A Decompose risk-of-ruin to individual stages

Proof of Theorem 3.1. The last claim is trivial: it is easy to verify that (1) and (2) hold if we let $\mathcal{T}_t = \emptyset$ for all $t \in [T]$. Now we prove the first and the second claim. The case for $T = 1$ is trivial. We assume $T \geq 2$. For any $t = 2, \dots, T$, we have that

$$
\begin{aligned}
\mathbb{P}(R_t \leq B) &= \mathbb{E}[\mathbb{P}(R_t \leq B \mid \mathcal{F}_{t-1})] \\
&= \mathbb{E}\Big[\mathbb{P}(R_t \leq B, R_{t-1} \leq B \mid \mathcal{F}_{t-1}) + \mathbb{P}(R_t \leq B, R_{t-1} > B \mid \mathcal{F}_{t-1})\Big] \\
&\overset{(a)}{\leq} \mathbb{P}(R_{t-1} \leq B) + \mathbb{E}[\mathbb{P}(R_t \leq B, R_{t-1} > B \mid \mathcal{F}_{t-1})] \\
&= \mathbb{P}(R_{t-1} \leq B) + \mathbb{E}\Big[\mathbb{I}(\mathbb{P}(R_{t-1} > B \mid \mathcal{F}_{t-1}) > 0) \\
&\qquad \times \mathbb{P}(R_t \leq B \mid R_{t-1} > B, \mathcal{F}_{t-1})\, \mathbb{P}(R_{t-1} > B \mid \mathcal{F}_{t-1}) \\
&\qquad + \mathbb{I}(\mathbb{P}(R_{t-1} > B \mid \mathcal{F}_{t-1}) = 0)\, \mathbb{P}(R_t \leq B, R_{t-1} > B \mid \mathcal{F}_{t-1})\Big] \\
&\overset{(b)}{=} \mathbb{P}(R_{t-1} \leq B) + \mathbb{E}\Big[\mathbb{I}(\mathbb{P}(R_{t-1} > B \mid \mathcal{F}_{t-1} > 0)) \\
&\qquad \times \mathbb{P}(R_t \leq B \mid R_{t-1} > B, \mathcal{F}_{t-1})\, \mathbb{P}(R_{t-1} > B \mid \mathcal{F}_{t-1})\Big] \\
&\overset{(c)}{\leq} \mathbb{P}(R_{t-1} \leq B) + \Delta_t \mathbb{E}\Big[\mathbb{I}(\mathbb{P}(R_{t-1} > B \mid \mathcal{F}_{t-1}) > 0)\, \mathbb{P}(R_{t-1} > B \mid \mathcal{F}_{t-1})\Big] \\
&= \mathbb{P}(R_{t-1} \leq B) + \Delta_t \mathbb{P}(R_{t-1} > B)
\end{aligned}
$$

where (b) used that $\mathbb{P}(R_{t-1} > B \mid \mathcal{F}_{t-1}) = 0 \Rightarrow \mathcal{T}_t = \emptyset$ and that $r_t((Y_{i,t})_{i \in \mathcal{N}_t}, \emptyset) = 0$, which implies that almost surely

$$
\begin{aligned}
&\mathbb{I}(\mathbb{P}(R_{t-1} > B \mid \mathcal{F}_{t-1}) = 0) \cdot \mathbb{P}(R_t \leq B, R_{t-1} > B \mid \mathcal{F}_{t-1}) \\
&= \mathbb{I}(\mathbb{P}(R_{t-1} > B \mid \mathcal{F}_{t-1}) = 0)\, \mathbb{P}\left(R_{t-1} + r_t\left((Y_{i,t})_{i \in \mathcal{N}}, \emptyset\right) \leq B, R_{t-1} > B \mid \mathcal{F}_{t-1}\right) \\
&= \mathbb{I}(\mathbb{P}(R_{t-1} > B \mid \mathcal{F}_{t-1}) = 0) \cdot \mathbb{P}(R_{t-1} \leq B, R_{t-1} > B \mid \mathcal{F}_{t-1}) \\
&= 0
\end{aligned}
$$

and (c) used that $b_t \geq B$, which implies that almost surely

$$
\mathbb{P}(R_t \leq B \mid R_{t-1} > B, \mathcal{F}_t) \leq \mathbb{P}(R_t \leq b_t \mid R_{t-1} > B, \mathcal{F}_t) \leq \Delta_t.
$$

