# A Additional Experimental Results

Table 3: Experimental results (%) on **BTAD**. We compare the performance between SAC-BL algorithm and SAC-SBL based on the same trained SAC network. ↑ indicates larger values are better and vice versa.

| Category | Method | Image-level | | | Pixel-level | | |
|---|---|---|---|---|---|---|---|
| | | TPR ↑ | FPR ↓ | F1-score ↑ | TPR ↑ | FPR ↓ | F1-score ↑ |
| 01 | SAC-BL | 100.0 | 18.37 | 81.35 | 93.41 | 42.90 | 92.17 |
| | SAC-SBL | 100.0 | 15.33 | 84.00 | 93.13 | 38.87 | 95.76 |
| 02 | SAC-BL | 100.0 | 68.50 | 30.46 | 99.80 | 91.73 | 91.86 |
| | SAC-SBL | 100.0 | 62.00 | 37.25 | 99.78 | 85.40 | 97.32 |
| 03 | SAC-BL | 99.00 | 14.63 | 98.75 | 99.70 | 78.95 | 96.68 |
| | SAC-SBL | 99.00 | 14.63 | 98.75 | 99.70 | 76.68 | 99.77 |
| Average | SAC-BL | 99.67 | 33.83 | 70.19 | 97.63 | 71.19 | 93.57 |
| | SAC-SBL | 99.67 | 30.65 | 73.33 | 97.54 | 66.98 | 97.61 |

# B Prrofs

## B.1 Proof of Theorem 3.2

*Proof.* For the observations $\tilde{T}_1, \tilde{T}_2, \cdots, \tilde{T}_n$, denote

$$\hat{\Psi}_0(t) = \frac{1}{n_0} \sum_{i \in \mathcal{H}_0} \mathbf{1}(\tilde{T}_i \geq t) \quad \hat{\Psi}_1(t) = \frac{1}{n_1} \sum_{i \in \mathcal{H}_1} \mathbf{1}(\tilde{T}_i \geq t),$$

where $n_0 = |\mathcal{H}_0| = n - n^{1-\beta}$ and $n_1 = |\mathcal{H}_1| = n^{1-\beta}$. Then, the empirical survival function can be expressed as

$$\hat{\Psi}(t) = \frac{n_0}{n} \hat{\Psi}_0(t) + \frac{n_1}{n} \hat{\Psi}_1(t) = (1-\eta)\hat{\Psi}_0(t) + \eta \hat{\Psi}_1(t), \tag{9}$$

where $\eta = \frac{1}{n^\beta}$.

Let $\tilde{T}_{(i)}$ be the $i$-th order statistic of $\tilde{T}_1, \tilde{T}_2, \cdots, \tilde{T}_n$ from the largest to the smallest. Hence, we have $\hat{\Psi}(\tilde{T}_{(i)}) = \frac{i}{n}$. It follows that

$$\begin{aligned} \left\{ p^\gamma_{(i)} \leq \frac{\delta i}{n} \alpha \right\} &= \left\{ \Psi^\gamma(\tilde{T}_{(i)}) \leq \delta\alpha \hat{\Psi}(\tilde{T}_{(i)}) \right\} \\ &= \left\{ \tilde{T}_{(i)} \geq \Psi^{-1}\left( (\delta\alpha\hat{\Psi}(\tilde{T}_{(i)}))^{\gamma^{-1}} \right) \right\}, \end{aligned}$$

where $\Psi^{-1}(\cdot)$ is the inverse function of $\Psi(\cdot)$. According to the definition of $k_{BL}$, we have $\tilde{T}_{(i)} \geq \Psi^{-1}\left( (\delta\alpha\hat{\Psi}(\tilde{T}_{(i)}))^{\gamma^{-1}} \right)$ if $i \leq k_{BL}$, and $\tilde{T}_{(i)} < \Psi^{-1}\left( (\delta\alpha\hat{\Psi}(\tilde{T}_{(i)}))^{\gamma^{-1}} \right)$ if $i > k_{BL}$.

