem $\mathcal{L}$, agent 2 will always choose $a_{i,2t} = 1$ at even steps to obtain $r_{2h} = 1$. And there will be no additional sharing since any additional sharing at timestep h will incur a communication cost $\kappa_h = 1 > \max \sum_{t=1}^{H^{\mathcal{P}}} \mathcal{R}_{2t-1}(s_{2t-1}, a_{2t-1})$, and thus it cannot achieve optimum. Therefore, state $s_h^2, h \in [H]$ are dummy states, and agents $2, 3$ are dummy agents. Then, any $(g_{1:H}^{a,*}, g_{1:H}^{m,*})$ being an $\frac{\epsilon}{H}$-team optimal strategy of $\mathcal{L}$ will directly give an ϵ-team-optimal strategy of $\mathcal{P}$ as $\{g_{1,2t-1}^{a,*}\}_{h \in [H^{\mathcal{P}}]}$. From Proposition B.7, we can complete the proof. □

B.3 Proof of Lemma B.4

Proof. We prove this result by showing a reduction from the Team Decision problem (Tsitsiklis & Athans, 1985).

Definition B.8 (Team decision problem (TDP))**.** Given finite sets Y_1, Y_2, U_1, U_2, a rational probability mass function $p : Y_1 \times Y_2 \to \mathbb{Q}$, and an integer cost function $c : Y_1 \times Y_2 \times U_1 \times U_2 \to \mathbb{N}$, find decision rules $\gamma_i : Y_i \to U_i, i = 1, 2$ that minimize the expected cost

$$J(\gamma_1, \gamma_2) = \sum_{y_1 \in Y_1, y_2 \in Y_2} c(y_1, y_2, \gamma_1(y_1), \gamma_2(y_2))p(y_1, y_2). \tag{B.1}$$

We show the `NP-hardness` of solving LTC from the problem TDP. Given any TDP $\mathcal{TD} = (\widetilde{Y}_1, \widetilde{Y}_2, \widetilde{U}_1, \widetilde{U}_2, \widetilde{c}, \widetilde{p}, \widetilde{J})$ with $|\widetilde{U}_1| = |\widetilde{U}_2| = 2$, let $\widetilde{U}_1 = \{1, 2\}, \widetilde{U}_2 = \{1, 2\}$, then we can construct an $H = 4$ and 2-agent LTC $\mathcal{L}$ with two parameters $n_1 \in \mathbb{N}, \alpha_1 \in \mathbb{R}, \alpha_2 \in (0, 1)$ (to be specified later) such that:

- Number of agents: $n = 2$; length of episode: $H = 4$.
- Underlying state: $\mathcal{S} = [2]^4$. For each $s_1 \in \mathcal{S}$, we can split s_1 into 4 parts as $s_1 = (s_1^1, s_1^2, s_1^3, s_1^4)$, where $s_1^1, s_1^2, s_1^3, s_1^4 \in [2]$. Similarly, $s_2, s_3, s_4 \in \mathcal{S}$ can be split in the same way.
- Initial state distribution: $\forall s_1 \in \mathcal{S}, \mu_1(s_1) = \frac{1}{16}$.
- Control action space: For the first 2 timesteps, $\forall i = 1, 2, \mathcal{A}_{i,1} = \mathcal{A}_{i,2} = \{\emptyset\}$; for $h = 3, \mathcal{A}_{1,3} = [2], \mathcal{A}_{2,3} = \{\emptyset\}$; for $h = 4, \mathcal{A}_{2,4} = [2], \mathcal{A}_{1,4} = \{\emptyset\}$.
- Transition: $\forall s \in \mathcal{S}, a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2, a_3 \in \mathcal{A}_3, \mathbb{T}_1(s \mid s, a_1) = \mathbb{T}_2(s \mid s, a_2) = \mathbb{T}_3(s \mid s, a_3) = 1$. Note that under the transition dynamics above, $s_1 = s_2 = s_3 = s_4$ always holds, for any $s_1 \in \mathcal{S}$.
- Observation space: $\mathcal{O}_{1,1} = \mathcal{O}_{2,1} = \mathcal{O}_{1,2} = \mathcal{O}_{2,2} = [2] \times \mathcal{S}, \mathcal{O}_{1,3} = \widetilde{Y_1} \times \mathcal{S}, \mathcal{O}_{2,3} = \widetilde{Y_2} \times \mathcal{S}, \mathcal{O}_{1,4} = \mathcal{O}_{2,4} = \mathcal{S}$; For each $i \in [2], h \in [2], o_{i,h} \in \mathcal{O}_{i,h}$, we can split $o_{i,h}$ into 2 parts as $o_{i,h} = (o_{i,h}^1, o_{i,h}^2)$, where $o_{i,h}^1 \in [2], o_{i,h}^2 \in \mathcal{S}$. For each $i \in [n], o_{i,3} \in \mathcal{O}_{i,3}$, similarly, we can split $o_{i,3}$ into 2 parts as $o_{i,3} = (o_{i,3}^1, o_{i,3}^2)$, where $o_{i,3}^1 \in \widetilde{Y}_i, o_{i,3}^2 \in \mathcal{S}$.
- The baseline sharing is null.
- Communication action space: For $i \in [2], h \in \{1, 2, 4\}, \mathcal{M}_{i,h} = \{0, 1\}^h, \mathcal{M}_{i,3} = \{1, 2\}$; For each $i \in [2], \phi_{i,h}$ is defined as $\forall h \in \{1, 2, 4\}, \phi_{i,h}(p_{i,h^-}, m_{i,h}) = \{o_{i,k} \mid k \leq h, k\text{-th digit of } m_{i,h} \text{ is 1 and } o_{i,k} \in p_{i,h^-}\}$; For $h = 3$, if $m_{i,3} = 1$, then $\phi_{i,h}(p_{i,3^-}, m_{i,3}) = \{o_{i,1}, o_{i,3}, m_{i,3}\}$; if $m_{i,3} = 2$, then $\phi_{i,h}(p_{i,h^-}, m_{i,3}) = \{o_{i,2}, o_{i,3}, m_{i,3}\}$.
- Emission matrix: For any $i \in [2], h \in [2], s_h \in \mathcal{S}, o_{i,h} \in \mathcal{O}_{i,h}, \mathbb{O}_h(o_h \mid s_h) = \Pi_{i=1}^2 \mathbb{O}_{i,h}(o_{i,h} \mid s_h)$ and $\mathbb{O}_{i,h}(o_{i,h} \mid s_h)$ is defined as:

$$\mathbb{O}_{i,h}(o_{i,h} \mid s_h) = \begin{cases} \frac{1-\alpha_2}{16} & o_{i,h}^1 = s_h^{i+2h-2}, o_{i,h}^2 \neq s_h \\ \frac{1-\alpha_2}{16} + \alpha_2 & o_{i,h}^1 = s_h^{i+2h-2}, o_{i,h}^2 = s_h \\ 0 & \text{o.w.} \end{cases}.$$

  For $i \in [2], s_3 \in \mathcal{S}, o_3 \in \mathcal{O}_3, \mathbb{O}_3(o_3 \mid s_3) = \mathbb{O}_3^1(o_3^1 \mid s_3)\mathbb{O}_3^2(o_3^2 \mid s_3), \mathbb{O}_3^2 = \Pi_{i=1}^2 \mathbb{O}_{i,3}^2(o_{i,3}^2 \mid s_3)$ is defined as:

$$\mathbb{O}_3^1(o_3^1 \mid s_3) = \widetilde{p}(o_{1,3}^1, o_{2,3}^1)$$
$$\mathbb{O}_{i,3}^2(o_3^2 \mid s_3) = \begin{cases} \frac{1-\alpha_2}{16} & o_{i,3}^2 \neq s_3 \\ \frac{1-\alpha_2}{16} + \alpha_2 & o_{i,3}^2 = s_3 \end{cases}.$$

  And for $i \in [2], s_4 \in \mathcal{S}, o_{i,4} \in \mathcal{O}_{i,4}, \mathbb{O}_4(o_4 \mid s_h) = \Pi_{i=1}^2 \mathbb{O}_{i,4}(o_{i,4} \mid s_4)$ and $\mathbb{O}_{i,4}(o_{i,4} \mid s_4)$ is defined as:

$$\mathbb{O}_{i,4}(o_{i,4} \mid s_4) = \begin{cases} \frac{1-\alpha_2}{16} & o_{i,4} \neq s_4 \\ \frac{1-\alpha_2}{16} + \alpha_2 & o_{i,4} = s_4 \end{cases}.$$

Such an emission matrix means that for each $h \in [2]$ and $i \in [2]$, agent i will accurately observe part of the underlying state s_h^{i+2h-2} and vaguely observe the whole underlying state s_h. For $h = 4, i \in [2]$, agent i can only vaguely observe the whole underlying state s_h. Such design is to make the problem satisfying Assumption 3.1. The reward functions are defined as:

$$\mathcal{R}_1(s_1, a_1) = \mathcal{R}_2(s_2, a_2) = 0, \ \ \forall s_1, s_2 \in \mathcal{S}, a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2;$$
$$\mathcal{R}_3(s_3, a_3) = \begin{cases} 1 & \text{if } a_{1,3} = s_3^2 \text{ or } a_{1,3} = s_3^4 \\ 0 & \text{o.w.} \end{cases};$$
$$\mathcal{R}_4(s_4, a_4) = \begin{cases} 1 & \text{if } a_{2,4} = s_4^1 \text{ or } a_{2,4} = s_4^3 \\ 0 & \text{o.w.} \end{cases}.$$

The communication cost functions are defined as:

$$\forall h \in \{1, 2, 4\}, z_h^a \in \mathcal{Z}_h^a, \mathcal{K}_h(z_h^a) = 1 \text{ if } z_h^a \neq \{m_{1,h}, m_{2,h}\} \text{ else } 0;$$
$$\mathcal{K}_3(z_3^a) = \begin{cases} \widetilde{c}(o_{1,3}^1, o_{2,3}^1, 1, 1)/\alpha_1 & \text{if } \{o_{1,1}, o_{2,1}\} \subseteq z_3^a \text{ and } \{o_{1,2}, o_{2,2}\} \cap z_3^a = \emptyset \\ \widetilde{c}(o_{1,3}^1, o_{2,3}^1, 2, 1)/\alpha_1 & \text{if } \{o_{1,2}, o_{2,1}\} \subseteq z_3^a \text{ and } \{o_{1,1}, o_{2,2}\} \cap z_3^a = \emptyset \\ \widetilde{c}(o_{1,3}^1, o_{2,3}^1, 1, 2)/\alpha_1 & \text{if } \{o_{1,1}, o_{2,2}\} \subseteq z_3^a \text{ and } \{o_{1,2}, o_{2,1}\} \cap z_3^a = \emptyset \\ \widetilde{c}(o_{1,3}^1, o_{2,3}^1, 2, 2)/\alpha_1 & \text{if } \{o_{1,2}, o_{2,2}\} \subseteq z_3^a \text{ and } \{o_{1,1}, o_{2,1}\} \cap z_3^a = \emptyset \end{cases}.$$

Let $\alpha_0 = \max_{y_1, y_2, u_1, u_2} \widetilde{c}(y_1, y_2, u_1, u_2)$, and set $\alpha_1 = 2\alpha_0$. Under such a construction, $\mathcal{L}$ satisfies the following conditions:

- Problem $\mathcal{L}$ is QC: For $\forall i_1, i_2 \in [2], h_1, h_2 \in [4]$, agent (i_1, h_1) does not influence (i_2, h_2) because agent (i_1, h_1) cannot influence the observation of agent (i_2, h_2), and baseline sharing is null.
- Problem $\mathcal{L}$ satisfies Assumptions 3.1 and 3.4: We prove this by showing that each agent $i \in [2]$ satisfies γ-observability. For $\forall i \in [2], h \in [2], b_1, b_2 \in \Delta(\mathcal{S})$, let

$$\begin{aligned} ||\mathbb{O}_{i,h}^\top(b_1 - b_2)||_1 &= \sum_{o_{i,h}^1 \in [2]} \sum_{o_{i,h}^2 \in \mathcal{S}} |\sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h))\mathbb{O}_{i,h}((o_{i,h}^1, o_{i,h}^2) \mid s_h)| \\ &\geq \sum_{o_{i,h}^2 \in \mathcal{S}} |\sum_{o_{i,h}^1 \in [2]} \sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h))\mathbb{O}_{i,h}((o_{i,h}^1, o_{i,h}^2) \mid s_h)| \\ &= \sum_{o_{i,h}^2 \in \mathcal{S}} |\sum_{s_h \in \mathcal{S}} \sum_{o_{i,h}^1 \in [2]} (b_1(s_h) - b_2(s_h))\mathbb{1}[o_{i,h}^1 = s_h^{i+2h-2}](\frac{1-\alpha_2}{16} + \alpha_2 \mathbb{1}[o_{i,h}^2 = s_h])| \\ &= \sum_{o_{i,h}^2 \in \mathcal{S}} |\sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h))(\frac{1-\alpha_2}{16} + \alpha_2 \mathbb{1}[o_{i,h}^2 = s_h])| \\ &= \sum_{o_{i,h}^2 \in \mathcal{S}} |\frac{1-\alpha_2}{16}(\sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h))) + \alpha_2(b_1(o_{i,h}^2) - b_2(o_{i,h}^2))| \\ &= \sum_{o_{i,h}^2 \in \mathcal{S}} \alpha_2 |b_1(o_{i,h}^2) - b_2(o_{i,h}^2)| = \alpha_2 ||b_1 - b_2||_1. \end{aligned}$$

  For $\forall i \in [2], h = 3, 4$, the proof is similar, by replacing $o_{i,h}^1 \in [2]$ with $o_{i,h}^1 \in \widetilde{Y}_i$ for $h = 3$ and replacing the space $o_{i,h}^1 \in [2]$ with $\emptyset$ for $h = 4$.
- Problem $\mathcal{L}$ satisfies Assumption 3.3, because control actions $a_{1:4}$ does not influence underlying states and we restrict the communication and control strategies do not use them as input.
- Problem $\mathcal{L}$ satisfies Assumption 4.3 since it satisfies the **Turn-based structures** condition in §G, with $ct(1) = ct(2) = ct(3) = 1, ct(4) = 2$.

We will show as follows that computing a team-optimal strategy can give us a team-optimal strategy in $\mathcal{TD}$. Given $(g_{1:4}^{a,*}, g_{1:4}^{m,*})$ to be a team optimal strategy of $\mathcal{L}$, firstly it will have no additional sharing at timesteps $h = 1, 2, 4$, namely, for $h = 1, 2, 4, \mathbb{P}(z_h^a \neq \{m_{1,h}, m_{2,h}\} \mid g_{1:4}^{a,*}, g_{1:4}^{m,*}) = 1$, since any additional sharing at timesteps $h = 1, 2, 4$ will incur a cost as high as 1, and cannot achieve the optimum. Also, for the additional sharing at timestep $h = 3$, agent i will definitely share $o_{i,3}$ and choose to share $o_{i,1}$ or $o_{i,2}$. Then $\forall \tau_{1,3^+} \in \mathcal{T}_{1,3^+}, g^{a,*}_{1,3}(\tau_{1,3^+}) = \begin{cases} o_{2,1} & \text{if } o_{2,1} \in \tau_{1,3^+} \\ o_{2,2} & \text{if } o_{2,2} \in \tau_{1,3^+} \end{cases}$ and $\forall \tau_{2,4^+} \in \mathcal{T}_{2,4^+}, g^{a,*}_{2,4}(\tau_{2,4^+}) = \begin{cases} o_{1,1} & \text{if } o_{1,1} \in \tau_{2,4^+} \\ o_{1,2} & \text{if } o_{1,2} \in \tau_{2,4^+} \end{cases}$, since such action can achieve the optimal reward $r_3 = r_4 = 1$. Therefore, $J_{\mathcal{L}}(g^{a,*}_{1:H}, g^{m,*}_{1:H}) = \mathbb{E}[\sum_{h=1}^4 r_h - \kappa_h \,|\, g^{a,*}_{1:H}, g^{m,*}_{1:H}] = 2 - \mathbb{E}[\kappa_3 \,|\, g^{a,*}_{1:H}, g^{m,*}_{1:H}] = 2 - \mathbb{E}[\widetilde{c}(o^1_{1,3}, o^1_{2,3}, m_{1,3}, m_{2,3})]$, where $m_{1,3} = g^{m,*}_{1,3}(\{o_{1,1}, o_{1,2}, o_{1,3}\})$. Since κ_3 is independent of $o_{1,1}, o_{1,2}, o^1_{1,3}$, $o_{1,1}, o_{1,2}, o^1_{1,3}$ are useless information for agent 1 to choose $m_{1,3}$ and minimize the κ. Therefore, not using them in $g^{m,*}_{1,3}$ does not lose any optimality. Hence, we can consider the $g^{m,*}_{1,3}$ that only has $o^1_{1,3}$ as input. In the same way, we consider the $g^{m,*}_{2,3}$ that has $o^1_{2,3}$ as input. Therefore, $J_{\mathcal{L}}(g^{a,*}_{1:H}, g^{m,*}_{1:H}) = 2 - \sum_{o^1_{1,3}, o^2_{1,3}, m_{1,3}, m_{2,3}} \frac{\widetilde{c}(o^1_{1,3}, o^1_{2,3}, m_{1,3}, m_{2,3})}{\alpha_1} g^{m,*}_{1,3}(m_{1,3} \,|\, o^1_{1,3}) g^{m,*}_{2,3}(m_{2,3} \,|\, o^1_{2,3}) \widetilde{p}(o^1_{1,3}, o^1_{2,3})$. Then we can construct $\gamma_1 = g^{m,*}_{1,3}, \gamma_2 = g^{m,*}_{2,3}$, which minimize $\widetilde{J}$. Therefore, we can conclude that computing a team optimal strategy of $\mathcal{L}$ can give us a team optimal strategy of $\mathcal{TD}$. From the `NP-hardness` of the TDP problem (Tsitsiklis & Athans, 1985), we complete our proof. $\square$

B.4 Proof of Lemma B.5

Proof of Lemma B.5. We prove this result by showing a reduction from the Team Decision problem. Given any TDP $\mathcal{TD} = (\widetilde{Y}_1, \widetilde{Y}_2, \widetilde{U}_1, \widetilde{U}_2, \widetilde{c}, \widetilde{p}, \widetilde{J})$ with $|\widetilde{U}_1| = |\widetilde{U}_2| = 2$, let $\widetilde{U}_1 = \{1, 2\}, \widetilde{U}_2 = \{1, 2\}$, then we can construct an $H = 5$ and 2 agents LTC $\mathcal{L}$ as follows:

- Underlying state: $\mathcal{S} = [2]^4$. For each $s_1 \in \mathcal{S}$, we can split s_1 into 4 parts as $s_1 = (s_1^1, s_1^2, s_1^3, s_1^4)$, where $s_1^1, s_1^2, s_1^3, s_1^4 \in [2]$. Similarly, $s_2, s_3, s_4, s_5 \in \mathcal{S}$ can be split in the same way.
- Initial state distribution: $\forall s_1 \in \mathcal{S}, \mu_1(s_1) = \frac{1}{16}$.
- Control action space: For $\forall i = 1, 2$, for $h = 1, 2$, $\mathcal{A}_{i,1} = \mathcal{A}_{i,2} = \{\emptyset\}$; For $h = 3$, $\mathcal{A}_{i,3} = \{(0, x), (x, 0) \,|\, x \in [2]\}$; We can write $a_{i,3} = (a^1_{i,3}, a^2_{i,3}), a^1_{i,3}, a^2_{i,3} \in \{0, 1, 2\}$. For $h = 4, \mathcal{A}_{1,4} = [2], \mathcal{A}_{2,4} = \{\emptyset\}$; For $h = 5, \mathcal{A}_{2,5} = [2], \mathcal{A}_{1,5} = \{\emptyset\}$.
- Transition: $\forall s \in \mathcal{S}, a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2, a_3 \in \mathcal{A}_3, a_4 \in \mathcal{A}_4, \mathbb{T}_1(s \,|\, s, a_1) = \mathbb{T}_2(s \,|\, s, a_2) = \mathbb{T}_3(s \,|\, s, a_3) = \mathbb{T}_4(s \,|\, s, a_4) = 1$. Note that under the transition dynamics above, $s_1 = s_2 = s_3 = s_4 = s_5$ always holds, for any $s_1 \in \mathcal{S}$.
- Observation space: $\mathcal{O}_{1,1} = \mathcal{O}_{2,1} = \mathcal{O}_{1,2} = \mathcal{O}_{2,2} = [2] \times \mathcal{S}, \mathcal{O}_{1,3} = \widetilde{Y}_1 \times \mathcal{S}, \mathcal{O}_{2,3} = \widetilde{Y}_2 \times \mathcal{S}, \mathcal{O}_{1,4} = \mathcal{O}_{2,4} = \mathcal{O}_{1,5} = \mathcal{O}_{2,5} = \mathcal{S}$; For each $i \in [2], h \in [2], o_{i,h} \in \mathcal{O}_{i,h}$, we can split $o_{i,h}$ into 2 parts as $o_{i,h} = (o^1_{i,h}, o^2_{i,h})$, where $o^1_{i,h} \in [2], o^2_{i,h} \in \mathcal{S}$. For each $i \in [2], o_{i,3} \in \mathcal{O}_{i,3}$, similarly, we can split $o_{i,3}$ into 2 parts as $o_{i,3} = (o^1_{i,3}, o^2_{i,3})$, where $o^1_{i,3} \in \widetilde{Y}_i, o^2_{i,3} \in \mathcal{S}$.
- The baseline sharing is null.
- Communication action space: For $i \in [2], h \in \{1, 2, 3, 5\}, \mathcal{M}_{i,h} = \{0, 1\}^{2h-1}$ and $\phi_{i,h}$ is defined as $\phi_{i,h}(p_{i,h^-}, m_{i,h}) = \{o_{i,k} \in p_{i,h^-} \,|\, k \leq h, (2k-1)^{\text{th}} \text{ digit of } m_{i,h} \text{ is } 1\} \cup \{a_{i,k} \in p_{i,h^-} \,|\, k \leq h-1, 2k^{\text{th}} \text{ digit of } m_{i,h} \text{ is } 1\} \cup \{m_{i,h}\}$; For $h = 4, \mathcal{M}_{i,4} = \{1, 2\}, \phi_{i,h}(p_{i,h^-}, 1) = \{o_{i,3}, m_{i,h}\}, \phi_{i,h}(p_{i,h^-}, 2) = \{o_{i,3}, a_{i,3}, m_{i,h}\}$.
- Emission matrix: For any $i \in [2], h \in [2], s_h \in \mathcal{S}, o_{i,h} \in \mathcal{O}_{i,h}, \mathbb{O}_h(o_h \,|\, s_h) = \Pi_{i=1}^2 \mathbb{O}_{i,h}(o_{i,h} \,|\, s_h)$ and $\mathbb{O}_{i,h}(o_{i,h} \,|\, s_h)$ is defined as:

$$\mathbb{O}_{i,h}(o_{i,h} \,|\, s_h) = \begin{cases} \frac{1-\alpha_2}{16} & o^1_{i,h} = s_h^{i+2h-2}, o^2_{i,h} \neq s_h \\ \frac{1-\alpha_2}{16} + \alpha_2 & o^1_{i,h} = s_h^{i+2h-2}, o^2_{i,h} = s_h \\ 0 & \text{o.w.} \end{cases}.$$

For $i \in [2], s_3 \in \mathcal{S}, o_3 \in \mathcal{O}_3, \mathbb{O}_3(o_3 \mid s_3) = \mathbb{O}_3^1(o_3^1 \mid s_3)\mathbb{O}_3^2(o_3^2 \mid s_3), \mathbb{O}_3^2 = \Pi_{i=1}^2 \mathbb{O}_{i,3}^2(o_{i,3}^2 \mid s_3)$ is defined as:

$$\mathbb{O}_3^1(o_3^1 \mid s_3) = \widetilde{p}(o_{1,3}^1, o_{2,3}^1)$$
$$\mathbb{O}_{i,3}^2(o_3^2 \mid s_3) = \begin{cases} \frac{1-\alpha_2}{16} & o_{i,3}^2 \neq s_3 \\ \frac{1-\alpha_2}{16} + \alpha_2 & o_{i,3}^2 = s_3 \end{cases}.$$

And for $i \in [2], h = 4$ or $5, s_h \in \mathcal{S}, o_{i,h} \in \mathcal{O}_{i,h}, \mathbb{O}_h(o_h \mid s_h) = \Pi_{i=1}^2 \mathbb{O}_{i,h}(o_{i,h} \mid s_h)$ and $\mathbb{O}_{i,h}(o_{i,h} \mid s_h)$ is defined as:

$$\mathbb{O}_{i,h}(o_{i,h} \mid s_h) = \begin{cases} \frac{1-\alpha_2}{16} & o_{i,h} \neq s_h \\ \frac{1-\alpha_2}{16} + \alpha_2 & o_{i,h} = s_h \end{cases}.$$

- Reward functions:

$$\mathcal{R}_1(s_1, a_1) = \mathcal{R}_2(s_2, a_2) = \mathcal{R}_3(s_3, a_3) = 0, \ \ \forall s_1, s_2, s_3 \in \mathcal{S}, a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2, a_3 \in \mathcal{A}_3;$$
$$\mathcal{R}_4(s_4, a_4) = \begin{cases} 1 & \text{if } a_{1,4} = s_4^2 \text{ or } a_{1,4} = s_4^4 \\ 0 & \text{o.w.} \end{cases};$$
$$\mathcal{R}_5(s_5, a_5) = \begin{cases} 1 & \text{if } a_{2,5} = s_5^1 \text{ or } a_{2,5} = s_5^3 \\ 0 & \text{o.w.} \end{cases}.$$

- Communication cost functions:

$$\forall h \in \{1,2,3,5\}, z_h^a \in \mathcal{Z}_h^a, \mathcal{K}_h(z_h^a) = 1 \text{ if } z_h^a \neq \{m_{1,h}, m_{2,h}\} \text{ else } 0;$$
$$\mathcal{K}_4(z_4^a) = \begin{cases} \widetilde{c}(o_{1,3}^1, o_{2,3}^1, 1, 1)/\alpha_1 & \text{if } a_{1,3}, a_{2,3} \in z_3^a, a_{1,3}^1 = 0, a_{2,3}^1 = 0 \\ \widetilde{c}(o_{1,3}^1, o_{2,3}^1, 2, 1)/\alpha_1 & \text{if } a_{1,3}, a_{2,3} \in z_3^a, a_{1,3}^2 = 0, a_{2,3}^1 = 0 \\ \widetilde{c}(o_{1,3}^1, o_{2,3}^1, 1, 2)/\alpha_1 & \text{if } a_{1,3}, a_{2,3} \in z_3^a, a_{1,3}^1 = 0, a_{2,3}^2 = 0 \\ \widetilde{c}(o_{1,3}^1, o_{2,3}^1, 2, 2)/\alpha_1 & \text{if } a_{1,3}, a_{2,3} \in z_3^a, a_{1,3}^2 = 0, a_{2,3}^2 = 0 \\ 1 & \text{o.w.} \end{cases};$$

Let $\alpha_0 = \max_{y_1, y_2, u_1, u_2} \widetilde{c}(y_1, y_2, u_1, u_2)$, set $\alpha_1 = 2\alpha_0$, and restrict agents to decide their communication strategy only based on their common information. Under such a construction, $\mathcal{L}$ satisfies the following conditions:

- Problem $\mathcal{L}$ is QC: For $\forall i_1, i_2 \in [2], h_1, h_2 \in [4]$, agent (i_1, h_1) does not influence (i_2, h_2) because agent (i_1, h_1) cannot influence the observation of agent (i_2, h_2), and the baseline sharing is null.

- Problem $\mathcal{L}$ satisfies Assumptions 3.1 and 3.4: We prove this by showing that each agent $i \in [2]$ satisfies γ-observability. For $\forall i \in [2], h \in [2], b_1, b_2 \in \Delta(\mathcal{S})$, let

$$
\begin{aligned}
||\mathbb{O}_{i,h}^\top(b_1 - b_2)||_1 &= \sum_{o_{i,h}^1 \in [2]} \sum_{o_{i,h}^2 \in \mathcal{S}} |\sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h))\mathbb{O}_{i,h}((o_{i,h}^1, o_{i,h}^2) \mid s_h)| \\
&\geq \sum_{o_{i,h}^2 \in \mathcal{S}} |\sum_{o_{i,h}^1 \in [2]} \sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h))\mathbb{O}_{i,h}((o_{i,h}^1, o_{i,h}^2) \mid s_h)| \\
&= \sum_{o_{i,h}^2 \in \mathcal{S}} |\sum_{s_h \in \mathcal{S}} \sum_{o_{i,h}^1 \in [2]} (b_1(s_h) - b_2(s_h))\mathbb{1}[o_{i,h}^1 = s_h^{i+2h-2}](\frac{1-\alpha_2}{16} + \alpha_2 \mathbb{1}[o_{i,h}^2 = s_h])| \\
&= \sum_{o_{i,h}^2 \in \mathcal{S}} |\sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h))(\frac{1-\alpha_2}{16} + \alpha_2 \mathbb{1}[o_{i,h}^2 = s_h])| \\
&= \sum_{o_{i,h}^2 \in \mathcal{S}} |\frac{1-\alpha_2}{16}(\sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h))) + \alpha_2(b_1(o_{i,h}^2) - b_2(o_{i,h}^2))| \\
&= \sum_{o_{i,h}^2 \in \mathcal{S}} \alpha_2 |b_1(o_{i,h}^2) - b_2(o_{i,h}^2)| = \alpha_2 ||b_1 - b_2||_1.
\end{aligned}
$$

  For $\forall i \in [2], h = 3, 4$, the proof is similar, by replacing $o_{i,h}^1 \in [2]$ with $o_{i,h}^1 \in \widetilde{Y}_i$ for $h = 3$ and replacing the space $o_{i,h}^1 \in [2]$ with $\{\emptyset\}$ for $h = 4, 5$.
- Problem $\mathcal{L}$ satisfies Assumption 3.2 since we restrict agents to decide their communication strategies only based on common information.
- Problem $\mathcal{L}$ satisfies Assumption 4.3 since it satisfies the **Turn-based structures** condition in §G, with $ct(1) = ct(2) = ct(3) = ct(4) = 1, ct(5) = 2$.