Rearranging this, we obtain a recurrence relation: for any $t = 2, ..., T$,

$$
\mathbb{P}(R_t > B) \geq (1 - \Delta_t) \cdot \mathbb{P}(R_{t-1} > B). \tag{14}
$$

Using the recurrence relation repeatedly for all $t \in [T]$, we obtain

$$
\begin{aligned}
\mathbb{P}(R_T > B) \geq \prod_{i=2}^{T} (1 - \Delta_t) \cdot \mathbb{P}(R_1 > b_1) \geq \prod_{t=1}^{T} (1 - \Delta_t) \\
\implies \mathbb{P}(R_T \leq B) \leq 1 - \prod_{t=1}^{T} (1 - \Delta_t) \leq \delta
\end{aligned}
$$

as required. To prove the second claim, observe that equality is attained in all of the above inequalities if equality is attained in (14), $(i), (ii)$ and (iii), and that equality is attained in (14) if equality is attained in (a) and (c). Finally, note that equality in (a) is attained if $r_t \leq 0, \forall t \in [T]$ and equality in (c) is attained if equality is attained in (i) and (iv). $\square$

B Stochastic domination

Lemma B.1 (Stochastic domination under truncation)**.** *For any two independent real random variable X, Z and real number $a, t \in \mathbb{R}$ such that $\mathbb{P}(X < a) > 0$, we have that*

$$
\mathbb{P}(X + Z \geq t \mid X < a) \leq \mathbb{P}(X + Z \geq t).
$$

Proof of Lemma B.1 . Assume that $\mathbb{P}(X \geq a) > 0$, or else the proof is trivial. We first claim that $\mathbb{P}(X + Z \geq t \mid X < a) \leq \mathbb{P}(X + Z \geq t \mid X \geq a)$. Note that this holds if and only if

$$\frac{\mathbb{P}(X \geq t - Z, X < a)}{\mathbb{P}(X < a)} \leq \frac{\mathbb{P}(X \geq t - Z, X \geq a)}{\mathbb{P}(X \geq a)}.$$

The above holds since its lhs and rhs satisfies

$$\frac{\mathbb{P}(X \geq t - Z, X < a)}{\mathbb{P}(X < a)} = \frac{\mathbb{P}(X \geq t - Z, X < a, a \geq t - Z)}{\mathbb{P}(X < a)} \leq \mathbb{P}(a \geq t - Z)$$
$$\frac{\mathbb{P}(X \geq t - Z, X \geq a)}{\mathbb{P}(X \geq a)} = \frac{\mathbb{P}(X \geq t - Z, X \geq a, a < t - Z)}{\mathbb{P}(X \geq a)} + \mathbb{P}(a \geq t - Z)$$

It then follows from law of total probability that

$$\begin{aligned}\mathbb{P}(X + Z \geq t) &= \mathbb{P}(X + Z \geq t \mid X < a)\mathbb{P}(X < a) + \mathbb{P}(X + Z \geq t \mid X \geq a)\mathbb{P}(X \geq a) \\ &\geq \mathbb{P}(X + Z \geq t \mid X < a)\mathbb{P}(X < a) + \mathbb{P}(X + Z \geq t \mid X < a)\mathbb{P}(X \geq a) \\ &= \mathbb{P}(X + Z \geq t \mid X < a)\end{aligned}$$

as required. □

Proof of Lemma 3.2. If $M_{t-1}^{(1)} = 0$, (8) holds with equality since $S_{t-1}^{\mathcal{T}}(0) < S_{t-1}^{\mathcal{T}}(1) - B \iff B < 0$. So, assume $M_{t-1}^{(1)} > 0$ from now on. By (15f) and the conditional distributions of multivariate Gaussian, we have

