A Additional Comments on Experiments

Here we slightly expand on the comment about computational complexity in the main text, and give more details about the cancer simulation we use from Bica et al. (2020); Geng et al. (2017); Seedat et al. (2022); Vanderschueren et al. (2023).

A comment on computational complexity: As commented in the main text, the per-iteration runtime of EDQ is similar to that of FQE, which is a common tool in large-scale offline RL problems; for example, Paine et al. (2020); Voloshin et al. (2021) use it in benchmarks and evaluations. The difference in computation times between EDQ and FQE is due to sampling from the target policy, or more accurately $\tilde{P}^a_t$, in order to draw the treatments used in the Q-update, i.e., δ and $\widetilde{\mathcal{H}}_{t+\delta}$ in algorithm 2. In most applications, the added complexity due to this difference is small relative to the cost of evaluating the Q-function and its gradients. In turn, the cost of function evaluation is the same for FQE and EDQ. The computational complexity of sampling from $\tilde{P}^a_t$ depends on how it is represented and implemented. For instance, we may specify policies by allowing evaluations of $\lambda^a(u|\mathcal{H}_u)$, and sample using the thinning algorithm (Lewis and Shedler, 1979; Ogata, 1981); with neural networks that allow sampling the time-to-next-event (e.g., see (McDermott et al., 2023; Nagpal et al., 2021) for examples of event time prediction); or with closed-form decision rules. For instance, in the simulation that follows, we sample exponential variables from times once a feature crosses a certain threshold.

B Proofs

We begin with some notation and additional definitions, in appendix B.2 we prove the consistency result for our method, and in appendix B.3 we give its discrete-time version. To avoid cluttered notation and longer proof, we will give the proof of theorem 1 for unmarked processes. Adding a distribution of marks is a trivial extension that does not alter the main steps of the derivation.

B.1 Notation and Definitions

For a multivariate point process we denote by $\lambda_\bullet(\cdot)$ the sum $\sum_k \lambda_k(\cdot)$, in our case this will include the components $\{a, x, y\}$. For any distribution or intensity, e.g. λ, and $s > t$, we will use the conditioning $\lambda(\cdot|\mathcal{H}_s = \mathcal{H}_t)$ to denote the event that jumps until time t are those that are in $\mathcal{H}_t$ and no events occur in the interval $(t, s]$. $\mathcal{H}_t \cup \{(t + \delta, k)\}$ is the event where up until time $t + \delta$ all jumps until time t agree with $\mathcal{H}_t$, and the next jump after that happens in N_k at time $t + \delta$. We assume that all processes have well-defined densities and intensity functions, and that $P \ll P_{\text{obs}}$ so that the conditional expectations we use in the derivation are well defined. The expectation $\mathbb{E}_{\delta_k \sim Q(\cdot|\mathcal{H}_t)}[\cdot]$ for $k \in \{x, a, y\}$ and a process Q denotes an expectation that draws the interval for next jump time $t + \delta_k$ of the N_k process, conditioned on $\mathcal{H}_t$, the history up to time t. We also adopt the convention where for the for the k-th event time T_k it holds that $\lim_{k\to\infty} T_k = \infty$ with probability 1. That is, the number of events with time smaller than T is countable and all events after T take on time ∞ (Andersen et al., 2012). We also use the notation $\mathbf{1}[\cdot]$ for the indicator function returning 1 if the condition inside it is satisfied and 0 otherwise.

Given a trajectory $\mathcal{H}$ and time t, we define $\delta_x(t) = \min\{s - t : s > t, (s, \cdot) \in \mathcal{H}^x\}$ as the first jump time for process N_x in trajectory $\mathcal{H}$ after time t, and likewise for y. We denote the first jump time for N_a by $\delta^{\text{obs}}_a(t)$, and a trajectory $\widetilde{\mathcal{H}}$ sampled from $\widetilde{P}^a_t(\cdot|\mathcal{H})$ we use $\delta_a(t)$ to refer to its first treatment jump time, i.e. $\delta_a(t) = \min\{s - t : s > t, (s, \cdot) \in \widetilde{\mathcal{H}}^a\}$. Since x, y play the same role throughout the derivation, as the parts of the process that are not part of the intervention, we will shorten notation to $\lambda^v(t + \delta|\mathcal{H}_{t+\delta})\mathbb{E}_P[Y|\mathcal{H}_t \cup \{(t + \delta, v)\}] := \lambda^x(t + \delta|\mathcal{H}_{t+\delta})\mathbb{E}_P[Y|\mathcal{H}_t \cup \{(t + \delta, x)\}] + \lambda^y(t + \delta|\mathcal{H}_{t+\delta})\mathbb{E}_P[Y|\mathcal{H}_t \cup \{(t + \delta, y)\}]$ and $\delta_v := \delta_x \wedge \delta_y$.

B.2 Proof of Formal Results

Below we prove theorem 1 where the result, eq. (2), implies that performing dynamic programming using the Q-function from the earliest disagreement time between observed data, and the data sampled from the target distribution, results in a correct estimator. We start with a lemma that the rest of the derivation relies on, which is similar to a tower property of conditional expectations with respect to the first jump that occurs in any component of the process.

Lemma 1. *Let P, P_{obs} be multivariate marked decision point processes, $t \in [0, T)$, and $\mathcal{H}$ a trajectory of events, where $\mathcal{H} \in \text{supp}(P)$. It holds that*

$$\begin{aligned}\mathbb{E}_P[Y|\mathcal{H}_t] = \mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H})}\Big[&\mathbf{1}\left[\delta_a(t) < \delta_v(t) \wedge \delta_a^{\text{obs}}(t)\right]\mathbb{E}_P\left[Y|\mathcal{H}_t \cup (t+\delta_a(t), a)\right] + \\ &\mathbf{1}\left[\delta_v(t) < \delta_a(t) \wedge \delta_a^{\text{obs}}(t)\right]\mathbb{E}_P\left[Y|\mathcal{H}_t \cup (t+\delta_v(t), v)\right] + \\ &\mathbf{1}\left[\delta_a^{\text{obs}}(t) < \delta_a(t) \wedge \delta_v(t)\right]\mathbb{E}_P\left[Y|\mathcal{H}_{t+\delta_a^{\text{obs}}(t)} = \mathcal{H}_t\right] + \\ &\mathbf{1}\left[\delta_a^{\text{obs}}(t) \wedge \delta_a(t) \wedge \delta_v(t) > T-t\right]\mathbb{E}_P\left[Y|\mathcal{H}_T = \mathcal{H}_t\right]\Big]\Big]\end{aligned} \tag{3}$$

Proof. Note that all the conditional expectations in the above expression exist since $P \ll P_{\text{obs}}$. From the law of total probability, since either no jumps occur in $(t, T]$, or $N_{x,y}$ jumps first, or N_a jumps first,

$$\begin{aligned}\mathbb{E}_P\left[Y|\mathcal{H}_t\right] = \exp\left\{-\int_t^T \lambda_\bullet(s|\mathcal{H}_s = \mathcal{H}_t)\right\}\mathbb{E}_P\left[Y|\mathcal{H}_T = \mathcal{H}_t\right] + \\ \int_0^{T-t}\exp\left\{-\int_t^{t+\delta}\lambda_\bullet(s|\mathcal{H}_s = \mathcal{H}_t)\mathrm{d}s\right\} \\ \Big(\lambda^a(t+\delta|\mathcal{H}_{t+\delta} = \mathcal{H}_t)\mathbb{E}_P\left[Y|\mathcal{H}_t \cup \{(t+\delta, a)\}\right] + \\ \lambda^x(t+\delta|\mathcal{H}_{t+\delta} = \mathcal{H}_t)\mathbb{E}_P\left[Y|\mathcal{H}_t \cup \{(t+\delta, x)\}\right] + \\ \lambda^y(t+\delta|\mathcal{H}_{t+\delta} = \mathcal{H}_t)\mathbb{E}_P\left[Y|\mathcal{H}_t \cup \{(t+\delta, y)\}\right]\Big)\mathrm{d}\delta.\end{aligned} \tag{4}$$