Now, we show that any team optimal strategy of $\mathcal{L}$ will give us the decision rules γ_1, γ_2 solving $\mathcal{TD}$. Let $(g_{1:5}^{a,*}, g_{1:5}^{m,*})$ be a team optimal strategy. First, $\forall \tau_{i,4^-} \in \mathcal{T}_{i,4^-}, g_{i,4}^{m,*}(\tau_{i,4^-}) = 2$, otherwise it will have communication cost $\kappa_{i,3} = 1$, and can cannot achieve the team optimum. Define $\overline{g}_{1:5}^a, \overline{g}_{1:5}^m$ as

$$
\begin{aligned}
&\forall \tau_{i,3^+} \in \mathcal{T}_{i,3^+}, \overline{g}_{i,3^+}^a(\tau_{i,3^+}) = \begin{cases} (o_{i,1}^1, 0) & \text{if } a_{i,3} = g_{i,3^+}^{a,*}(\tau_{i,3^+}), a_{i,3}^1 = 0 \\ (0, o_{i,2}^1) & \text{o.w.} \end{cases} \\
&\forall \tau_{1,4^+} \in \mathcal{T}_{1,4^+}, \overline{g}_{1,4^+}^a(\tau_{1,4^+}) = \begin{cases} a_{2,4}^1 & \text{if } a_{2,4}^1 \neq 0 \\ a_{2,4}^2 & \text{o.w.} \end{cases} \\
&\overline{g}_{1:5}^m = g_{1:5}^{m,*}, \overline{g}_{1:2}^a = g_{1:2}^{a,*}, \overline{g}_{4:5}^a = g_{4:5}^{a,*}.
\end{aligned}
$$

Then, $J_{\mathcal{L}}(\overline{g}_{1:5}^a, \overline{g}_{1:5}^m) - J_{\mathcal{L}}(g_{1:5}^{a,*}, g_{1:5}^{m,*}) \geq 0$. Hence $(\overline{g}_{1:5}^a, \overline{g}_{1:5}^m)$ is a team optimal strategy. Then, $J_{\mathcal{L}}(\overline{g}_{1:5}^a, \overline{g}_{1:5}^m) = 2 - \mathbb{E}[\kappa_4 \mid \overline{g}_{1:5}^a, \overline{g}_{1:5}^m] = 2 - \mathbb{E}[\kappa_4 \mid \overline{g}_3^a]$, where $\overline{g}_3^a$ minimizes κ_4. Note that $\tau_{i,3^+} = \{o_{i,1}, o_{i,2}, o_{i,3}\}$. Since κ_4 is independent of $o_{i,1}, o_{i,2}, o_{i,3}^2$, they are useless information for agent i to choose $a_{i,3}$ and minimize κ_4. Therefore, only using $o_{i,3}^1$ to determine $a_{i,3}$ does not lose any optimality, and we can consider $g_{1,3}^{a,*}$ that has only $o_{i,3}^1$ as input. In the same way, we consider $g_{2,3}^{a,*}$ that has only $o_{i,3}^1$ as input. Then, we can construct $\gamma_1 = g_{1,3}^{a,*}, \gamma_2 = g_{2,3}^{a,*}$ as decision rules that minimize $\widetilde{J}$. Therefore, we can conclude that computing a team optimal strategy of $\mathcal{L}$ can give us a team optimal strategy of $\mathcal{TD}$. From the `NP-hardness` of the TDP problem (Tsitsiklis & Athans, 1985), we complete our proof. □

B.5 Proof of Lemma B.6

Proof. We prove this by showing a reduction from the hardness of finding an ϵ-optimal strategy in POMDP. Given any POMDP $\mathcal{P} = (\mathcal{S}^{\mathcal{P}}, \mathcal{A}^{\mathcal{P}}, \mathcal{O}^{\mathcal{P}}, \{\mathbb{O}_h^{\mathcal{P}}\}_{h \in [H^{\mathcal{P}}]}, \{\mathbb{T}_h^{\mathcal{P}}\}_{h \in [H^{\mathcal{P}}]}, \{\mathcal{R}_h^{\mathcal{P}}\}_{h \in [H^{\mathcal{P}}]}, \mu_1^{\mathcal{P}})$, we can construct a LTC $\mathcal{L}$ with 2 agents as follows:

- Number of agents: $n = 2$; length of episode: $H = H^{\mathcal{P}}$.
- $\mathcal{S} = \mathcal{S}^{\mathcal{P}}, \forall s \in \mathcal{S}$.
- Initial state distribution: $\forall s_1 \in \mathcal{S}, \mu_1(s_1) = \mu_1^{\mathcal{P}}(s_1)$.
- Control action space: $\forall h \in [H], \mathcal{A}_{1,h} = \mathcal{A}_h^{\mathcal{P}}, \mathcal{A}_{2,h} = \{\emptyset\}$.
- Transition: $\forall s_h, s_{h+1} \in \mathcal{S}, a_h \in \mathcal{A}_h, \mathbb{T}_h(s_{h+1} \mid s_h, a_h) = \mathbb{T}_h^{\mathcal{P}}(s_{h+1} \mid s_h, a_{1,h})$.
- Observation space: $\forall h \in [H], \mathcal{O}_{1,h} = \mathcal{O}^{\mathcal{P}}, \mathcal{O}_{2,h} = \mathcal{S}$.
- Emission matrix: For any $h \in [H], \forall o_h \in \mathcal{O}_h, s_h \in \mathcal{S}, \mathbb{O}_h(o_h \mid s_h) = \mathbb{O}_h^{\mathcal{P}}(o_{1,h} \mid s_h)\mathbb{1}[o_{2,h} = s_h]$.
- Reward functions: For any $h \in [H], i \in [2], s_h \in \mathcal{S}, a_h \in \mathcal{A}_h, \mathcal{R}_h(s_h, a_h) = \mathcal{R}^{\mathcal{P}}(s_h, a_{1,h})/H$.
- The baseline sharing: For any $h \in [H], z_h^b = \{o_{1,h}, a_{1,h-1}\}$.
- Communication action space: For any $h \in [H], \mathcal{M}_{1,h} = \{\emptyset\}, \mathcal{M}_{2,h} = \{0,1\}^h$. For any $p_{1,h^-} \in \mathcal{P}_{1,h^-}, p_{2,h^-} \in \mathcal{P}_{2,h^-}, m_h \in \mathcal{M}_h, \phi_{1,h}(p_{1,h^-}, m_{1,h}) = \{m_{1,h}\}, \phi_{2,h}(p_{2,h^-}, m_{2,h}) = \{o_{2,k} \mid k\text{-th digit of } p_{2,h^-} \text{ is 1 and } o_{2,k} \in p_{i,h^-}\} \cup \{m_{2,h}\}$.
- Communication cost functions: For any $h \in [H], z_h^a \in \mathcal{Z}_h^a, \mathcal{K}_h(z_h^a) = \mathbb{1}[z_h^a \neq \{m_h\}]$. It means the communication cost is 1 unless there is no additional sharing.
- We restrict that the communication strategy can only use c_h as input, and remove $a_{2,t}$ in τ_h for any $h > t$.

We first verify that $\mathcal{L}$ is QC and satisfies Assumptions 3.1, 3.2, 3.3, and 4.3.

- $\mathcal{L}$ is QC: For any $\forall h_1 < h_2 \leq H$, agent $(2, h_1)$ does not influence agent $(1, h_2)$ under baseline sharing since agent $(2, h_1)$ does not influence $s_h^1, \forall h \in [H]$, then does not influence $o_{1,h}, \forall h \in [H]$, and thus not influencing agent $(1, h_1)$. For any $\forall h_1 < h_2 \leq H$, under baseline sharing, $p_{1,h^-} = \emptyset$. Then $\sigma(\tau_{1,h_1^-}) \subseteq \sigma(c_{h_1^-}) \subseteq \sigma(c_{h_2^-}) \subseteq \sigma(\tau_{2,h_2^-})$.
- $\mathcal{L}$ satisfies Assumption 3.1: For any $h \in [H], b_1, b_2 \in \Delta(\mathcal{S}), \mathbb{O}_h$ satisfies

$$
\begin{aligned}
||\mathbb{O}_h^{\top}(b_1 - b_2)||_1 &= \sum_{o_{1,h} \in \mathcal{O}^{\mathcal{P}}} \sum_{o_{2,h} \in \mathcal{S}} |\sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h))\mathbb{O}_h((o_{1,h}, o_{2,h}) \mid s_h)| \\
&\geq \sum_{o_{2,h} \in \mathcal{S}} |\sum_{o_{1,h} \in \mathcal{O}^{\mathcal{P}}} \sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h))\mathbb{O}_{1,h}(o_{1,h} \mid s_h)\mathbb{O}_{2,h}(o_{2,h} \mid s_h)| \\
&= \sum_{o_{2,h} \in \mathcal{S}} |\sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h))\mathbb{O}_{2,h}(o_{2,h} \mid s_h) \sum_{o_{1,h} \in \mathcal{O}^{\mathcal{P}}} \mathbb{O}_{1,h}(o_{1,h} \mid s_h)| \\
&= \sum_{o_{2,h} \in \mathcal{S}} |\sum_{s_h \in \mathcal{S}} (b_1(s_h) - b_2(s_h))\mathbb{1}[o_{2,h} = s_h]| \\
&= \sum_{o_{2,h} \in \mathcal{S}} |b_1(o_{2,h}) - b_2(o_{2,h})| = ||b_1 - b_2||_1.
\end{aligned}
$$

- $\mathcal{L}$ satisfies Assumption 3.2: For any $h \in [H]$, we restrict that each agent decides $m_{i,h}$ based on c_h.
- $\mathcal{L}$ satisfies Assumption 3.3: For any $h \in [H], a_{2,h}$ does not influence s_{h+1}, and it is removed from τ.
- $\mathcal{L}$ satisfies Assumption 4.3 since it satisfies the **Turn-based structures** condition in §G, with $ct(h) = 1$ for any $h \in [H]$.

Agent 2 will share nothing through additional sharing, otherwise it will suffer the communication cost $\kappa_h = 1 > \max \sum_{h=1}^H \mathcal{R}_h(s_h, a_h)$ and cannot achieve optimum. Hence, Agent 2 is the dummy player. Therefore, any $(g_{1:H}^{a,*}, g_{1:H}^{m,*})$ be an ϵ/H-team optimal strategy of $\mathcal{L}$ will directly gives the ϵ-optimal of $\mathcal{P}$ as $\{g_{1,1:H}^{a,*}\}_{h \in [H]}$. From Proposition B.7, we can complete our proof. $\square$

C Deferred Details of §4

C.1 Reformulation of $\mathcal{L}$

Given an LTC problem $\mathcal{L}$, we can reformulate it as a Dec-POMDP $\mathcal{D}_{\mathcal{L}}$ defined as $\langle \widetilde{H}, \widetilde{\mathcal{S}}, \{\widetilde{\mathcal{A}}_{i,h}\}_{i\in[n],h\in[\widetilde{H}]}, \{\widetilde{\mathcal{O}}_{i,h}\}_{i\in[n],h\in[\widetilde{H}]}, \widetilde{\mathbb{T}}, \widetilde{\mathbb{O}}, \widetilde{\mu}_1, \{\widetilde{\mathcal{R}}_h\}_{h\in[\widetilde{H}]}\rangle$ as follows

$$\begin{aligned}&\widetilde{H} = 2H,\ \ \widetilde{\mathcal{S}} = \mathcal{S},\ \ \widetilde{s}_{2h-1} = \widetilde{s}_{2h} = s_h,\ \ \widetilde{\mathcal{A}}_{i,2h-1} = \mathcal{M}_{i,h},\ \ \widetilde{\mathcal{A}}_{i,2h} = \mathcal{A}_{i,h},\ \ \widetilde{a}_{i,2h-1} = m_{i,h},\\ &\widetilde{a}_{i,2h} = a_{i,h},\ \ \widetilde{\mathcal{O}}_{i,2h-1} = \mathcal{O}_{i,h},\ \ \widetilde{\mathcal{O}}_{i,2h} = \{\emptyset\},\ \ \widetilde{o}_{i,2h-1} = o_{i,h},\ \ \widetilde{o}_{i,2h} = \emptyset,\\ &\widetilde{\mathbb{T}}_{2h-1}(\widetilde{s}_{2h}\,|\,\widetilde{s}_{2h-1}, \widetilde{a}_{2h-1}) = \mathbb{1}[\widetilde{s}_{2h} = \widetilde{s}_{2h-1}],\ \ \widetilde{\mathbb{T}}_{2h}(\widetilde{s}_{2h+1}\,|\,\widetilde{s}_{2h}, \widetilde{a}_{2h}) = \mathbb{T}_h(\widetilde{s}_{2h+1}\,|\,\widetilde{s}_{2h}, \widetilde{a}_{2h}),\\ &\widetilde{\mu}_1 = \mu_1,\ \ \widetilde{\mathcal{R}}_{2h-1} = -\mathcal{K}_h,\ \ \widetilde{\mathcal{R}}_{2h} = \mathcal{R}_h,\ \ \widetilde{p}_{i,2h-1} = p_{i,h^-},\ \ \widetilde{p}_{i,2h} = p_{i,h^+},\ \ \widetilde{c}_{2h-1} = c_{h^-},\\ &\widetilde{c}_{2h} = c_{h^+},\ \ \widetilde{z}_{2h-1} = z_h^b,\ \ \widetilde{z}_{2h} = z_h^a,\ \ \widetilde{\tau}_{i,2h-1} = c_{h^-},\ \ \widetilde{\tau}_{i,2h} = \tau_{i,h^+},\end{aligned} \tag{C.1}$$

Note that, at the odd timestep $2h-1$, we set $\widetilde{\tau}_{i,2h-1} = c_{h^-}$ under Assumption 3.2, i.e., in $\mathcal{D}_{\mathcal{L}}$, each agent only uses the common information so far for decision-making at timestep $2h-1$. Correspondingly, for any $h \in [\widetilde{H}], i \in [n]$, we denote by $\widetilde{g}_{i,h}, \widetilde{g}_h$ the (joint) strategy and by $\widetilde{\mathcal{G}}_{i,h}, \widetilde{\mathcal{G}}_h$ the (joint) strategy spaces. Similarly, the objective of $\mathcal{D}_{\mathcal{L}}$ is defined as $J_{\mathcal{D}_{\mathcal{L}}}(\widetilde{g}_{1:\widetilde{H}}) = \mathbb{E}_{\mathcal{D}_{\mathcal{L}}}[\sum_{h=1}^{\widetilde{H}} \widetilde{r}_h\,|\,\widetilde{g}_{1:\widetilde{H}}]$. Essentially, this reformulation splits the H-step decision-making and communication procedure into a $2H$-step one. A similar splitting of the timesteps was also used in Sudhakara et al. (2021); Kartik et al. (2022). In comparison, we consider a more general setting, where the state is not decoupled, and agents are allowed to share the observations and actions at the *previous* timesteps, due to the generality of our LTC formulation. The equivalence between $\mathcal{L}$ and $\mathcal{D}_{\mathcal{L}}$ is more formally stated as follows.

Proposition C.1 (Equivalence between $\mathcal{L}$ and $\mathcal{D}_{\mathcal{L}}$)**.** Let $\mathcal{D}_{\mathcal{L}}$ be the reformulated Dec-POMDP from $\mathcal{L}$, then the solutions of the two problems are equivalent, in the sense that $\forall g^m_{1:H} \in \mathcal{G}^m_{1:H}, g^a_{1:H} \in \mathcal{G}^a_{1:H}, i \in [n]$, let $\widetilde{g}_{1:\widetilde{H}} = (g^m_1, g^a_1, \cdots, g^m_H, g^a_H)$, then $J_{\mathcal{D}_{\mathcal{L}}}(\widetilde{g}_{1:\widetilde{H}}) = J_{\mathcal{L}}(g^m_{1:H}, g^a_{1:H})$. Also, $\forall \widetilde{g}_{1:\widetilde{H}} \in \widetilde{\mathcal{G}}_{1:\widetilde{H}}, i \in [n]$, let $g^m_{1:H} = (\widetilde{g}_1, \widetilde{g}_3, \cdots, \widetilde{g}_{\widetilde{H}-1}), g^a_{1:H} = (\widetilde{g}_2, \widetilde{g}_4, \cdots, \widetilde{g}_{\widetilde{H}})$, then $J_{\mathcal{L}}(g^m_{1:H}, g^a_{1:H}) = J_{\mathcal{D}_{\mathcal{L}}}(\widetilde{g}_{1:\widetilde{H}})$.

C.2 Proof of Theorem 4.1

Proof. We prove the following lemma first.

Lemma C.2. Let the $\mathcal{L}$ be the QC LTC problem satisfying Assumptions 3.3, 3.4, and 3.5, and $\mathcal{D}_{\mathcal{L}}$ be the reformulated Dec-POMDP. Then for $i_1, i_2 \in [n], t_1, t_2 \in [H]$, if agent $(i_1, 2t_1)$ influences agent $(i_2, 2t_2)$ in $\mathcal{D}_{\mathcal{L}}$, then $\sigma(\tau_{i_1,t_1^-}) \subseteq \sigma(\tau_{i_2,t_2^-})$ in $\mathcal{L}$. Moreover, if $\mathcal{L}$ is sQC, then $\sigma(a_{i_1,t_1}) \subseteq \sigma(\tau_{i_2,t_2^-})$.

Proof. We prove this by cases.

- If a_{i_1,t_1} influences the underlying state s_{t_1+1}, then from Assumption 3.4, agent (i_1, t_1) influences o_{-i_1,t_1+1}, so there must exist $i_3 \neq i_1$, such that agent (i_1, t_1) influences o_{i_3,t_1+1}. From part (e) of Assumption 2.1 and $t_1 < t_2$, we know $o_{i_3,t_1+1} \in \tau_{i_3,(t_1+1)^-} \subseteq \tau_{i_3,t_2^-}$ even under no additional sharing, and then we get agent (i_1, t_1) influences agent (i_3, t_2) in $\overline{\mathcal{D}}_{\mathcal{L}}$ (the Dec-POMDP induced by $\mathcal{L}$). From Lemma B.2, it holds that $\sigma(\tau_{i_1,t_1^-}) \subseteq \sigma(\tau_{i_3,t_2^-})$. From Assumption 3.5 and $i_3 \neq i_1$, we know $\sigma(\tau_{i_1,t_1^-}) \subseteq \sigma(c_{t_2^-}) \subseteq \sigma(\tau_{i_2,t_2^-})$. (Similarly, if $\mathcal{L}$ is sQC, we have $\sigma(a_{i_1,t_1}) \subseteq \sigma(\tau_{i_3,t_2^-})$ from Assumption 3.5, and $\sigma(a_{i_1,t_1}) \subseteq \sigma(c_{t_2^-}) \subseteq \sigma(\tau_{i_2,t_2^-})$ from Assumption 3.5).
- If a_{i_1,t_1} does not influence s_{t_1+1}, from Assumption 3.3, $\forall t > t_1, a_{i_1,t_1} \notin \tau_{t^-}$ and $a_{i_1,t_1} \notin \tau_{t^+}$. Then in $\mathcal{D}_{\mathcal{L}}$, agent $(i_1, 2t_1)$ does not influence $\widetilde{\tau}_{i,2t_1+1}, \forall i \in [n]$, hence it does not influence $\widetilde{a}_{i,2t_1+1}, \forall i \in [n]$. Then it does not influence $\widetilde{z}_{2t_1+1}$, and further does not influence $\widetilde{\tau}_{i,2t_1+2}$ and $\widetilde{a}_{i,2t_1+2}, \forall i \in [n]$. From induction, we know agent $(i_1, 2t_1)$ does not influence agent $(i_2, 2t_2)$, which leads to a contradiction.

This completes the proof of this lemma. □

We now go back to prove the theorem. Firstly, we prove the QC cases. To show $\mathcal{D}_{\mathcal{L}}$ is QC, we need to prove $\forall i_1, i_2 \in [n], h_1, h_2 \in [\widetilde{H}]$, if agent (i_1, h_1) influences agent (i_2, h_2) with $h_1 < h_2$, then $\sigma(\widetilde{\tau}_{i_1,h_1}) \subseteq \sigma(\widetilde{\tau}_{i_2,h_2})$, where we use $\widetilde{\tau}_{i,h}$ to denote the available information of agent (i, h) in $\mathcal{D}_{\mathcal{L}}$. We prove this by considering the following cases:

1. If $h_1 = 2t_1 - 1$ with $t_1 \in [H]$, by the construction of $\mathcal{D}_{\mathcal{L}}$ and Assumption 3.2, we have $\widetilde{\tau}_{i_1,h_1} = \widetilde{c}_{h_1} = c_{t_1^-} \subseteq \widetilde{\tau}_{i_2,h_2}$, since common information accumulates over time by definition, and will always be included in the available information $\widetilde{\tau}_{i,h}$ in later steps. Thus, $\sigma(\widetilde{\tau}_{i_1,h_1}) \subseteq \sigma(\widetilde{\tau}_{i_2,h_2})$.
2. If $h_1 = 2t_1, h_2 = 2t_2$ with $t_1, t_2 \in [H]$, then $\widetilde{\tau}_{i_1,h_1} = \tau_{i_1,t_1^+} = \tau_{i_1,t_1^-} \cup z_{t_1}^a$ by definition. Consider agent (i_1, t_1) and (i_2, t_2) in $\mathcal{L}$. From Lemma C.2, we know $\sigma(\tau_{i_1,t_1^-}) \subseteq \sigma(\tau_{i_2,t_2^-}) \subseteq \sigma(\tau_{i_2,t_2^+})$. Also, $z_{t_1}^a \subseteq c_{t_1^+} \subseteq c_{t_2^+} \subseteq \tau_{i_2,t_2^+} = \widetilde{\tau}_{i_2,h_2}$ by the accumulation of c_{h^+} over time. Thus, we have $\sigma(\widetilde{\tau}_{i_1,h_1}) \subseteq \sigma(\widetilde{\tau}_{i_2,h_2})$.
3. If $h_1 = 2t_1, h_2 = 2t_2 - 1, t_1, t_2 \in [H]$, then $\widetilde{\tau}_{i_2,h_2} = \widetilde{c}_{h_2}$, then $\exists i_3 \in [n], i_3 \neq i_1, \widetilde{\tau}_{i_2,h_2} \subseteq \widetilde{c}_{h_2+1} \subseteq \widetilde{\tau}_{i_3,h_2+1}$. From agent (i_1, h_1) influences (i_2, h_2), we know agent (i_1, h_1) also influences agent $(i_3, h_2 + 1)$ in $\mathcal{D}_{\mathcal{L}}$, hence agent (i_1, t_1) influences agent (i_2, t_2) in $\mathcal{L}$. Since $\mathcal{L}$ is QC, we know $\sigma(\tau_{i_1,t_1^-}) \subseteq \sigma(\tau_{i_3,t_2^-})$. From Assumption 3.5 and $i_1 \neq i_3$, we know $\sigma(\widetilde{\tau}_{i_1,h_1}) = \sigma(\tau_{i_1,t_1^-}) \subseteq \sigma(c_{t_2^-}) = \sigma(\widetilde{\tau}_{i_2,h_2})$.

Second, we prove the sQC case. In $\mathcal{D}_{\mathcal{L}}$, for $\forall i_1, i_2 \in [n], h_1, h_2 \in [\widetilde{H}]$, agent (i_1, h_1) influences (i_2, h_2). From the proof above, we know $\sigma(\widetilde{\tau}_{i_1,h_1}) \subseteq \sigma(\widetilde{\tau}_{i_2,h_2})$. We only need to prove $\sigma(\widetilde{a}_{i_1,h_1}) \subseteq \sigma(\widetilde{\tau}_{i_2,h_2})$.

1. If $h_1 = 2t_1 - 1$ with $t_1 \in [H]$, then we know $\widetilde{a}_{i_1,h_1} = m_{i_1,t}$. From Assumption 2.1, we know that $m_{i_1,t} \subseteq z_{i_1,t}^a$. Then we get $\sigma(\widetilde{a}_{i_1,h_1}) \subseteq \sigma(\widetilde{z}_{i_1,h_1+1}) \subseteq \sigma(\widetilde{c}_{h_2}) \subseteq \sigma(\widetilde{\tau}_{i_2,h_2})$.
2. If $h_1 = 2t_1, h_2 = 2t_2$ with $t_1, t_2 \in [H]$, then from Lemma C.2, we know that $\sigma(\widetilde{a}_{i_1,h_1}) \subseteq \sigma(\widetilde{\tau}_{i_2,h_2})$.
3. If $h_1 = 2t_1, h_2 = 2t_2 - 1, t_1, t_2 \in [H]$, then $\widetilde{\tau}_{i_2,h_2} = \widetilde{c}_{h_2}$, then $\exists i_3 \in [n], i_3 \neq i_1, \widetilde{\tau}_{i_2,h_2} \subseteq \widetilde{c}_{h_2+1} \subseteq \widetilde{\tau}_{i_3,h_2+1}$. From agent (i_1, h_1) influences (i_2, h_2), we know agent (i_1, h_1) also influences agent $(i_3, h_2 + 1)$ in $\mathcal{D}_{\mathcal{L}}$, hence agent (i_1, t_1) influences agent (i_2, t_2) in $\mathcal{L}$. Since $\mathcal{L}$ is sQC, we know $\sigma(a_{i_1,t_1^-}) \subseteq \sigma(\tau_{i_3,t_2^-})$. From Assumption 3.5 and $i_1 \neq i_3$, we know $\sigma(\widetilde{a}_{i_1,h_1}) = \sigma(a_{i_1,t_1}) \subseteq \sigma(c_{t_2^-}) = \sigma(\widetilde{\tau}_{i_2,h_2})$.

This completes the proof. □

Lemma C.3. If $\mathcal{D}_{\mathcal{L}}$ is QC, then $\mathcal{D}_{\mathcal{L}}^{\dagger}$ is sQC.

C.3 Proof of Lemma C.3

Proof. From the construction of $\mathcal{D}_{\mathcal{L}}^{\dagger}$, since $\mathcal{D}_{\mathcal{L}}^{\dagger}$ requires agent to share more than $\mathcal{D}_{\mathcal{L}}$, it is easy to observe the fact that $\forall h \in [\widetilde{H}], i \in [n], \widetilde{c}_h \subseteq \breve{c}_h, \widetilde{\tau}_{i,h} \subseteq \breve{\tau}_{i,h}$.
Let $i_1, i_2 \in [n], h_1, h_2 \in [\widetilde{H}], h_1 < h_2$, and agent (i_1, h_1) influences agent (i_2, h_2) in $\mathcal{D}_{\mathcal{L}}^{\dagger}$.

- If $h_1 = 2t_1 - 1$ with $t_1 \in [H]$, then h_1 is communication step. So $\breve{\tau}_{i_1,h_1} = \breve{c}_{h_1} \subseteq \breve{c}_{h_2}$, and $\widetilde{a}_{i_1,h_1} = m_{i_1,t_1} \subseteq \breve{c}_{h_1+1} \subseteq \breve{c}_{h_2}$ from Assumption 2.1. Therefore, we have $\sigma(\breve{\tau}_{i_1,h_1}) \cup \sigma(\breve{a}_{i_1,h_1}) \subseteq \sigma(\breve{c}_{h_1}) \subseteq \sigma(\breve{\tau}_{i_2,h_2})$.
- If $h_1 = 2t_1, h_2 = 2t_2 - 1$ with $t_1, t_2 \in [H]$, then $\breve{\tau}_{i_2,h_2} = \breve{c}_{h_2}$. If agent (i_1, h_1) does not influence (i_2, h_2) in $\mathcal{D}_{\mathcal{L}}$, but agent (i_1, h_1) influences (i_2, h_2) in $\mathcal{D}_{\mathcal{L}}^{\dagger}$, then it means $\breve{a}_{i_1,h_1} \in \breve{\tau}_{i_2,h_2}$ but $\widetilde{a}_{i_1,h_1} \notin \widetilde{\tau}_{i_2,h_2}$. This can only happen when $\sigma(\widetilde{\tau}_{i_1,h_1}) \subseteq \sigma(\widetilde{c}_{h_2}) \subseteq \sigma(\breve{c}_{h_2})$, and $\widetilde{a}_{i_1,h_1} \subseteq \breve{c}_{h_2}$. Also, from the construction of $\mathcal{D}_{\mathcal{L}}^{\dagger}$, we know that $\breve{\tau}_{i_1,h_1} \backslash \widetilde{\tau}_{i_1,h_1} \subseteq \breve{c}_{h_1}$. Therefore, we have $\sigma(\breve{\tau}_{i_1,h_1}) \cup \sigma(\widetilde{a}_{i_1,h_1}) \subseteq \sigma(\breve{c}_{h_2}) \subseteq \sigma(\breve{\tau}_{i_2,h_2})$.