$$\left[s_t^{\mathcal{T}}(0)\middle|S_{t-1}^{\mathcal{T}}(0), \mathcal{F}_{t-1}\right] = \left[\mu_2 + V_{21}V_{11}^{-1}(S_{t-1}^{\mathcal{T}}(0) - \mu_1) + (V_{22} - V_{21}V_{11}^{-1}V_{12})^{1/2}Z\middle|S_{t-1}^{\mathcal{T}}(0), \mathcal{F}_{t-1}\right]$$

where $Z \sim N(0,1)$ is independent of $S_{t-1}^{\mathcal{T}}(0)$ conditioned on $\mathcal{F}_{t-1}$ and μ, V are defined in (15f). Here, we used that $V_{11} > 0$ since $\sigma_{p,t}(0)^2, \sigma(0)^2 > 0$ by Definition 3.1, and $M_{t-1}^{(1)} \neq 0$. Using the above and that $S_t^{\mathcal{T}}(0) = s_t^{\mathcal{T}}(0) + S_{t-1}^{\mathcal{T}}(0)$, we have

$$\begin{aligned}\left[S_t^{\mathcal{T}}(0)\middle|S_{t-1}^{\mathcal{T}}(0), \mathcal{F}_{t-1}\right] = \Big[&(V_{21}V_{11}^{-1} + 1)S_{t-1}^{\mathcal{T}}(0) + \mu_2 - V_{21}V_{11}^{-1}\mu_1 \\ &+ (V_{22} - V_{21}V_{11}^{-1}V_{12})^{1/2}Z\Big|S_{t-1}^{\mathcal{T}}(0), \mathcal{F}_{t-1}\Big].\end{aligned}$$

Since $V_{21}V_{11}^{-1} + 1 > 0$ in the above, using also that $b_t - S_{t-1}^{\mathcal{T}}(1), S_{t-1}^{\mathcal{T}}(1) - B \in \mathcal{F}_{t-1}$ and that $s_t^{\mathcal{T}}(1)$ is independent of $S_{t-1}^{\mathcal{T}}(0), S_t^{\mathcal{T}}(0)$, (8) follows from Lemma B.1. □

C Derivation of the decision rule

Proof of these facts follows from standard Bayesian analysis (see e.g. [15])

Lemma C.1 (Posterior distributions)**.** *We have for* $w = 0, 1, t \in [T]$

$$\mu_{p,t}(1) := \mathbb{E}\left[\mu_{\text{true}}(1)\middle|\mathcal{F}_{t-1}\right] = \frac{1}{\frac{1}{\sigma_0(1)^2} + \frac{M_{t-1}^{(1)}}{\sigma(0)^2}}\left(\frac{\mu_0(1)}{\sigma_0(1)^2} + \frac{S_{t-1}^{\mathcal{T}}(1)}{\sigma(1)^2}\right) \tag{15a}$$

$$\mu_{p,t}(0) := \mathbb{E}\left[\mu_{\text{true}}(0)\middle|\mathcal{F}_{t-1}\right] = \frac{1}{\frac{1}{\sigma_0(0)^2} + \frac{M_{t-1}^{(0)}}{\sigma(0)^2}}\left(\frac{\mu_0(0)}{\sigma_0(0)^2} + \frac{S_{t-1}^{\mathcal{C}}(0)}{\sigma(0)^2}\right) \tag{15b}$$

$$\sigma_{p,t}(w)^2 := \mathbb{V}\left[\mu_{\text{true}}(w)\middle|\mathcal{F}_{t-1}\right] = \left(\frac{1}{\sigma_0(w)^2} + \frac{M_{t-1}(w)}{\sigma(w)^2}\right)^{-1} \tag{15c}$$

$$\left[\mu_{\text{true}}(w)\middle|\mathcal{F}_{t-1}\right] \sim N\big(\mu_{p,t}(w),\ \sigma_{p,t}(w)^2\big) \tag{15d}$$

$$\left[s_t^{\mathcal{T}}(1)\middle|\mathcal{F}_{t-1}\right] \sim N\left(\mu_{p,t}(1) \cdot m_t,\ m_t^2 \cdot \sigma_{p,t}(1)^2 + m_t \cdot \sigma(0)^2\right) \tag{15e}$$