Next we write down each item in eq. (3),

$$\begin{aligned}&\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H})}\Big[\mathbf{1}\left[\delta_a(t) < \delta_v(t) \wedge \delta_a^{\text{obs}}(t)\right] \cdot \mathbb{E}\left[Y|\mathcal{H}_t \cup (t+\delta_a(t), a)\right]\Big]\Big] = \\ &\int_0^{T-t}\lambda_a(t+\delta|\mathcal{H}_{t+\delta} = \mathcal{H}_t)\exp\{-\int_t^{t+\delta}\lambda_a(s|\mathcal{H}_s = \mathcal{H}_t)\mathrm{d}s\} \\ &\qquad\exp\{-\int_t^{t+\delta}\lambda_\bullet^{\text{obs}}(s|\mathcal{H}_s = \mathcal{H}_t)\mathrm{d}s\}\mathbb{E}\left[Y|\mathcal{H}_t \cup (t+\delta_a, a)\right]\mathrm{d}\delta = \\ &\int_0^{T-t}\lambda_a(t+\delta|\mathcal{H}_{t+\delta} = \mathcal{H}_t)\exp\{-\int_t^{t+\delta}\lambda_\bullet(s|\mathcal{H}_s = \mathcal{H}_t)\mathrm{d}s\} \\ &\qquad\exp\{-\int_t^{t+\delta}\lambda_a^{\text{obs}}(s|\mathcal{H}_s = \mathcal{H}_t)\mathrm{d}s\}\mathbb{E}\left[Y|\mathcal{H}_t \cup (t+\delta_a, a)\right]\mathrm{d}\delta = \\ &\int_0^{T-t}\lambda_a(t+\delta|\mathcal{H}_{t+\delta} = \mathcal{H}_t)\exp\{-\int_t^{t+\delta}\lambda_\bullet(s|\mathcal{H}_s = \mathcal{H}_t)\mathrm{d}s\}\mathbb{E}\left[Y|\mathcal{H}_t \cup (t+\delta_a, a)\right] \\ &\qquad\cdot(1 - 1 + \exp\{-\int_t^{t+\delta}\lambda_a^{\text{obs}}(s|\mathcal{H}_s = \mathcal{H}_t)\mathrm{d}s\})\mathrm{d}\delta.\end{aligned} \tag{5}$$

The first equality simply expands the expectation as an integration over all possible stopping times for N_a (according to the definition of $\widetilde{P}$, see definition 4). The second equality holds since the intensities $\lambda_x^{\text{obs}}, \lambda_y^{\text{obs}}$ are equal to λ_x, λ_y respectively. Then finally we simply add and subtract 1 from the last item. Similarly, for the second item in eq. (3)

$$\begin{aligned}&\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H})}\Big[\mathbf{1}\left[\delta_v(t) < \delta_a(t) \wedge \delta_a^{\text{obs}}(t)\right] \cdot \mathbb{E}\left[Y|\mathcal{H}_t \cup (t+\delta_v(t), v)\right]\Big]\Big] = \\ &\int_0^{T-t}\lambda_v(t+\delta|\mathcal{H}_{t+\delta} = \mathcal{H}_t)\mathbb{E}\left[Y|\mathcal{H}_t \cup (t+\delta_v, v)\right]\exp\{-\int_t^{t+\delta}\lambda_\bullet(s|\mathcal{H}_s = \mathcal{H}_t)\mathrm{d}s\}\end{aligned}$$

$$\cdot\,(1 - 1 + \exp\{-\int_t^{t+\delta} \lambda_v(s|\mathcal{H}_s = \mathcal{H}_t)\mathrm{d}s\})\mathrm{d}\delta. \tag{6}$$

The last item in eq. (3) is

$$\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\left[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}^a_t(\cdot|\mathcal{H})}\left[\mathbf{1}\left[\delta^{\text{obs}}_a(t)\wedge\delta_a(t)\wedge\delta_v(t) > T-t\right]\cdot\mathbb{E}_P\left[Y|\mathcal{H}_T=\mathcal{H}_t\right]\right]\right] =$$
$$\exp\left\{-\int_t^T \lambda^{\text{obs}}_a(s|\mathcal{H}_s=\mathcal{H}_t)\right\}\exp\left\{-\int_t^T \lambda_\bullet(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\right\}\mathbb{E}_P\left[Y|\mathcal{H}_T=\mathcal{H}_t\right] \tag{7}$$

Adding up eq. (5), eq. (6) and eq. (7), and matching the items with eq. (4), we get

$$\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}^a_t(\cdot|\mathcal{H})}\Big[\mathbf{1}\left[\delta_a(t)<\delta_v(t)\wedge\delta^{\text{obs}}_a(t)\right]\cdot\mathbb{E}_P\left[Y|\mathcal{H}_t\cup(t+\delta_a,a)\right]+$$
$$\mathbf{1}\left[\delta_v(t)<\delta_a(t)\wedge\delta^{\text{obs}}_a(t)\right]\cdot\mathbb{E}_P\left[Y|\mathcal{H}_t\cup(t+\delta_v,v)\right]+$$
$$\mathbf{1}\left[\delta^{\text{obs}}_a(t)\wedge\delta_a(t)\wedge\delta_v(t)>T-t\right]\cdot\mathbb{E}_P\left[Y|\mathcal{H}_T=\mathcal{H}_t\right]\Big]\Big] =$$
$$\mathbb{E}_P\left[Y|\mathcal{H}_t\right]+ \tag{8}$$
$$-\left(1-\exp\left\{-\int_t^T \lambda^{\text{obs}}_a(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\right\}\right)\exp\left\{-\int_t^T \lambda_\bullet(s|\mathcal{H}_s=\mathcal{H}_t)\right\}\mathbb{E}_P[Y|\mathcal{H}_T=\mathcal{H}_t]$$
$$-\int_0^{T-t}\Bigg[\Big(\lambda_x(t+\delta|\mathcal{H}_{t+\delta}=\mathcal{H}_t)\mathbb{E}_P[Y|\mathcal{H}_t\cup(t+\delta,x)]+$$
$$\lambda_a(t+\delta|\mathcal{H}_{t+\delta}=\mathcal{H}_t)\mathbb{E}_P[Y|\mathcal{H}_t\cup(t+\delta,a)]\Big)\cdot$$
$$\exp\{-\int_t^{t+\delta}\lambda_\bullet(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\}\cdot\left(1-\exp\{-\int_t^{t+\delta}\lambda^{\text{obs}}_a(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\}\right)\Bigg]\mathrm{d}\delta. \tag{9}$$

Note that we have,

$$1-\exp\{-\int_t^{t+\delta}\lambda^{\text{obs}}_a(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\} =$$
$$\int_t^{t+\delta}\lambda^{\text{obs}}_a(s|\mathcal{H}_s=\mathcal{H}_t)\exp\{-\int_t^s \lambda^{\text{obs}}_a(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}u\}\mathrm{d}s, \tag{10}$$

because the left-hand-side is 1 minus the probability that N^{obs}_a does not jump in the interval $(t, t+\delta]$, and the integration on the right hand side is the probability that the process jumps at least once (where the first jump is at time s).