Note that $\hat{\Psi}(\cdot)$ is constant between two observations. Hence, there exists $\zeta_n$[1] $\in (\tilde{T}_{k_{BS}+1},\ \tilde{T}_{k_{BS}})$ such that

$$\zeta_n = \min\left\{ t : t \geq \Psi^{-1}\left( (\delta\alpha\hat{\Psi}(t))^{\gamma^{-1}} \right) \right\} = \min\left\{ t : t = \Psi^{-1}\left( (\delta\alpha\hat{\Psi}(t))^{\gamma^{-1}} \right) \right\} \tag{10}$$

In terms of notation $\zeta_n$, the FPR of SAC-BL algorithm can be represented as

$$\begin{aligned} \frac{|\mathcal{R}^c \cap \mathcal{H}_1|}{|\mathcal{H}_1|} &= \frac{1}{n_1} \sum_{i \in \mathcal{H}_1} \mathbf{1}\left( p^\gamma_{(i)} > \frac{\delta k_{BL}}{n} \alpha \right) \\ &= \frac{1}{n_1} \sum_{i \in \mathcal{H}_1} \mathbf{1}\left( \tilde{T}_i < \zeta_n \right) = 1 - \frac{1}{n_1} \sum_{i \in \mathcal{H}_1} \mathbf{1}\left( \tilde{T}_i \geq \zeta_n \right) = 1 - \hat{\Psi}_1(\zeta_n). \end{aligned}$$

The above analysis shows that we just need to prove that $\hat{\Psi}_1(\zeta_n)$ converges to 1 in probability. For this purpose, the following theorem is required.

[1] $\zeta_n$ depends on the observations $\tilde{T}_1, \tilde{T}_2, \cdots, \tilde{T}_n$.

**Theorem B.1** (Eicker (1979)). *Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables with the continuous survival function $\Psi(\cdot)$. Denote by $\hat{\Psi}(\cdot)$ the empirical survival function. Then, as $n \to \infty$, we have*

$$\frac{1}{\upsilon_n} \sup_t \frac{\hat{\Psi}(t) - \Psi(t)}{\sqrt{\Psi(t)(1-\Psi(t))}} \to 1$$

where $\upsilon_n = \sqrt{\frac{2\log\log n}{n}}$.

By Theorem B.1, we obtain

$$\hat{\Psi}_1(t) = \Psi(t) + O_p(\upsilon_{n_0}\sqrt{\Psi(t)(1-\Psi(t))})$$
$$\hat{\Psi}_1(t) = \Psi(t-\mu) + O_p(\upsilon_{n_1}\sqrt{\Psi(t-\mu)(1-\Psi(t-\mu))})$$

Then, the empirical function $\hat{\Psi}(t)$ in Eq. (9) can be represented as

$$\hat{\Psi}(t) = (1-\eta)\Psi(t) + \eta\Psi(t-\mu) + Q_n(t),$$

where

$$Q_n(t) = O_p\left((1-\eta)\upsilon_{n_0}\sqrt{\Psi(t)(1-\Psi(t))}\right) + O_p\left(\eta\upsilon_{n_1}\sqrt{\Psi(t-\mu)(1-\Psi(t-\mu))}\right).$$

Particularly, we conclude

$$\hat{\Psi}_1(\zeta_n) = \Psi(\zeta_n - \mu) + o_p(1),$$

since $\upsilon_{n_1}$ converges to 0 and $\sqrt{\Psi(\zeta_n-\mu)(1-\Psi(\zeta_n-\mu))}$ is bounded. Therefore, the rest is to show $\Psi(\zeta_n - \mu)$ converges to 1 in probability, equivalently, $\zeta_n - \mu \to -\infty$ in probability. Next, we focus on proving that for $\varepsilon > 0$,

$$\mathbb{P}(\zeta_n - \mu < -\varepsilon) \to 1$$

as $n \to \infty$.

Recall that $r > \beta$, take a number $\nu \in (\beta, r)$ such that $\gamma\nu > \beta$, and define $\rho_n = (\tau\nu\log n)^{1/\tau}$, then $\rho_n - \mu \to \infty$. Denote two events:

$$\mathrm{A} = \{\rho_n - \mu < -\varepsilon\}, \quad \mathrm{B} = \{\zeta_n \le \rho_n\}.$$