If agent (i_1, h_1) influences (i_2, h_2) in $\mathcal{D}_{\mathcal{L}}$, then from QC of $\mathcal{D}_{\mathcal{L}}$, we know that $\sigma(\widetilde{\tau}_{i_1,h_1}) \subseteq \sigma(\widetilde{c}_{h_2})$, then from the construction of $\mathcal{D}_{\mathcal{L}}^{\dagger}$, we have $\widetilde{a}_{i_1,h_1} \in \breve{c}_{h_2}$. Still, we have $\breve{\tau}_{i_1,h_1} \backslash \widetilde{\tau}_{i_1,h_1} \subseteq \breve{c}_{h_1}$. Therefore, $\sigma(\breve{\tau}_{i_1,h_1}) \cup \sigma(\widetilde{a}_{i_1,h_1}) \subseteq \sigma(\breve{\tau}_{i_2,h_2})$.

- If $h_1 = 2t_1, h_2 = 2t_2$ with $t_1, t_2 \in [H]$. If agent (i_1, h_1) does not influence (i_2, h_2) in $\mathcal{D}_{\mathcal{L}}$, then it means sharing $\widetilde{a}_{i_1,h_1}$ leads to the influence. Then, $\sigma(\widetilde{\tau}_{i_1,h_1}) \subseteq \sigma(\widetilde{c}_{h_2}) \subseteq \sigma(\breve{c}_{h_2})$, and $\widetilde{a}_{i_1,h_1} \subseteq \breve{c}_{h_2}$. We can conclude $\sigma(\breve{\tau}_{i_1,h_1}) \cup \sigma(\widetilde{a}_{i_1,h_1}) \subseteq \sigma(\breve{c}_{h_2}) \subseteq \sigma(\breve{\tau}_{i_2,h_2})$.

  Now we consider the case that agent (i_1, h_1) influences (i_2, h_2) in $\mathcal{D}_{\mathcal{L}}$. If $i_1 \neq i_2$, then we have $\widetilde{\tau}_{i_1,h_1} \subseteq \widetilde{\tau}_{i_2,h_2}$. From Assumption 3.5, and $i_1 \neq i_2$, we know $\widetilde{\tau}_{i_1,h_1} \subseteq \widetilde{c}_{h_2}$. Then, from the construction of $\mathcal{D}_{\mathcal{L}}^{\dagger}$, we have $\widetilde{a}_{i_1,h_1} \subseteq \breve{c}_{h_2}$. Finally, we have $\sigma(\breve{\tau}_{i_1,h_1}) \cup \sigma(\widetilde{a}_{i_1,h_1}) \subseteq \sigma(\breve{\tau}_{i_2,h_2})$.

  If $i_1 = i_2$, then from the perfect recall of $\mathcal{L}$, we know that $\widetilde{\tau}_{i_1,h_1} \cup \widetilde{a}_{i_1,h_1} \subseteq \widetilde{\tau}_{i_2,h_2}$. From $\breve{\tau}_{i_1,h_1} \backslash \widetilde{\tau}_{i_1,h_1} \subseteq \breve{c}_{h_1}$, we conclude $\sigma(\breve{\tau}_{i_1,h_1}) \cup \sigma(\widetilde{a}_{i_1,h_1}) \subseteq \sigma(\breve{\tau}_{i_2,h_2})$.

This completes the proof. □

Theorem C.4. Let $\mathcal{D}_{\mathcal{L}}$ be the QC Dec-POMDP reformulated from a QC LTC $\mathcal{L}$, and $\mathcal{D}_{\mathcal{L}}^{\dagger}$ be the sQC expansion of $\mathcal{D}_{\mathcal{L}}$. Then, for any ϵ-team-optimal strategy $\breve{g}^*_{1:\breve{H}}$ of $\mathcal{D}_{\mathcal{L}}^{\dagger}$, there exists a function φ such that $\widetilde{g}^*_{1:\widetilde{H}} = \varphi(\breve{g}^*_{1:\breve{H}}, \mathcal{D}_{\mathcal{L}})$ is an ϵ-team-optimal strategy of $\mathcal{D}_{\mathcal{L}}$, with $J_{\mathcal{D}_{\mathcal{L}}}(\widetilde{g}^*_{1:\widetilde{H}}) = J_{\mathcal{D}_{\mathcal{L}}^{\dagger}}(\breve{g}^*_{1:\breve{H}})$.

C.4 Proof of Theorem C.4

Proof. We firstly prove that given any strategy $\breve{g}_{1:H}$ and $\widetilde{g}_{1:H} = \varphi(\breve{g}_{1:H}, \mathcal{D}_{\mathcal{L}})$, $J_{\mathcal{D}_{\mathcal{L}}^{\dagger}}(\breve{g}_{1:H}) = J_{\mathcal{D}_{\mathcal{L}}}(\widetilde{g}_{1:H})$, where the function φ is shown in Algorithm 3. Since $\mathcal{D}_{\mathcal{L}}^{\dagger}$ only changes what to share, $\widetilde{\tau}_h = \breve{\tau}_h$ always hold. Then, for any $i \in [n], h \in [\widetilde{H}], \widetilde{\tau}_h \in \widetilde{\mathcal{T}}_h$, let $\widetilde{\tau}_{i,h}, \breve{\tau}_{i,h}$ be the corresponding information of agent i in $\mathcal{D}_{\mathcal{L}}, \mathcal{D}_{\mathcal{L}}^{\dagger}$, respectively. From Algorithm 3, we know that $\widetilde{g}_{i,h}(\widetilde{\tau}_{i,h}) = \breve{g}_{i,h}(\breve{\tau}_{i,h})$. This is because, for any $\widetilde{a}_{j,t} \in \breve{\tau}_{i,h} \backslash \widetilde{\tau}_{i,h}, j \in [n], t < h$, there must holds that $\sigma(\widetilde{\tau}_{j,t}) \subseteq \sigma(\widetilde{c}_{i,h})$. Therefore, we can always recover $\widetilde{a}_{j,t}$ from $\widetilde{\tau}_{i,h}$ and $\widetilde{g}_{i,h}$. As a result, we can have $J_{\mathcal{D}_{\mathcal{L}}^{\dagger}}(\breve{g}_{1:H}) = J_{\mathcal{D}_{\mathcal{L}}}(\widetilde{g}_{1:H})$.
Since $\mathcal{D}_{\mathcal{L}}^{\dagger}$ has larger strategy spaces, i.e., $\max_{\widetilde{g}_{1:\widetilde{H}} \in \widetilde{G}_{1:\widetilde{H}}} J_{\mathcal{D}_{\mathcal{L}}}(\widetilde{g}_{1:\widetilde{H}}) \leq \max_{\breve{g}_{1:\breve{H}} \in \breve{G}_{1:\breve{H}}} J_{\mathcal{D}_{\mathcal{L}}^{\dagger}}(\breve{g}_{1:\breve{H}})$. Let $\breve{g}^*_{1:\breve{H}}$ be the strategy satisfying $J_{\mathcal{D}_{\mathcal{L}}^{\dagger}}(\breve{g}^*_{1:\breve{H}}) \geq \max_{\breve{g}_{1:\breve{H}} \in \breve{G}_{1:\breve{H}}} J_{\mathcal{D}_{\mathcal{L}}^{\dagger}}(\breve{g}_{1:\breve{H}}) - \epsilon$. Then, we have $J_{\mathcal{D}_{\mathcal{L}}}(\varphi(\breve{g}^*_{1:\breve{H}}, \mathcal{D}_{\mathcal{L}})) = J_{\mathcal{D}_{\mathcal{L}}^{\dagger}}(\breve{g}^*_{1:\breve{H}}) \geq \max_{\breve{g}_{1:\breve{H}} \in \breve{G}_{1:\breve{H}}} J_{\mathcal{D}_{\mathcal{L}}^{\dagger}}(\breve{g}_{1:\breve{H}}) - \epsilon \geq \max_{\widetilde{g}_{1:\widetilde{H}} \in \widetilde{G}_{1:\widetilde{H}}} J_{\mathcal{D}_{\mathcal{L}}}(\widetilde{g}_{1:\widetilde{H}}) - \epsilon$. Then $\varphi(\breve{g}^*_{1:\breve{H}}, \mathcal{D}_{\mathcal{L}})$ is an ϵ-team optimal strategy of $\mathcal{D}_{\mathcal{L}}$. □

Theorem C.5. Let $\mathcal{D}_{\mathcal{L}}^{\dagger}$ be an sQC Dec-POMDP generated from $\mathcal{L}$ after reformulation and strict expansion, then $\mathcal{D}_{\mathcal{L}}^{\dagger}$ has *strategy-independent common-information-based beliefs* (Nayyar et al., 2013a; Liu & Zhang, 2023). More formally, for any $h \in [\breve{H}]$, any two different joint strategies $\breve{g}_{1:h-1}$ and $\breve{g}'_{1:h-1}$, and any common information $\breve{c}_h$ that can be reached under strategy $\breve{g}_{1:h-1}$, for any joint private information $\breve{p}_h \in \breve{\mathcal{P}}_h$ and state $\breve{s}_h \in \breve{\mathcal{S}}$,

$$\mathbb{P}_h^{\mathcal{D}_{\mathcal{L}}^{\dagger}}(\breve{s}_h, \breve{p}_h \mid \breve{c}_h, \breve{g}_{1:h-1}) = \mathbb{P}_h^{\mathcal{D}_{\mathcal{L}}^{\dagger}}(\breve{s}_h, \breve{p}_h \mid \breve{c}_h, \breve{g}'_{1:h-1}). \tag{C.2}$$

C.5 Proof of Theorem C.5

Proof. To prove that $\mathcal{D}_{\mathcal{L}}^{\dagger}$ has SI-CIB, it is sufficient to prove that for any $h = 2, \cdots, \breve{H}$, fix any $h_1 \in [h-1], i_1 \in [n]$, and for any $\breve{g}_{1:h-1} \in \breve{\mathcal{G}}_{1:h-1}, \breve{g}'_{i_1,h_1} \in \breve{\mathcal{G}}_{i_1,h_1}$, let $\breve{g}'_{h_1} := (\breve{g}_{1,h_1}, \cdots, \breve{g}_{i_1,h_1}, \cdots, \breve{g}_{n,h_1})$ and $\breve{g}'_{1:h-1} := (\breve{g}_1, \cdots, \breve{g}'_{h_1}, \cdots, \breve{g}_{h-1})$, the following holds

$$\mathbb{P}(\breve{s}_h, \breve{p}_h \mid \breve{c}_h, \breve{g}_{1:h-1}) = \mathbb{P}(\breve{s}_h, \breve{p}_h \mid \breve{c}_h, \breve{g}'_{1:h-1}). \tag{C.3}$$

We prove this case-by-case as follows:

1. If there exists some $i_3 \neq i_1$ such that $\sigma(\breve{\tau}_{i_1,h_1}) \subseteq \sigma(\breve{\tau}_{i_3,h}), \sigma(\breve{a}_{i_1,h_1}) \subseteq \sigma(\breve{\tau}_{i_3,h})$, then from Assumption 3.5, we know that $\sigma(\breve{\tau}_{i_1,h_1}) \subseteq \sigma(\breve{c}_h), \sigma(\breve{a}_{i_1,h_1}) \subseteq \sigma(\breve{c}_h)$. Therefore, there exist

deterministic functions α_1, α_2 such that $\breve{\tau}_{i_1,h_1} = \alpha_1(\breve{c}_h), \breve{a}_{i_1,h_1} = \alpha_2(\breve{c}_h)$, and further it holds that

$$\mathbb{P}(\breve{s}_h, \breve{p}_h \,|\, \breve{c}_h, \breve{g}_{1:h-1}) = \mathbb{P}(\breve{s}_h, \breve{p}_h \,|\, \alpha_1(\breve{c}_h), \alpha_2(\breve{c}_h), \breve{c}_h, \breve{g}_{1:h-1})$$
$$= \mathbb{P}(\breve{s}_h, \breve{p}_h \,|\, \breve{\tau}_{i_1,h_1}, \breve{a}_{i_1,h_1}, \breve{c}_h, \breve{g}_{1:h-1}) = \mathbb{P}(\breve{s}_h, \breve{p}_h \,|\, \breve{\tau}_{i_1,h_1}, \breve{a}_{i_1,h_1}, \breve{c}_h, \breve{g}'_{1:h-1}).$$

The last equality is due to the fact that the input and output of $\breve{g}_{i_1,h_1}$ are $\breve{\tau}'_{i_1,h_1}$ and $\breve{a}'_{i_1,h_1}$, respectively.

2. If there does not exist any $i_2 \neq i_1$ such that $\sigma(\breve{\tau}_{i_1,h_1}) \not\subseteq \sigma(\breve{\tau}_{i_2,h})$ or $\sigma(\breve{a}_{i_1,h_1}) \not\subseteq \sigma(\breve{\tau}_{i_2,h})$, then agent (i_1, h_1) does not influence agent (i_2, h) for any $i_2 \neq i_1$ in $\mathcal{D}^{\dagger}_{\mathcal{L}}$ because $\mathcal{D}^{\dagger}_{\mathcal{L}}$ is sQC, and $h_1 = 2k_1$ with $k_1 \in [n]$. (If h_1 is odd, then $\breve{\tau}_{i_1,h_1} = \breve{c}_{h_1} \subseteq \breve{c}_h \subseteq \breve{\tau}_{i_2,h}$, and $\breve{a}_{i_1,h_1} = m_{i_1, \frac{h_1+1}{2}} \in z^a_{\frac{h_1+1}{2}} = \breve{z}_{h_1+1} \subseteq \breve{c}_h$ based on Assumption 2.1(b), which leads to a contradiction.) Now, we claim that agent (i_1, h_1) does not influence state $\breve{s}_h$, and does not influences $\breve{\tau}_{i_1,h}$, and prove this case-by-case as below:

   (a) If $h = 2k-1, k \in [n]$, then $\breve{p}_h = \emptyset$. If agent (i_1, h_1) influences $\breve{s}_h$ in $\mathcal{D}^{\dagger}_{\mathcal{L}}$, then agent (i_1, h_1) influences $\widetilde{s}_h$ in $\mathcal{D}_{\mathcal{L}}$ (because strict expansion does not change system dynamics). From Assumption 3.4, we know that she also influences $\widetilde{o}_{-i_1,h}$. Then there exists $i_3 \neq i_1$ such that agent (i_1, h_1) influences $\widetilde{o}_{i_3,h}$ in $\mathcal{D}_{\mathcal{L}}$. From Assumption 2.1 (e), it holds $\widetilde{o}_{i_3,h} \in \widetilde{\tau}_{i_3,h+1}$. Therefore, agent (i_1, h_1) influences agent $(i_3, h+1)$ in the problem $\mathcal{D}_{\mathcal{L}}$. From Lemma C.2, we know $\sigma(\tau_{i_1,k_1^-}) \subseteq \sigma(\tau_{i_3,k^-})$ in $\mathcal{L}$. Furthermore, from Assumption 3.5 and $i_3 \neq i_1$, it holds $\sigma(\tau_{i_1,k_1^-}) \subseteq \sigma(c_{k^-})$. Also, from the reformulation, it holds $\widetilde{\tau}_{i_1,h_1} = \tau_{i_1,k_1^+} = \tau_{i_1,k_1^-} \cup z^a_{k_1}$ and $z^a_{k_1} = \widetilde{z}_{h_1} \subseteq \widetilde{c}_h$. Then, we have $\sigma(\widetilde{\tau}_{i_1,h_1}) \subseteq \sigma(\widetilde{c}_h) = \sigma(\widetilde{\tau}_{i_3,h})$. Based on the strict expansion from $\mathcal{D}_{\mathcal{L}}$ to $\mathcal{D}^{\dagger}_{\mathcal{L}}$, we can get $\breve{\tau}_{i_1,h_1} \backslash \widetilde{\tau}_{i_1,h_1} \subseteq \breve{c}_{i_1,h_1} \subseteq \breve{\tau}_{i_3,h}$, and $\breve{a}_{i_1,h_1} \in \breve{c}_h$. Then, it holds that $\sigma(\breve{\tau}_{i_1,h_1}) \subseteq \sigma(\breve{\tau}_{i_3,h}), \sigma(\breve{a}_{i_1,h_1}) \subseteq \sigma(\breve{\tau}_{i_3,h})$, which leads to contradition of $\sigma(\breve{\tau}_{i_1,h_1}) \not\subseteq \sigma(\breve{\tau}_{i_2,h})$ or $\sigma(\breve{a}_{i_1,h_1}) \not\subseteq \sigma(\breve{\tau}_{i_2,h})$. Hence, we know agent (i_1, h_1) does not influence state $\breve{s}_h$. Additionally, for any $i_2 \neq i_1$, since agent (i_1, h_1) does not influences agent (i_2, h), and $\breve{\tau}_{i_1,h} = \breve{c}_h = \breve{\tau}_{i_2,h}$, then we know that agent (i_1, h_1) does not influence $\breve{\tau}_{i_1,h}$.

   (b) If $h = 2k, k \in [n]$. If agent (i_1, h_1) influences $\breve{s}_{h_1+1}$, then from Assumption 3.4, agent (i_1, h_1) influences $\breve{o}_{-i_1,h_1+1}$, and then there exists $i_3 \neq i_1$ such that agent (i_1, h_1) influence $\breve{o}_{i_3,h_1+1}$. Howver, from Assumption 2.1 (e), we know that $\breve{o}_{i_3,h_1+1} \in \breve{\tau}_{i_3,h}$, which means agent (i_1, h_1) influences agent (i_3, h) and leads to a contradiction. Therefore, we know that agent (i_1, h_1) does not influence $\breve{s}_{h_1+1}$, and further does not influence $\breve{s}_h$. Also, from the Assumption 3.3, $\breve{a}_{i_1,h_1} \notin \breve{\tau}_{i_1,h'}, \forall h' > h_1$, and agent (i_1, h_1) does not influence $\breve{s}_{h_1+1}$. This means she does not influence any element in $\breve{\tau}_{i_1,h_1+1}$. Therefore, agent (i_1, h_1) does not influence $\breve{\tau}_{i_1,h_1+1}$, and hence does not influence $\breve{a}_{i_1,h_1+1}$. In the same way, we know that agent (i_1, h_1) does not $\breve{\tau}_{i_1,h'}$ and $\breve{a}_{i_1,h'}$ for any $h' > h_1$. Finally, we conclude that agent (i_1, h_1) does not influence $\breve{\tau}_{i_1,h}$/

Therefore, we know agent (i_1, h_1) does not influence $\breve{s}_h$, and does not influence $\breve{\tau}_{i,h}, \forall i \in [n]$.

$$\mathbb{P}(\breve{s}_h, \breve{p}_h \,|\, \breve{c}_h, \breve{g}_{1:h-1}) = \mathbb{P}(\breve{s}_h, \breve{p}_h, \breve{c}_h \,|\, \breve{c}_h, \breve{g}_{1:h-1}) = \mathbb{P}(\breve{s}_h, \breve{\tau}_h \,|\, \breve{c}_h, \breve{g}_{1:h-1})$$
$$= \mathbb{P}(\breve{s}_h, \{\breve{\tau}_{i,h}\}_{i\in[n]} \,|\, \breve{c}_h, \breve{g}_{1:h-1}) = \mathbb{P}(\breve{s}_h, \{\breve{\tau}_{i,h}\}_{i\in[n]} \,|\, \breve{c}_h, \breve{g}'_{1:H}) = \mathbb{P}(\breve{s}_h, \breve{p}_h \,|\, \breve{c}_h, \breve{g}'_{1:h-1}).$$

This completes the proof.

□

C.6 Proof of Theorem 4.2

Proof. Firstly, from the construction of $\mathcal{D}'_{\mathcal{L}}$ and strategy space $\overline{\mathcal{G}}_{1:\overline{H}}$, we know that for any $h \in [H], i \in [n], \overline{\mathcal{C}}_{2h-1} = \breve{\mathcal{C}}_{2h-1}, \overline{\mathcal{A}}_{i,2h-1} = \breve{\mathcal{A}}_{i,2h-1}, \overline{\mathcal{T}}_{i,2h} = \breve{\mathcal{T}}_{i,2h}, \overline{\mathcal{A}}_{i,2h} = \breve{\mathcal{A}}_{i,2h}$. Therefore, $\overline{\mathcal{G}}_{1:\overline{H}} = \breve{\mathcal{G}}_{1:\breve{H}}$, and finding a team optimal strategy of $\mathcal{D}'_{\mathcal{L}}$ in the strategy space $\overline{\mathcal{G}}_{1:\overline{H}}$ is equivalent to

finding a team-optimum of $\mathcal{D}^{\dagger}_{\mathcal{L}}$ in the strategy space $\breve{\mathcal{G}}_{1:\breve{H}}$.
Secondly, we will prove that the Dec-POMDP $\mathcal{D}'_{\mathcal{L}}$ satisfies the information evolution rules in the theorem. For each $t \in [H]$, we define the random variable $\widehat{p}_{i,2t-1} = p_{i,t^-}, \widehat{p}_{2t-1} = p_{t^-}$. Recall that in the reformulation, $\widetilde{p}_{i,2t-1} = \emptyset$ rather than p_{i,t^-}. Then, from the $2H$-reformulation and Assumption 2.1, it holds that, for any $i \in [n], h \in [\overline{H}]$, if $h = 2t-1$ with $t \in [2 : H]$

$$\widetilde{z}_h = \chi_t(\widetilde{p}_{h-1}, \widetilde{a}_{h-1}, \widetilde{o}_h), \qquad \widehat{p}_{i,h} = \xi_{i,t}(\widetilde{p}_{i,h-1}, \widetilde{a}_{i,h-1}, \widetilde{o}_{i,h});$$

if $h = 2t$ with $t \in [H]$, then

$$\widetilde{z}_h = \phi_t(\widehat{p}_{h-1}, \widetilde{a}_{h-1}), \qquad \widetilde{p}_{i,h} = \widehat{p}_{i,h-1} \backslash \phi_{i,t}(\widehat{p}_{i,h-1}, \widetilde{a}_{i,h-1}),$$

where $\chi_t, \xi_{i,t}$ are fixed transformations and $\phi_h, \phi_{i,h}$ are additional-sharing functions. Then, we can construct the $\{\overline{\chi}_{h+1}\}_{h \in [\overline{H}]}, \{\overline{\xi}_{i,h+1}\}_{i \in [n], h \in [\overline{H}]}$ accordingly as follows:

- If $h = 2t-1$ with $t \in [H]$, for any $\overline{p}_{h-1}, \overline{a}_{h-1}, \overline{o}_h$, since $\overline{p}_{h-1} = \breve{p}_{h-1}$ from construction of $\mathcal{D}'_{\mathcal{L}}$, we can select a $\widetilde{p}_{h-1}$ that $\breve{p}_{h-1}$ can be generated from $\widetilde{p}_{h-1}$ through expansion (such $\widetilde{p}_{h-1}$ might not be unique). Then, define $\overline{\chi}_h(\overline{p}_{h-1}, \overline{a}_{h-1}, \overline{o}_h) = \chi_t(\widetilde{p}_{h-1}, \overline{a}_{h-1}, \overline{o}_h) \cup \{\overline{a}_{j,h_1} \,|\, j \in [n], h_1 < h, \overline{a}_{j,h_1} \in \overline{p}_{h-1}, \sigma(\widetilde{\tau}_{j,h_1}) \subseteq \sigma(\widetilde{c}_h)\} \backslash (\widetilde{p}_{h-1} \backslash \overline{p}_{h-1})$. Since χ_t is a fixed transformation and we remove the $\widetilde{p}_{h-1} \backslash \overline{p}_{h-1}$ part from $\overline{z}_h$, the value $\overline{\chi}_h(\overline{p}_{h-1}, \overline{a}_{h-1}, \overline{o}_h)$ is the same no matter what $\widetilde{p}_{h-1}$ we select, and thus such $\overline{\chi}_h$ is well-defined. Similarly, we can define $\overline{\xi}_{i,h}(\overline{p}_{i,h-1}, \overline{a}_{i,h-1}, \overline{o}_{i,h-1}) = \xi_{i,t}(\widetilde{p}_{i,h-1}, \overline{a}_{i,h-1}, \overline{o}_{i,h}) \backslash \{\overline{a}_{i,h_1} \,|\, h_1 < h, \overline{a}_{i,h_1} \in \overline{p}_{i,h-1}, \sigma(\widetilde{\tau}_{i,h_1}) \subseteq \sigma(\widetilde{c}_h)\} \backslash (\widetilde{p}_{i,h-1} \backslash \overline{p}_{i,h-1})$.
- If $h = 2t$ with $t \in [H]$, for any $\overline{p}_{h-1}, \overline{a}_{h-1}$, from the construction of $\mathcal{D}'_{\mathcal{L}}$, we can select a $\widehat{p}_{h-1}$ that $\overline{p}_{h-1}$ can be generated from $\widehat{p}_{h-1} = p_{t^-}$ through expansion (such $\widehat{p}_{h-1}$ might not be unique). Also, it holds that $\overline{o}_h = \emptyset$, then define $\overline{\chi}_h(\overline{p}_{h-1}, \overline{a}_{h-1}, \overline{o}_h) = \phi_t(\widehat{p}_{h-1}, \overline{a}_{h-1}) \cup \{\overline{a}_{j,h_1} \,|\, j \in [n], h_1 < h, \overline{a}_{j,h_1} \in \overline{p}_{h-1}, \sigma(\widetilde{\tau}_{j,h_1}) \subseteq \sigma(\widetilde{c}_h)\} \backslash (\widehat{p}_{h-1} \backslash \overline{p}_{h-1})$. Still, since ϕ_t is the addition-sharing function, which part of $\widehat{p}_{h-1}$ to share only depends on $\overline{a}_{h-1}$, and not depends on the value of $\widehat{p}_{h-1}$, and we remove the $\widehat{p}_{h-1} \backslash \overline{p}_{h-1}$ part from $\overline{z}_h$, the value of $\overline{\chi}_h(\overline{p}_{h-1}, \overline{a}_{h-1}, \overline{o}_h)$ is the same no matter what $\widehat{p}_{h-1}$ we select, and thus such $\overline{\chi}_h$ is well-defined. Similarly, we can define $\overline{\xi}_{i,h}(\overline{p}_{i,h-1}, \overline{a}_{i,h-1}, \overline{o}_{i,h-1}) = \overline{p}_{i,h-1} \backslash \{\overline{a}_{i,h_1} \,|\, h_1 < h, \overline{a}_{i,h_1} \in \overline{p}_{i,h-1}, \sigma(\widetilde{\tau}_{i,h_1}) \subseteq \sigma(\widetilde{c}_h)\} \backslash \phi_{i,t}(\widehat{p}_{i,h-1}, \overline{a}_{i,h-1})$.