Next, we write the first item of eq. (3) to see that it cancels the residual above in eq. (9).

$$\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\left[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}^a_t(\cdot|\mathcal{H})}\left[\mathbf{1}\left[\delta^{\text{obs}}_a(t)<\delta_a(t)\wedge\delta_v(t)\right]\cdot\mathbb{E}\left[Y|\mathcal{H}_{t+\delta^{\text{obs}}_a}=\mathcal{H}_t\right]\right]\right] =$$
$$\int_0^{T-t}\lambda^{\text{obs}}_a(t+\delta|\mathcal{H}_{t+\delta}=\mathcal{H}_t)\exp\{-\int_t^{t+\delta}\lambda^{\text{obs}}_a(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\}$$
$$\exp\{-\int_t^{t+\delta}\lambda_\bullet(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\}\mathbb{E}_P\left[Y|\mathcal{H}_{t+\delta}=\mathcal{H}_t\right]\mathrm{d}\delta. \tag{11}$$

We expand $\mathbb{E}_P\left[Y|\mathcal{H}_{t+\delta}=\mathcal{H}_t\right]$ again by towering expectations w.r.t to the first jump after $t+\delta$,

$$\mathbb{E}_P[Y|\mathcal{H}_{t+\delta}=\mathcal{H}_t] = \int_{t+\delta}^T\Big(\lambda_a(s|\mathcal{H}_s=\mathcal{H}_t)\mathbb{E}_P\left[Y|\mathcal{H}_s=\mathcal{H}_t\cup(s,a)\right]+$$
$$\lambda_v(s|\mathcal{H}_s=\mathcal{H}_t)\mathbb{E}_P\left[Y|\mathcal{H}_s=\mathcal{H}_t\cup(s,v)\right]\Big)\exp\{-\int_{t+\delta}^s\lambda_\bullet(u|\mathcal{H}_u=\mathcal{H}_t)\mathrm{d}u\}\mathrm{d}s$$
$$+\,\mathbb{E}_P[Y|\mathcal{H}_T=\mathcal{H}_t]\exp\{-\int_{t+\delta}^T\lambda_\bullet(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\}$$

Plugging this into eq. (11) and rearranging the integration order we arrive at

$$\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\left[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}_t^a(\cdot|\mathcal{H})}\left[\mathbf{1}\left[\delta_a^{\text{obs}}(t)<\delta_a(t)\wedge\delta_v(t)\right]\mathbb{E}\left[Y|\mathcal{H}_{t+\delta_a^{\text{obs}}}=\mathcal{H}_t\right]\right]\right]=$$
$$\int_0^{T-t}\Big(\lambda_a(t+\delta|\mathcal{H}_{t+\delta}=\mathcal{H}_t)\mathbb{E}_P\left[Y|\mathcal{H}_{t_\delta}=\mathcal{H}_t\cup(t_\delta,a)\right]+$$
$$\lambda_x(t+\delta|\mathcal{H}_{t+\delta}=\mathcal{H}_t)\mathbb{E}_P\left[Y|\mathcal{H}_{t+\delta}=\mathcal{H}_t\cup(t+\delta,x)\right]\Big)$$
$$\exp\{-\int_t^{t+\delta}\lambda_\bullet(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\}\left(\int_t^{t+\delta}\lambda_a^{\text{obs}}(s|\mathcal{H}_s\mathcal{H}_t)\exp\{-\int_t^s\lambda_a^{\text{obs}}(u|\mathcal{H}_u=\mathcal{H}_t)\mathrm{d}u\}\mathrm{d}s\right)\mathrm{d}\delta$$
$$+\left(\int_0^{T-t}\lambda_a^{\text{obs}}(t+\delta|\mathcal{H}_{t+\delta}=\mathcal{H}_t)\exp\{-\int_t^{t+\delta}\lambda_a^{\text{obs}}(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\}\right)\cdot$$
$$\mathbb{E}_P[Y|\mathcal{H}_T=\mathcal{H}_t]\exp-\int_t^T\lambda_\bullet(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s=$$
$$\int_0^{T-t}\Big(\lambda_a(t+\delta|\mathcal{H}_{t+\delta}=\mathcal{H}_t)\mathbb{E}_P\left[Y|\mathcal{H}_{t_\delta}=\mathcal{H}_t\cup(t_\delta,a)\right]+$$
$$\lambda_x(t+\delta|\mathcal{H}_{t+\delta}=\mathcal{H}_t)\mathbb{E}_P\left[Y|\mathcal{H}_{t+\delta}=\mathcal{H}_t\cup(t+\delta,x)\right]\Big)$$
$$\exp\{-\int_t^{t+\delta}\lambda_\bullet(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\}\left(1-\exp\{-\int_t^{t+\delta}\lambda_a^{\text{obs}}(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\}\right)\mathrm{d}\delta$$
$$+\left(1-\exp\{-\int_t^T\lambda_a^{\text{obs}}(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\}\right)\cdot\mathbb{E}_P[Y|\mathcal{H}_T=\mathcal{H}_t]\exp\{-\int_t^T\lambda_\bullet(s|\mathcal{H}_s=\mathcal{H}_t)\mathrm{d}s\}.$$

In the last equality we plugged in eq. (10). Now it can be seen that the above expression cancels with the residual of eq. (9), which means that eq. (3) holds as claimed. ☐

Next we prove a lemma from which theorem 1 follows directly. We first state the lemma below.

Lemma 2. *For any $\mathcal{H}$ and $t\in[0,T)$, define $\delta_{\mathcal{H}}^k(t)>0$ such that $t+\delta_{\mathcal{H}}^k(t)$ is the time of the k-th event in $\mathcal{H}$ after t, and defining $\delta_{\mathcal{H}_v}^0(t)=0$ as an edge case. That is, assuming $\mathcal{H}=\{(t_j,v_j)\}_{j\in\mathbb{N}}$ then $\delta_{\mathcal{H}}^k(t):=\min\{t_j-t:t_{j-k+1}>t\}$. For all $d\in\mathbb{N}_+$ we have that*

$$\mathbb{E}_P[Y|\mathcal{H}_t]=\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}_t^a(\cdot|\mathcal{H})}\Big[\tag{12}$$
$$\sum_{k=1}^d\Big(\mathbf{1}\left[\delta_{\mathcal{H}_v}^{k-1}(t)\le\delta_{\mathcal{H}_a}(t)<\delta_{\mathcal{H}_v}^k(t)\wedge\delta_{\mathcal{H}_a}^{\text{obs}}(t)\wedge T-t\right]\mathbb{E}_P\left[Y|\widetilde{\mathcal{H}}_{t+\delta_{\mathcal{H}_a}(t)}\right]$$
$$+\mathbf{1}\left[\delta_{\mathcal{H}_v}^{k-1}(t)\le\delta_{\mathcal{H}_a}^{\text{obs}}(t)<t+\delta_{\mathcal{H}_v}^k(t)\wedge\delta_{\mathcal{H}_a}(t)\wedge T-t\right]\mathbb{E}_P\left[Y|\widetilde{\mathcal{H}}_{t+\delta_{\mathcal{H}_a^{\text{obs}}}(t)}\right]$$
$$+\mathbf{1}\left[T-t<\delta_{\mathcal{H}_v}^k(t)\wedge\delta_{\mathcal{H}_a}^{\text{obs}}(t)\wedge\delta_{\mathcal{H}_a}(t)\right]\mathbb{E}_P[Y|\mathcal{H}_T=\mathcal{H}_t]\Big)$$
$$+\mathbf{1}\left[\delta_{\mathcal{H}_v}^d(t)<\delta_{\mathcal{H}_a}^{\text{obs}}(t)\wedge\delta_{\mathcal{H}_a}(t)\wedge T-t\right]\mathbb{E}_P\left[Y|\widetilde{\mathcal{H}}_{t+\delta_{\mathcal{H}_v}^d(t)}\right]\Big]\Bigg].$$

Now let us recall theorem 1 and prove it, assuming that lemma 2 holds. Then we will prove the lemma and complete the proofs of our claims.

Theorem 1. *Let P,P_{obs} be multivariate marked decision point processes, $t\in[0,T)$, and $\mathcal{H}_t$ a list of events up to time t. For any trajectory $\mathcal{H}$, we let $\widetilde{P}_t^a(\cdot|\mathcal{H}),\delta_{\widetilde{\mathcal{H}}}(t),\delta_{\mathcal{H}}(t)$ as in definition 4. Under Assumptions 1 and 2, we have that*

$$\mathbb{E}_P\left[Y|\mathcal{H}_t\right]=\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\left[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}_t^a(\cdot|\mathcal{H})}\left[\mathbb{E}_P\left[Y\Big|\widetilde{\mathcal{H}}_{t+\delta_{\mathcal{H}}(t)\wedge\delta_{\widetilde{\mathcal{H}}}(t)}\right]\right]\right]. \tag{2}$$

Proof of theorem 1. Examining eq. (12), we observe that the multipliers of the first 3 indicator functions condition on the histories at times of the first treatment after time t, being taken either from

$\mathcal{H}$ or $\widetilde{\mathcal{H}}$ (or conditioning on the complete trajectory, in case there were no treatments after time t and then $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ coincide). Taking $d \to \infty$, the probability of the event $\delta^d_{\mathcal{H}_v}(t) < \delta^{\text{obs}}_{\mathcal{H}_a}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T - t$ approaches 0 since we assumed the number of events in a trajectory is countable w.p. 1. Hence we are left with the summation over products of indicators on earliest treatment times and corresponding conditional expectations. definition 4 denotes $\delta_{\widetilde{\mathcal{H}}}(t) = \min\{s - t : s > t, (s, \cdot) \in \widetilde{\mathcal{H}}^a\}$ as the time of the earliest treatment in $\widetilde{\mathcal{H}}$ after time t, and likewise for $\delta_{\mathcal{H}}(t)$, hence eq. (2) coincides with eq. (12) as $d \to \infty$. □

Next, let us complete the proof of the required lemma that we assumed to hold.