Obviously, we have $\mathbb{P}(A) = 1$ and $\mathrm{A} \cap \mathrm{B} \subset \{\zeta_n - \mu < -\varepsilon\}$. The above arguments indicate $\mathbb{P}(B) \le \mathbb{P}(\zeta_n - \mu < -\varepsilon)$, thus, it suffices to prove that $\mathbb{P}(\zeta_n \le \rho_n)$ converges to 1. Since $\Psi(\cdot) \in \mathcal{F}$, we have

$$\Psi(\rho_n) = \frac{1}{n^{\nu + o(1)}}.$$

Note that $\rho_n - \mu \to -\infty$, so $\Psi(\rho_n - \mu) \to 1$. It follows that

$$(1-\eta)\Psi(\rho_n) + \eta\Psi(\rho_n - \mu) \sim \frac{1}{n^\beta}$$

where $\eta = \frac{1}{n^\beta}$. Similarly, the simple analysis can derive that

$$O_p\left(\eta\upsilon_{n_1}\sqrt{\Psi(\rho_n-\mu)(1-\Psi(\rho_n-\mu))}\right) = o_p(\frac{1}{n^\beta})$$
$$O_p\left((1-\eta)\upsilon_{n_0}\sqrt{\Psi(\rho_n)(1-\Psi(\rho_n))}\right) = o_p(\frac{1}{n^\beta})$$

since $\kappa_{n_0} = o(\frac{1}{n^{\beta/2}})$. The above arguments indicate that $\hat{\Psi}(\rho_n) = \frac{1}{n^\beta} + o_p(\frac{1}{n^\beta})$. Note that $(\gamma)\nu - \beta > 0$ and

$$\frac{\frac{1}{n^\beta} + o_p(\frac{1}{n^\beta})}{\frac{1}{n^{\beta + o_p(\frac{1}{n^\beta})}}} = \frac{\frac{1}{n^\beta} + o_p(\frac{1}{n^\beta})}{\frac{1}{n^\beta}} \cdot \frac{\frac{1}{n^\beta}}{\frac{1}{n^{\beta + o_p(\frac{1}{n^\beta})}}} \xrightarrow{P} 1,$$

we conclude that

$$\frac{\Psi^\gamma(\rho_n)}{\hat{\Psi}(\rho_n)} = \frac{\frac{1}{n^{\gamma\nu + o_p(1)}}}{\frac{1}{n^{\beta + o_p(\frac{1}{n^\beta})}}} \cdot \frac{\frac{1}{n^{\beta + o_p(\frac{1}{n^\beta})}}}{\frac{1}{n^\beta} + o_p(\frac{1}{n^\beta})}$$
$$= \frac{1}{n^{\gamma\nu - \beta + o_p(1) + o_p(\frac{1}{n^\beta})}} \cdot \frac{\frac{1}{n^{\beta + o_p(\frac{1}{n^\beta})}}}{\frac{1}{n^\beta} + o_p(\frac{1}{n^\beta})} \xrightarrow{P} 0,$$

namely, for any $\varepsilon > 0$, we have

$$\mathbb{P}\left(\frac{\Psi^{\gamma}(\rho_n)}{\hat{\Psi}(\rho_n)} \le \varepsilon\right) \to 1. \tag{11}$$

Denote

$$\mathcal{S} := \left\{t : t \ge \Psi^{-1}\left((\delta\alpha\hat{\Psi}(t))^{\gamma^{-1}}\right)\right\} = \left\{t : \Psi^{\gamma}(t) \le \delta\alpha\hat{\Psi}(t)\right\}$$

By (11),

$$\mathbb{P}\left(\frac{\Psi^{\gamma}(\rho_n)}{\hat{\Psi}(\rho_n)} \le \delta\alpha\right) \to 1,$$

and $\mathbb{P}(\rho_n \in \mathcal{S}) = \mathbb{P}\left(\Psi^{\gamma}(\rho_n)\hat{\Psi}(\rho_n) \le \delta\alpha\hat{\Psi}(\rho_n)\right) \to 1,$. According to the definition of $\zeta_n$ in (10), if $\rho_n \in \mathcal{S}$, we have $\zeta_n \le \rho_n$. It follows that

$$1 \ge \mathbb{P}(\zeta_n \le \rho_n) \ge \mathbb{P}(\rho_n \in \mathcal{S}) \to 1,$$