Therefore, the common and private information of $\mathcal{D}'_{\mathcal{L}}$ satisfies that

$$\overline{c}_{h+1} = \overline{c}_h \cup \overline{z}_{h+1}, \overline{z}_{h+1} = \overline{\chi}_{h+1}(\overline{p}_h, \overline{a}_h, \overline{o}_{h+1})$$
$$\text{for each } i \in [n], \overline{p}_{i,h+1} = \overline{\xi}_{i,h+1}(\overline{p}_{i,h}, \overline{a}_{i,h}, \overline{o}_{i,h+1}),$$

with some functions $\{\overline{\chi}_{h+1}\}_{h \in [\overline{H}]}, \{\overline{\xi}_{i,h+1}\}_{i \in [n], h \in [\overline{H}]}$.
Thirdly, we prove that such a Dec-POMDP $\mathcal{D}'_{\mathcal{L}}$ is SI with respect to the strategy space $\overline{\mathcal{G}}_{1:\overline{H}}$. This is equivalent to that for any $h \in [2 : \overline{H}], \overline{s}_h \in \overline{\mathcal{S}}, \overline{p}_h \in \overline{\mathcal{P}}_h, \overline{c}_h \in \overline{\mathcal{C}}_h, i_1 \in [n], h_1 < h, \overline{g}_{1:h-1}, \overline{g}'_{i_1,h_1} \in \overline{\mathcal{G}}_{i_1:h_1}$, let $\overline{g}'_{1:h-1} = (\overline{g}_{1,1}, \cdots, \overline{g}_{i_1-1,h_1}, \overline{g}'_{i_1,h_1}, \cdots, \overline{g}_{n,h-1})$, it holds that

$$\mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_h(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}_{1:h-1}) = \mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_h(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}'_{1:h-1}). \tag{C.4}$$

We prove this case by case. If $h = 2t$ with $t \in [H]$, then from the result of Theorem C.5, it holds that

$$\mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_h(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}_{1:h-1}) = \mathbb{P}^{\mathcal{D}^{\dagger}_{\mathcal{L}}}_h(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}_{1:h-1})$$
$$= \mathbb{P}^{\mathcal{D}^{\dagger}_{\mathcal{L}}}_h(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}'_{1:h-1}) = \mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_h(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}'_{1:h-1}).$$

If $h = 2t-1$ with $t \in [H]$, and $h_1 = 2t_1 - 1$ with $t_1 \in [H]$, which means that $\overline{a}_{h_1}$ corresponds to the communication action in previously $\mathcal{L}$. Then it holds that $\overline{c}_{h_1} \subseteq \overline{c}_h, \overline{a}_{i_1,h_1} = m_{i_1,\frac{h_1+1}{2}} \in \overline{c}_h$, then

$$\mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_h(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}_{1:h-1}) = \mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_h(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_{h_1}, \overline{a}_{i_1,h_1}, \overline{c}_h, \overline{g}_{1:h-1})$$
$$= \mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_h(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_{h_1}, \overline{a}_{i_1,h_1}, \overline{c}_h, \overline{g}_{1:h-1} \backslash \overline{g}_{i_1,h_1}) = \mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_h(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}'_{1:h-1}),$$

where the second equality is because the input and output of $\overline{g}_{i_1,h_1}$ are $\overline{c}_{h_1}$ and $\overline{a}_{i_1,h_1}$.
If $h = 2t-1$ with $t \in [H]$, and $h_1 = 2t_1$ with $t_1 \in [H]$, which means that h_1 is in the control timestep, then if agent (i_1, h_1) influences the underlying state $\overline{s}_{h_1+1}$, then from Assumption 3.4, we know that there exists $i_2 \neq i_1$ that, agent (i_1, t_1) influences $o_{i_2,t}$, and thus influences agent (i_2, t) in problem $\mathcal{L}$ even there is no additional sharing. From QC of $\mathcal{L}$ and Assumption 3.5, we know that $\sigma(\tau_{i_1,t_1^-}) \subseteq \sigma(\tau_{i_2,t^-}) \subseteq \sigma(c_t)$. Also, from $\tau_{i_1,t^-} \backslash \tau_{i_1,t_1^+} \subseteq c_{t^+}$, we get $\sigma(\tau_{i_1,t_1^+}) \subseteq \sigma(c_t)$. After reformulation, we have $\sigma(\widetilde{\tau}_{i_1,h_1}) \subseteq \sigma(\widetilde{c}_h)$. From the definition of strict expansion in Eq. (4.1), we have $\overline{a}_{i_1,h_1} \in \overline{c}_h$, and $\sigma(\overline{\tau}_{i_1,h_1}) \subseteq \sigma(\overline{c}_h)$. Then, we conclude

$$\begin{aligned}\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}_{1:h-1}) &= \mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h, \overline{p}_h \,|\, \overline{\tau}_{i_1,h_1}, \overline{a}_{i_1,h_1}, \overline{c}_h, \overline{g}_{1:h-1}) \\ &= \mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h, \overline{p}_h \,|\, \overline{\tau}_{i_1,h_1}, \overline{a}_{i_1,h_1}, \overline{c}_h, \overline{g}_{1:h-1} \backslash \overline{g}_{i_1,h_1}) = \mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}'_{1:h-1}),\end{aligned}$$

where the second equal sign is because the input and output of $\overline{g}_{i_1,h_1}$ are $\overline{\tau}_{i_1,h_1}$ and $\overline{a}_{i_1,h_1}$.
If agent (i_1, h_1) does not influence the underlying state $\overline{s}_{h_1+1}$, then from Assumption 3.3, $\overline{a}_{i_1,h_1} \notin \overline{\tau}_{h_2}$ for any $h_2 > h_1$. Then, agent (i_1, h_1) will not influence $\overline{s}_h$ and $\overline{p}_h$. Then, it directly holds that

$$\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}_{1:h-1}) = \mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}'_{1:h-1}),$$

which completes the proof. □

C.7 Important Definitions of SI Dec-POMDP

Given a Dec-POMDP SI $\mathcal{D}'_{\mathcal{L}}$ obtained from $\mathcal{L}$ after reformulation, strict expansion and refinement. In this part, we only need to discuss how to solve this $\mathcal{D}'_{\mathcal{L}}$. Recall that we use $^-$ for the notation of the elements and quantities in $\mathcal{D}'_{\mathcal{L}}$.

First, we define the following quantities.

Definition C.6 (Value function)**.** For each $i \in [n]$ and $h \in [\overline{H}]$, given common information $\overline{c}_h$ and strategy $\overline{g}_{1:H}$, the value function conditioned on the common information is defined as:

$$V_h^{\overline{g},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_h) := \mathbb{E}_{\overline{g}}^{\mathcal{D}'_{\mathcal{L}}}[\sum_{h'=h}^{\overline{H}} \overline{\mathcal{R}}_{h'}(\overline{s}_{h'}, \overline{a}_{h'}, \overline{p}_{h'}) \,|\, \overline{c}_h], \tag{C.5}$$

where $\overline{\mathcal{R}}_{h'}$ takes $\overline{s}_{h'}, \overline{a}_{h'}, \overline{p}_{h'}$ as input, since after reformulation, the reward may come from communication cost, which is a function of $\overline{p}_{h'}$ and $\overline{a}_{h'}$.

Definition C.7 (Prescription and Q-Value function)**.** Prescription is an important concept in the common-information-based framework (Nayyar et al., 2013b;a). The prescription of agent i at the timestep h is defined as $\gamma_{i,h} : \overline{\mathcal{P}}_{i,h} \to \overline{\mathcal{A}}_{i,h}$. We use γ_h to denote the joint prescription and $\Gamma_{i,h}, \Gamma_h$ to denote the prescription space. The prescriptions are the marginalization of strategy $\overline{g}_h$, i.e., $\gamma_{i,h}(\cdot \,|\, \overline{p}_{i,h}) = \overline{g}_{i,h}(\cdot \,|\, \overline{c}_h, \overline{p}_{i,h})$. Then we can define the Q-value function as

$$Q_h^{\overline{g},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_h, \gamma_h) := \mathbb{E}_{\overline{g}}^{\mathcal{D}'_{\mathcal{L}}}\left[\sum_{h'=h}^{\overline{H}} \overline{\mathcal{R}}_{h'}(\overline{s}'_h, \overline{a}'_h, \overline{p}'_h) \,|\, \overline{c}_h, \gamma_h\right]. \tag{C.6}$$

Remark C.8. In this paper, for any Dec-POMDP $\mathcal{D}'_{\mathcal{L}}$ generated by an $\mathcal{L}$ after reformulation, strict expansion, and refinement, we only consider the strategy spaces at odd timesteps as $\overline{\mathcal{G}}_{i,2t-1} : \overline{\mathcal{C}}_{2t-1} \to \overline{\mathcal{A}}_{i,2t-1}$ and aim to find the optimal strategy in these classes. Therefore, we define the prescription spaces at odd timesteps as $\forall h \in [H], i \in [n], \Gamma_{i,2h-1} = \overline{\mathcal{A}}_{i,2h-1}, \Gamma_{2h-1} = \overline{\mathcal{A}}_{2h-1}$.

Definition C.9 (Expected approximate common information model)**.** We define an expected approximate common information model of $\mathcal{D}'_{\mathcal{L}}$ as

$$\mathcal{M} := \left(\{\widehat{\mathcal{C}}_h\}_{h\in[\overline{H}]}, \{\widehat{\phi}_h\}_{h\in[\overline{H}]}, \{\mathbb{P}_h^{\mathcal{M},z}\}_{h\in[\overline{H}]}, \Gamma, \{\widehat{\mathcal{R}}_h^{\mathcal{M}}\}_{h\in[\overline{H}]}\right), \tag{C.7}$$

where Γ is the joint prescription space, $\widehat{\mathcal{C}}_h$ is the space of approximate common information at step h. $\mathbb{P}_h^{\mathcal{M},z}: \widehat{\mathcal{C}}_h \times \Gamma_h \to \Delta(\overline{Z}_{h+1})$ gives the probability of $\overline{z}_{h+1}$ under $\widehat{c}_h$ and γ_h. $\widehat{\mathcal{R}}_h^{\mathcal{M}}: \widehat{\mathcal{C}}_h \times \Gamma_h \to [0,1]$ gives the reward at timestep h given $\widehat{c}_h$ and γ_h. Then, we call that $\mathcal{M}$ is an $(\epsilon_r(\mathcal{M}), \epsilon_z(\mathcal{M}))$-expected-approximate common information model of $\mathcal{D}'_{\mathcal{L}}$ with some compression function Compress_h such that $\widehat{c}_h = \text{Compress}_h(\overline{c}_h)$ satisfies the following:

- There exists a transformation function $\widehat{\phi}_h$ such that
$$\widehat{c}_h = \widehat{\phi}_h(\widehat{c}_{h-1}, \overline{z}_h), \tag{C.8}$$
where $\overline{z}_h = \overline{c}_h \backslash \overline{c}_{h-1}$ in $\mathcal{D}'_{\mathcal{L}}$.
- For any $\overline{g}_{1:h-1}$ and any prescription $\gamma_h \in \Gamma_h$, it holds that
$$\mathbb{E}^{\mathcal{D}'_{\mathcal{L}}}_{\overline{a}_{1:h-1}, \overline{o}_{1:h} \sim \overline{g}_{1:h-1}} |\mathbb{E}^{\mathcal{D}'_{\mathcal{L}}}[\overline{\mathcal{R}}_h(\overline{s}_h, \overline{a}_h, \overline{p}_h) \,|\, \overline{c}_h, \gamma_h] - \widehat{\mathcal{R}}_h^{\mathcal{M}}(\widehat{c}_h, \gamma_h)| \leq \epsilon_r(\mathcal{M}). \tag{C.9}$$
- For any $\overline{g}_{1:h-1}$ and any prescription $\gamma_h \in \Gamma_h$, it holds that
$$\mathbb{E}^{\mathcal{D}'_{\mathcal{L}}}_{\overline{a}_{1:h-1}, \overline{o}_{1:h} \sim \overline{g}_{1:h-1}} ||\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\cdot \,|\, \overline{c}_h, \gamma_h) - \mathbb{P}_h^{\mathcal{M},z}(\cdot \,|\, \widehat{c}_h, \gamma_h)||_1 \leq \epsilon_z(\mathcal{M}). \tag{C.10}$$

Definition C.10 (Value function under $\mathcal{M}$)**.** Given an Dec-POMDP $\mathcal{D}'_{\mathcal{L}}$and its expected approximate common information model $\mathcal{M}$. For any strategy $\overline{g}_{1:\overline{H}} \in \overline{\mathcal{G}}_{1:\overline{H}}, h \in [H]$, we define the value function as

$$\begin{aligned} V_h^{\overline{g}_{1:\overline{H}}, \mathcal{M}}(\overline{c}_h) =& \widehat{\mathcal{R}}_h^{\mathcal{M}}(\text{Compress}_h(\overline{c}_h), \{\overline{g}_{j,h}(\cdot \,|\, \overline{c}_h, \cdot)\}_{j\in[n]}) \\ &+ \mathbb{E}^{\mathcal{M}}[V_h^{\overline{g}_{1:\overline{H}}, \mathcal{M}}(\overline{c}_{h+1}) \,|\, \text{Compress}_h(\overline{c}_h), \{\overline{g}_{j,h}(\cdot \,|\, \overline{c}_h, \cdot)\}_{j\in[n]}]. \end{aligned} \tag{C.11}$$

Definition C.11 (Model-belief consistency)**.** We say the expected approximate common information model $\mathcal{M}$ is *consistent with* some belief $\{\mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h, \overline{p}_h \,|\, \widehat{c}_h)\}_{h\in[H]}$ if it satisfies the following for all $i \in [n]$, $h \in [H]$:

$$\mathbb{P}_h^{\mathcal{M},z}(\overline{z}_{h+1} \,|\, \widehat{c}_h, \gamma_h) = \sum_{\substack{\overline{s}_h, \overline{p}_h, \overline{a}_h, \overline{o}_{h+1}: \\ \chi_{h+1}(\overline{p}_h, \overline{a}_h, \overline{o}_{h+1}) = \overline{z}_{h+1}}} \tag{C.12}$$
$$\Big(\mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h, \overline{p}_h \,|\, \widehat{c}_h) \mathbb{1}[\overline{a}_h = \gamma_h(\overline{p}_h)] \sum_{s_{h+1}} \overline{\mathbb{T}}_h(\overline{s}_{h+1} \,|\, \overline{s}_h, \overline{a}_h)] \overline{\mathbb{O}}_{h+1}(\overline{o}_{h+1} \,|\, \overline{s}_{h+1})\Big), \tag{C.13}$$
$$\widehat{\mathcal{R}}_h^{\mathcal{M}}(\widehat{c}_h, \gamma_h) = \sum_{\overline{s}_h, \overline{p}_h, \overline{a}_h} \mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h, \overline{p}_h \,|\, \widehat{c}_h) \mathbb{1}[\overline{a}_h = \gamma_h(\overline{p}_h)] \overline{\mathcal{R}}_h(\overline{s}_h, \overline{a}_h). \tag{C.14}$$

Definition C.12 (Strategy-dependent approximate common information model)**.** Given a model $\widetilde{\mathcal{M}}$ (as in Definition C.9) and H joint strategies $g^{1:H}$, where each $g^h \in \overline{\mathcal{G}}_{1:H}$ for $h \in [H]$, we say $\widetilde{\mathcal{M}}$ is a *strategy-dependent expected approximate common information model*, denoted as $\widetilde{\mathcal{M}}(\pi^{1:H})$, if it is consistent with the *strategy-dependent* belief $\{\mathbb{P}_h^{\pi^h, \mathcal{D}'_{\mathcal{L}}}(s_h, p_h \,|\, \widehat{c}_h)\}_{h\in[H]}$ (as per C.11). we say $\widetilde{\mathcal{M}}$ is a *strategy-dependent expected approximate common information model*, denoted as $\widetilde{\mathcal{M}}(g^{1:H})$, if it is consistent with the *strategy-dependent* belief $\{\mathbb{P}_h^{g^h, \mathcal{D}'_{\mathcal{L}}}(\overline{s}_h, \overline{p}_h \,|\, \widehat{c}_h)\}_{h\in[H]}$ (as per C.11).

Definition C.13 (Length of approximate common information)**.** Given the compression functions $\{\text{Compress}_h\}_{h\in[H+1]}$, we define the integer $\widehat{L} > 0$ as the minimum length such that there exists a mapping $\widehat{f}_h: \overline{\mathcal{A}}_{\max\{1, h-\widehat{L}\}:h-1} \times \overline{\mathcal{O}}_{\max\{1, h-\widehat{L}+1\}, h} \to \widehat{\mathcal{C}}_h$ such that for each $h \in [H+1]$ and joint history $\{\overline{o}_{1:h}, \overline{a}_{1:h-1}\}$, we have $\widehat{f}_h(x_h) = \widehat{c}_h$, where $x_h = \{\overline{a}_{\max\{h-\widehat{L},1\}}, \overline{o}_{\max\{h-\widehat{L},1\}+1}, \cdots, \overline{a}_{h-1}, \overline{o}_h\}$.

C.8 Main Results for Planning in QC LTC

Finally, we provide the formal guarantees for planning in QC LTC.

Theorem C.14. Given any QC LTC problem $\mathcal{L}$ satisfying Assumptions 3.1, 3.2, 3.3, 3.4, and 4.3, we can construct an SI Dec-POMDP problem $\mathcal{D}'_{\mathcal{L}}$ such that for any $\epsilon > 0$, solving an ϵ-team optimal strategy in $\mathcal{D}'_{\mathcal{L}}$ can give us an ϵ-team optimal strategy of $\mathcal{L}$, and the following holds. Fix $\epsilon_r, \epsilon_z > 0$ and given any (ϵ_r, ϵ_z)-expected-approximate common information model $\mathcal{M}$ for $\mathcal{D}'_{\mathcal{L}}$ that is consistent with some given approximate belief $\{\mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h, \overline{p}_h \mid \widehat{c}_h)\}_{h\in[\overline{H}]}$, Algorithm 1 can compute a $(2\overline{H}\epsilon_r + \overline{H}^2\epsilon_z)$-team optimal strategy for the original LTC problem $\mathcal{L}$ with time complexity $\max_{h\in[\overline{H}]} |\widehat{\mathcal{C}}_h| \cdot \texttt{poly}(|\mathcal{S}|, |\mathcal{A}_h|, |\mathcal{P}_h|, \overline{H})$. In particular, for fixed $\epsilon > 0$, if $\mathcal{L}$ has any one of baseline sharing protocols as in §A, one can construct a $\mathcal{M}$ and apply Algorithm 1 to compute an ϵ-team optimal strategy for $\mathcal{L}$ in quasi-polynomial time.

Proof. We divide the proof into the following three **Parts**.

Part I: Given any QC LTC problem $\mathcal{L}$ satisfying Assumptions 3.1, 3.2, 3.3, and 3.4, we can construct an SI Dec-POMDP problem $\mathcal{D}'_{\mathcal{L}}$ such that finding an ϵ-team optimal strategy can give us an ϵ-team optimal strategy of $\mathcal{L}$, as shown in Algorithm 1.
We can construct a Dec-POMDP $\mathcal{D}'_{\mathcal{L}}$ from $\mathcal{L}$ through Algorithm 1. From Proposition C.1 and Theorems C.4, C.5. We know that $\mathcal{D}'_{\mathcal{L}}$ is SI and an ϵ-team-optimal strategy of $\mathcal{D}'_{\mathcal{L}}$ can give us an ϵ-team optimal strategy of $\mathcal{L}$.

Part II: Given any ϵ-expected-approximate common information model $\mathcal{M}$ of the Dec-POMDP $\mathcal{D}'_{\mathcal{L}}$, there exists an algorithm, Algorithm 6, that can output an ϵ-team optimal strategy of $\mathcal{D}'_{\mathcal{L}}$.
First, we need to prove that solving $\mathcal{M}$ can get the ϵ-team optimal strategy of $\mathcal{D}'_{\mathcal{L}}$. We prove the following 2 lemmas first.

Lemma C.15. For any strategy $\overline{g}_{1:\overline{H}}$, and $h \in [\overline{H}]$, we have

$$\mathbb{E}_{\overline{g}_{1:\overline{H}}}^{\mathcal{D}'_{\mathcal{L}}}[|V_h^{\overline{g}_{1:\overline{H}},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_h) - V_h^{\overline{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_h)|] \leq (\overline{H} - h + 1)\epsilon_r + \frac{(\overline{H} - h + 1)(\overline{H} - h)}{2}\epsilon_z. \tag{C.15}$$

Proof. We prove it by induction. For $h = \overline{H} + 1$, we have $V_h^{\overline{g}_{1:\overline{H}},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_h) = V_h^{\overline{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_h) = 0$.
For the step $h \leq \overline{H}$, we have

$$\begin{aligned}
&\mathbb{E}_{\overline{g}_{1:\overline{H}}}^{\mathcal{D}'_{\mathcal{L}}}[|V_h^{\overline{g}_{1:\overline{H}},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_h) - V_h^{\overline{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_h)|] \\
\leq &\mathbb{E}_{\overline{g}_{1:\overline{H}}}^{\mathcal{D}'_{\mathcal{L}}}\left[|\mathbb{E}^{\mathcal{D}_{\mathcal{L}}}[\overline{\mathcal{R}}_h(\overline{s}_h, \overline{a}_h, \overline{p}_h) \mid \overline{c}_h, \{\overline{g}_{j,h}(\cdot \mid \overline{c}_h, \cdot)\}_{j\in[n]}] - \widehat{\mathcal{R}}_h^{\mathcal{M}}(\widehat{c}_h, \{\overline{g}_{j,h}(\cdot \mid \overline{c}_h, \cdot)\}_{j\in[n]})|\right] \\
&+ \mathbb{E}_{\overline{g}_{1:\overline{H}}}^{\mathcal{D}'_{\mathcal{L}}}\Big[|\mathbb{E}_{\overline{z}_{h+1}\sim\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\cdot \mid \overline{c}_h, \{\overline{g}_{j,h}(\cdot \mid \overline{c}_h, \cdot)\}_{j\in[n]})}[V_h^{\overline{g}_{1:\overline{H}},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_h \cup \overline{z}_{h+1})] \\
&- \mathbb{E}_{\overline{z}_{h+1}\sim\mathbb{P}_h^{\mathcal{M},z}(\cdot \mid \widehat{c}_h, \{\overline{g}_{j,h}(\cdot \mid \overline{c}_h, \cdot)\}_{j\in[n]})}[V_h^{\overline{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_h \cup \overline{z}_{h+1})]|\Big] \\
\leq &\epsilon_r + (\overline{H} - h)\mathbb{E}_{\overline{a}_{1:h-1},\overline{o}_{1:h}\sim\overline{g}_{1:h-1}}^{\mathcal{D}'_{\mathcal{L}}}||\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\cdot \mid \overline{c}_h, \gamma_h) - \mathbb{P}_h^{\mathcal{M},z}(\cdot \mid \widehat{c}_h, \gamma_h)||_1 \\
&+ \mathbb{E}_{\overline{a}_{1:h-1},\overline{o}_{1:h}\sim\overline{g}_{1:h-1}}^{\mathcal{D}'_{\mathcal{L}}}\left[|V_{h+1}^{\overline{g}_{1:\overline{H}},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_{h+1}) - V_{h+1}^{\overline{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_{h+1})|\right] \\
\leq &\epsilon_r + (\overline{H} - h)\epsilon_z + (\overline{H} - h)\epsilon_r + \frac{(\overline{H} - h)(\overline{H} - h - 1)}{2}\epsilon_z \\
\leq &(\overline{H} - h + 1)\epsilon_r + \frac{(\overline{H} - h)(\overline{H} - h + 1)}{2}\epsilon_z.
\end{aligned}$$

The proof mainly follows from the proof of Lemma 2 in (Liu & Zhang, 2023). But the difference is that $\mathcal{D}'_{\mathcal{L}}$ may not satisfy Assumption 2.1. In the third line of this proof, we had

$\overline{z}_{h+1} \sim \mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\cdot \,|\, \overline{c}_h, \{\overline{g}_{j,h}(\cdot \,|\, \overline{c}_h, \cdot)\}_{j\in[n]})$, where $\overline{z}_{h+1}$ is generated as

$$\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{z}_{h+1} \,|\, \overline{c}_h, \gamma_h) = \sum_{\overline{s}_h \in \overline{\mathcal{S}}, \overline{p}_h \in \overline{\mathcal{P}}_h} \mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h,\ \overline{p}_h \,|\, \overline{c}_h)$$
$$\sum_{\overline{s}_{h+1} \in \overline{\mathcal{S}}, \overline{o}_{h+1} \in \overline{\mathcal{O}}_{h+1}} \overline{\mathbb{T}}_{h+1}(\overline{s}_{h+1} \,|\, \overline{s}_h, \gamma_h(\overline{p}_h)) \overline{\mathbb{O}}_{h+1}(\overline{o}_{h+1} \,|\, \overline{s}_{h+1}) \mathbb{1}[\overline{\chi}_{h+1}(\overline{p}_h, \gamma_h(\overline{p}_h), \overline{o}_{h+1})],$$

with $\gamma_h = \{\overline{g}_{j,h}(\cdot \,|\, \overline{c}_h, \cdot)\}_{j\in[n]}$. $\square$

Lemma C.16. Let $\widehat{g}^*_{1:\overline{H}}$ be the strategy output by Algorithm 6, then for any $h \in [\overline{H}], \overline{c}_h \in \overline{\mathcal{C}}_h, \overline{g}_{1:\overline{H}} \in \overline{\mathcal{G}}_{1:\overline{H}}$, it holds that

$$V_h^{\overline{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_h) \leq V_h^{\widehat{g}^*_{1:\overline{H}},\mathcal{M}}(\overline{c}_h). \tag{C.16}$$

Proof. We prove it by induction. For $h = \overline{H}+1$, we have $V_h^{\overline{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_h) = V_h^{\widehat{g}^*_{1:\overline{H}},\mathcal{M}}(\overline{c}_h) = 0$. For the timestep $h \leq H$, we have

$$\begin{aligned}
V_h^{\overline{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_h) &= \mathbb{E}^{\mathcal{M}}[\widehat{r}_h^{\mathcal{M}}(\widehat{c}_h) + V_{h+1}^{\overline{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_{h+1}) \,|\, \widehat{c}_h, \overline{g}_{1:\overline{H}}] \\
&\leq \mathbb{E}^{\mathcal{M}}[\widehat{r}_h^{\mathcal{M}}(\widehat{c}_h) + V_{h+1}^{\widehat{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_{h+1}) \,|\, \widehat{c}_h, \overline{g}_{1:\overline{H}}] \\
&= Q_h^{\widehat{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_h, \{\overline{g}_{j,h}(\cdot \,|\, \overline{c}_h)\}_{j\in[n]}) \\
&\leq Q_h^{\widehat{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_h, \{\overline{g}_{j,h}(\cdot \,|\, \overline{c}_h)\}_{j\in[n]}) \\
&= V_h^{\widehat{g}^*_{1:\overline{H}},\mathcal{M}}(\overline{c}_h).
\end{aligned}$$

For the first inequality, we use the induction hypothesis. For the second inequality sign, we use the property of argmax in algorithm and $V_h^{\widehat{g}^*_{1:\overline{H}},\mathcal{M}}(\overline{c}_h) = V_h^{\widehat{g}^*_{1:\overline{H}},\mathcal{M}}(\widehat{c}_h)$. By induction, we complete the proof. $\square$

We now go back to the proof of the theorem. Let $\widehat{g}^*_{1:\overline{H}}$ be the solution output by Algorithm 6, then for any $\overline{g}_{1:\overline{H}} \in \overline{\mathcal{G}}_{1:\overline{H}}, h \in [\overline{H}], \overline{c}_h \in \overline{\mathcal{C}}_h$, we have

$$\begin{aligned}
&\mathbb{E}_{\overline{g}_{1:\overline{H}}}^{\mathcal{D}'_{\mathcal{L}}}\left[V_h^{\overline{g}_{1:\overline{H}},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_h) - V_h^{\widehat{g}^*_{1:\overline{H}},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_h)\right] \\
&\quad = \mathbb{E}_{\overline{g}_{1:\overline{H}}}^{\mathcal{D}'_{\mathcal{L}}}\left[\left(V_h^{\overline{g}_{1:\overline{H}},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_h) - V_h^{\widehat{g}^*_{1:\overline{H}},\mathcal{M}}(\overline{c}_h)\right) + \left(V_h^{\widehat{g}^*_{1:\overline{H}},\mathcal{M}}(\overline{c}_h) - V_h^{\widehat{g}^*_{1:\overline{H}},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_h)\right)\right] \\
&\quad \leq \mathbb{E}_{\overline{g}_{1:\overline{H}}}^{\mathcal{D}'_{\mathcal{L}}}\left[\left(V_h^{\overline{g}_{1:\overline{H}},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_h) - V_h^{\overline{g}_{1:\overline{H}},\mathcal{M}}(\overline{c}_h)\right) + \left(V_h^{\widehat{g}^*_{1:\overline{H}},\mathcal{M}}(\overline{c}_h) - V_h^{\widehat{g}^*_{1:\overline{H}},\mathcal{D}'_{\mathcal{L}}}(\overline{c}_h)\right)\right] \\
&\quad \leq (\overline{H} - h + 1)\epsilon_r + \frac{(\overline{H} - h)(\overline{H} - h + 1)}{2}\epsilon_z + (\overline{H} - h + 1)\epsilon_r + \frac{(\overline{H} - h)(\overline{H} - h + 1)}{2}\epsilon_z \\
&\quad = 2(\overline{H} - h + 1)\epsilon_r + (\overline{H} - h)(\overline{H} - h + 1)\epsilon_z.
\end{aligned} \tag{C.17}$$

For the first inequality, we use Lemma C.16. For the second inequality sign, we use Lemma C.15. Then apply $h = 1$, we have $J_{\mathcal{D}'_{\mathcal{L}}}(\overline{g}_{1:\overline{H}}) \leq J_{\mathcal{D}'_{\mathcal{L}}}(\widehat{g}^*_{1:\overline{H}}) + 2\overline{H}\epsilon_r + \overline{H}^2\epsilon_z$. This completes the proof of **Part II**.

Part III: If the baseline sharing of $\mathcal{L}$ is one of the 4 cases in §A, we can construct an expected-approximate common information model of $\mathcal{D}'_{\mathcal{L}}$.
3kkk We first prove following lemmas: We aim to bound (ϵ_r, ϵ_z) using the following lemma.

Lemma C.17. Given any belief $\{\mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h,\overline{p}_h\}_{h\in[\overline{H}]}$ consistent with the expected-approximate-common-information model $\mathcal{M}$, it holds that for any $h\in[\overline{H}],\overline{\mathcal{C}}_h,\gamma_h\in\Gamma_h$:

$$||\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\cdot\,|\,\overline{c}_h,\gamma_h)-\mathbb{P}_h^{\mathcal{M},z}(\cdot\,|\,\widehat{c}_h,\gamma_h)||_1\leq||\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\cdot,\cdot\,|\,\overline{c}_h)-\mathbb{P}_h^{\mathcal{M},c}(\cdot,\cdot\,|\,\widehat{c}_h)||_1,$$
$$|\mathbb{E}^{\mathcal{D}'_{\mathcal{L}}}[\mathcal{R}_h(\overline{s}_h,\overline{a}_h,\overline{p}_h)\,|\,\overline{c}_h,\gamma_h]-\widehat{\mathcal{R}}_h^{\mathcal{M}}(\widehat{c}_h,\gamma_h)|\leq||\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\cdot,\cdot\,|\,\overline{c}_h)-\mathbb{P}_h^{\mathcal{M},c}(\cdot,\cdot\,|\,\widehat{c}_h)||_1,$$

where $\widehat{c}_h$ =Compress$_h(\overline{c}_h)$.