Proof of lemma 2. For $d = 1$, eq. (12) is exactly eq. (3) which we already proved in lemma 1, and we will proceed by induction. Assume for some $d - 1 > 1$ that eq. (12) holds, and shorten the notation $t + \delta^{d-1}_{\mathcal{H}_v}(t)$ to t_{d-1} from now on for convenience. Using lemma 1 again, it also holds that

$$
\begin{aligned}
&\mathbb{E}_P\left[Y|\mathcal{H}_{t_{d-1}}\right] = \\
&\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_{t_{d-1}})}\Big[\qquad (13)\\
&\quad \mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}(\cdot|\mathcal{H})}\Big[\mathbf{1}\left[\delta_a(t_{d-1}) < \delta_v(t_{d-1}) \wedge \delta^{\text{obs}}_a(t_{d-1})\right]\mathbb{E}_P\left[Y|\mathcal{H}_t \cup (t + \delta_a(t_{d-1}), a)\right] + \\
&\qquad \mathbf{1}\left[\delta_v(t_{d-1}) < \delta_a(t_{d-1}) \wedge \delta^{\text{obs}}_a(t_{d-1})\right]\mathbb{E}_P\left[Y|\mathcal{H}_t \cup (t + \delta_v(t_{d-1}), v)\right] + \\
&\qquad \mathbf{1}\left[\delta^{\text{obs}}_a(t_{d-1}) < \delta_a(t_{d-1}) \wedge \delta_v(d-1)\right]\mathbb{E}_P\left[Y|\mathcal{H}_{t+\delta^{\text{obs}}_a(t_{d-1})} = \mathcal{H}_t\right] + \\
&\qquad \mathbf{1}\left[\delta^{\text{obs}}_a \wedge \delta_a \wedge \delta_v(d-1) > T - t\right]\mathbb{E}_P\left[Y|\mathcal{H}_T = \mathcal{H}_t\right]\Big]\Big] \qquad (14)
\end{aligned}
$$

Now let us write down the induction hypothesis,

$$
\begin{aligned}
\mathbb{E}_P[Y|\mathcal{H}_t] = \mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H})}\Big[\qquad (15)\\
\sum_{k=1}^{d-1}\Big(\mathbf{1}\left[\delta^{k-1}_{\mathcal{H}_v}(t) \le \delta_{\mathcal{H}_a}(t) < \delta^k_{\mathcal{H}_v}(t) \wedge \delta^{\text{obs}}_{\mathcal{H}_a}(t) \wedge T - t\right]\mathbb{E}_P\left[Y|\widetilde{\mathcal{H}}_{t+\delta_{\mathcal{H}_a}(t)}\right] \\
+ \mathbf{1}\left[\delta^{k-1}_{\mathcal{H}_v}(t) \le \delta^{\text{obs}}_{\mathcal{H}_a}(t) < t + \delta^k_{\mathcal{H}_v}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T - t\right]\mathbb{E}_P\left[Y|\widetilde{\mathcal{H}}_{t+\delta_{\mathcal{H}^{\text{obs}}_a}(t)}\right] \\
+ \mathbf{1}\left[T - t < \delta^k_{\mathcal{H}_v}(t) \wedge \delta^{\text{obs}}_{\mathcal{H}_a}(t) \wedge \delta_{\mathcal{H}_a}(t)\right]\mathbb{E}_P[Y|\mathcal{H}_T = \mathcal{H}_t]\Big) \\
+ \mathbf{1}\left[\delta^{d-1}_{\mathcal{H}_v}(t) \le \delta^{\text{obs}}_{\mathcal{H}_a}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T - t\right]\mathbb{E}_P\left[Y|\widetilde{\mathcal{H}}_{t+\delta^{d-1}_{\mathcal{H}_v}(t)}\right]\Big]\Big]
\end{aligned}
$$

Since the argument of the expectation $\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H})}[\cdot]]$ only contains events that occur in $(t, t + \delta^{d-1}_v(t)]$, we can condition the inner expectation on $\mathcal{H}$ stopped at $t + \delta^{d-1}_v(t)$, i.e. $\mathbb{E}_{\widetilde{\mathcal{H}}_{t+\delta^{d-1}_v(t)}\sim\widetilde{P}^a_t(\cdot|\mathcal{H}_{t+\delta^{d-1}_v(t)})}[\cdot]$ (this also requires noticing that $\widetilde{P}^a_t$ samples increments in $(t, t + \delta^{d-1}_v(t)]$ independently of events that occur after $t + \delta^{d-1}_v(t)$). Now for a similar reason we can sample $\mathcal{H}$ from P_{obs} and stop at the $d - 1$-th jump of N_v. Overall, below we replace $\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H})}[\cdot]]$ with $\mathbb{E}_{\mathcal{H}_{t+\delta^{d-1}_{\mathcal{H}_v}}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}[\mathbb{E}_{\widetilde{\mathcal{H}}_{t+\delta^{d-1}_{\mathcal{H}_v}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H}_{t+\delta^{d-1}_{\mathcal{H}_v}})}[\cdot]]$ in the last summand taken from eq. (15) and change again to the notation t_{d-1} used at the beginning of the proof to make terms a bit more compact.

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H})}\Big[\mathbf{1}\left[\delta^{d-1}_{\mathcal{H}_v}(t) \le \delta^{\text{obs}}_{\mathcal{H}_a}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T\right]\mathbb{E}_P\left[Y|\widetilde{\mathcal{H}}_{t+\delta^{d-1}_{\mathcal{H}_v}(t)}\right]\Big]\Big] \\
=&\mathbb{E}_{\mathcal{H}_{t+\delta^{d-1}_{\mathcal{H}_v}}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}_{t+\delta^{d-1}_{\mathcal{H}_v}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H}_{t+\delta^{d-1}_{\mathcal{H}_v}})}\Big[\\
&\mathbf{1}\left[\delta^{d-1}_{\mathcal{H}_v}(t) \le \delta^{\text{obs}}_{\mathcal{H}_a}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T\right]\mathbb{E}_P\left[Y|\mathcal{H}_{t+\delta^{d-1}_{\mathcal{H}_v}(t)}\right]\Big]\Big] \\
=&\mathbb{E}_{\mathcal{H}_{t_{d-1}}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}_{t_{d-1}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H}_{t_{d-1}})}\Big[\mathbf{1}\left[\delta^{d-1}_{\mathcal{H}_v}(t) \le \delta^{\text{obs}}_{\mathcal{H}_a}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T\right]\mathbb{E}_P\left[Y|\mathcal{H}_{t_{d-1}}\right]\Big]\Big].
\end{aligned}
$$

The first transition also replaces conditioning on $\widetilde{\mathcal{H}}_{t+\delta_v^{d-1}(t)}$ with $\mathcal{H}_{t+\delta_v^{d-1}(t)}$ which holds since when $\mathbf{1}\left[\delta_{\mathcal{H}_v}^{d-1}(t) \le \delta_{\mathcal{H}_a}^{\text{obs}}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T\right] = 1$ we know that N_a, N_a^{obs} do not jump in $(t, t_{\delta_{\mathcal{H}_v}^{d-1}}]$, and hence these histories are equal. Next we plug-in eq. (13) into the last item.