Therefore, we have

$$\mathbb{P}(\zeta_n \le \rho_n) \to 1,$$

which completes the proof of Theorem 3.2. □

### B.2 Proof of Theorem 3.3

*Proof.* According to Theorem 3.2, we know

$$\frac{|\mathcal{R}^c \cap \mathcal{H}_1|}{|\mathcal{H}_1|} \to 0 \quad \text{in probability}$$

as $n \to \infty$. Note that $\frac{|\mathcal{R}^c \cap \mathcal{H}_1|}{|\mathcal{H}_1|} \le 1$ almost surely, Then, by bounded convergence theorem, we have

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{|\mathcal{R}^c \cap \mathcal{H}_1|}{|\mathcal{H}_1|}\right) = 0.$$

If $lim_{n\to\infty}\mathbb{E}\left(\frac{|\mathcal{R}^c \cap \mathcal{H}_1|}{|\mathcal{H}_1|}\right) = 0$, by Markov's inequality, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{|\mathcal{R}^c \cap \mathcal{H}_1|}{|\mathcal{H}_1|} > \varepsilon\right) \le \frac{\mathbb{E}\left(\frac{|\mathcal{R}^c \cap \mathcal{H}_1|}{|\mathcal{H}_1|}\right)}{\varepsilon} \to 0.$$

Hence, $\frac{|\mathcal{R}^c \cap \mathcal{H}_1|}{|\mathcal{H}_1|}$ converges 0 in probability. The above arguments demonstrate that

$$\frac{|\mathcal{R}^c \cap \mathcal{H}_1|}{|\mathcal{H}_1|} \xrightarrow{P} 0$$

if and only if

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{|\mathcal{R}^c \cap \mathcal{H}_1|}{|\mathcal{H}_1|}\right) = 0.$$

□

### B.3 Prrof of Theorem 4.3

*Proof.* According to the Definition 4.1, for any $p_i$, we have

$$\left\{\Upsilon_{\kappa(i)}(p_i) \le \xi_{\kappa(i)}\right\} \Leftrightarrow \left\{\Lambda \le \frac{\xi_{\kappa(i)}}{p_i^{\gamma}}\right\}.$$

It is easy to verify that

$$\left\{\xi_{\kappa(j)} \le \xi_i\right\} \Leftrightarrow \{\epsilon_j \le i\},$$

where $j \in [n]$ and $i \in \mathcal{B} := \{\kappa(1), \kappa(2), \ldots, \kappa(n)\}$. Define $\mathcal{G}_i := \{j \in [n] : \kappa(j) \le i\}$ where $i \in \mathcal{B}$. According to the definition of $\kappa(i)$, we have $|\mathcal{G}_i| = i$ where $i \in \mathcal{B}$ and $\mathcal{G}_i \subset \mathcal{G}_j$ for $i < j$

where $i, j \in \mathcal{B}$. For any $j \in [n]$ satisfying $p_j \leq p_{(i)}$, we have that $\kappa(j) < i$. Then, for any $i \notin \mathcal{B}$, we have that

$$\left|\left\{j \in [n] : \Upsilon_{\kappa(j)}(p_j) \geq \xi_i\right\}\right| < i$$

It follows that the number of hypotheses rejected by the SAC-BL algorithm in Definition 4.1 belongs to $\mathcal{B}$, i.e., $i^*_{SBL} \in \mathcal{B}$. Define

$$\mathcal{B}_i = \min\left\{j \in \mathcal{B} : j \geq i\right\}.$$

It follows that

$$\{i^*_{SBL} \geq i\} \Leftrightarrow \{i^*_{SBL} \geq \mathcal{B}_i\}.$$

Based on the above analysis, we have the following relationship:

$$\begin{aligned}
\{i \leq i^*_{SBL}\} &\Leftrightarrow \left\{\Upsilon_{\kappa(j)}(p_j) \leq \xi_{\kappa(j)} \text{for } j \in \mathcal{G}_i\right\} \\
&\Leftrightarrow \left\{\Lambda \leq \frac{\xi_{\kappa(j)}}{p_j^\gamma} \text{for } j \in \mathcal{G}_i\right\} \\
&\Leftrightarrow \left\{\Lambda \leq \frac{\xi_i}{p_{(i)}^\gamma}\right\}. \qquad (12)
\end{aligned}$$