Proof. Adapted from Lemma 3 in (Liu & Zhang, 2023) by changing the reward function of $r_{i,h}(s_h,a_h)$ to $\mathcal{R}_h(\overline{s}_h,\overline{a}_h,\overline{p}_h)$. Note that the latter can still be evaluated given the common-information-based belief, $\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h,\overline{p}_h\,|\,\overline{c}_h)$. □

Then we define the belief states following the notation in (Golowich et al., 2023; Liu & Zhang, 2023) as $\overline{\boldsymbol{b}}_1(\emptyset)=\mu_1$, $\overline{\boldsymbol{b}}_h(\overline{a}_{1:h-1},\overline{o}_{1:h})=\mathbb{P}(\overline{s}_h=\cdot\,|\,\overline{o}_{1:h},\overline{a}_{1:h-1}),\overline{\boldsymbol{b}}_h(\overline{a}_{1:h-1},\overline{o}_{1:h-1})=\mathbb{P}(\overline{s}_h=\cdot\,|\,\overline{o}_{1:h-1},\overline{a}_{1:h-1})$, where $\overline{\boldsymbol{b}}\in\Delta(\mathcal{S})$. Also, we define the approximate belief state using the most recent L-step history, that

$$\overline{\boldsymbol{b}}_h'(\overline{a}_{h-L:h-1},\overline{o}_{h-L+1:h-1})=\mathbb{P}(\overline{s}_h=\cdot\,|\,\overline{s}_{h-L}\sim\text{Unif}(\mathcal{S}),\overline{a}_{h-L:h-1},\overline{o}_{h-L+1:h})$$
$$\overline{\boldsymbol{b}}_h'(\overline{a}_{h-L:h-1},\overline{o}_{h-L+1:h-1})=\mathbb{P}(\overline{s}_h=\cdot\,|\,\overline{s}_{h-L}\sim\text{Unif}(\mathcal{S}),\overline{a}_{h-L:h-1},\overline{o}_{h-L+1:h}).$$

Also, for any set $N\subseteq[n]$, we define $\overline{a}_{N,h}=\{\overline{a}_{i,h}\}_{i\in N}$, and the same for $\overline{o}_{N,h}$. We can also define the belief of states given historical observations and actions as follows: for any $N\subseteq[n]$,

$$\overline{\boldsymbol{b}}_h(\overline{a}_{1:h-1},\overline{o}_{1:h-1},\overline{o}_{N,h})=\mathbb{P}(\overline{s}_h=\cdot\,|\,\overline{a}_{1:h-1},\overline{o}_{1:h-1},\overline{o}_{N,h})$$
$$\overline{\boldsymbol{b}}_h'(\overline{a}_{h-L:h-1},\overline{o}_{h-L+1:h-1},\overline{o}_{N,h})=\mathbb{P}_h(\overline{s}_h=\cdot\,|\,\overline{s}_{h-L}\sim\text{Unif}(\mathcal{S}),\overline{a}_{h-L:h-1},\overline{o}_{h-L+1:h-1},\overline{o}_{N,h}).$$

Then, we have the following lemma.

Lemma C.18. There is a constant $C\geq 1$ such that the following holds. Given any LTC problem $\mathcal{L}$ satisfying Assumption 3.1, and let $\mathcal{D}'_{\mathcal{L}}$ be the Dec-POMDP after reformulation, strict expansion and refinement. Let $\epsilon\geq 0$, fix a strategy $\overline{g}_{1:\overline{H}}$ and indices $1\leq h-L<h-1\leq\overline{H}$. If $L\geq C\gamma^{-4}\log(\frac{S}{\epsilon})$, then the following set of inequalities hold

$$\mathbb{E}_{\overline{a}_{1:h-1},\overline{o}_{1:h}\sim\overline{g}_{1:\overline{H}}}||\overline{\boldsymbol{b}}_h(\overline{a}_{1:h-1},\overline{o}_{1:h})-\overline{\boldsymbol{b}}_h'(\overline{a}_{h-L:h-1},\overline{o}_{h-L+1:h})||_1\leq\epsilon \quad (C.18)$$
$$\mathbb{E}_{\overline{a}_{1:h-1},\overline{o}_{1:h}\sim\overline{g}_{1:\overline{H}}}||\overline{\boldsymbol{b}}_h(\overline{a}_{1:h-1},\overline{o}_{1:h-1})-\overline{\boldsymbol{b}}_h'(\overline{a}_{h-L:h-1},\overline{o}_{h-L+1:h-1})||_1\leq\epsilon \quad (C.19)$$
$$\mathbb{E}_{\overline{a}_{1:h-1},\overline{o}_{1:h}\sim\overline{g}_{1:\overline{H}}}||\overline{\boldsymbol{b}}_h(\overline{a}_{1:h-1},\overline{o}_{1:h-1},\overline{o}_{N,h})-\overline{\boldsymbol{b}}_h'(\overline{a}_{h-L:h-1},\overline{o}_{h-L+1:h-1},\overline{o}_{N,h})||_1\leq\epsilon. \quad (C.20)$$

Proof. Given any LTC problem $\mathcal{L}$, we can construct a Dec-POMDP $\check{\mathcal{D}}$ that the transition and observation functions of $\check{\mathcal{D}}$ are the same as $\mathcal{L}$. And the information of $\check{\mathcal{D}}$ is fully sharing, which means it shares all the $o_{1:h-1},a_{1:h}$ as common information at timestep h. Since $\mathcal{D}'_{\mathcal{L}}$ is reformulated from $\mathcal{L}$, we have

$$\overline{\boldsymbol{b}}_h(\overline{a}_{1:h-1},\overline{o}_{1:h})=\boldsymbol{b}_{\lfloor\frac{h+1}{2}\rfloor}(a_{1:\lfloor\frac{h-1}{2}\rfloor},o_{1:\lfloor\frac{h+1}{2}\rfloor})=\check{\boldsymbol{b}}_{\lfloor\frac{h+1}{2}\rfloor}(\check{a}_{1:\lfloor\frac{h-1}{2}\rfloor},\check{o}_{1:\lfloor\frac{h+1}{2}\rfloor})$$
$$\overline{\boldsymbol{b}}_h(\overline{a}_{1:h-1},\overline{o}_{1:h-1})=\boldsymbol{b}_{\lfloor\frac{h+1}{2}\rfloor}(a_{1:\lfloor\frac{h-1}{2}\rfloor},o_{1:\lfloor\frac{h}{2}\rfloor})=\check{\boldsymbol{b}}_{\lfloor\frac{h+1}{2}\rfloor}(\check{a}_{1:\lfloor\frac{h-1}{2}\rfloor},\check{o}_{1:\lfloor\frac{h}{2}\rfloor}).$$

And for the approximate belief state, we have

$$\begin{aligned}&\overline{b}_{h+1}'(\overline{a}_{h-L:h},\overline{o}_{h-L+1:h})=\boldsymbol{b}'_{\lfloor\frac{h+2}{2}\rfloor}(a_{\lfloor\frac{h-L}{2}\rfloor:\lfloor\frac{h}{2}\rfloor},o_{\lfloor\frac{h-L+2}{2}\rfloor:\lfloor\frac{h+1}{2}\rfloor})\\&=\check{\boldsymbol{b}}'_{\lfloor\frac{h+2}{2}\rfloor}(\check{a}_{\lfloor\frac{h-L}{2}\rfloor:\lfloor\frac{h}{2}\rfloor},\check{o}_{\lfloor\frac{h-L+2}{2}\rfloor:\lfloor\frac{h+1}{2}\rfloor})\\&\overline{b}_h'(\overline{a}_{h-L:h-1},\overline{o}_{h-L+1:h})\\&=\boldsymbol{b}'_{\lfloor\frac{h+1}{2}\rfloor}(a_{\lfloor\frac{h-L}{2}\rfloor:\lfloor\frac{h-1}{2}\rfloor},o_{\lfloor\frac{h-L+2}{2}\rfloor:\lfloor\frac{h+1}{2}\rfloor})=\check{\boldsymbol{b}}'_{\lfloor\frac{h+1}{2}\rfloor}(\check{a}_{\lfloor\frac{h-L}{2}\rfloor:\lfloor\frac{h-1}{2}\rfloor},\check{o}_{\lfloor\frac{h-L+2}{2}\rfloor:\lfloor\frac{h}{2}\rfloor}).\end{aligned}$$

Also, since for any $t \in [H], \overline{a}_{2t-1}$ are communication actions, $\overline{o}_{2t} = \emptyset$ is null, and $\overline{s}_{2t-1} = \overline{s}_{2t}$ always holds. Then we can write Eq. (C.18) and Eq. (C.19) as

$$\mathbb{E}_{\{\overline{a}_{2t}\}_{t=1}^{\lfloor\frac{h-1}{2}\rfloor}, \{\overline{o}_{2t-1}\}_{t=1}^{\lfloor\frac{h+1}{2}\rfloor} \sim \overline{g}_{1:\overline{H}}} ||\overline{\boldsymbol{b}}_h(\overline{a}_{1:h-1}, \overline{o}_{1:h}) - \overline{\boldsymbol{b}}'_h(\overline{a}_{h-L:h-1}, \overline{o}_{h-L+1:h})||_1 \leq \epsilon \tag{C.21}$$

$$\mathbb{E}_{\{\overline{a}_{2t}\}_{t=1}^{\lfloor\frac{h-1}{2}\rfloor}, \{\overline{o}_{2t-1}\}_{t=1}^{\lfloor\frac{h+1}{2}\rfloor} \sim \overline{g}_{1:\overline{H}}} ||\overline{\boldsymbol{b}}_h(\overline{a}_{1:h-1}, \overline{o}_{1:h-1}) - \overline{\boldsymbol{b}}'_h(\overline{a}_{h-L:h-1}, \overline{o}_{h-L+1:h-1})||_1 \leq \epsilon. \tag{C.22}$$

Since $\check{\mathcal{D}}$ has a fully-sharing IS, for any $i \in [n], h \in [\overline{H}]$ and information $\overline{\tau}_{i,h}, \overline{\tau}_{i,2h}$, we have $\sigma(\overline{\tau}_{i,h}) \subseteq \sigma(\check{\tau}_{i,\lfloor\frac{h+1}{2}\rfloor})$. Therefore, given any strategy $\overline{g}_{1:\overline{H}}$, we can construct a strategy $\check{g}_{1:H}$ such that, for any $\overline{a}_{1:h-1}, \overline{o}_{1:h}$

$$\mathbb{P}(\{\overline{a}_{2t}\}_{t=1}^{\lfloor\frac{h-1}{2}\rfloor}, \{\overline{o}_{2t-1}\}_{t=1}^{\lfloor\frac{h+1}{2}\rfloor} \,|\, \overline{g}_{1:\overline{H}}) = \mathbb{P}(\check{a}_{1:\lfloor\frac{h-1}{2}\rfloor}, \check{o}_{1:\lfloor\frac{h+1}{2}\rfloor} \,|\, \check{g}_{1:H}).$$

Since $\check{\mathcal{D}}$ satisfies Assumption 3.1, we can apply the Theorem 10 in (Liu & Zhang, 2023) with $\check{g}_{1:H}$ to get the result that there is a constant $C_0 \geq 1$ such that if $L' \geq C_0\gamma^{-4}\log(\frac{S}{\epsilon})$, the following holds

$$\mathbb{E}_{\check{a}_{1:\lfloor\frac{h-1}{2}\rfloor}, \check{o}_{1:\lfloor\frac{h+1}{2}\rfloor} \sim \check{g}_{1:H}} \tag{C.23}$$

$$||\check{\boldsymbol{b}}_{\lfloor\frac{h+1}{2}\rfloor}(\check{a}_{1:\lfloor\frac{h-1}{2}\rfloor}, \check{o}_{1:\lfloor\frac{h+1}{2}\rfloor}) - \check{\boldsymbol{b}}'_{\lfloor\frac{h+1}{2}\rfloor}(\check{a}_{\lfloor\frac{h}{2}\rfloor-L':\lfloor\frac{h-1}{2}\rfloor}, \check{o}_{\lfloor\frac{h+1}{2}\rfloor-L'+1:\lfloor\frac{h+1}{2}\rfloor})||_1 \leq \epsilon \tag{C.24}$$

$$\mathbb{E}_{\check{a}_{1:\lfloor\frac{h-1}{2}\rfloor}, \check{o}_{1:\lfloor\frac{h+1}{2}\rfloor} \sim \check{g}_{1:H}} \tag{C.25}$$

$$||\check{\boldsymbol{b}}_{\lfloor\frac{h+1}{2}\rfloor}(\check{a}_{1:\lfloor\frac{h-1}{2}\rfloor}, \check{o}_{1:\lfloor\frac{h}{2}\rfloor}) - \check{\boldsymbol{b}}'_{\lfloor\frac{h+1}{2}\rfloor}(\check{a}_{\lfloor\frac{h}{2}\rfloor-L':\lfloor\frac{h-1}{2}\rfloor}, \check{o}_{\lfloor\frac{h+1}{2}\rfloor-L'+1:\lfloor\frac{h}{2}\rfloor})||_1 \leq \epsilon. \tag{C.26}$$

We choose $C = 3C_0, L = 2L' + 1$. If $L \geq C\gamma^{-4}\log(\frac{S}{\epsilon})$, there must have $L' \geq C_0\gamma^{-4}\log(\frac{S}{\epsilon})$. Therefore, we directly get Eq. (C.21) and Eq. (C.22).
For Eq. (C.20), we cannot directly apply Theorem 10 in (Liu & Zhang, 2023), but we can slightly change the Eq. (E.11) of Theorem 10 in (Liu & Zhang, 2023) as

$$\mathbb{E}^{\mathcal{D}'_{\mathcal{L}}}_{a_{1:h-1}, o_{1:h} \sim g_{1:\overline{H}}} ||\overline{\boldsymbol{b}}_h(a_{1:h-1}, o_{1:h-1}, o_{N,h}) - \overline{\boldsymbol{b}}'_h(a_{h-L:h-1}, o_{h-L+1:h-1}, o_{N,h})||_1 \leq \epsilon. \tag{C.27}$$

It still holds if the posterior update $F^q(P : o_{1,h})$ is changed to $F^q(P : o_{N,h})$, when applying Lemma 9 in the proof of Theorem 10 of (Liu & Zhang, 2023). Therefore, we can use the same arguments to prove Eq. (C.20) from Eq. (C.27) as above, and this completes the proof. $\square$

Then we can compress the common information using a finite-memory truncation. Here, we discuss case-by-case how to compress it for the 8 examples of QC LTC given in §A. Note that after reformulation, strict expansion, and refinement, **Examples 5** and **6** will be the same as **Example 1**, and **Examples 7** and **8** will be the same as **Example 2**. Therefore, we can categorize the examples in §A into 4 types.

Type 1: Baseline sharing of $\mathcal{L}$ is one of **Examples 1, 5, 6** in §A. Then, common information should be that for any $t \in [H], \overline{c}_{2t-1} = \{\overline{o}_{1:2t-2}, \overline{a}_{1:2t-2}\}, \overline{c}_{2t} = \{\overline{o}_{1:2t-2}, \overline{a}_{1:2t-1}, \overline{o}_{N,2t-1}\}, N \subseteq [n]$, where N is the set of agents choose to share their observations through additional sharing, and N can be inferred from $\overline{c}_{2t}$. Then we have that $\mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_{2t-1}(\overline{s}_{2t-1}, \overline{p}_{2t-1} \,|\, \overline{c}_{2t-1}) = \overline{\boldsymbol{b}}_{2t-1}(\overline{a}_{1:2t-2}, \overline{o}_{1:2t-2})(\overline{s}_{2t-1})\overline{\mathbb{O}}_{2t-1}(\overline{o}_{2t-1} \,|\, \overline{s}_{2t-1})$. Fix compress length $L > 0$, we define the approximate common information as $\widehat{c}_{2t-1} = \{\overline{a}_{2t-1-L:2t-2}, \overline{o}_{2t-L:2t-2}\}$, and the common information conditioned belief as $\mathbb{P}^{\mathcal{M},c}_{2t-1}(\overline{s}_{2t-1}, \overline{p}_{2t-1} \,|\, \widehat{c}_{2t-1}) = \overline{\boldsymbol{b}}_{2t-1}(\overline{a}_{2t-1-L:2t-2}, \overline{o}_{2t-L:2t-2})(\overline{s}_{2t-1})\overline{\mathbb{O}}_{2t-1}(\overline{o}_{2t-1} \,|\, \overline{s}_{2t-1})$. Also, we have $\mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_{2t}(\overline{s}_{2t}, \overline{p}_{2t} \,|\, \overline{c}_{2t}) = \overline{\boldsymbol{b}}_{2t-1}(\overline{a}_{1:2t-1}, \overline{o}_{1:2t-2}, \overline{o}_{N,2t-1})(\overline{s}_{2t-1})\mathbb{P}_{2t-1}(\overline{o}_{-N,2t-1} \,|\, \overline{s}_{2t-1}, \overline{o}_{N,2t-1})$. Fix compress length $L > 0$, we define the approximate common information a $\widehat{c}_{2t} = \{\overline{a}_{2t-1-L:2t-2}, \overline{o}_{2t-L:2t-2}, \overline{o}_{N,2t-1}\}$, and the common information conditioned belief as $\mathbb{P}^{\mathcal{M},c}_{2t}(\overline{s}_{2t}, \overline{p}_{2t} \,|\, \widehat{c}_{2t}) = \overline{\boldsymbol{b}}'_{2t-1}(\overline{a}_{2t-1-L:2t-2}, \overline{o}_{2t-L:2t-2}, \overline{o}_{N,2t-1})(\overline{s}_{2t-1})\mathbb{P}_{2t-1}(\overline{o}_{-N,2t-1} \,|\, \overline{s}_{2t-1}, \overline{o}_{N,2t-1})$, where

$\mathbb{P}_{2t-1}(\overline{o}_{-N,2t-1}\,|\,\overline{s}_{2t-1},\overline{o}_{N,2t-1}) = \frac{\overline{\mathbb{O}}_{2t-1}(\overline{o}_{N,2t-1},\overline{o}_{-N,2t-1}\,|\,\overline{s}_{2t-1})}{\sum_{\overline{o}'_{-N,2t-1}}\overline{\mathbb{O}}_{2t-1}(\overline{o}_{N,2t-1},\overline{o}'_{-N,2t-1}\,|\,\overline{s}_{2t-1})}$. Now, we need to verify that Definition C.9 is satisfied.

- The $\{\widehat{c}_h\}_{h\in[\overline{H}]}$ satisfied the Eq. (C.8) since for any $h\in[H]$, $\widehat{c}_{h+1}\subseteq\widehat{c}_h\cup\overline{z}_h$.
- Note that for any $\overline{c}_{2t-1}$ and the corresponding $\widehat{c}_{2t-1}$ constructed above:

$$\begin{aligned}
&||\mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_{2t-1}(\cdot,\cdot\,|\,\overline{c}_h)-\mathbb{P}^{\mathcal{M},c}_{2t-1}(\cdot,\cdot\,|\,\widehat{c}_h)||_1\\
&=\sum_{\overline{s}_{2t-1},\overline{o}_{2t-1}}|\overline{\boldsymbol{b}}_{2t-1}(\overline{a}_{1:2t-2},\overline{o}_{1:2t-2})(\overline{s}_{2t-1})\overline{\mathbb{O}}_{2t-1}(\overline{o}_{2t-1}\,|\,\overline{s}_{2t-1})\\
&\qquad-\overline{\boldsymbol{b}}'_{2t-1}(\overline{a}_{2t-1-L:2t-2},\overline{o}_{2t-L:2t-1})(\overline{s}_{2t-1})\overline{\mathbb{O}}_{2t-1}(\overline{o}_{2t-1}\,|\,\overline{s}_{2t-1})|\\
&=||\overline{\boldsymbol{b}}_{2t-1}(\overline{a}_{1:2t-2},\overline{o}_{1:2t-2})-\overline{\boldsymbol{b}}'_{2t-1}(\overline{a}_{2t-1-L:2t-2},\overline{o}_{2t-L:2t-1})||_1.
\end{aligned}$$

For any $\overline{c}_{2t}$ and the corresponding $\widehat{c}_{2t}$ constructed above:

$$\begin{aligned}
&||\mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_{2t}(\cdot,\cdot\,|\,\overline{c}_h)-\mathbb{P}^{\mathcal{M},c}_{2t}(\cdot,\cdot\,|\,\widehat{c}_h)||\\
&=\sum_{\overline{s}_{2t-1},\overline{o}_{-N,2t-1}}|\overline{\boldsymbol{b}}_{2t-1}(\overline{a}_{1:2t-1},\overline{o}_{1:2t-2},\overline{o}_{N,2t-1})(\overline{s}_{2t-1})\mathbb{P}_{2t-1}(\overline{o}_{-N,2t-1}\,|\,\overline{s}_{2t-1},\overline{o}_{N,2t-1})\\
&\qquad-\overline{\boldsymbol{b}}'_{2t-1}(\overline{a}_{2t-1-L:2t-2},\overline{o}_{2t-L:2t-2},\overline{o}_{N,2t-1})(\overline{s}_{2t-1})\mathbb{P}_{2t-1}(\overline{o}_{-N,2t-1}\,|\,\overline{s}_{2t-1},\overline{o}_{N,2t-1})|\\
&=||\overline{\boldsymbol{b}}_{2t-1}(\overline{a}_{1:2t-1},\overline{o}_{1:2t-2},\overline{o}_{N,2t-1})-\overline{\boldsymbol{b}}'_{2t-1}(\overline{a}_{2t-1-L:2t-2},\overline{o}_{2t-L:2t-2},\overline{o}_{N,2t-1})||_1.
\end{aligned}$$

If we choose $L\geq C\gamma^{-4}\log(\frac{S}{\epsilon})$, then we have that for any $h\in[\overline{H}]$

$$\mathbb{E}_{\overline{a}_{1:h-1},o_{1:h}\sim\overline{g}_{1:\overline{H}}}||\mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_h(\cdot,\cdot\,|\,\overline{c}_h)-\mathbb{P}^{\mathcal{M},c}_h(\cdot,\cdot\,|\,\widehat{c}_h)||_1\leq\epsilon.$$

Therefore, such a model is an ϵ-expected-approximate common information model.

Type 2: Baseline sharing of $\mathcal{L}$ is **Example 3** in §A. Then, common information common information should be that for any $t\in[H],\overline{c}_{2t-1}=\{\overline{o}_{1:2t-2},\overline{a}_{1:2t-2},\overline{o}_{1:2t-1}\},\overline{c}_{2t}=\{\overline{o}_{1:2t-2},\overline{a}_{1:2t-1},\overline{o}_{N,2t-1}\},N\subseteq[n],1\in N$. Here N is the same as defined in case 1, but it must satisfy that $1\in N$. Then we similarly as case 1, we construct $\widehat{c}_{2t-1}=\{\overline{o}_{2t-L:2t-2},\overline{a}_{2t-L-1:2t-2},\overline{o}_{1:2t-1}\},\widehat{c}_{2t}=\{\overline{a}_{2t-1-L:2t-2},\overline{o}_{2t-L:2t-2},\overline{o}_{N,2t-1}\}$, and approximate common information conditioned belief as $\mathbb{P}^{\mathcal{M},c}_{2t-1}(\overline{s}_{2t-1},\overline{p}_{2t-1}\,|\,\widehat{c}_{2t-1})=\overline{\boldsymbol{b}}_{2t-1}(\overline{a}_{2t-1-L:2t-2},\overline{o}_{2t-L:2t-2},\overline{o}_{1,2t-1})(\overline{s}_{2t-1})\mathbb{P}_{2t-1}(\overline{o}_{-1,2t-1}\,|\,\overline{s}_{2t-1},\overline{o}_{1,2t-1}),\mathbb{P}^{\mathcal{M},c}_{2t}(\overline{s}_{2t},\overline{p}_{2t}\,|\,\widehat{c}_{2t})=\overline{\boldsymbol{b}}'_{2t-1}(\overline{a}_{2t-1-L:2t-2},\overline{o}_{2t-L:2t-2},\overline{o}_{N,2t-1})(\overline{s}_{2t-1})\mathbb{P}_{2t-1}(\overline{o}_{-N,2t-1}\,|\,\overline{s}_{2t-1},\overline{o}_{N,2t-1})$. Now, we need to verify Definition C.9 is satisfied.

- The $\{\widehat{c}_h\}_{h\in[\overline{H}]}$ satisfies the Eq. (C.8) since for any $h\in[H]$, $\widehat{c}_{h+1}\subseteq\widehat{c}_h\cup\overline{z}_h$.
- Note that for any $\overline{c}_{2t-1}$ and the corresponding $\widehat{c}_{2t-1}$ constructed above:

$$\begin{aligned}
&||\mathbb{P}^{\mathcal{D}'_{\mathcal{L}}}_{2t-1}(\cdot,\cdot\,|\,\overline{c}_h)-\mathbb{P}^{\mathcal{M},c}_{2t-1}(\cdot,\cdot\,|\,\widehat{c}_h)||_1\\
&=\sum_{\overline{s}_{2t-1},\overline{o}_{-1,2t-1}}|\overline{\boldsymbol{b}}_{2t-1}(\overline{a}_{1:2t-1},\overline{o}_{1:2t-2},\overline{o}_{1,2t-1})(\overline{s}_{2t-1})\mathbb{P}_{2t-1}(\overline{o}_{-1,2t-1}\,|\,\overline{s}_{2t-1},\overline{o}_{1,2t-1})\\
&\qquad-\overline{\boldsymbol{b}}'_{2t-1}(\overline{a}_{2t-1-L:2t-2},\overline{o}_{2t-L:2t-2},\overline{o}_{1,2t-1})(\overline{s}_{2t-1})\mathbb{P}_{2t-1}(\overline{o}_{-1,2t-1}\,|\,\overline{s}_{2t-1},\overline{o}_{1,2t-1})|\\
&=||\overline{\boldsymbol{b}}_{2t-1}(\overline{a}_{1:2t-1},\overline{o}_{1:2t-2},\overline{o}_{1,2t-1})-\overline{\boldsymbol{b}}'_{2t-1}(\overline{a}_{2t-1-L:2t-2},\overline{o}_{2t-L:2t-2},\overline{o}_{1,2t-1})||_1.
\end{aligned}$$

For any $\overline{c}_{2t}$ and the corresponding $\widehat{c}_{2t}$ constructed above:

$$\begin{aligned}
&||\mathbb{P}_{2t}^{\mathcal{D}'_{\mathcal{L}}}(\cdot,\cdot\,|\,\overline{c}_h) - \mathbb{P}_{2t}^{\mathcal{M},c}(\cdot,\cdot\,|\,\widehat{c}_h)||_1 \\
&\quad = \sum_{\overline{s}_{2t-1},\overline{o}_{-N,2t-1}} |\overline{\boldsymbol{b}}_{2t-1}(\overline{a}_{1:2t-1},\overline{o}_{1:2t-2},\overline{o}_{N,2t-1})(\overline{s}_{2t-1})\mathbb{P}_{2t-1}(\overline{o}_{-N,2t-1}\,|\,\overline{s}_{2t-1},\overline{o}_{N,2t-1}) \\
&\qquad\qquad - \overline{\boldsymbol{b}}'_{2t-1}(\overline{a}_{2t-1-L:2t-2},\overline{o}_{2t-L:2t-2},\overline{o}_{N,2t-1})(\overline{s}_{2t-1})\mathbb{P}_{2t-1}(\overline{o}_{-N,2t-1}\,|\,\overline{s}_{2t-1},\overline{o}_{N,2t-1})| \\
&\quad = ||\overline{\boldsymbol{b}}_{2t-1}(\overline{a}_{1:2t-1},\overline{o}_{1:2t-2},\overline{o}_{N,2t-1}) - \overline{\boldsymbol{b}}'_{2t-1}(\overline{a}_{2t-1-L:2t-2},\overline{o}_{2t-L:2t-2},\overline{o}_{N,2t-1})||_1.
\end{aligned}$$

If we choose $L \geq C\gamma^{-4}\log(\frac{S}{\epsilon})$, then from Lemma C.18 we have, for any $h \in [\overline{H}]$

$$\mathbb{E}_{\overline{a}_{1:h-1},o_{1:h}\sim\overline{g}_{1:\overline{H}}}||\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\cdot,\cdot\,|\,\overline{c}_h) - \mathbb{P}_h^{\mathcal{M},c}(\cdot,\cdot\,|\,\widehat{c}_h)||_1 \leq \epsilon.$$

Therefore, such a model is an ϵ-expected-approximate common information model.