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}_t^a(\cdot|\mathcal{H})}\Big[\mathbf{1}\big[\delta_{\mathcal{H}_v}^{d-1}(t) \le \delta_{\mathcal{H}_a}^{\text{obs}}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T-t\big]\mathbb{E}_P\big[Y|\widetilde{\mathcal{H}}_{t+\delta_{\mathcal{H}_v}^{d-1}(t)}\big]\Big]\Big] = \\
&\mathbb{E}_{\mathcal{H}_{t_{d-1}}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}_{t_{d-1}}\sim \widetilde{P}_t^a(\cdot|\mathcal{H}_{t_{d-1}})}\Big[\mathbf{1}\big[\delta_{\mathcal{H}_v}^{d-1}(t) \le \delta_{\mathcal{H}_a}^{\text{obs}}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T-t\big]\cdot \\
&\quad \mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_{t_{d-1}})}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}_{t_{d-1}}^a(\cdot|\mathcal{H})}\Big[\\
&\quad \mathbf{1}\big[\delta_a(t_{d-1}) < \delta_v(t_{d-1}) \wedge \delta_a^{\text{obs}}(t_{d-1}) \wedge T - t_{d-1}\big]\cdot \mathbb{E}_P\left[Y|\mathcal{H}_t \cup (t+\delta_a(t_{d-1}), a)\right] + \\
&\quad \mathbf{1}\big[\delta_v(t_{d-1}) < \delta_a(t_{d-1}) \wedge \delta_a^{\text{obs}}(t_{d-1}) \wedge T - t_{d-1}\big]\cdot \mathbb{E}_P\left[Y|\mathcal{H}_t \cup (t+\delta_v(t_{d-1}), v)\right] + \\
&\quad \mathbf{1}\big[\delta_a^{\text{obs}}(t_{d-1}) < \delta_a(t_{d-1}) \wedge \delta_v(d-1) \wedge T - t_{d-1}\big]\cdot \mathbb{E}_P\left[Y|\mathcal{H}_{t+\delta_a^{\text{obs}}(t_{d-1})} = \mathcal{H}_{t_{d-1}}\right] + \\
&\quad \mathbf{1}\big[\delta_a^{\text{obs}} \vee \delta_a \vee \delta_v(t_{d-1}) > T - t_{d-1}\big]\cdot \mathbb{E}_P\left[Y|\mathcal{H}_T = \mathcal{H}_{t_{d-1}}\right]\Big]\Big]\Big]\Big]
\end{aligned}
$$

Next we pull the expectations $\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_{t_{d-1}})}[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}_{t_{d-1}}^a(\cdot|\mathcal{H})}[\cdot]]$ outside.

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}_t^a(\cdot|\mathcal{H})}\Big[\mathbf{1}\big[\delta_{\mathcal{H}_v}^{d-1}(t) \le \delta_{\mathcal{H}_a}^{\text{obs}}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T-t\big]\mathbb{E}_P\big[Y|\widetilde{\mathcal{H}}_{t+\delta_{\mathcal{H}_v}^{d-1}(t)}\big]\Big]\Big] = \\
&\mathbb{E}_{\mathcal{H}_{t_{d-1}}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_{t_{d-1}})}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}_{t_{d-1}}^a(\cdot|\mathcal{H})}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}_{t_{d-1}}\sim \widetilde{P}_t^a(\cdot|\mathcal{H}_{t_{d-1}})}\Big[\\
&\quad \mathbf{1}\big[\delta_{\mathcal{H}_v}^{d-1}(t) \le \delta_{\mathcal{H}_a}^{\text{obs}}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T-t\big]\cdot\Big(\\
&\quad \mathbf{1}\big[\delta_a(t_{d-1}) < \delta_v(t_{d-1}) \wedge \delta_a^{\text{obs}}(t_{d-1}) \wedge T - t_{d-1}\big]\cdot \mathbb{E}_P\left[Y|\mathcal{H}_t \cup (t+\delta_a(t_{d-1}), a)\right] + \\
&\quad \mathbf{1}\big[\delta_v(t_{d-1}) < \delta_a(t_{d-1}) \wedge \delta_a^{\text{obs}}(t_{d-1}) \wedge T - t_{d-1}\big]\cdot \mathbb{E}_P\left[Y|\mathcal{H}_t \cup (t+\delta_v(t_{d-1}), v)\right] + \\
&\quad \mathbf{1}\big[\delta_a^{\text{obs}}(t_{d-1}) < \delta_a(t_{d-1}) \wedge \delta_v(d-1) \wedge T - t_{d-1}\big]\cdot \mathbb{E}_P\left[Y|\mathcal{H}_{t+\delta_a^{\text{obs}}(t_{d-1})} = \mathcal{H}_{t_{d-1}}\right] + \\
&\quad \mathbf{1}\big[\delta_a^{\text{obs}} \vee \delta_a \vee \delta_v(t_{d-1}) > T - t_{d-1}\big]\cdot \mathbb{E}_P\left[Y|\mathcal{H}_T = \mathcal{H}_{t_{d-1}}\right]\Big)\Big]\Big]\Big]\Big] = \\
&\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}_t^a(\cdot|\mathcal{H})}\Big[\\
&\quad \mathbf{1}\big[\delta_{\mathcal{H}_v}^{d-1}(t) \le \delta_{\mathcal{H}_a}^{\text{obs}}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T-t\big]\cdot\Big(\\
&\quad \mathbf{1}\big[\delta_a(t_{d-1}) < \delta_v(t_{d-1}) \wedge \delta_a^{\text{obs}}(t_{d-1}) \wedge T - t_{d-1}\big]\cdot \mathbb{E}_P\left[Y|\mathcal{H}_t \cup (t+\delta_a(t_{d-1}), a)\right] + \\
&\quad \mathbf{1}\big[\delta_v(t_{d-1}) < \delta_a(t_{d-1}) \wedge \delta_a^{\text{obs}}(t_{d-1}) \wedge T - t_{d-1}\big]\cdot \mathbb{E}_P\left[Y|\mathcal{H}_t \cup (t+\delta_v(t_{d-1}), v)\right] + \\
&\quad \mathbf{1}\big[\delta_a^{\text{obs}}(t_{d-1}) < \delta_a(t_{d-1}) \wedge \delta_v(d-1) \wedge T - t_{d-1}\big]\cdot \mathbb{E}_P\left[Y|\mathcal{H}_{t+\delta_a^{\text{obs}}(t_{d-1})} = \mathcal{H}_{t_{d-1}}\right] + \\
&\quad \mathbf{1}\big[\delta_a^{\text{obs}} \vee \delta_a \vee \delta_v(t_{d-1}) > T - t_{d-1}\big]\cdot \mathbb{E}_P\left[Y|\mathcal{H}_T = \mathcal{H}_{t_{d-1}}\right]\Big)\Big]\Big].
\end{aligned}
$$

Now it is easy to simplify the multiples of indicator functions as conjunctions, for instance by observing that,

$$
\begin{aligned}
&\mathbf{1}\big[\delta_{\mathcal{H}_v}^{d-1}(t) \le \delta_{\mathcal{H}_a}^{\text{obs}}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T-t\big]\cdot \\
&\quad \mathbf{1}\big[\delta_a(t_{d-1}) < \delta_v(t_{d-1}) \wedge \delta_a^{\text{obs}}(t_{d-1}) \wedge T - t_{d-1}\big] = \mathbf{1}\big[\delta_v^{d-1} \le \delta_a(t) \le \delta_v^d \wedge \delta_a^{\text{obs}}(t) \wedge T - t\big].
\end{aligned}
$$