According to the Definition 4.2, we have

$$\left\{\frac{1}{p_{(i)}^\gamma} \geq \frac{n\Lambda}{\delta\alpha i}\right\} \Leftrightarrow \left\{\Lambda \leq \frac{\xi_i}{p_{(i)}^\gamma}\right\}. \qquad (13)$$

Combining Eq. (12) and Eq. (13), we conclude that the SBL algorithm in Definition 4.1 is equivalent to the one in Definition 4.2. □

### B.4 Proof of Theorem 4.4

*Proof.* Note that

$$\left\{p_i^\gamma \cdot \mathbf{1}\left(p_i^\gamma \leq \xi_i\right) \leq \frac{\delta\alpha i}{n}\right\} \Leftrightarrow \left\{\frac{\delta}{p_i^\gamma} \cdot \mathbf{1}\left(\frac{\delta}{p_i^\gamma} \geq \frac{n}{\alpha i}\right) \geq \frac{n}{\alpha i}\right\}$$

and

$$\left\{\xi_i \cdot \mathbf{1}\left(\Lambda\xi_i < \Lambda p_i^\gamma \leq \xi_i\right) \leq \frac{\delta\alpha i}{n}\right\} \Leftrightarrow \left\{\frac{n}{\alpha i} \cdot \mathbf{1}\left(\frac{n}{\alpha i} > \frac{\delta}{p_i^\gamma} \geq \frac{n}{\alpha i}\Lambda\right) \geq \frac{n}{\alpha i}\right\}.$$

For simplicity, define $\tilde{\xi}_i = \frac{n}{\alpha i}$ and

$$\tilde{\Upsilon}_i(x) = \frac{\delta}{x^\gamma} \cdot \mathbf{1}\left(\frac{\delta}{x^\gamma} \geq \tilde{\xi}_i\right) + \tilde{\xi}_i \cdot \mathbf{1}\left(\tilde{\xi}_i > \frac{\delta}{x^\gamma} \geq \tilde{\xi}_i\Lambda\right).$$

Then we conclude that

$$\left\{\Upsilon_{\kappa(i)}(p_i) \leq \frac{\delta\alpha}{n} i^*_{BL}\right\} \Leftrightarrow \left\{\tilde{\Upsilon}_{\kappa(i)}(p_i) \geq \frac{n}{\alpha i^*_{BL}}\right\}.$$

In other words, the null Hypothesis $H_{i;0}$ is rejected by the SBL algorithm if and only if $\tilde{\Upsilon}_{\kappa(i)}(p_i) \geq \frac{n}{\alpha i^*_{BL}}$.

According to the double expectation theorem, we have

$$\begin{aligned}
\mathbb{E}\left(\tilde{\xi}_{\kappa(i)} \cdot \mathbf{1}\left(\tilde{\xi}_{\kappa(i)} > \frac{\delta}{p_i^\gamma} \geq \tilde{\xi}_{\kappa(i)}\Lambda\right)\right) &= \mathbb{E}\left(\mathbb{E}\left(\tilde{\xi}_{\kappa(i)} \cdot \mathbf{1}\left(\tilde{\xi}_{\kappa(i)} > \frac{\delta}{p_i^\gamma} \geq \tilde{\xi}_{\kappa(i)}\Lambda\right) |p_1, \cdots, p_n\right)\right) \\
&= \mathbb{E}\left(\tilde{\xi}_{\kappa(i)} \cdot \mathbf{1}(\frac{\delta}{p_i^\gamma} < \tilde{\xi}_{\kappa(i)}) \cdot \mathbb{P}(\frac{\delta}{p_i^\gamma} \geq \Lambda\tilde{\xi}_{\kappa(i)}|p_1, \cdots, p_n)\right) \\
&= \mathbb{E}\left(\tilde{\xi}_{\kappa(i)} \cdot \mathbf{1}(\frac{\delta}{p_i^\gamma} < \tilde{\xi}_{\kappa(i)}) \cdot \mathbb{P}(\frac{1}{\tilde{\xi}_{\kappa(i)}} \cdot \frac{\delta}{p_i^\gamma} \geq \Lambda|p_1, \cdots, p_n)\right) \\
&= \mathbb{E}\left(\tilde{\xi}_{\kappa(i)} \cdot \mathbf{1}(\frac{\delta}{p_i^\gamma} < \tilde{\xi}_{\kappa(i)}) \cdot \min\left\{\frac{1}{\tilde{\xi}_{\kappa(i)}} \cdot \frac{\delta}{p_i^\gamma}, 1\right\}\right) \\
&= \mathbb{E}\left(\frac{\delta}{p_i^\gamma} \cdot \mathbf{1}(\frac{\delta}{p_i^\gamma} < \tilde{\xi}_{\kappa(i)})\right).
\end{aligned}$$