Type 3: Baseline sharing of $\mathcal{L}$ is one of **Examples 2, 7, 8** in §A. Then the common information should be that, for any $h \in [\overline{H}]$, $\overline{c}_h = \{\overline{o}_{1:h-2d}, \overline{a}_{1,1:h-1}, \{\overline{a}_{-1,2t-1}\}_{t=\lfloor\frac{h-2d+1}{2}\rfloor}^{\lfloor\frac{h}{2}\rfloor}, \overline{o}_{1,h-2d+1:h}, \overline{o}_M\}$, where $M \subset \{(i,t)\,|\,1 < i \leq n, h-2d+1 \leq t \leq h\}$ and $\overline{o}_M = \{o_{i,t}\,|\,(i,t) \in M\}$, and corresponding $\overline{p}_h = \{\overline{o}_{i,t}\,|\,1 < i \leq n, h-2d < t \leq h, (i,t) \notin M\}$. Actually, $\overline{o}_M$ are the observations shared by the additional sharing in $\mathcal{L}$. Denote $f_{\tau,h-2d} = \{\overline{a}_{1:h-2d-1}, \overline{o}_{h-2d}, \{\overline{a}_{-1,2t-1}\}_{t=\lfloor\frac{h-2d+1}{2}\rfloor}^{\lfloor\frac{h}{2}\rfloor}\}$, $f_a = \{\overline{a}_{1,h-2d:h-1}\}$, $f_o = \{\overline{o}_{1,h-2d+1:h}, \overline{o}_M\}$. We can compute the common-information-based belief as

$$\begin{aligned}
\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h,\overline{p}_h\,|\,\overline{c}_h) &= \sum_{\overline{s}_{h-2d}} \mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h,\overline{p}_h\,|\,\overline{s}_{h-2d},f_a,f_o)\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_{h-2d}\,|\,f_{\tau,h-2d},f_a,f_o) \\
&= \sum_{\overline{s}_{h-2d}} \mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h,\overline{p}_h\,|\,\overline{s}_{h-2d},f_a,f_o)\frac{\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_{h-2d},f_a,f_o\,|\,f_{\tau,h-2d})}{\sum_{\overline{s}'_{h-2d}}\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}'_{h-2d},f_a,f_o\,|\,f_{\tau,h-2d})}.
\end{aligned}$$

Denote the probability $P_h(f_o\,|\,\overline{s}_{h-2d},f_a) := \Pi_{t=1}^{2d}\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{o}_{1,h-2d+t},\overline{o}_{M_{h-2d+t}}\,|\,\overline{s}_{h-2d},\overline{a}_{1,h-2d:h-2d+t})$, where $M_{h-2d+t} = \{(i,h-2d+t)\,|\,(i,h-2d+t) \in M\}$ denotes the set of observations at timestep $h-2d+t$ and shared through additional sharing. With such notation, we have

$$\begin{aligned}
\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_{h-2d}\,|\,f_{\tau,h-2d},f_a,f_o) =& \frac{\overline{\boldsymbol{b}}_{h-2d}(\overline{a}_{1:h-2d-1},\overline{o}_{1:h-2d})(\overline{s}_{h-2d})P_h(f_o\,|\,\overline{s}_{h-2d},f_a)}{\sum_{\overline{s}'_{h-2d}}\overline{\boldsymbol{b}}_{h-2d}(\overline{a}_{1:h-2d-1},\overline{o}_{1:h-2d})(\overline{s}'_{h-2d})P_h(f_o\,|\,\overline{s}'_{h-2d},f_a)} \\
=& F^{P_h(\cdot\,|\,\cdot,f_a)}(\overline{\boldsymbol{b}}_{h-2d}(\overline{a}_{1:h-2d-1},\overline{o}_{1:h-2d});f_o)(\overline{s}_{h-2d}),
\end{aligned}$$

where $F^{P_h(\cdot\,|\,\cdot,f_a)}(\cdot;f_o) : \Delta(\mathcal{S}) \to \Delta(\mathcal{S})$ is the posterior belief update function. The formal definition is shown in Lemma 9 in (Liu & Zhang, 2023).

Then, we define the approximate common information as $\widehat{c}_h := \{\overline{o}_{1,h-2d-L+1:h}, \overline{a}_{1,h-2d-L:h-1}, \overline{o}_M\}$ and corresponding approximate common information conditioned belief as

$$\mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h,\overline{p}_h\,|\,\widehat{c}_h) = \sum_{\overline{s}_{h-2d}}$$

$$\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h,\overline{p}_h\,|\,\overline{s}_{h-2d},f_a,f_o)F^{P_h(\cdot\,|\,\cdot,f_a)}(\overline{\boldsymbol{b}}'_{h-2d}(\overline{a}_{h-2d-L:h-2d-1},\overline{o}_{h-2d-L+1:h-2d});f_o)(\overline{s}_{h-2d}).$$

Now we verify that Definition C.9 is satisfied.

- Obviously, the $\{\widehat{c}_h\}_{h\in[\overline{H}]}$ satisfies Eq. (C.8).
- For any $\overline{c}_h$ and the corresponding $\widehat{c}_h$ constructed above:

$$\begin{aligned}
||\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\cdot,\cdot\,|\,\overline{c}_h) - \mathbb{P}_h^{\mathcal{M},c}(\cdot,\cdot\,|\,\widehat{c}_h)||_1 \leq ||&F^{P(\cdot\,|\,\cdot,f_a)}(\overline{\boldsymbol{b}}_{h-2d}(\overline{a}_{1:h-2d-1},\overline{o}_{1:h-2d});f_o) - \\
&F^{P(\cdot\,|\,\cdot,f_a)}(\overline{\boldsymbol{b}}'_{h-2d}(\overline{a}_{h-2d-L:h-2d-1},\overline{o}_{h-2d-L+1:h-2d});f_o)||_1.
\end{aligned}$$

If we choose $L \geq C\gamma^{-4}\log(\frac{S}{\epsilon})$, then for any strategy $\overline{g}_{1:\overline{H}}$, by taking expectations over $f_{\tau,h-2d}, f_a, f_o$, from Lemma C.18 and Lemma 9 in (Liu & Zhang, 2023), we have, for any $h \in [\overline{H}]$

$$\mathbb{E}_{\overline{a}_{1:h-1},o_{1:h}\sim\overline{g}_{1:\overline{H}}}||\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\cdot,\cdot\,|\,\overline{c}_h) - \mathbb{P}_h^{\mathcal{M},c}(\cdot,\cdot\,|\,\widehat{c}_h)||_1 \leq \epsilon.$$

Therefore, such a model is an ϵ-expected-approximate common information model.

Type 4: Baseline sharing of $\mathcal{L}$ is **Example 4** in §A. Then, for any $h \in [H]$, the common information should be $\widehat{c}_h = \{\overline{o}_{1:h-2d}, \{\overline{a}_{2t-1}\}_{t=1}^{\lfloor\frac{h}{2}\rfloor}, \overline{o}_M\}$, where $M = \{(i,t)\,|\,i \in [n], h-2d+1 \leq t \leq h\}$. Then, still we denote $f_{\tau,h-2d} = \{\overline{o}_{1:h-2d}, \{\overline{a}_{2t-1}\}_{t=1}^{\lfloor\frac{h}{2}\rfloor}\}, f_o = \{\overline{o}_M\}$. We can compute the common information-based belief as

$$\begin{aligned}\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h,\overline{p}_h\,|\,\overline{c}_h) &= \sum_{\overline{s}_{h-2d}} \mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h,\overline{p}_h\,|\,\overline{s}_{h-2d}, f_o)\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_{h-2d}\,|\,f_{\tau,h-2d}, f_o)\\ &= \sum_{\overline{s}_{h-2d}} \mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h,\overline{p}_h\,|\,\overline{s}_{h-2d}, f_o)\frac{\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_{h-2d}, f_o\,|\,f_{\tau,h-2d})}{\sum_{\overline{s}'_{h-2d}}\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}'_{h-2d}, f_o\,|\,f_{\tau,h-2d})}.\end{aligned}$$

Denote the probability $P_h(f_o\,|\,\overline{s}_{h-2d}) := \Pi_{t=1}^{2d}\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{o}_{1,h-2d+t}, \overline{o}_{M_{h-2d+t}}\,|\,\overline{s}_{h-2d})$, where $M_{h-2d+t} = \{(i,h-2d+t)\,|\,(i,h-2d+t) \in M\}$ denotes the set of observations at timestep $h-2d+t$ and shared through additional sharing. Since the actions do not influence underlying states, here we use the belief notation $\overline{\boldsymbol{b}}_k(\overline{o}_{1:k}), \overline{\boldsymbol{b}}_k(\overline{o}_{k-L:k}), \forall k \in [\overline{H}], L < k$. With such notation, we have

$$\begin{aligned}&\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_{h-2d}\,|\,f_{\tau,h-2d}, f_o)\\ &= \frac{\overline{\boldsymbol{b}}_{h-2d}(\overline{o}_{1:h-2d})(\overline{s}_{h-2d})P_h(f_o\,|\,\overline{s}_{h-2d})}{\sum_{\overline{s}'_{h-2d}}\overline{\boldsymbol{b}}_{h-2d}(\overline{o}_{1:h-2d})(\overline{s}'_{h-2d})P_h(f_o\,|\,\overline{s}'_{h-2d})} = F^{P_h(\cdot\,|\,\cdot)}(\overline{\boldsymbol{b}}_{h-2d}(\overline{o}_{1:h-2d}); f_o)(\overline{s}_{h-2d}),\end{aligned}$$

where $F^{P_h(\cdot\,|\,\cdot)}(\cdot; f_o) : \Delta(\mathcal{S}) \to \Delta(\mathcal{S})$ is the posterior belief update function, the same as mentioned in **Type 3**.
Then, we define the approximate common information as $\widehat{c}_h := \{\overline{o}_{h-2d-L+1:h}, \overline{o}_M\}$ and corresponding approximate common information conditioned belief as

$$\mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h,\overline{p}_h\,|\,\widehat{c}_h) = \sum_{\overline{s}_{h-2d}} \mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\overline{s}_h,\overline{p}_h\,|\,\overline{s}_{h-2d}, f_o)F^{P_h(\cdot\,|\,\cdot)}(\overline{\boldsymbol{b}}'_{h-2d}(\overline{o}_{h-2d-L+1:h-2d}); f_o)(\overline{s}_{h-2d}).$$

Now we verify that Definition C.9 is satisfied.

- Obviously, the $\{\widehat{c}_h\}_{h\in[\overline{H}}$ satisfies Eq.(C.8).
- For any $\overline{c}_h$ and corresponding $\widehat{c}_h$ constructed above:

$$\begin{aligned}&||\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\cdot,\cdot\,|\,\overline{c}_h) - \mathbb{P}_h^{\mathcal{M},c}(\cdot,\cdot\,|\,\widehat{c}_h)||_1\\ &\quad\leq ||F^{P(\cdot\,|\,\cdot)}(\overline{\boldsymbol{b}}_{h-2d}(\overline{o}_{1:h-2d}); f_o) - F^{P(\cdot\,|\,\cdot)}(\overline{\boldsymbol{b}}'_{h-2d}(\overline{a}_{h-2d-L:h-2d-1}, \overline{o}_{h-2d-L+1:h-2d}); f_o)||_1.\end{aligned}$$

  If we choose $L \geq C\gamma^{-4}\log(\frac{S}{\epsilon})$, then for any strategy $\overline{g}_{1:\overline{H}}$, by taking expectations over $f_{\tau,h-2d}, f_o$, from Lemma C.18 and Lemma 9 in (Liu & Zhang, 2023), we have, for any $h \in [\overline{H}]$

$$\mathbb{E}_{\overline{a}_{1:h-1},o_{1:h}\sim\overline{g}_{1:\overline{H}}}||\mathbb{P}_h^{\mathcal{D}'_{\mathcal{L}}}(\cdot,\cdot\,|\,\overline{c}_h) - \mathbb{P}_h^{\mathcal{M},c}(\cdot,\cdot\,|\,\widehat{c}_h)||_1 \leq \epsilon.$$

Therefore, such a model is an ϵ-expected-approximate common information model.

Combining **Parts I, II, III**, we complete the proof. □

Remark C.19. Let $\mathcal{L}$ be an LTC problem satisfying Assumptions 3.1, 3.2, 3.3, and 3.4, and $\mathcal{D}'_{\mathcal{L}}$ be the Dec-POMDP after reformulation, strict expansion and refinement. Then, if $\mathcal{L}$ has any one of baseline sharing protocols as in Appendix A, and $\mathcal{L}$ satisfies the conditions as follows, then $\mathcal{D}'_{\mathcal{L}}$ satisfies Assumption 4.3.

- If $\mathcal{L}$ has baseline sharing protocol as one of **Examples 1, 5, 6** in A, $\mathcal{L}$ needs to satisfy the part (1) of **Factorized structure** in G.
- If $\mathcal{L}$ has baseline sharing protocol as one of **Examples 2, 7, 8** in A, $\mathcal{L}$ needs to satisfy $\mathcal{R}_h(\cdot\,|\,s_h, a_{1,h}, a_{-1,h}) = \mathcal{R}_h(\cdot\,|\,s_h, a_{1,h}, a'_{-1,h})$ for any $h \in [H], s_h \in \mathcal{S}, a_{1,h} \in \mathcal{A}_{1,h}, a_{-1,h}, a'_{-1,h} \in \mathcal{A}_{-1,h}$.
- If $\mathcal{L}$ has baseline sharing protocol as one of **Examples 3, 4** in A, it does not need additional condition.

Actually, such condition is also considered in (Liu & Zhang, 2023). For $\mathcal{L}$ with baseline sharing protocols as one of examples in A and satisfying the conditions as above, we can construct expected common information model $\mathcal{M}$ of $\mathcal{D}'_{\mathcal{L}}$ as mentioned in the proof of Theorem C.14. If the baseline sharing protocol of $\mathcal{L}$ is one of **Examples 1, 5, 6**, then $\mathcal{D}'_{\mathcal{L}}$ and $\mathcal{M}$ satisfy **Factorized structures** condition in G; If the baseline sharing protocol of $\mathcal{L}$ is one of **Examples 2, 7, 8**, then $\mathcal{D}'_{\mathcal{L}}$ and $\mathcal{M}$ satisfy **Turn-based structures** condition in G; If the baseline sharing protocol of $\mathcal{L}$ is one of **Examples 3, 4**, then $\mathcal{D}'_{\mathcal{L}}$ and $\mathcal{M}$ satisfy **Nested private information** condition in G. From Lemma G.1, we can conclude that Assumption 4.3 holds.

C.9 Main Results for Learning in QC LTC

Here we provide a full version of Theorem 4.4 as follows.

Theorem C.20. Given any QC LTC problem $\mathcal{L}$ satisfying Assumptions 3.1, 3.2, 3.3, 3.4, and 4.3, we can construct an SI-CIB Dec-POMDP problem $\mathcal{D}'_{\mathcal{L}}$ such that the following holds. Given a strategy $\overline{g}^{1:\overline{H}}$, $\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})$, and $\widehat{L}$, where each $\overline{g}^h$ is a complete strategy with $\overline{g}^h_{h-\widehat{L}:h} = \text{Unif}(\mathcal{A})$ for $h \in [\overline{H}]$, we define the statistical error for estimating $\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})$ as $\epsilon_{apx}(\overline{g}^{1:\overline{H}}, \widehat{L}, \zeta_1, \zeta_2, \theta_1, \theta_2, \phi)$ for some parameters $\delta_1, \zeta_1, \zeta_2, \theta_1, \theta_2, \phi > 0$. Then, there exists an algorithm that can learn an ϵ-team-optimal strategy for $\mathcal{L}$ with probability at least $1-\delta_1$, using a sample complexity $N_0 = \texttt{poly}(\max_{h\in[\overline{H}]} |\mathcal{P}_h|, \max_{h\in[\overline{H}]} |\widehat{\mathcal{C}}_h|, H, \max_{h\in[\overline{H}]} |\mathcal{A}_h|, \max_{h\in[\overline{H}]} |\mathcal{O}_h|, 1/\zeta_1, 1/\zeta_2, 1/\theta_1, 1/\theta_2) \cdot \log(1/\delta_1)$, where $\epsilon := \overline{H}\epsilon_r(\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})) + \overline{H}^2\epsilon_z(\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})) + (\overline{H}^2 + \overline{H})\epsilon_{apx}(\overline{g}^{1:\overline{H}}, \widehat{L}, \zeta_1, \zeta_2, \theta_1, \theta_2, \phi)$. Specifically, if $\mathcal{L}$ has the baseline sharing protocols as in §A, there exists an algorithm that learns an ϵ-team optimal strategy for $\mathcal{L}$ with both quasi-polynomial time and sample complexities.

Proof. Firstly, given any LTC problem $\mathcal{L}$, we can apply Algorithm 2 to solve such problem. From the proof of C.14, we know that Algorithm 6 can output the team optimal strategy of $\widehat{\mathcal{M}}(\overline{g}^{1:\overline{H},j})$ for each $j \in [K]$. Then, from Theorem 4 in (Liu & Zhang, 2023), it can guarantee that $\overline{g}^*_{1:\overline{H}}$ is an ϵ-team optimum of $\mathcal{D}'_{\mathcal{L}}$ with probability at least $1-\delta_1$, where $\epsilon = \overline{H}\epsilon_r(\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})) + \overline{H}^2\epsilon_z(\widetilde{\mathcal{M}}(\overline{g}^{1:\overline{H}})) + (\overline{H}^2 + \overline{H})\epsilon_{apx}(\overline{g}^{1:\overline{H}}, \widehat{L}, \zeta_1, \zeta_2, \theta_1, \theta_2, \phi) + \overline{H}\epsilon_e$. Then, from the proof of Theorem C.14, we have that $(g^{m,*}_{1:H}, g^{a,*}_{1:H})$ is an ϵ-team optimal strategy of $\mathcal{L}$ is $\overline{g}^*_{1:\overline{H}}$ is an ϵ-team optimal strategy of $\mathcal{D}'_{\mathcal{L}}$. Therefore, we complete the proof. $\square$

D Deferred Details of §5

In the following, we will use $^-$ to denote the elements and random variables in the Dec-POMDP $\mathcal{D}$. We first introduce the notion of *perfect recall* (Kuhn, 1953):

Definition D.1 (Perfect recall)**.** We say that agent i has perfect recall if $\forall h \in 2, \cdots, \overline{H}$, it holds that $\tau_{i,h-1} \cup \{a_{i,h-1}\} \subseteq \tau_{i,h}$. If for any $i \in [n]$, agent i has perfect recall, we call that the Dec-POMDP has a perfect recall property.

D.0.1 Proof of Theorem 5.1

Proof. sQC $\Rightarrow$ SI-CIB:
Let $\mathcal{D}$ be the Dec-POMDP with an sQC information structure, and $\mathcal{D}$ satisfy Assumptions 3.3, 3.4, and 3.5. To prove that $\mathcal{D}$ has SI-CIB, it is sufficient to prove that for any $h = 2, \cdots, \overline{H}$, fix any $h_1 \in [h-1], i_1 \in [n]$, and for any $\overline{g}_{1:h-1} \in \overline{\mathcal{G}}_{1:h-1}, \overline{g}'_{i_1,h_1} \in \overline{\mathcal{G}}_{i_1,h_1}$, let $\overline{g}'_{h_1} := (\overline{g}_{1,h_1}, \cdots, \overline{g}'_{i_1,h_1}, \cdots, \overline{g}_{n,h_1})$ and $\overline{g}'_{1:h-1} := (\overline{g}_1, \cdots, \overline{g}'_{h_1}, \cdots, \overline{g}_{h-1})$, the following holds

$$\mathbb{P}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}_{1:h-1}) = \mathbb{P}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}'_{1:h-1}). \tag{D.1}$$

We prove this case-by-case as follows:

1. If there exists some $i_3 \neq i_1$ such that $\sigma(\overline{\tau}_{i_1,h_1}) \cup \sigma(\overline{a}_{i_1,h_1}) \subseteq \sigma(\overline{\tau}_{i_3,h})$, then from Assumption 3.5, we know that $\sigma(\overline{\tau}_{i_1,h_1}) \cup \sigma(\overline{a}_{i_1,h_1}) \subseteq \sigma(\overline{c}_h)$. Therefore, there exist deterministic functions β_1, β_2 such that $\overline{\tau}_{i_1,h_1} = \beta_1(\overline{c}_h), \overline{a}_{i_1,h_1} = \beta_2(\overline{c}_h)$, and further it holds that

$$\begin{aligned}&\mathbb{P}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}_{1:h-1}) = \mathbb{P}(\overline{s}_h, \overline{p}_h \,|\, \beta_1(\overline{c}_h), \beta_2(\overline{c}_h), \overline{c}_h, \overline{g}_{1:h-1})\\&\quad = \mathbb{P}(\overline{s}_h, \overline{p}_h \,|\, \overline{\tau}_{i_1,h_1}, \overline{a}_{i_1,h_1}, \overline{c}_h, \overline{g}_{1:h-1}) = \mathbb{P}(\overline{s}_h, \overline{p}_h \,|\, \overline{\tau}_{i_1,h_1}, \overline{a}_{i_1,h_1}, \overline{c}_h, \overline{g}'_{1:h-1}).\end{aligned}$$

   The last equality is due to the fact that the input and output of $\overline{g}_{i_1,h_1}$ are $\overline{\tau}_{i_1,h_1}$ and $\overline{a}_{i_1,h_1}$, respectively.

2. If there does not exist any $i_2 \neq i_1$ such that $\sigma(\overline{\tau}_{i_1,h_1}) \cup \sigma(\overline{a}_{i_1,h_1}) \subseteq \sigma(\overline{\tau}_{i_2,h})$, i.e., for all $i_2 \neq i_1$, either $\sigma(\overline{\tau}_{i_1,h_1}) \nsubseteq \sigma(\overline{\tau}_{i_2,h})$ or $\sigma(\overline{a}_{i_1,h_1}) \nsubseteq \sigma(\overline{\tau}_{i_2,h})$, then agent (i_1, h_1) does not influence agent (i_2, h) for any $i_2 \neq i_1$, since $\mathcal{D}$ is sQC. Now, we first claim that agent (i_1, h_1) does not influence $\overline{s}_{h_1+1}$: since if it influences, from Assumption 3.4, there exists some $i_3 \neq i_1$ such that agent (i_1, h_1) influences $\overline{o}_{i_3,h_1+1}$; however, from Assumption 2.1 (e), we know $\overline{o}_{i_3,h_1+1} \in \overline{\tau}_{i_3,h_1+1} \subseteq \overline{\tau}_{i_3,h}$; therefore, agent (i_1, h_1) influences agent (i_3, h), contradicting the argument above that the former does not influence (i_2, h) for any $i_2 \neq i_1$. Hence, we further have that agent (i_1, h_1) does not influence $\overline{s}_{h_2}$ for any $h_2 > h_1$. Therefore, by Assumption 3.3, for any $h_2 > h_1$, $\overline{a}_{i_1,h_1} \notin \overline{\tau}_{h_2}$.

   Second, we claim that agent (i_1, h_1) does not influence $\overline{\tau}_{i_4,h_2}$, for any $i_4 \in [n]$ and $h_2 > h_1$. This is because of the fact that agent (i_1, h_1) does not influence $\overline{s}_{h_1+1}$ and thus not $\overline{o}_{i_4,h_1+1}$ for any $i_4 \in [n]$, together with the fact proved above that $\overline{a}_{i_1,h_1} \notin \overline{\tau}_{h_1+1}$, implies that agent (i_1, h_1) does not influence any element in $\overline{\tau}_{i_4,h_1+1}$ for any $i_4 \in [n]$, either directly or indirectly. Since $\overline{\tau}_{i_4,h_1+1}$ is the input of agent i_4's strategy at timestep h_1+1 to decide $\overline{a}_{i_4,h_1+1}$, agent (i_1, h_1) thus does not influence $\overline{a}_{i_4,h_1+1}$ for any $i_4 \in [n]$, either, which, together with the fact that it does not influence $\overline{s}_{h_1+2}$ and thus not $\overline{o}_{i_4,h_1+2}$ for any $i_4 \in [n]$, further implies that it does not influence any element in $\overline{\tau}_{i_4,h_1+2}$ for any $i_4 \in [n]$. By recursion, agent (i_1, h_1) does not influence $\overline{\tau}_{i_4,h_2}$ for any $i_4 \in [n]$ and $h_2 > h_1$.

   Therefore, agent (i_1, h_1) does not influence $\overline{c}_h = \cap_{i_4=1}^n \overline{\tau}_{i_4,h}$ nor $\overline{p}_h = \overline{\tau}_h \backslash \overline{c}_h$, which thus leads to

$$\mathbb{P}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}_{1:h-1}) = \mathbb{P}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}'_{1:h-1}).$$

SI-CIB $\Rightarrow$ sQC:
Since $\mathcal{D}$ has perfect recall and has SI-CIB, i.e., $\forall i \in [n], h \in [\overline{H}], \forall \overline{g}_{1:h-1}, \overline{g}'_{1:h-1} \in \overline{\mathcal{G}}_{1:h-1}, \overline{c}_h \in \overline{\mathcal{C}}_h, \overline{s}_h \in \overline{\mathcal{S}}, \overline{p}_h \in \overline{\mathcal{P}}_h$, the following holds

$$\mathbb{P}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}_{1:h-1}) = \mathbb{P}(\overline{s}_h, \overline{p}_h \,|\, \overline{c}_h, \overline{g}'_{1:h-1}).$$

Our goal is to prove that $\mathcal{D}$ is sQC (up to null sets). In particular, we meant to prove that if agent (i_1, h_1) influences agent (i_2, h_2) in the intrinsic model of the Dec-POMDP (cf. §F), then under any strategy $\overline{g}_{1:\overline{H}} \in \overline{\mathcal{G}}_{1:\overline{H}}$, $\sigma(\overline{\tau}_{i_1,h_1}) \subseteq \sigma(\overline{\tau}_{i_2,h_2})$ except the null sets generated by $\overline{g}_{1:\overline{H}}$.

We prove this by contradiction. If this is not true, then there exists some strategy $\overline{g}_{1:\overline{H}}$ and $i_1, i_2 \in [n], h_1, h_2 \in [\overline{H}]$, such that agent (i_1, h_1) influences agent (i_2, h_2), but either

$\sigma(\overline{\tau}_{i_1,h_1}) \nsubseteq \sigma(\overline{\tau}_{i_2,h_2})$ or $\sigma(\overline{a}_{i_1,h_1}) \nsubseteq \sigma(\overline{\tau}_{i_2,h_2})$ (up to the null sets generated by $\overline{g}_{1:\overline{H}}$). First, we can assume $i_2 \neq i_1$, since otherwise it always holds that $\overline{\tau}_{i_1,h_1} \subseteq \overline{\tau}_{i_1,h_2}$ and $\overline{a}_{i_1,h_1} \in \overline{\tau}_{i_1,h_2}$, due to the assumption that the agents in $\mathcal{D}$ have perfect recall.

Then, we discuss the following different cases. Note that in the following discussion, when it comes to σ-algebra inclusion, we meant it up to the null sets generated by $\overline{g}_{1:\overline{H}}$.

1. If $\sigma(\overline{a}_{i_1,h_1}) \nsubseteq \sigma(\overline{\tau}_{i_2,h_2})$, then it implies that $\sigma(\overline{a}_{i_1,h_1}) \nsubseteq \sigma(\overline{c}_{h_2})$ because $\overline{c}_{h_2} \subseteq \overline{\tau}_{i_2,h_2}$. This also implies that $\overline{a}_{i_1,h_1} \notin \overline{c}_{h_2}$, and thus $\overline{a}_{i_1,h_1} \in \overline{p}_{i_1,h_2}$ due to perfect recall. Note that there must exist some realizations $\overline{c}_{h_2} \in \overline{\mathcal{C}}_{h_2}, \overline{p}_{h_2} \in \overline{\mathcal{P}}_{h_2}, \overline{s}_{h_2} \in \overline{\mathcal{S}}$ such that $\overline{c}_{h_2}$ has non-zero probability under $\overline{g}_{1:h_2-1}$, and $\mathbb{P}(\overline{s}_{h_2}, \overline{p}_{h_2} \,|\, \overline{c}_{h_2}, \overline{g}_{1:h_2-1}) \neq 0$. Meanwhile, there must exist another different action realization $\overline{a}'_{i_1,h_1}$ such that
$$\mathbb{P}(\overline{s}_{h_2}, \overline{p}_{h_2} \backslash \{\overline{a}_{i_1,h_1}\} \cup \{\overline{a}'_{i_1,h_1}\} \,|\, \overline{c}_{h_2}, \overline{g}_{1:h_2-1}) \neq 0, \tag{D.2}$$
since otherwise it holds that $\sigma(\overline{a}_{i_1,h_1}) \subseteq \sigma(\overline{c}_{h_2})$. Actually, this means that there are some non-zero probability trajectories containing $\overline{a}_{i_1,h_1}$ and $\overline{c}_{h_2}$, and some non-zero probability trajectories containing $\overline{a}'_{i_1,h_1}$ and $\overline{c}_{h_2}$. Then, we define another strategy $\overline{g}'_{i_1,h_1}$ as:
$$\forall \overline{\tau}_{i_1,h_1} \in \overline{\mathcal{T}}_{i_1,h_1}, \quad \overline{g}'_{i_1,h_1}(\overline{\tau}_{i_1,h_1}) = \overline{a}'_{i_1,h_1}, \tag{D.3}$$
and we let $\overline{g}'_{h_1} := (\overline{g}_{1,h_1}, \cdots, \overline{g}'_{i_1,h_1}, \cdots, \overline{g}_{n,h_1})$ and $\overline{g}'_{1:h_2-1} := (\overline{g}_1, \cdots, \overline{g}'_{h_1}, \cdots, \overline{g}_{h_2-1})$. Now we claim that $\overline{c}_{h_2}$ has non-zero probability under $\overline{g}'_{1:h_2-1}$. From that $\overline{c}_{h_2}$ has non-zero probability under $\overline{g}_{1:h_2-1}$, and $\mathbb{P}(\overline{s}_{h_2}, \overline{p}_{h_2} \backslash \{\overline{a}_{i_1,h_1}\} \cup \{\overline{a}'_{i_1,h_1}\} \,|\, \overline{c}_{h_2}, \overline{g}_{1:h_2-1}) \neq 0$, we can get $\mathbb{P}(\overline{a}'_{i_1,h_1}, \overline{c}_{h_2} \,|\, \overline{g}_{1:h_2-1}) > 0$. Since $\overline{g}'_{1:h_2-1}$ only differs from $\overline{g}_{1:h_2-1}$ in the strategy of agent (i_1, h_1), and $\overline{g}'_{i_1,h_1}$ always chooses $\overline{a}'_{i_1,h_1}$, then we get $\mathbb{P}(\overline{a}'_{i_1,h_1}, \overline{c}_{h_2} \,|\, \overline{g}'_{1:h_2-1}) \geq \mathbb{P}(\overline{a}'_{i_1,h_1}, \overline{c}_{h_2} \,|\, \overline{g}_{1:h_2-1}) > 0$ because $\overline{g}_{1:h_2-1}$ and $\overline{g}'_{1:h_2-1}$ are the same in those trajectories containing $\overline{a}'_{i_1,h_1}$ and $\overline{c}_{h_2}$, and thus $\mathbb{P}(\overline{c}_{h_2} \,|\, \overline{g}'_{1:h_2-1}) > 0$. Hence, we prove our claim.