Simplifying the rest of the terms in a similar manner we arrive at

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}_t^a(\cdot|\mathcal{H})}\Big[\mathbf{1}\big[\delta_{\mathcal{H}_v}^{d-1}(t) \le \delta_{\mathcal{H}_a}^{\text{obs}}(t) \wedge \delta_{\mathcal{H}_a}(t) \wedge T-t\big]\mathbb{E}_P\big[Y|\widetilde{\mathcal{H}}_{t+\delta_{\mathcal{H}_v}^{d-1}(t)}\big]\Big]\Big] = \\
&\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Big[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}_t^a(\cdot|\mathcal{H})}\Big[
\end{aligned}
$$

$$\mathbf{1}\left[\delta_v^{d-1}(t) \leq \delta_a(t) \leq \delta_v^d(t) \wedge \delta_a^{\text{obs}}(t) \wedge T-t\right] \cdot \mathbb{E}_P\left[Y|\mathcal{H}_{t_{d-1}} \cup (t+\delta_a(t), a)\right] +$$
$$\mathbf{1}\left[\delta_v^{d}(t) \leq \delta_a(t) \wedge \delta_a^{\text{obs}}(t) \wedge T-t\right] \cdot \mathbb{E}_P\left[Y|\mathcal{H}_{t_{d-1}} \cup (t_{d-1}+\delta_v(t_{d-1}), v)\right] +$$
$$\mathbf{1}\left[\delta_v^{d-1}(t) \leq \delta_a^{\text{obs}}(t) < \delta_v^d(t) \wedge \delta_a(t) \wedge T-t\right] \cdot \mathbb{E}_P\left[Y|\mathcal{H}_{t_{d-1}} \cup (t+\delta_a^{\text{obs}}(t), a)\right] +$$
$$\mathbf{1}\left[\delta_a^{\text{obs}}(t_{d-1}) \vee \delta_a(t_{d-1}) \vee \delta_v(t_{d-1}) > T-t_{d-1}\right] \cdot \mathbb{E}_P\left[Y|\mathcal{H}_T = \mathcal{H}_{t_{d-1}}\right]\Bigg]\Bigg] =$$

$$\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}_t^a(\cdot|\mathcal{H})}\Bigg[$$
$$\mathbf{1}\left[\delta_v^{d-1}(t) \leq \delta_a(t) \leq \delta_v^d(t) \wedge \delta_a^{\text{obs}}(t) \wedge T-t\right] \cdot \mathbb{E}_P\left[Y|\mathcal{H}_{t_{d-1}} \cup (t+\delta_a(t), a)\right] +$$
$$\mathbf{1}\left[\delta_v^{d}(t) \leq \delta_a(t) \wedge \delta_a^{\text{obs}}(t) \wedge T-t\right] \cdot \mathbb{E}_P\left[Y|\mathcal{H}_{t_{d-1}} \cup (t_{d-1}+\delta_v(t_{d-1}), v)\right] +$$
$$\mathbf{1}\left[\delta_v^{d-1}(t) \leq \delta_a^{\text{obs}}(t) < \delta_v^d(t) \wedge \delta_a(t) \wedge T-t\right] \cdot \mathbb{E}_P\left[Y|\mathcal{H}_{t_{d-1}} \cup (t+\delta_a^{\text{obs}}(t), a)\right] +$$
$$\mathbf{1}\left[\delta_a^{\text{obs}}(t) \vee \delta_a(t) \vee \delta_v^d(t) > T-t\right] \cdot \mathbb{E}_P\left[Y|\mathcal{H}_T = \mathcal{H}_{t_{d-1}}\right]\Bigg]\Bigg].$$

Plugging this into the induction hypothesis at eq. (15) we arrive at the desired identity. □

B.3 Discrete Time Version

For the discrete time version we keep a similar notation, but take time increments of 1 and call the target policy π, which takes a history of the process and outputs a distribution over possible treatments. The trajectory $\mathcal{H}$ now simplifies to the form $\{(\mathbf{x}_1, \mathbf{y}_1, \mathbf{a}_1), (\mathbf{x}_2, \mathbf{y}_2, \mathbf{a}_2), \ldots, (\mathbf{x}_T, \mathbf{y}_T, \mathbf{a}_T)\}$ and similarly for the history $\mathcal{H}_t$. The analogous claim to theorem 1 for these decision processes follows from the lemma we prove below by setting $d = T$.

Lemma 3. *For any $\mathcal{H}$, $t \in [0, T)$ and $1 \leq d \leq T-t$ such that $\mathcal{H}_t \in \operatorname{supp}(P)$ we have that*

$$\mathbb{E}_P[Y|\mathcal{H}_t] = \mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}_t^a(\cdot|\mathcal{H})}\Bigg[\sum_{k=1}^{d}\Bigg(\mathbf{1}_{\tilde{a}_{t+k}\neq a_{t+k}}\prod_{i=1}^{k-1}\mathbf{1}_{\tilde{a}_{t+k-i}=a_{t+k-i}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+k}]\Bigg) + \prod_{k=1}^{d}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+d}]\Bigg]\Bigg].$$

Note that for $d = 1$, we define $\prod_{i=1}^{k-1}\mathbf{1}_{\tilde{a}_{t+k-i}=a_{t+k-i}} = 1$.

Proof. We will prove this by induction on d. The base case for $d = 1$ follows from some simple manipulations and the equality $P(X_{t+1}, Y_{t+1}|\mathcal{H}_t) = P_{\text{obs}}(X_{t+1}, Y_{t+1}|\mathcal{H}_t)$. To see this we first claim that

$$\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}_t^a(\cdot|\mathcal{H})}\Bigg[\mathbf{1}_{\tilde{a}_{t+1}\neq a_{t+1}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}] + \mathbf{1}_{\tilde{a}_{t+1}=a_{t+1}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}]\Bigg]\Bigg] = \tag{16}$$
$$\mathbb{E}_{\mathcal{H}_{t+1}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}\sim \widetilde{P}_t^a(\cdot|\mathcal{H}_{t+1})}\Bigg[\mathbf{1}_{\tilde{a}_{t+1}\neq a_{t+1}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}] + \mathbf{1}_{\tilde{a}_{t+1}=a_{t+1}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}]\Bigg]\Bigg].$$

The difference between the two sides is that we switch $\mathcal{H}$ with $\mathcal{H}_{t+1}$ both for sampling $\mathcal{H}_{t+1} \sim P_{\text{obs}}(\cdot|\mathcal{H}_t)$, and also in the conditioning $\widetilde{P}_t^a(\cdot|\mathcal{H}_{t+1})$. This equality holds because the only treatment sampled from $\widetilde{P}_t^a$ that appears inside the expectation is $\tilde{a}_{t+1}$, and it is sampled independently from the future of $\mathcal{H}$ beyond time $t+1$. Hence we can drop the conditioning on events after time $t+1$, then the future of $\mathcal{H}$ after time $t+1$ marginalizes with the expectation $\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}[\cdot]$.

Next we note that the sampling $\widetilde{\mathcal{H}} \sim \widetilde{P}_t^a(\cdot|\mathcal{H}_{t+1})$ can be replaced with $a_{t+1} \sim \pi(\cdot|\mathcal{H}_t, X_t, Y_t)$. Again, this is since only $\tilde{a}_{t+1}$ appears inside the expectation, and because a_{t+1} is sample independently from a_{t+1}^{obs} sampled from the outside expectation. This leaves us with the left hand side of the following equality,

$$\mathbb{E}_{\mathbf{x}_{t+1},\mathbf{y}_{t+1}\sim P(\cdot|\mathcal{H}_t), a_{t+1}^{\text{obs}}\sim \pi_{\text{obs}}(\cdot|\mathcal{H}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1})}\Bigg[$$

$$\mathbb{E}_{\tilde{a}_{t+1}\sim\pi(\cdot|\mathcal{H}_t,\mathbf{x}_{t+1},\mathbf{y}_{t+1})}\left[\mathbf{1}_{\tilde{a}_{t+1}\neq a_{t+1}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}]+\mathbf{1}_{\tilde{a}_{t+1}=a_{t+1}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}]\right]\Bigg] =$$

$$\mathbb{E}_{\mathbf{x}_{t+1},\mathbf{y}_{t+1}\sim P(\cdot|\mathcal{H}_t),\tilde{a}_{t+1}\sim\pi(\cdot|\mathcal{H}_t,\mathbf{x}_{t+1},\mathbf{y}_{t+1})}\Bigg[$$

$$\mathbb{E}_{a^{\text{obs}}_{t+1}\sim\pi_{\text{obs}}(\cdot|\mathcal{H}_t,\mathbf{x}_{t+1},\mathbf{y}_{t+1})}\left[\mathbf{1}_{\tilde{a}_{t+1}\neq a_{t+1}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}]+\mathbf{1}_{\tilde{a}_{t+1}=a_{t+1}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}]\right]\Bigg]$$