It follows that

$$\mathbb{E}\left(\tilde{\Upsilon}_{\kappa(i)}(p_i)\right) = \mathbb{E}\left(\frac{\delta}{p_i^\gamma}\cdot \mathbf{1}(\frac{\delta}{p_i^\gamma} \geq \tilde{\xi}_{\kappa(i)})\right) + \mathbb{E}\left(\frac{\delta}{p_i^\gamma}\cdot \mathbf{1}(\frac{\delta}{p_i^\gamma} < \tilde{\xi}_{\kappa(i)})\right)$$
$$= \mathbb{E}\left(\frac{\delta}{p_i^\gamma}\right).$$

Recall that the definition of p-value in 2.1, for any $c \in (0,\ 1)$, we have $\mathbb{P}(p_i \leq c) \leq c$. Denote by $\Lambda$ the random variable uniformly distributed on $(0, 1)$. If $c \geq 1$, then $\mathbb{P}(p_i \leq c) = \mathbb{P}(\Lambda \leq c) = 1$. Otherwise, if $0 < c < 1$, we get $\mathbb{P}(p_i \leq c) \leq c = \mathbb{P}(\Lambda \leq c)$. Therefore, we conclude that

$$\mathbb{P}(p_i \leq c) \leq \mathbb{P}(\Lambda \leq c). \tag{14}$$

Then, it follows that

$$\begin{aligned}\mathbb{P}\left(\frac{\delta}{p_i^\gamma} > c\right) &\leq \mathbb{P}\left(p_i \leq (\frac{\delta}{c})^{1/\gamma}\right)\\ &\leq \mathbb{P}\left(\Lambda \leq (\frac{\delta}{c})^{1/\gamma}\right) = \mathbb{P}\left(\frac{\delta}{\Lambda^\gamma} > c\right).\end{aligned} \tag{15}$$

Note that for any non-negative random variable $Y$, its expectation satisfies

$$\mathbb{E}(Y) = \int_0^\infty y f(y)\, dy = \int_0^\infty \mathbb{P}(Y > y)\, dy.$$

Then we obtain

$$\mathbb{E}\left[\frac{\delta}{p_i^\gamma}\right] = \int_0^\infty \mathbb{P}\left(\frac{\delta}{p_i^\gamma} \geq x\right)\, dx \leq \int_0^\infty \mathbb{P}\left(\frac{\delta}{X^\gamma} \geq x\right)\, dx$$
$$= \mathbb{E}\left[\frac{\delta}{\Lambda^\gamma}\right] = \int_0^1 \frac{\delta}{x^\gamma}\cdot 1\, dx \leq \int_0^1 \frac{1-\gamma}{x^\gamma}\, dx = 1.$$

The above analysis indicates that

$$\mathbb{E}\left(\tilde{\Upsilon}_{\kappa(i)}(p_i)\right) \leq 1.$$

According to the SAC-BL algorithm, a null hypothesis $H_i$ is rejected by the SAC-BL algorithm if the corresponding p-value $p_i$ satisfies

$$\tilde{\Upsilon}_{\kappa(i)}(p_i) \geq \frac{n}{i^*_{SBL}\cdot\alpha}.$$

Note that

$$\mathbf{1}\left(\tilde{\Upsilon}_{\kappa(i)}(p_i) \geq \frac{n}{i^*_{SBL}\cdot\alpha}.\right) = \mathbf{1}\left(\tilde{\Upsilon}_{\kappa(i)}(p_i)\cdot\frac{i^*_{SBL}\alpha}{n} \geq 1\right) \leq \tilde{\Upsilon}_{\kappa(i)}(p_i)\cdot\frac{i^*_{SBL}\alpha}{n}.$$