   Meanwhile, due to (D.3), notice that
$$\mathbb{P}(\overline{s}_{h_2}, \overline{p}_{h_2} \,|\, \overline{c}_{h_2}, \overline{g}'_{1:h_2-1}) = 0 \neq \mathbb{P}(\overline{s}_{h_2}, \overline{p}_{h_2} \,|\, \overline{c}_{h_2}, \overline{g}_{1:h_2-1}), \tag{D.4}$$
which leads to a contradiction to the fact that $\mathcal{D}$ has SI-CIB.

2. If $\sigma(\overline{a}_{i_1,h_1}) \subseteq \sigma(\overline{\tau}_{i_2,h_2})$, then it implies that $\sigma(\overline{\tau}_{i_1,h_1}) \nsubseteq \sigma(\overline{\tau}_{i_2,h_2})$, and further implies that $\sigma(\overline{\tau}_{i_1,h_1}) \nsubseteq \sigma(\overline{c}_{h_2})$ since $\overline{c}_{h_2} \subseteq \overline{\tau}_{i_2,h_2}$. Note that there must exist some realizations $\overline{c}_{h_2} \in \overline{\mathcal{C}}_{h_2}, \overline{\tau}_{i_2,h_2} \in \overline{\mathcal{T}}_{i_2,h_2}$ such that $\overline{\tau}_{i_2,h_2}$ has non-zero probability under $\overline{g}_{1:h_2-1}$ and $\overline{c}_{h_2} \subseteq \overline{\tau}_{i_2,h_2}$, and there exist two realizations $\overline{\tau}_{i_1,h_1}, \overline{\tau}'_{i_1,h_1} \in \overline{\mathcal{T}}_{i_1,h_1}$ such that $\mathbb{P}(\overline{\tau}_{i_1,h_1} \,|\, \overline{\tau}_{i_2,h_2}) > 0, \mathbb{P}(\overline{\tau}'_{i_1,h_1} \,|\, \overline{\tau}_{i_2,h_2}) > 0$, since otherwise, it holds that $\sigma(\overline{\tau}_{i_1,h_1}) \subseteq \sigma(\overline{c}_{h_2})$. Furthermore, we know that there exist $\overline{s}_{h_2} \in \overline{\mathcal{S}}, \overline{p}_{h_2} \in \overline{\mathcal{P}}_{h_2}$ such that $\mathbb{P}(\overline{s}_{h_2}, \overline{p}_{h_2} \,|\, \overline{c}_{h_2}, \overline{g}_{1:h_2-1}) > 0$ and $\overline{\tau}'_{i_2,h_2} \subseteq \overline{c}_{h_2} \cup \overline{p}_{h_2}$. Since $\sigma(\overline{a}_{i_1,h_1}) \subseteq \sigma(\overline{\tau}_{i_2,h_2})$, we know that there exists $\overline{a}_{i_1,h_1}$ that $\mathbb{P}(\overline{a}_{i_1,h_1} \,|\, \overline{\tau}_{i_2,h_2}) = 1$. Let $\tau := \overline{\tau}_{i_1,h_1} \backslash \overline{c}_{h_2}$ and $\tau' := \overline{\tau}'_{i_1,h_1} \backslash \overline{c}_{h_2}$. and consider another action $\overline{a}'_{i_1,h_1} \neq \overline{a}_{i_1,h_1}$ and strategy $\overline{g}'_{i_1,h_1}$ defined such that
$$\overline{g}'_{i_1,h_1}(\overline{\tau}_{i_1,h_1}) = \overline{a}'_{i_1,h_1}, \quad \overline{g}'_{i_1,h_1}(\overline{\tau}'_{i_1,h_1}) = \overline{a}_{i_1,h_1}, \tag{D.5}$$
and keeps $g'_{i_1,h_1}(\overline{\tau}''_{i_1,h_1})$ the same as $g_{i_1,h_1}(\overline{\tau}''_{i_1,h_1})$ for any other $\overline{\tau}''_{i_1,h_1}$. We denote $\overline{g}'_{h_1} := (\overline{g}_{1,h_1}, \cdots, \overline{g}'_{i_1,h_1}, \cdots, \overline{g}_{n,h_1})$ and $\overline{g}'_{1:h_2-1} := (\overline{g}_1, \cdots, \overline{g}'_{h_1}, \cdots, \overline{g}_{h_2-1})$. Since $(\overline{\tau}'_{i_1,h_1}, \overline{\tau}_{i_2,h_2})$ has non-zero probability under $\overline{g}_{1:h_2-1}$ and $\mathbb{P}(\overline{a}_{i_1,h_1} \,|\, \overline{\tau}_{i_2,h_2})$, then we know $(\overline{\tau}'_{i_1,h_1}, \overline{\tau}_{i_2,h_2})$ has non-zero probability under $\overline{g};_{1:h_2-1}$. Hence, we know that $\overline{c}_{h_2}$ has non-zero probability under $\overline{g};_{1:h_2-1}$. Meanwhile, it holds that
$$\begin{aligned}\mathbb{P}(\overline{s}_{h_2}, \overline{p}_{h_2} \,|\, \overline{c}_{h_2}, \overline{g}'_{1:h_2-1}) &= \frac{\mathbb{P}(\overline{s}_{h_2}, \overline{p}_{h_2}, \overline{c}_{h_2} \,|\, \overline{g}'_{1:h_2-1})}{\mathbb{P}(\overline{c}_{h_2} \,|\, \overline{g}'_{1:h_2-1})} \\ &= \frac{\mathbb{P}(\overline{s}_{h_2}, \overline{\tau}_{h_2} \,|\, \overline{g}'_{1:h_2-1})}{\mathbb{P}(\overline{c}_{h_2} \,|\, \overline{g}'_{1:h_2-1})} = 0 \neq \mathbb{P}(\overline{s}_{h_2}, \overline{p}_{h_2} \,|\, \overline{c}_{h_2}, \overline{g}_{1:h_2-1}),\end{aligned} \tag{D.6}$$

where the third equal sign is because $\overline{a}_{i_1,h_1} \in \overline{\tau}_{h_2}, \overline{\tau}_{i_1,h_1} \subseteq \overline{\tau}_{h_2}$ from perfect recall, and $\overline{a}_{i_1,h_1}, \overline{\tau}_{i_1,h_1}$ can never happen together under $\overline{g}'_{1:h_2-1}$ due to (D.5). Therefore, (D.6) leads to a contradiction to the fact that $\mathcal{D}$ has SI-CIB and thus completes the proof.

□

E Collection of Algorithm Pseudocodes

Here we collect both our planning and learning algorithms as pseudocodes in Algorithms 1, 2, 3, 4, 5, and 6.

Algorithm 1 Planning in QC LTC Problems

Require: LTC $\mathcal{L}$, accuracy levels $\epsilon_r, \epsilon_z > 0$
- Reformulate $\mathcal{L}$ to $\mathcal{D}_{\mathcal{L}}$ based on Eq. (C.1).
- Expand $\mathcal{D}_{\mathcal{L}}$ to $\mathcal{D}^{\dagger}_{\mathcal{L}}$ based on Eq. (4.1).
- Refine $\mathcal{D}^{\dagger}_{\mathcal{L}}$ to $\mathcal{D}'_{\mathcal{L}}$ based on $\mathcal{L}$.
- Construct expected Approximate Common-information Model $\mathcal{M}$ from $\mathcal{D}'_{\mathcal{L}}$ with error ϵ_r, ϵ_z.
- $\overline{g}^*_{1:\widetilde{H}} \leftarrow$ Algorithm 6($\mathcal{M}$)
- $\widetilde{g}^*_{1:\widetilde{H}} \leftarrow \varphi(\overline{g}^*_{1:\widetilde{H}}, \mathcal{D}_{\mathcal{L}})$
- $g^{m,*}_{1:H} \leftarrow \{\widetilde{g}^*_1, \widetilde{g}^*_3, \cdots, \widetilde{g}^*_{2H-1}\}$
- $g^{a,*}_{1:H} \leftarrow \{\widetilde{g}_2, \widetilde{g}_4, \cdots, \widetilde{g}_{2H}\}$
- **Return** $(g^{m,*}_{1:H}, g^{a,*}_{1:H})$

Algorithm 2 Learning in QC LTC Problems

Require: Underlying environment LTC $\mathcal{L}$, iteration number K.
- Reformulate $\mathcal{L}$ to $\mathcal{D}_{\mathcal{L}}$ based on Eq. (C.1).
- Refine $\mathcal{D}_{\mathcal{L}}$ to $\mathcal{D}'_{\mathcal{L}}$ based on Eq. (4.1).
- Obtain $\{\overline{g}^{1:\overline{H},j}\}_{j=1}^{K}$ by calling Algorithm 3 of (Golowich et al., 2022).
- **for** $j = 1$ to K **do**
  - Construct $\widehat{\mathcal{M}}(\overline{g}^{1:\overline{H},j})$ by calling Algorithm 5 of (Liu & Zhang, 2023) with the underlying environment $\mathcal{D}'_{\mathcal{L}}$ and $\overline{g}^{1:\overline{H},j}$.
  - $\overline{g}^{j,*}_{1:\overline{H}} \leftarrow$ Algorithm 6($\widehat{\mathcal{M}}(\overline{g}^{1:\overline{H},j})$)
- **end for**
- $\overline{g}^*_{1:\overline{H}} \leftarrow$ Algorithm 8($\{\overline{g}^{j,*}_{1:\overline{H}}\}_{j=1}^{K}$) of (Liu & Zhang, 2023).
- $\widetilde{g}^*_{1:\widetilde{H}} \leftarrow \varphi(\overline{g}^*_{1:\overline{H}}, \mathcal{D}_{\mathcal{L}})$
- $g^{m,*}_{1:H} \leftarrow \{\widetilde{g}^*_1, \widetilde{g}^*_3, \cdots, \widetilde{g}^*_{2H-1}\}$
- $g^{a,*}_{1:H} \leftarrow \{\widetilde{g}_2, \widetilde{g}_4, \cdots, \widetilde{g}_{2H}\}$
- **Return** $(g^{m,*}_{1:H}, g^{a,*}_{1:H})$

F Decentralized POMDPs (with Information Sharing)

A Dec-POMDP with n agents and potential information sharing can be characterized by a tuple

$$\mathcal{D} = \langle H, \mathcal{S}, \{\mathcal{A}_{i,h}\}_{i\in[n],h\in[H]}, \{\mathcal{O}_{i,h}\}_{i\in[n],h\in[H]}, \{\mathbb{T}_h\}_{h\in[H]}, \{\mathbb{O}_h\}_{h\in[H]}, \mu_1, \{\mathcal{R}_h\}_{h\in[H]}\rangle,$$

where H denotes the length of each episode, $\mathcal{S}$ denotes state space, and $\mathcal{A}_{i,h}$ denotes the *control action* spaces of agent i at timestep h. We denote by $s_h \in \mathcal{S}$ the state and by $a_{i,h}$ the control action of agent i at timestep h. We use $a_h := (a_{1,h}, \cdots, a_{n,h}) \in \mathcal{A}_h := \mathcal{A}_{1,h} \times \mathcal{A}_{2,h} \times \cdots \mathcal{A}_{n,h}$ to denote the joint control action for all the n agents at timestep h, with $\mathcal{A}_h$ denoting the joint control action space at timestep h. We denote $\mathbb{T} = \{\mathbb{T}_h\}_{h\in[H]}$ the collection of transition functions, where

Algorithm 3 Vanilla Realization of $\varphi(\breve{g}_{1:\breve{H}}, \mathcal{D}_{\mathcal{L}})$

Require: Strategy $\breve{g}_{1:\breve{H}}$, QC Dec-POMDP $\mathcal{D}_{\mathcal{L}}$
  $\widetilde{g}_{1:\breve{H}} \leftarrow \emptyset$
  for $h_2 = 1$ to $\breve{H}$, $i_2 = 1$ to n, $\widetilde{\tau}_{i_2,h_2} \in \widetilde{\mathcal{T}}_{i_2,h_2}$ **do**
    $\breve{\tau}_{i_2,h_2} \leftarrow \widetilde{\tau}_{i_2,h_2}$
    for $h_1 = 1$ to $h_2 - 1$, $i_1 = 1$ to n **do**
      if $\sigma(\widetilde{\tau}_{i_1,h_1}) \subseteq \sigma(\widetilde{\tau}_{i_2,h_2})$ in $\mathcal{D}_{\mathcal{L}}$ **then**
        $\widetilde{a}_{i_1,h_1} \leftarrow \widetilde{g}_{i_1,h_1}(\widetilde{\tau}_{i_1,h_1})$
        $\breve{\tau}_{i_2,h_2} \leftarrow \breve{\tau}_{i_2,h_2} \cup \{\widetilde{a}_{i_1,h_1}\}$
      end if
    end for
    $\widetilde{g}_{i_2,h_2}(\widetilde{\tau}_{i_2,h_2}) \leftarrow \breve{g}_{i_2,h_2}(\breve{\tau}_{i_2,h_2})$
  end for
  Return $\widetilde{g}_{1:\widetilde{H}}$

Algorithm 4 Efficient Implementation of $\varphi(\breve{g}_{1:\breve{H}}, \mathcal{D}_{\mathcal{L}})$

Require: Strategy $\breve{g}_{1:\breve{H}}$, QC Dec-POMDP $\mathcal{D}_{\mathcal{L}}$
  for $h = 1$ to $\breve{H}$ **do**
    for $i = 1$ to n **do**
      Agent i receives $\widetilde{\tau}_{i,h}$
      $\breve{\tau}_{i,h} \leftarrow$ Recover$(\widetilde{\tau}_{i,h}, \breve{g}_{1:h-1}, \mathcal{D}_{\mathcal{L}})$ \\ Defined in Algorithm 5
      Agent i chooses $\breve{g}_{i,h}(\breve{\tau}_{i,h})$ as $\widetilde{a}_{i,h}$
    end for
  end for

Algorithm 5 Recover $\breve{\tau}_{i,h}$ from $\widetilde{\tau}_{i,h}$

Require: Information $\widetilde{\tau}_{i,h}$, Strategy $\breve{g}_{1:h-1}$, QC Dec-POMDP $\mathcal{D}_{\mathcal{L}}$
  $\breve{\tau}_{i,h} \leftarrow \widetilde{\tau}_{i,h}$
  for $j = 1$ to n, $h' = 1$ to $h - 1$ **do**
    if $\sigma(\widetilde{\tau}_{j,h'}) \subseteq \sigma(\widetilde{c}_h)$ in $\mathcal{D}_{\mathcal{L}}$ and $\widetilde{a}_{j,h'} \notin \breve{\tau}_{i,h}$ **then**
      $\breve{\tau}_{j,h'} \leftarrow$ Recover$(\widetilde{\tau}_{j,h'}, \breve{g}_{1:h'-1}, \mathcal{D}_{\mathcal{L}})$
      $\widetilde{a}_{j,h'} \leftarrow \breve{g}_{j,h'}(\breve{\tau}_{j,h})$
      $\breve{\tau}_{i,h} \leftarrow \breve{\tau}_{j,h} \cup \{\widetilde{a}_{j,h'}\}$
    end if
  end for
  Return $\breve{\tau}_{i,h}$

Algorithm 6 Planning in Dec-POMDP with expected Approximate Common-information Model

Require: Expected Approximate Common-information Model $\mathcal{M}$.
  for $i \in [n]$ and $\widehat{c}_{\overline{H}+1} \in \widehat{\mathcal{C}}_{\overline{H}+1}$ **do**
    $V^{*,\mathcal{M}}_{i,\overline{H}+1}(\widehat{c}_{\overline{H}+1}) \leftarrow 0$
  end for
  for $h = \overline{H}$ to 1 **do**
    for $\widehat{c}_h \in \widehat{\mathcal{C}}_h$ **do**
      Define $Q^{*,\mathcal{M}}_h(\widehat{c}_h, \gamma_{1,h}, \cdots, \gamma_{n,h}) := \widehat{\mathcal{R}}^{\mathcal{M}}_h(\widehat{c}_h, \gamma_h) + \mathbb{E}^{\mathcal{M}}\left[V^{*,\mathcal{M}}_{h+1}(\widehat{c}_{h+1}) \,|\, \widehat{c}_h, \gamma_h\right]$

$$\left(\widehat{g}^*_{1,h}(\cdot \,|\, \widehat{c}_h, \cdot), \cdots, \widehat{g}^*_{n,h}(\cdot \,|\, \widehat{c}_h, \cdot)\right) \leftarrow \operatorname*{argmax}_{\gamma_{1:n,h} \in \Gamma_h} Q^{*,\mathcal{M}}_h(\widehat{c}_h, \gamma_{1,h}, \cdots, \gamma_{n,h}) \tag{E.1}$$

    end for
    $V^{*,\mathcal{M}}_h(\widehat{c}_h) \leftarrow \max_{\gamma_{1:n,h}} Q^{*,\mathcal{M}}_h(\widehat{c}_h, \gamma_{1,h}, \cdots, \gamma_{n,h})$
  end for
  Return $\widehat{g}^*_{1:\overline{H}}$

$\mathbb{T}_h(\cdot \,|\, s_h, a_h) \in \Delta(\mathcal{S})$ gives the transition probability to the next state s_{h+1} when taking the joint control action a_h at state s_h. We use $\mu_1 \in \Delta(\mathcal{S})$ to denote the distribution of the initial state s_1. We denote by $\mathcal{O}_{i,h}$ the observation space and by $o_{i,h} \in \mathcal{O}_{i,h}$ the observation of agent i at timestep h. We use $o_h := (o_{1,h}, o_{2,h}, \cdots, o_{n,h}) \in \mathcal{O}_h := \mathcal{O}_{1,h} \times \mathcal{O}_{2,h} \times \cdots \mathcal{O}_{n,h}$ to denote the joint observation of all the n agents at timestep h, with $\mathcal{O}_h$ denoting the joint observation space at timestep h. We use $\{\mathbb{O}_h\}_{h\in[H]}$ to denote the collection of emission matrices, where $o_h \sim \mathbb{O}_h(\cdot \,|\, s_h) \in \Delta(\mathcal{O}_h)$ at timestep h under state $s_h \in \mathcal{S}$. For notational convenience, we adopt the matrix convention, where $\mathbb{O}_h$ is a matrix with each row $\mathbb{O}_h(\cdot \,|\, s_h)$ for all $s_h \in \mathcal{S}$. Also, we denote by $\mathbb{O}_{i,h}$ the marginalized emission for agent i at timestep h. Finally, $\{\mathcal{R}_h\}_{h\in[H]}$ is a collection of reward functions among all agents, where $\mathcal{R}_h : \mathcal{S} \times \mathcal{A}_h \to [0, 1]$.

At timestep h, each agent i in the Dec-POMDP has access to some information $\tau_{i,h}$, a subset of historical joint observations and actions, namely, $\tau_{i,h} \subseteq \{o_1, a_1, o_2, \cdots, a_{h-1}, o_h\}$, and the collection of all possible such available information is denoted by $\mathcal{T}_{i,h}$. We use τ_h to denote the *joint* available information at timestep h. Meanwhile, agents may *share* part of the history with each other. The *common information* $c_h = \cup_{t=1}^h z_t$ at timestep h is thus a subset of the joint history τ_h, where z_h is the information shared at timestep h. We use $\mathcal{C}_h$ to denote the collection of all possible c_h at timestep h, and use $\mathcal{T}_{i,h}$ to denote the collection of all possible $\tau_{i,h}$ of agent i at timestep h. Besides the common information c_h, each agent also has her *private information* $p_{i,h} = \tau_{i,h} \backslash c_h$, where the collection of $p_{i,h}$ is denoted by $\mathcal{P}_{i,h}$. We also denote by p_h the *joint* private information, and by $\mathcal{P}_h$ the collection of all possible p_h at timestep h. We refer to the above the *state-space model* of the Dec-POMDP (with information sharing).

Each agent i at timestep h chooses the control action $a_{i,h}$ based on some strategy $g_{i,h} : \mathcal{T}_{i,h} \to \mathcal{A}_{i,h}$. We denote by $g_h := (g_{1,h}, g_{2,h}, \cdots, g_{n,h})$ the joint control strategy of all the agents, and by $g_{1:h} := (g_1, g_2, \cdots, g_h), \forall h \in [H]$ the sequence of joint strategies from timestep 1 to h. We use $\mathcal{G}_{i,h}$ to denote the strategy space of $g_{i,h}$, and use $\mathcal{G}_h, \mathcal{G}_{1:h}$ to denote joint strategy spaces, correspondingly.

Next, we introduce some background on the intrinsic model and information structure of Dec-POMDPs.

F.1 Intrinsic Model

In an intrinsic model (Witsenhausen, 1975), we regard the agent i at different timesteps as *different agents*, and each agent only acts *once* throughout. Any Dec-POMDP $\mathcal{D}$ with n agents can be formulated within the intrinsic-model framework, and can be characterized by a tuple $\langle(\Omega, \mathscr{F}), N, \{(\mathbb{U}_l, \mathscr{U}_l)\}_{l=1}^N, \{(\mathbb{I}_l, \mathscr{I}_l)\}_{l=1}^N\rangle$ (Mahajan et al., 2012), where $(\Omega, \mathcal{F})$ is a measurable space of the environment, $N = n \times H$ is the number of agents in the intrinsic model. By a slight abuse of notation, we write $[N] := [n] \times [H]$, and write $l := (i, h) \in [N]$ for notational convenience. This way, any agent $l \in [N]$ corresponds to an agent $i \in [n]$ at timestep $h \in [H]$ in the state-space model. We denote by $\mathbb{U}_l$ the measurable action space of agent l and by $\mathscr{U}_l$ the σ-algebra over $\mathbb{U}_l$. For $A \subseteq [N]$, let $\mathbb{H}_A := \Omega \times \prod_{l \in A} \mathbb{U}_l$ and $\mathbb{H} := \mathbb{H}_{[N]}$. For any σ-algebra $\mathscr{C}$ over $\mathbb{H}_A$, let $\langle \mathscr{C} \rangle$ denote the cylindrical extension of $\mathscr{C}$ on $\mathbb{H}$. Let $\mathscr{H}_A := \langle \mathscr{F} \otimes (\otimes_{l \in A} \mathscr{U}_l) \rangle$ and $\mathscr{H} = \mathscr{H}_{[N]}$. We denote by $\mathbb{I}_l$ the space of *information available* to agent l, and by $\mathscr{I}_l$ the σ-algebra over $\mathbb{H}$. For $l \in [N]$, we denote by I_l the information of agent l, and U_l the action of agent l. The spaces and random variables of agent $l = (i, h)$ in the intrinsic model are related to those in the state-space model as follows: $\forall l = (i, h) \in [N], \mathbb{U}_l = \mathcal{A}_{i,h}, \mathbb{I}_l = \mathcal{T}_{i,h}, U_l = a_{i,h}, I_l = \tau_{i,h}$.

F.2 Information Structures of Dec-POMDPs

An important class of IS is the *quasi-classical* one, which is defined as follows (Witsenhausen, 1975; Mahajan et al., 2012; Yüksel & Başar, 2023).

Definition F.1 (Quasi-classical Dec-POMDPs)**.** We call a Dec-POMDP problem *QC* if each agent in the intrinsic model knows the information available to the agents who influence her, directly or indirectly, i.e. $\forall l_1, l_2 \in [N], l_1 = (i_1, h_1), l_2 = (i_2, h_2), i_1, i_2 \in [n], h_1, h_2 \in [H]$, if agent l_1 influences agent l_2, then $\mathscr{I}_{l_1} \subseteq \mathscr{I}_{l_2}$.

Furthermore, *strictly* quasi-classical IS (Witsenhausen, 1975; Mahajan & Yüksel, 2010), as a subclass of QC IS, is defined as follows.

Definition F.2 (Strictly quasi-classical Dec-POMDPs)**.** We call a Dec-POMDP problem *sQC* if each agent in the intrinsic model knows the information *and* actions available to the agents who influence her, directly or indirectly. That is, $\forall l_1, l_2 \in [N], l_1 = (i_1, h_1), l_2 = (i_2, h_2), i_1, i_2 \in [n], h_1, h_2 \in [H]$, if agent l_1 influences agent l_2, then $\mathscr{I}_{l_1} \cup \langle \mathscr{U}_{l_1} \rangle \subseteq \mathscr{I}_{l_2}$.

F.3 Intrinsic Model of LTC Problems

Firstly, we formally define the Dec-POMDP induced by LTC as follows

Definition F.3 (Dec-POMDP (with information sharing) induced by LTC)**.** For an LTC $\mathcal{L}$, we call a Dec-POMDP (with information sharing) $\overline{\mathcal{D}}_{\mathcal{L}}$ *the Dec-POMDP (with information sharing) induced by* $\mathcal{L}$ if the agents share information only following the baseline sharing protocol of $\mathcal{L}$, i.e., without additional sharing. We may refer to it as the *Dec-POMDP induced by LTC* or the *induced Dec-POMDP* for short.

Given any LTC $\mathcal{L}$ of the state-space-model form defined in §2.1, we define the intrinsic model of $\mathcal{L}$ as a tuple $\langle (\Omega, \mathscr{F}), N, \{(\mathbb{U}_l, \mathscr{U}_l)\}_{l=1}^N, \{(\mathbb{M}_l, \mathscr{M}_l)\}_{l=1}^N, \{(\mathbb{I}_{l^-}, \mathscr{I}_{l^-})\}_{l=1}^N,$ $\{(\mathbb{I}_{l^+}, \mathscr{I}_{l^+})\}_{l=1}^N \rangle$, where $(\Omega, \mathscr{F})$ is the measure space representing all the uncertainty in the system; $N = n \times H$ is the number of agents in the intrinsic model. By a slight abuse of notation, we write $[N] := [n] \times [H]$, and write $l := (i, h) \in [N]$ for convenience. This way, any agent $l \in [N]$ corresponds to an agent $i \in [n]$ at timestep $h \in [H]$ in the state-space model, and we thus define $l^- := (i, h^-)$ and $l^+ := (i, h^+)$ accordingly. We denote by $\mathbb{U}_l$ and $\mathbb{M}_l$ the measurable control and communication action spaces of agent l, and by $\mathscr{U}_l$ and $\mathscr{M}_l$ the σ-algebra over $\mathbb{U}_l$ and $\mathbb{M}_l$, respectively. For any $A \subseteq [N]$, let $\mathbb{H}_A := \Omega \times \prod_{l \in A} (\mathbb{U}_l \times \mathbb{M}_l)$ and $\mathbb{H} := \mathbb{H}_{[N]}$. For any σ-algebra $\mathscr{C}$ over $\mathbb{H}_A$, let $\langle \mathscr{C} \rangle$ denote the cylindrical extension of $\mathscr{C}$ on $\mathbb{H}$. Let $\mathscr{H}_A := \langle \mathscr{F} \otimes (\otimes_{l \in A} \mathscr{U}_l) \otimes (\otimes_{l \in A} \mathscr{M}_l) \rangle$, $\mathscr{H} = \mathscr{H}_{[N]}$. We denote by $\mathbb{I}_{l^-}$ and $\mathbb{I}_{l^+}$ the spaces of *information available* to agent l *before* and *after* additional sharing, respectively, and by $\mathscr{I}_{l^-} \subseteq \mathscr{H}$ and $\mathscr{I}_{l^+} \subseteq \mathscr{H}$ the associated σ-algebra. The spaces and random variables of agent $l = (i, h)$ in the intrinsic model are related to those in the state-space model as follows: $\forall l = (i, h) \in [N], \mathbb{U}_l = \mathcal{A}_{i,h}, \mathbb{M}_l = \mathcal{M}_{i,h}, \mathbb{I}_{l^-} = \mathcal{T}_{i,h^-}, \mathbb{I}_{l^+} = \mathcal{T}_{i,h^+}, U_l = a_{i,h}, M_l = m_{i,h}, I_{l^-} = \tau_{i,h^-}, I_{l^+} = \tau_{i,h^+}$. For notational convenience, for any random variable B in LTC and the σ-algebra $\mathscr{B}$ generated by B, we overload $\sigma(B)$ to denote the cylindrical extension of $\mathscr{B}$ on $\mathbb{H}$, i.e., $\sigma(B) = \langle \mathscr{B} \rangle$.

G Conditions Leading to Assumption 4.3

As a minimal requirement for computational tractability (for both Dec-POMDPs and LTCs), Assumption 4.3 is needed for the one-step tractability of the team-decision problem involved in the value iteration in Algorithm 6. We now adapt several such structural conditions from (Liu & Zhang, 2023) to the LTC setting, which lead to this assumption and have been studied in the literature. Note that since we need to do planning in the approximate model $\mathcal{M}$, which is oftentimes constructed based on the original problem $\mathcal{L}$ and approximate belief $\{\mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h, \overline{p}_h \mid \widehat{c}_h)\}_{h\in[\overline{H}]}$, we necessarily need assumptions on these two models $\mathcal{L}$ and $\mathcal{M}$, for which we refer to as the **Part (1)** and **Part (2)** of the conditions below, respectively.