The equality holds since we can switch the order of expectations, as $\tilde{a}_{t+1}$ is not sampled conditionally on a_{t+1}. Finally we note that clearly for any two $\tilde{a}_{t+1}, a_{t+1}$ it either holds that $\mathbf{1}_{\tilde{a}_{t+1}\neq a_{t+1}}=1$, or $\mathbf{1}_{\tilde{a}_{t+1}=a_{t+1}}=1$. Hence the inside of the expectation is simplified to $\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}]$. Equating this to the left hand side of eq. (16), we arrive at the desired result.

$$\begin{aligned}
&\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\left[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H})}\left[\mathbf{1}_{\tilde{a}_{t+1}\neq a_{t+1}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}]+\mathbf{1}_{\tilde{a}_{t+1}=a_{t+1}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}]\right]\right]=\\
&\mathbb{E}_{\mathbf{x}_{t+1},\mathbf{y}_{t+1}\sim P(\cdot|\mathcal{H}_t),\tilde{a}_{t+1}\sim\pi(\cdot|\mathcal{H}_t,\mathbf{x}_{t+1},\mathbf{y}_{t+1})}\left[\mathbb{E}_{a^{\text{obs}}_{t+1}\sim\pi_{\text{obs}}(\cdot|\mathcal{H}_t,\mathbf{x}_{t+1},\mathbf{y}_{t+1})}\left[\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}]\right]\right]=\\
&\mathbb{E}_{\mathcal{H}_{t+1}\sim P(\cdot|\mathcal{H}_t)}\left[\mathbb{E}_{a^{\text{obs}}_{t+1}\sim\pi_{\text{obs}}(\cdot|\mathcal{H}_t,\mathbf{x}_{t+1},\mathbf{y}_{t+1})}\left[\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}]\right]\right]=\\
&\mathbb{E}_{\mathcal{H}_{t+1}\sim P(\cdot|\mathcal{H}_t)}\left[\left[\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+1}]\right]\right]=\mathbb{E}_P\left[\mathbb{E}_P[Y|\mathcal{H}_t]\right].
\end{aligned}$$

Next, assume the claim holds for some $d-1$. From the same considerations we gave for the base case, we have that

$$\begin{aligned}
\mathbb{E}_P[Y|\mathcal{H}_{t+d-1}]=\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_{t+d-1})}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_{t+d-1}(\cdot|\mathcal{H})}\Bigg[&\mathbb{E}_P\left[Y|\mathcal{H}_{t+d}\right]\cdot\mathbf{1}_{\tilde{a}_{t+d}\neq a_{t+d}}\\
&+\mathbb{E}_P\left[Y|\mathcal{H}_{t+d}\right]\cdot\mathbf{1}_{\tilde{a}_{t+d}=a_{t+d}}\Bigg]\Bigg].
\end{aligned}\tag{17}$$

Now let us write down the induction hypothesis,

$$\begin{aligned}
\mathbb{E}_P[Y|\mathcal{H}_t]=\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H})}\Bigg[&\sum_{k=1}^{d-1}\left(\mathbf{1}_{\tilde{a}_{t+k}\neq a_{t+k}}\prod_{i=1}^{k-1}\mathbf{1}_{\tilde{a}_{t+k-i}=a_{t+k-i}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+k}]\right)\\
&+\prod_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+d-1}]\Bigg]\Bigg].
\end{aligned}\tag{18}$$

First we note that in eq. (18), the second appearance of the expectation $\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+d-1}]$ (i.e. the last item in the equation) is multiplied by $\prod_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}$, hence we have $\widetilde{\mathcal{H}}_{t+d-1}=\mathcal{H}_{t+d-1}$ whenever the item is non-zero. The first equality we will write below uses this, and then we plug in eq. (17).

$$\begin{aligned}
&\prod_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+d-1}]=\prod_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\mathbb{E}_P[Y|\mathcal{H}_{t+d-1}]=\\
&\quad\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_{t+d-1})}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_{t+d-1}(\cdot|\mathcal{H})}\Bigg[\mathbb{E}_P\left[Y|\widetilde{\mathcal{H}}_{t+d}\right]\cdot\mathbf{1}_{\tilde{a}_{t+d}\neq a_{t+d}}\prod_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}+\\
&\qquad\qquad\mathbb{E}_P\left[Y|\widetilde{\mathcal{H}}_{t+d}\right]\cdot\prod_{k=1}^{d}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\Bigg]\Bigg].
\end{aligned}\tag{19}$$

As we did for the first step, we now notice that for time larger than $t+d-1$, the only treatments and features that appear inside the first two expectations are from time $t+d$ (i.e. no later times appear). Hence we can condition only on $\mathcal{H}_{t+d}$ in the second expectation, i.e. $\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_{t+d-1}(\cdot|\mathcal{H})}[\cdot] = \mathbb{E}_{\widetilde{\mathcal{H}}_{t+d}\sim\widetilde{P}^a_{t+d-1}(\cdot|\mathcal{H}_{t+d})}[\cdot]$ (as $\tilde{a}_{t+d}$ is sampled conditionally on the past and not the future). Furthermore, we can also sample just $\mathcal{H}_{t+d}$ in the outer expectation, i.e. $\mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_{t+d-1})}[\cdot] = \mathbb{E}_{\mathcal{H}_{t+d}\sim P_{\text{obs}}(\cdot|\widetilde{\mathcal{H}}_{t+d-1})}[\cdot]$. We start by replacing this in eq. (19),

$$
\begin{aligned}
&\prod_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+d-1}] = \\
&\quad \mathbb{E}_{\mathcal{H}_{t+d}\sim P_{\text{obs}}(\cdot|\mathcal{H}_{t+d-1})}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}_{t+d}\sim\widetilde{P}^a_{t+d-1}(\cdot|\mathcal{H}_{t+d})}\Bigg[\mathbb{E}_P\Big[Y|\widetilde{\mathcal{H}}_{t+d}\Big]\cdot\mathbf{1}_{\tilde{a}_{t+d}\neq a_{t+d}}\prod_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}} + \\
&\qquad \mathbb{E}_P\Big[Y|\widetilde{\mathcal{H}}_{t+d}\Big]\cdot\prod_{k=1}^{d}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\Bigg]\Bigg].
\end{aligned}
$$

Next we note that following the same reasoning we used for the last step, in the argument on removing future times from the expectations, also holds for eq. (18) and time $t+d-1$. Hence we also replace this in eq. (18),