Then, the FDR of SAC-SBL algorithm satisfies

$$\begin{aligned}FDR_{SAC-SBL} &= \sum_{i=1}^n \mathbb{E}\left(\frac{\mathbf{1}\left(\tilde{\Upsilon}_{\kappa(i)}(p_i) \geq \frac{n}{i^*_{SBL}\cdot\alpha}.\right)}{i^*_{SBL}}\right)\\ &\leq \sum_{i=1}^n \mathbb{E}\left(\frac{\tilde{\Upsilon}_{\kappa(i)}(p_i) i^*_{SBL}\alpha}{n\cdot i^*_{SBL}}\right)\\ &= \frac{\alpha}{n}\sum_{i=1}^n \mathbb{E}\left(\tilde{\Upsilon}_{\kappa(i)}(p_i)\right)\\ &\leq \frac{\alpha}{n}\cdot n = \alpha.\end{aligned}$$

Now, we aim to demonstrate that if a null hypothesis $H_i$ is rejected by the SAC-BL algorithm, then $H_i$ must be rejected by the SAC-SBL algorithm. Suppose $H_i$ is rejected by the SAC-BL algorithm, then, we have

$$\frac{\delta}{p_i^\gamma} \geq \frac{n}{\alpha\cdot i^*_{BL}}$$

and $\epsilon_i \leq i^*_{BL}$. By the definition of $\epsilon_i$ and $\kappa(i)$, we have

$$\frac{\delta\alpha i^*_{BL}}{np^{\gamma}_{(i^*_{BL})}} \leq \frac{\delta\alpha\kappa(i)}{np^{\gamma}_{(\kappa(i))}} \leq \frac{\delta\alpha\kappa(i)}{np^{\gamma}_{i}}$$

Recall the definition of $i^*_{BL}$, we have

$$\frac{\delta}{p^{\gamma}_{(i^*_{BL})}} \geq \frac{n}{\alpha \cdot i^*_{BL}}$$

Therefore, we conclude

$$\frac{\delta\alpha\kappa(i)}{np^{\gamma}_{i}} \geq \frac{\delta\alpha i^*_{BL}}{np^{\gamma}_{(i^*_{BL})}} \geq 1.$$

If $j > i^*_{BL}$, then

$$\frac{j\delta}{p^{\gamma}_{(j)}} < \frac{n}{\alpha} \leq \frac{i^*_{BL}\delta}{p_{(i^*_{BL})}}$$

So, we have $\kappa(i) \leq i^*_{BL}$. It follows that

$$\frac{\delta}{p^{\gamma}_{i}} \geq \frac{n}{\alpha\kappa(i)} \geq \frac{n}{\alpha i^*_{BL}}.$$

Further, we have

$$\tilde{\Upsilon}_{\kappa(i)}(p_i) \geq \frac{n}{\alpha i^*_{BL}}.$$

In other words, $H_i$ is rejected by the SAC-SBL algorithm. The above analysis show that $\mathcal{R}_{SAC-BL} \subset \mathcal{R}_{SAC-SBL}$ almost surely, i.e.,

$$\mathbb{P}\left(\mathcal{R}_{SAC-BL} \subset \mathcal{R}_{SAC-SBL}\right) = 1.$$

If there exists a p-value $p_i$ which satisfies $\mathcal{P}\left(p^{\gamma}_{i} < \frac{\delta\alpha i^*_{BL}}{n}\right) > 0$, then

$$\mathbb{P}\left(\tilde{\Upsilon}_{\kappa(i)}(p_i) \geq \frac{n}{\alpha i^*_{BL}}\right) > 0.$$

It follows that

$$\mathbb{P}\left(|\mathcal{R}_{SAC-BL}| < |\mathcal{R}_{SAC-SBL}|\right) > 0.$$

□

### B.5 Proof of Theorem 4.5

*Proof.* According the the definition of FPR, we have

$$FPR = \frac{FP}{FP+TN} = \frac{|\mathcal{R}^c \cup \mathcal{H}_1|}{|\mathcal{H}_1|}.$$

Therefore, larger $\mathcal{R}$ means smaller FPR. Based on the conclusions in Theorem 4.4, the results in Theorem 4.5 are obvious. □

## C Limitation

**Limitation**. Our results in Theorems 3.2 and 3.3 are based on the generalized Gaussian-like distribution family. In the future work, we will discuss the potential extensions of our finding in Theorems 3.2 and 3.3 to more general distribution.