- **Turn-based structures. Part (1):** At each timestep $h \in [H]$, there is only one agent, denoted as $ct(h) \in [n]$, that can affect the state transition. More concretely, the transition dynamics take the forms of $\mathbb{T}_h : \mathcal{S} \times \mathcal{A}_{ct(h)} \to \Delta(\mathcal{S})$. Additionally, we assume the reward function admits an additive structure such that $\mathcal{R}_h(s_h, a_h) = \sum_{i\in[n]} \mathcal{R}_{i,h}(s_h, a_{i,h})$ for some functions $\{\mathcal{R}_{i,h}\}_{i\in[n]}$. Meanwhile, since only agent $ct(h)$ takes the action, we assume the increment of the common information $z_{h+1}^b = \chi_{h+1}(p_{h^+}, a_{ct(h),h}, o_{h+1})$. **Part (2):** No additional requirement. Such a structure has been commonly studied in (fully observable) stochastic games and multi-agent RL (Filar & Vrieze, 2012; Bai & Jin, 2020).
- **Nested private information. Part (1):** No additional requirement. **Part (2):** At each timestep $h \in [\overline{H}]$, all the agents form a *hierarchy* according to the private information after $a_{i,h}$ they possess, in the sense that $\forall\ i, j \in [n], j < i, \overline{p}_{j,h} = Y_h^{i,j}(\overline{p}_{i,h})$ for some function $Y_h^{i,j}$. More formally, the approximate belief satisfies that $\mathbb{P}_h^{\mathcal{M},c}(\overline{p}_{j,h} = Y_h^{i,j}(\overline{p}_{i,h}) \mid \overline{p}_{i,h}, \widehat{c}_h) = 1$. Such a structure has been investigated in (Peralez et al., 2024) with heuristic search, and in (Liu & Zhang, 2023) with finite-time complexity analysis.
- **Factorized structures. Part (1):** At each timestep $h \in [H]$, the state s_h can be partitioned into n local states, i.e., $s_h = (s_{1,h}, s_{2,h}, \cdots, s_{n,h})$. Meanwhile, the transition kernel takes the product form of $\mathbb{T}_h(s_{h+1} \mid s_h, a_h) = \prod_{i=1}^n \mathbb{T}_{i,h}(s_{i,h+1} \mid s_{i,h}, a_{i,h})$, the emission also takes the product form of $\mathbb{O}_h(o_h \mid s_h) = \prod_{i=1}^n \mathbb{O}_{i,h}(o_{i,h} \mid s_{i,h})$, and the reward function can be decoupled into n terms such that $\mathcal{R}_h(s_h, a_h) = \sum_{i,h} \mathcal{R}_h(s_{i,h}, a_{i,h})$. **Part (2):** At each even timestep $h \in [\overline{H}]$, the approximate common information is also factorized so that $\widehat{c}_h = (\widehat{c}_{1,h}, \widehat{c}_{2,h}, \cdots, \widehat{c}_{n,h})$ and its evolution satisfies that $\widehat{c}_{i,h+1} = \widehat{\phi}_{i,h+1}(\widehat{c}_{i,h}, \overline{z}_{i,h})$ for some function $\widehat{\phi}_{i,h+1}$. Correspondingly, the approximate belief need to satisfy that $\mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h, \overline{p}_h \mid \widehat{c}_h) = \Pi_{i=1}^n \mathbb{P}_{i,h}^{\mathcal{M},c}(\overline{s}_{i,h}, \overline{p}_{i,h} \mid \widehat{c}_{i,h})$ for some functions $\{\mathbb{P}_{i,h}^{\mathcal{M},c}\}_{i\in[n],h\in[\overline{H}]}$ Such a structure, under general information sharing protocols, can lead to non-classical IS. In this case, it can be viewed an example of non-classical ISs where the agents have no incentive for signaling (Yüksel & Başar, 2023, §3.8.3).

Lemma G.1. Given any LTC problem $\mathcal{L}$ and $\mathcal{D}'_{\mathcal{L}}$ is the Dec-POMDP after reformulation and expansion. For any $\mathcal{M}$ to be the approximate model of $\mathcal{D}_{\mathcal{L}}$ and $\{\mathbb{P}_h^{\mathcal{M},c}\}_{h\in[\overline{H}]}$ to be the approximate belief, if they satisfy any of the 3 conditions above, then Eq. (E.1) in Algorithm 6 can be solved in polynomial time, i.e., Assumption 4.3 holds.

Proof. We prove the result case by case:

- **Turn-based structures:** For any $h = 2t, t \in [H], \gamma_{ct(h),h} \in \Gamma_{ct(h)}, \gamma_{-ct(h),h}, \gamma'_{-ct(h),h} \in \Gamma_{-ct(h),h}$, where $ct(h)$ is the controller, it holds for any $\widehat{c}_h$ that

$$\begin{aligned}
&Q_h^{*,\mathcal{M}}(\widehat{c}_h, \gamma_{ct(h),h}, \gamma_{-ct(h),h})\\
&= \sum_{\overline{s}_h, \overline{p}_h, \overline{s}_{h+1}, \overline{o}_{h+1}} \mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h, \overline{p}_h \mid \widehat{c}_h)\overline{\mathbb{T}}_h(\overline{s}_{h+1} \mid \overline{s}_h, \gamma_{ct(h),h}(\overline{p}_{ct(h),h})\gamma_{-ct(h),h}(\overline{p}_{-ct(h),h}))\\
&\overline{\mathbb{O}}_{h+1}(\overline{o}_{h+1} \mid \overline{s}_{h+1})[\overline{\mathcal{R}}_h(\overline{s}_h, \gamma_{ct(h),h}(\overline{p}_{ct(h),h})) + V_{h+1}^{*,\mathcal{M}}(\widehat{c}_{h+1})]\\
&= \sum_{\overline{s}_h, \overline{p}_h, \overline{s}_{h+1}, \overline{o}_{h+1}} \mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h, \overline{p}_h \mid \widehat{c}_h)\overline{\mathbb{T}}_h(\overline{s}_{h+1} \mid \overline{s}_h, \gamma_{ct(h),h}(\overline{p}_{ct(h),h})\\
&\overline{\mathbb{O}}_{h+1}(\overline{o}_{h+1} \mid \overline{s}_{h+1})[\overline{\mathcal{R}}_h(\overline{s}_h, \gamma_{ct(h),h}(\overline{p}_{ct(h),h})) + V_{h+1}^{*,\mathcal{M}}(\widehat{c}_{h+1})],
\end{aligned}$$

  where the last step is due to the fact that $\widehat{c}_{h+1} = \widehat{\phi}_{h+1}(\widehat{c}_h, \overline{z}_{h+1})$. And $\overline{z}_{h+1} = z^b_{\frac{h}{2}+1} = \chi_{\frac{h}{2}+1}(\overline{p}_h, \overline{a}_{ct(h),h}, \overline{o}_{h+1})$. Therefore, right-hand side does no depend on $\gamma_{-ct(h),h}$. Therefore, Eq. (E.1) with complexity poly$(\overline{\mathcal{S}}, \overline{\mathcal{P}}_{ct(h)}, \overline{\mathcal{A}}_{ct(h)})$.

- **Nested private information:** For any $i \in [n], h = 2t, t \in [H]$, we first define the $u_{i,h} \in \mathcal{U}_{i,h} := \{(\times_{j=1}^i \mathcal{P}_{j,h}) \times (\times_{j=1}^{i-1} \mathcal{A}_{j,h}) \to \mathcal{A}_{i,h}\}$ and slightly abuse the notation for $Q_h^{*,\mathcal{M}}$ as follows

$$\begin{aligned}
&Q_h^{*,\mathcal{M}}(\widehat{c}_h, u_{1,h}, \cdots, u_{n,h})\\
&:= \sum_{\overline{s}_h, \overline{p}_h, \overline{a}_h, \overline{s}_{h+1}, \overline{o}_{h+1}} \mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h, \overline{p}_h \mid \widehat{c}_h)\Pi_{i=1}^n \mathbb{1}[\overline{a}_{i,h} = u_{i,h}(\overline{p}_{1:i,h}, \overline{a}_{1:i-1,h})]\overline{\mathbb{T}}_h(\overline{s}_{h+1} \mid \overline{s}_h, \overline{a}_h)\\
&\overline{\mathbb{O}}_{h+1}(\overline{o}_{h+1} \mid \overline{s}_{h+1})[\overline{\mathcal{R}}_h(\overline{s}_h, \overline{a}_h) + V_{h+1}^{*,\mathcal{M}}(\widehat{c}_{h+1})]
\end{aligned}$$

  Since the space of $\mathcal{U}_{i,h}$ covers the space $\Gamma_{i,h}$, then for the $u^*_{1:n,h}$ be an optimal one that maximize the $Q_h^{*,\mathcal{M}}$, we have

$$\begin{aligned}
&Q_h^{*,\mathcal{M}}(\widehat{c}_h, u^*_{1,h}, \cdots, u^*_{n,h})\\
&= \max_{\{u_{i,h} \in \mathcal{U}_{i,h}\}_{i\in[n]}} Q_h^{*,\mathcal{M}}(\widehat{c}_h, u_{1,h}, \cdots, u_{n,h}) \geq \max_{\{\gamma_{i,h} \in \Gamma_{i,h}\}_{i\in[n]}} Q_h^{*,\mathcal{M}}(\widehat{c}_h, \gamma_{1,h}, \cdots, \gamma_{n,h}).
\end{aligned}$$

  Meanwhile, due to the nested private information condition, for any $\overline{p}_h \in \overline{\mathcal{P}}_h$, there must exists $\gamma'_{1:n,h}$ such that $\gamma'_{1:n,h}$ output the same actions as $u^*_{1:n,h}$ under $\overline{p}_h$. Therefore, we can conclude that

$$\max_{\{u_{i,h} \in \mathcal{U}_{i,h}\}_{i\in[n]}} Q_h^{*,\mathcal{M}}(\widehat{c}_h, u_{1,h}, \cdots, u_{n,h}) = \max_{\{\gamma_{i,h} \in \Gamma_{i,h}\}_{i\in[n]}} Q_h^{*,\mathcal{M}}(\widehat{c}_h, \gamma_{1,h}, \cdots, \gamma_{n,h})$$

  Therefore, we can solve Eq. (E.1) and compute $\gamma^*_{1:n,h}$ from computing $u^*_{1:n,h}$, which can be solved with complexity poly$(\overline{\mathcal{P}}_h, \overline{\mathcal{A}}_h, \overline{\mathcal{S}})$.

- **Factorized structures:** For any $h \in [\overline{H}], t \in [H]$, for any $\widehat{c}_h \in \widehat{\mathcal{C}}_h, \gamma_h \in \Gamma_h$ we use backward induction to prove that, there exist n functions $\{F_{i,h}\}_{i\in[n]}$ such that

$$Q_h^{*,\mathcal{M}}(\widehat{c}_h, \gamma_h) = \sum_{i=1}^n F_{i,h}(\widehat{c}_{i,h}, \gamma_{i,h})$$

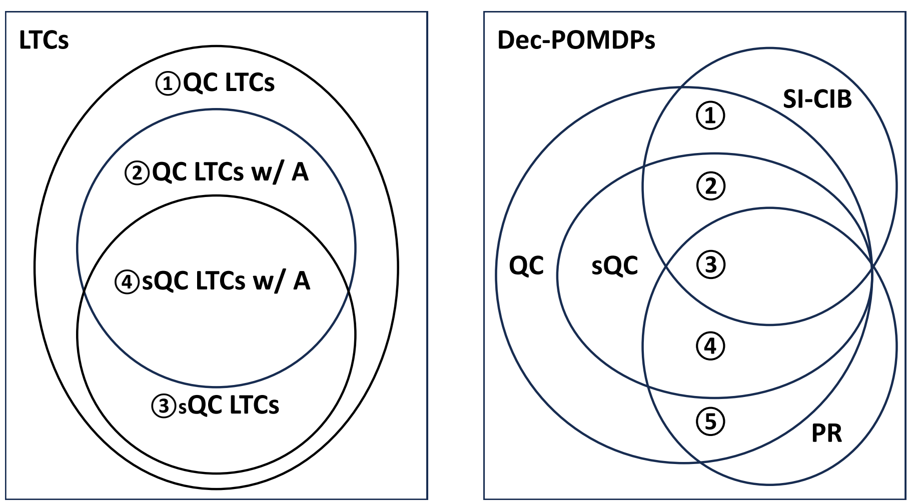
It holds for $h = \overline{H}+1$ obviously. For any $h \le \overline{H}$, it holds that

$$
\begin{aligned}
&Q_h^{*,\mathcal{M}}(\widehat{c}_h, \gamma_h) \\
&= \sum_{\overline{s}_h, \overline{p}_h, \overline{s}_{h+1}, \overline{o}_{h+1}} \mathbb{P}_h^{\mathcal{M},c}(\overline{s}_h, \overline{p}_h \,|\, \widehat{c}_h) \overline{\mathbb{T}}_h(\overline{s}_{h+1} \,|\, \overline{s}_h, \gamma_h(\overline{p}_h)) \overline{\mathbb{O}}_{h+1}(\overline{o}_{h+1} \,|\, \overline{s}_{h+1}) \\
&\qquad [\sum_{i=1}^n \overline{\mathcal{R}}_{i,h}(\overline{s}_{i,h}, \gamma_{i,h}(\overline{p}_{i,h})) + F_{i,h+1}(\widehat{c}_{i,h+1}, \widehat{g}^*_{i,h+1}(\widehat{c}_{i,h+1}))] \\
&= \sum_{i=1}^n \sum_{\overline{s}_{i,h}, \overline{p}_{i,h}, \overline{s}_{i,h+1}, \overline{o}_{i,h+1}} \mathbb{P}_{i,h}^{\mathcal{M},c}(\overline{s}_{i,h}, \overline{p}_{i,h} \,|\, \widehat{c}_{i,h}) \overline{\mathbb{T}}_h(\overline{s}_{i,h+1} \,|\, \overline{s}_{i,h}, \gamma_{i,h}(\overline{p}_{i,h})) \\
&\qquad \overline{\mathbb{O}}_{i,h+1}(\overline{o}_{i,h+1} \,|\, \overline{s}_{i,h+1}) [\overline{\mathcal{R}}_{i,h}(\overline{s}_{i,h}, \gamma_{i,h}(\overline{p}_{i,h})) + F_{i,h+1}(\widehat{c}_{i,h+1}, \widehat{g}^*_{i,h+1}(\widehat{c}_{i,h+1}))] \\
&=: \sum_{i=1}^n F_{i,h}(\widehat{c}_{i,h}, \gamma_{i,h}).
\end{aligned}
$$

Then, by induction, we know that it holds for any $h \in [\overline{H}]$. We can define $\widehat{g}^*_{i,h}(\widehat{c}_h) \in \operatorname{argmax}_{\gamma_{i,h} \in \Gamma_{i,h}} F_{i,h+1}(\widehat{c}_{i,h+1}, \gamma_{i,h})$, and thus solve Eq.(E.1) with complexity $\sum_{i=1}^n \mathrm{poly}(\overline{\mathcal{S}}_i, \overline{\mathcal{A}}_{i,h}, \overline{\mathcal{P}}_{i,h})$.

This completes the proof. $\square$

H Venn Diagrams of LTCs and General POSGs



(a) (b)

Figure 1: (a) Venn diagram of LTCs with different ISs: ① QC LTCs. ② QC LTCs satisfying Assumptions 3.2, 3.3, and 3.4. ③ sQC LTCs. ④ sQC LTCs satisfying Assumptions 3.2, 3.3, and 3.4, whose reformulated Dec-POMDPs have SI-CIB; (b) Venn diagram of general Dec-POMDPs with different ISs. PR denotes perfect recall. ③ denotes the Dec-POMDPs we mainly consider, e.g., the examples in (Nayyar et al., 2013a; Liu & Zhang, 2023).

Here, we show some examples of the areas ①-⑤ in the Venn diagram in Fig. 1b.

- **①: Multi-agent MDP (Boutilier, 1999) with historical states.** The Dec-POMDPs satisfying that for any $h \in [H], i \in [n], \mathcal{O}_{i,h} = \mathcal{S}, \mathbb{O}_{i,h}(s \,|\, s) = 1, c_h = s_{1:h}, p_h = \emptyset$ lie in the area ①.

- **②: Uncontrolled state process without any historical information.** The Dec-POMDPs satisfying that for any $h \in [H], i \in [n], s_h, a_h, a'_h, \mathbb{T}_h(\cdot \mid s_h, a_h) = \mathbb{T}_h(\cdot \mid s_h, a'_h), c_h = \emptyset, p_{i,h} = \{o_{i,h}\}$ lie in the area ②.
- **③: Dec-POMDPs with sQC information structure and perfect recall, and satisfying Assumptions 3.3 and 3.4.** This class is what we mainly considered in §5.
- **④: State controlled by one controller with no sharing and only observability of controller.** We consider a Dec-POMDP $\mathcal{D}$. The state dynamics are controller by only one agent (, for convenience, agent 1), and only agent 1 has observability, i.e. $\mathbb{T}_h(\cdot \mid s_h, a_{1,h}, a_{-1,h}) = \mathbb{T}_h(\cdot \mid s_h, a_{1,h}, a'_{-1,h})$ for all $s_h, a_{1,h}, a_{-1,h}, a'_{-1,h}$, and $\mathcal{O}_{-1,h} = \emptyset$. There is no information sharing, i.e. $c_h = \emptyset, p_{1,h} = \{o_{1:h}, a_{1:h-1}\}, p_{j,h} = \{a_{j,1:h-1}\}, \forall j \neq 1$. Then $\forall j \neq 1, h_1 < h_2 \in [H]$, agent $(1, h_1)$ does not influence (j, h_2), since $\tau_{j,h_2} = \{a_{j,1:h_2-1}\}$ is not influenced by agent $(1, h_1)$. Therefore, $\mathcal{D}$ is sQC and has perfect recall, $\mathcal{D}$ is not SI (underlying state s_h influenced by $g_{1,1:h-1}$). This is because $\mathcal{D}$ does not satisfy Assumption 3.4. Then $\mathcal{D}$ lies in the area ④.
- **⑤: One-step delayed observation sharing and two-step delayed action sharing.** The Dec-POMDPs satisfying that for any $h \in [H], i \in [n], c_h = \{o_{1:h-1}, a_{1:h-2}\}, p_{i,h} = \{a_{i,h-1}, o_{i,h}\}$ lie in the area ⑤.

I Experimental Results

For the experiments, we validate both the implementability and performance of our LTC algorithms, and conduct ablation studies for LTCs with different communication costs and horizons.

Experimental setup We conduct our experiments on two popular and modest-scale partially observable benchmarks, Dectiger (Nair et al., 2003) and Grid3x3 (Amato et al., 2009). We train the agents in each LTC problem in the two environments with 20 different random seeds and different communication cost functions, and execute them in problems with horizons $[4, 6, 8, 10]$. To fit the setting of LTC in our paper. We regularize the reward between [0,1] and set the base information structure as one-step-delay. As for the communication cost function, we set $\mathcal{K}_h(Z_h^a) = \alpha|Z_h^a|$, and set $\alpha \in [0.01, 0.05, 0.1]$ for the purpose of ablation study. Also, we study 2 baselines under the same environment with information structure of one-step delay and fully-sharing, respectively. The one-step-delay baseline can be regarded as an LTC problem with extremely high communication cost, thus no additional sharing. On the other hand, the fully-sharing baseline is the LTC problem with no communication cost.

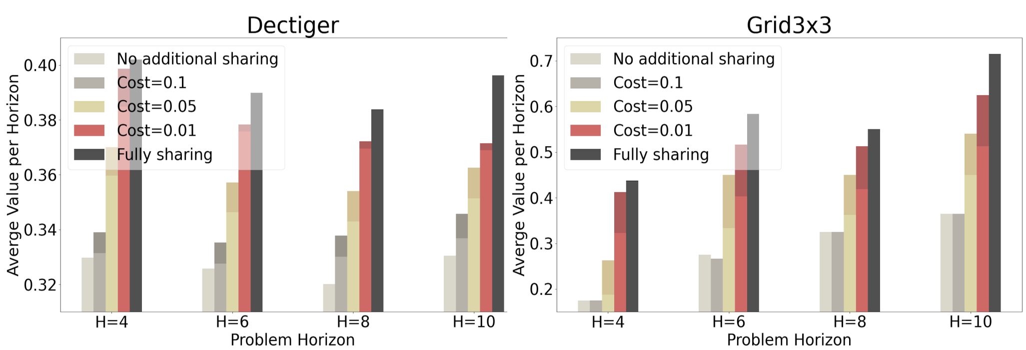


Figure 2: The average-values achieved under different communication costs and horizons. Each full bar, the dark part, and the light part denote the values associated with the reward, the communication cost, and the overall objective (reward minus cost) of the agents, respectively. Note that, as baselines, there is no communication cost in the *no additional sharing* and *fully sharing* cases.

Results and analysis The attained average-values are presented in Fig. 2, and the learning curves are shown in Fig. 3. Additionally, the results of different horizons and communications costs over

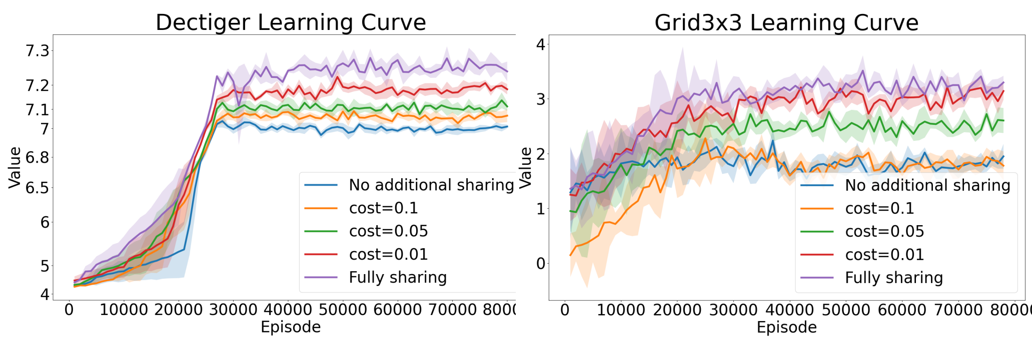


Figure 3: Learning curves with different communication costs.

Horizon/Cost	No Sharing	Cost=0.1	Cost=0.05	Cost=0.01	Fully Sharing
H=4 w/ cost	1.32±0.025	1.33±0.044	1.44±0.034	1.54±0.013	1.57±0.004
H=4 w/o cost	-	1.36±0.032	1.48±0.034	1.59±0.002	-
H=6 w/ cost	1.95±0.009	1.97±0.07	2.08±0.068	2.26±0.012	2.29±0.002
H=6 w/o cost	-	2.01±0.047	2.14±0.072	2.27±0.011	-
H=8 w/ cost	2.56±0.041	2.64±0.078	2.74±0.118	2.96±0.021	3.0±0.002
H=8 w/o cost	-	2.7±0.044	2.83±0.117	2.98±0.02	-
H=10 w/ cost	3.31±0.024	3.37±0.135	3.51±0.153	3.69±0.029	3.87±0.007
H=10 w/o cost	-	3.46±0.069	3.63±0.152	3.71±0.026	-

Table 1: Experimental results for Dectiger.

Horizon/Cost	No Sharing	Cost=0.1	Cost=0.05	Cost=0.01	Fully Sharing
H=4 w/ cost	0.14±0.003	0.14±0.019	0.15±0.002	0.26±0.028	-0.48±0.023
H=4 w/o cost	-	0.14±0.019	0.21±0.007	0.33±0.023	-
H=6 w/ cost	0.33±0.02	0.32±0.025	0.4±0.009	0.48±0.059	-0.38±0.075
H=6 w/o cost	-	0.32±0.025	0.54±0.02	0.62±0.075	-
H=8 w/ cost	0.52±0.084	0.52±0.051	0.58±0.072	0.67±0.031	-0.4±0.022
H=8 w/o cost	-	0.52±0.051	0.72±0.035	0.82±0.074	-
H=10 w/ cost	0.73±0.02	0.73±0.037	0.9±0.169	1.03±0.019	-0.15±0.188
H=10 w/o cost	-	0.73±0.037	1.08±0.14	1.25±0.062	-

Table 2: Experimental results for Grid3x3.

20 random seeds are shown in Tables 1 and 2. The results show that communication is beneficial for agents to obtain higher values with better sample efficiency. Also, cheaper communication costs can encourage agents to share more information, and jointly achieve a better strategy.

J Additional Figures

We provide a few figures to better illustrate the paradigms and algorithmic ideas of this paper. Fig. 4 and Fig. 5 illustrate the paradigm and the timeline of the LTC problems considered in this paper, and Fig. 6 illustrates how Algorithm 1 solves the LTC problems, including the subroutines presented in §4.

K Related Work

Communication-control joint optimization. The joint design of control and communication strategies has been studied in the control literature (Xiao et al., 2005; Yüksel, 2013; Sudhakara et al., 2021; Kartik et al., 2022). However, even with model knowledge, the computational complexity (and associated necessary conditions) of solving these models remains elusive, let alone the sample

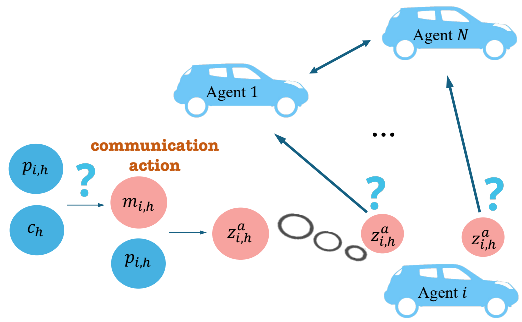


Figure 4: Illustrating the paradigm of the Learning-to-Communicate problem considered in this paper.

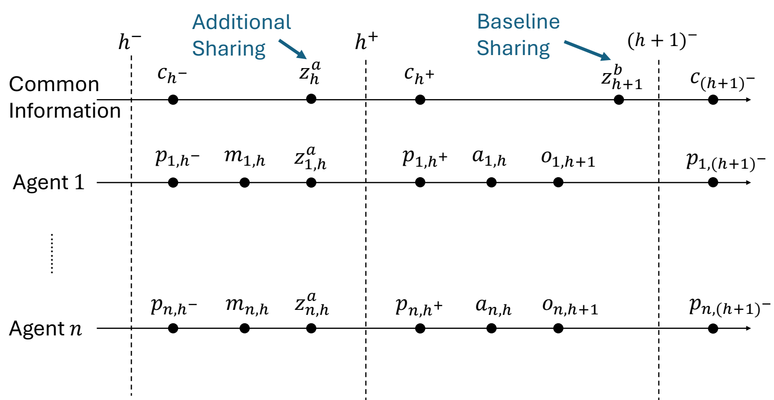


Figure 5: Timeline of the information sharing and evolution protocols in the Learning-to-Communicate problem considered in this paper.

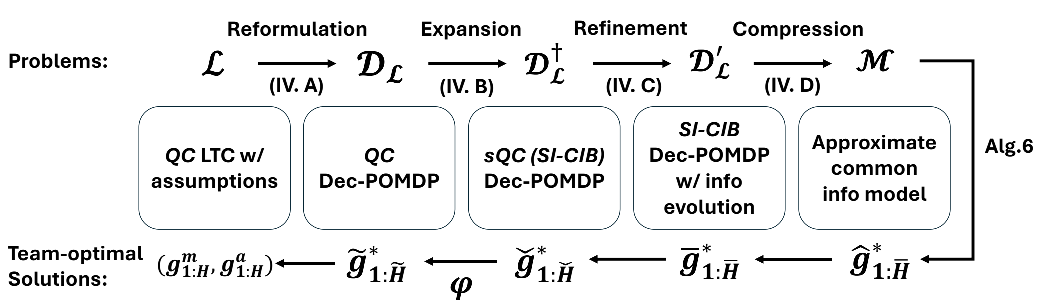


Figure 6: Illustrating the subroutines in §4 for solving the LTC problems.

complexity when it comes to learning. Moreover, these models mostly have more special structures, e.g., with linear systems (Xiao et al., 2005; Yüksel, 2013), or allowing to share only instantaneous observations (Sudhakara et al., 2021; Kartik et al., 2022).

Information sharing and information structures. Information structure has been extensively studied to characterize *who knows what and when* in decentralized control (Mahajan et al., 2012; Yüksel & Başar, 2023). Our paper aims to formally understand LTC through the lens of information structures. The common-information-based approaches to formalize *information sharing* in (Nayyar et al., 2013b;a) serve as the basis of our work. In comparison, these results focused on the *structural results*, without concrete computational (and sample) complexity analysis.

Partially observable MARL theory. Planning and learning in partially observable MARL are known to be hard (Papadimitriou & Tsitsiklis, 1987; Lusena et al., 2001; Jin et al., 2020; Bernstein et al., 2002). Recently, (Liu et al., 2022; Altabaa & Yang, 2024) developed polynomial-sample complexity algorithms for partially observable stochastic games, but with computationally intractable oracles; (Liu & Zhang, 2023) developed quasi-polynomial-time and sample algorithms for such models, leveraging information sharing. In contrast, our paper focuses on *optimizing/learning to share*, together with control strategy optimization/learning.

L Concluding Remarks

We formalized the learning-to-communicate problem under the Dec-POMDP framework, and proposed a few structural assumptions for LTCs with quasi-classical information structures, violating which can cause computational hardness in general. We then developed provable planning and learning algorithms for QC LTCs. Along the way, we also established some relationship between the strictly quasi-classical information structure and the condition of having strategy-independent common-information-based beliefs, as well as solving general Dec-POMDPs without computationally intractable oracles beyond those with the SI-CIB condition. Our work has opened up many future directions, including the formulation, together with the development of provable planning/learning algorithms, of LTC in non-cooperative (game-theoretic) settings, and the relaxation of (some of) the structural assumptions when it comes to equilibrium computation.

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