$$
\begin{aligned}
\mathbb{E}_P[Y|\mathcal{H}_t] &= \mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H})}\Bigg[\sum_{k=1}^{d-1}\Bigg(\mathbf{1}_{\tilde{a}_{t+k}\neq a_{t+k}}\prod_{i=1}^{k-1}\mathbf{1}_{\tilde{a}_{t+k-i}=a_{t+k-i}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+k}]\Bigg) \\
&\quad + \prod_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+d-1}]\Bigg]\Bigg] \\
&= \mathbb{E}_{\mathcal{H}_{t+d-1}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}_{t+d-1}\sim\widetilde{P}^a_t(\cdot|\mathcal{H}_{t+d-1})}\Bigg[\sum_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}\neq a_{t+k}}\prod_{i=1}^{k-1}\mathbf{1}_{\tilde{a}_{t+k-i}=a_{t+k-i}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+k}] \\
&\quad + \prod_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+d-1}]\Bigg]\Bigg] \\
&= \mathbb{E}_{\mathcal{H}_{t+d-1}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}_{t+d-1}\sim\widetilde{P}^a_t(\cdot|\mathcal{H}_{t+d-1})}\Bigg[\sum_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}\neq a_{t+k}}\prod_{i=1}^{k-1}\mathbf{1}_{\tilde{a}_{t+k-i}=a_{t+k-i}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+k}] \\
&\quad + \mathbb{E}_{\mathcal{H}_{t+d}\sim P_{\text{obs}}(\cdot|\mathcal{H}_{t+d-1})}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}_{t+d}\sim\widetilde{P}^a_{t+d-1}(\cdot|\mathcal{H}_{t+d})}\Bigg[\mathbb{E}_P\Big[Y|\widetilde{\mathcal{H}}_{t+d}\Big]\cdot\mathbf{1}_{\tilde{a}_{t+d}\neq a_{t+d}}\prod_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}} + \\
&\quad \mathbb{E}_P\Big[Y|\widetilde{\mathcal{H}}_{t+d}\Big]\cdot\prod_{k=1}^{d}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\Bigg]\Bigg]\Bigg]\Bigg] \\
&= \mathbb{E}_{\mathcal{H}_{t+d}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}_{t+d}\sim\widetilde{P}^a_t(\cdot|\mathcal{H}_{t+d})}\Bigg[\sum_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}\neq a_{t+k}}\prod_{i=1}^{k-1}\mathbf{1}_{\tilde{a}_{t+k-i}=a_{t+k-i}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+k}] \\
&\quad + \mathbb{E}_{\widetilde{\mathcal{H}}_{t+d}\sim\widetilde{P}^a_{t+d-1}(\cdot|\mathcal{H}_{t+d})}\Big[Y|\widetilde{\mathcal{H}}_{t+d}\Big]\cdot\mathbf{1}_{\tilde{a}_{t+d}\neq a_{t+d}}\prod_{k=1}^{d-1}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}} \\
&\quad + \mathbb{E}_P\Big[Y|\widetilde{\mathcal{H}}_{t+d}\Big]\cdot\prod_{k=1}^{d}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\Bigg]\Bigg] \\
&= \mathbb{E}_{\mathcal{H}_{t+d}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}_{t+d}\sim\widetilde{P}^a_t(\cdot|\mathcal{H}_{t+d})}\Bigg[\sum_{k=1}^{d}\mathbf{1}_{\tilde{a}_{t+k}\neq a_{t+k}}\prod_{i=1}^{k-1}\mathbf{1}_{\tilde{a}_{t+k-i}=a_{t+k-i}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+k}] \\
&\quad + \mathbb{E}_P\Big[Y|\widetilde{\mathcal{H}}_{t+d}\Big]\cdot\prod_{k=1}^{d}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\Bigg]\Bigg] \\
&= \mathbb{E}_{\mathcal{H}\sim P_{\text{obs}}(\cdot|\mathcal{H}_t)}\Bigg[\mathbb{E}_{\widetilde{\mathcal{H}}\sim\widetilde{P}^a_t(\cdot|\mathcal{H})}\Bigg[\sum_{k=1}^{d}\mathbf{1}_{\tilde{a}_{t+k}\neq a_{t+k}}\prod_{i=1}^{k-1}\mathbf{1}_{\tilde{a}_{t+k-i}=a_{t+k-i}}\mathbb{E}_P[Y|\widetilde{\mathcal{H}}_{t+k}]
\end{aligned}
$$

$$+\prod_{k=1}^{d}\mathbf{1}_{\tilde{a}_{t+k}=a_{t+k}}\cdot\mathbb{E}_P\left[Y|\widetilde{\mathcal{H}}_{t+d}\right]\Bigg]\Bigg].$$

The equality between the first and last expression is exactly our claim. The first equality is simply the induction hypothesis eq. (18); in the second one we replace the times in expectations as mentioned before the equation; the third equality plugs in eq. (19); in the fourth equality we pull out the expectations over $\mathcal{H}_{t+d}$ and $\widetilde{\mathcal{H}}_{t+d}$ and use the towering property of conditional expectations to write the expectations w.r.t $\mathcal{H}_{t+d}, \widetilde{\mathcal{H}}_{t+d}$ fully on the outside; the fifth equality simply gather items to a sum; finally, the last step adds back in the expectation on the future $\mathcal{H}, \widetilde{\mathcal{H}}$ after time $t+d$. The last step is simply done to get back the form of our claim, as discussed earlier it is valid due to the law of total probability and that no future features or treatments appear inside the expectation. □

C Additional Discussion on Related Work

As outlined in section 4, several techniques have been proposed for scalable estimation of causal effects in sequential decision-making, with more limited development in the case of irregular observation times. One set of approaches (Bica et al., 2020; Lim, 2018; Melnychuk et al., 2022), that only apply to discrete time processes and static policies, can be roughly characterized as follows. A prediction model $f(\mathcal{H}_t, \mathcal{H}^a_T; \boldsymbol{\theta})$ for the outcome Y is learned, where $\mathcal{H}_t$ is the observed history of events and $\mathcal{H}^a_T$ is the set of future treatments we would like to reason about. That is, in our notation we would like $f(\mathcal{H}_t, \mathcal{H}^a_T; \boldsymbol{\theta})$ to estimate $\mathbb{E}_{P_{\mathcal{H}^a_T}}[Y|\mathcal{H}_t]$, where $P_{\mathcal{H}^a_T}$ assigns the treatments in $\mathcal{H}^a_T$ w.p. 1. In potential outcomes notation, this corresponds to $\mathbb{E}\left[Y^{\mathbf{a}}|\mathcal{H}_t\right]$, where $Y^{\mathbf{a}}$ is a random variable that outputs the outcome under a set of static future treatments $\mathbf{a}$. All methods involve learning a representation of history $\mathbf{Z}_t = \phi(\mathcal{H}_t; \eta)$, and combine two important elements for achieving correct estimates.

1. To yield correct causal estimates under an observational distribution that is not sequentially randomized, methods either estimate products of propensity weights (Lim, 2018), or add a loss to make $\mathbf{Z}_t$ non-predictive of the treatment A_t, ϕ is then called a balancing representation.
2. To facilitate prediction of Y under a set of future treatments in the interval $(t, T]$, either ϕ is taken as a sequence model, or a separate "decoder" network is learned (Bica et al., 2020; Lim, 2018). A sequence model is trained with inputs where $\mathcal{H}^{x,y}_{i,T}\setminus\mathcal{H}^{x,y}_{i,t}$, i.e. the covariates in a projection interval $(t, T]$ are masked, while the decoder takes $\mathbf{Z}_t$ and $\mathcal{H}^a_T$ as inputs. Both are trained to predict the outcome Y and serve as an estimator for $\mathbb{E}_{P_{\text{obs}}}[Y|\mathbf{Z}_t, \mathcal{H}^a_T]$, which recovers the correct causal effect under sequential exchangeability. Notice that these techniques preclude estimation with dynamic treatments, i.e. policies.

For irregular sampling, Seedat et al. (2022) follow the same recipe but choose a neural CDE architecture. This interpolates the latent path $\mathbf{Z}_t$ in intervals between jump times of the processes, and is shown empirically to be more suitable when working with data that is subsampled from a complete trajectory of features in continuous time. The solution is not equipped to estimate interventions on continuous treatment times (in our notation, λ_a). As mentioned earlier, Vanderschueren et al. (2023) handle informative sampling times with inverse weighting based on the intensity λ. However, this is a different problem setting from ours, as they do not seek to intervene on sampling times but wish to solve a case where outcomes, features and treatments always jump simultaneously. In our setting, intervening on λ^a with such simultaneous jumps would result in $\lambda^{x,y}_{\text{obs}} \neq \lambda^{x,y}$, which is not the focus of our work. Finally, we also note the required assumption for causal validity that is claimed in these works is roughly $P_{\text{obs}}(A_t = a_t|\mathcal{H}_T) = P_{\text{obs}}(A_t = a_t|\mathcal{H}_t)$. The assumption is unreasonable since $\mathcal{H}_T$ includes future *factual* outcomes that depend on the taken action, instead of the more standard exchangeability assumption that posits independence of *potential* outcomes.

The G-estimation solution of (Li et al., 2021) for discrete time decision processes fits models for both $\pi_{\text{obs}}(A(t)|\mathcal{H}_{t-1}, X(t))$, and $P_{\text{obs}}(X(t), Y(t)|\mathcal{H}_{t-1})$. Then at inference time, they replace π_{obs} with the desired policy π and estimate trajectories or conditional expectations of Y with monte-carlo simulations. A straightforward generalization of this approach to decision point processes can be devised by fitting the intensities λ_{obs} and replacing λ_a for inference. While we believe that this is an interesting direction for future work, we do not pursue it further in our experiments, since developing architectures and methods for learning generative models under irregular sampling deserves a dedicated and in-depth exploration.