$$\left[\begin{pmatrix} S_{t-1}^{\mathcal{T}}(0) \\ s_t^{\mathcal{T}}(0) \end{pmatrix}\middle|\mathcal{F}_{t-1}\right] \sim N(\mu, V) \tag{15f}$$

where

$$\mu := \begin{pmatrix} \mu_{p,t}(0) \cdot M_{t-1}^{(1)} \\ \mu_{p,t}(0) \cdot m_t \end{pmatrix},$$

$$V := \begin{pmatrix} (M_{t-1}^{(1)})^2 \sigma_{p,t}(0)^2 + M_{t-1}^{(1)} \sigma(0)^2 & M_{t-1}^{(1)} m_t \sigma_{p,t}(0)^2 \\ M_{t-1}^{(1)} m_t \sigma_{p,t}(0)^2 & m_t^2 \sigma_{p,t}(0)^2 + m_t \sigma(0)^2 \end{pmatrix}.$$

D Robustness to non-identically distributed and non-Gaussian outcomes

Proof of Theorem 3.3. To show the experiment by Algorithm 1 is (δ, B)-RRC under Definition 3.4, it suffices to show that (1), (2) hold for each $t \geq 1$. Since (1), (2) hold for each $t \geq 1$ if $m_t = 0$, we only need to show that for each $t \geq 1$, if $m_t \neq 0$, almost surely

$$\mathbb{P}\left(S_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) > B \mid \mathcal{F}_t\right) > 0 \tag{16a}$$

$$\mathbb{P}\left(S_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) \leq b_t \mid \mathcal{F}_{t-1}, S_{t-1}^{\mathcal{T}}(1) - S_{t-1}^{\mathcal{T}}(0) > B\right) \leq \Delta_t. \tag{16b}$$

Note that for each $t \geq 1$, if $m_t \neq 0$,

$$\begin{aligned} \mathbb{P}\left(S_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) \leq b_t \mid \mathcal{F}_{t-1}\right) &= \mathbb{P}\left(\frac{s_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) - \tilde{\mu}_t}{\tilde{\sigma}_t} \leq z_t \mid \mathcal{F}_{t-1}\right) \\ &\overset{(a)}{\leq} \mathbb{P}\left(\frac{s_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) - \breve{\mu}_t}{\breve{\sigma}_t} \leq z_t \mid \mathcal{F}_{t-1}\right) \\ &\overset{(b)}{\leq} \Phi\left(z_t\right) \overset{(c)}{\leq} \Delta_t \end{aligned}$$

where we used first inequality in (13) in (a), second inequality in (13) in (b), and (11) in (c).

We now show (16a) by induction. For $t = 1$, if $m_1 \neq 0$, Algorithm 1 ensures that

$$\mathbb{E}\left(\mathbb{P}\left(s_1^{\mathcal{T}}(1) - s_1^{\mathcal{T}}(0) \leq b_1 \mid \mathcal{F}_1\right)\right) = \mathbb{P}\left(s_1^{\mathcal{T}}(1) - s_1^{\mathcal{T}}(0) \leq b_1\right) \leq \Delta_1 < 1$$

by construction, which implies that

$$\mathbb{P}\left(S_1^{\mathcal{T}}(1) - S_1^{\mathcal{T}}(0) > B \mid \mathcal{F}_1\right) \geq \mathbb{P}\left(s_1^{\mathcal{T}}(1) - s_1^{\mathcal{T}}(0) > b_1 \mid \mathcal{F}_1\right) > 0$$

almost surely. If $m_1 = 0$, then $\mathbb{P}\left(S_1^{\mathcal{T}}(1) - S_1^{\mathcal{T}}(0) > B \mid \mathcal{F}_1\right) = 1$ since $B < 0$. This proves the base case. For the inductive case, if $m_t \neq 0$, Algorithm 1 ensures that

$$\mathbb{E}\left(\mathbb{P}\left(S_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) \leq b_t \mid \mathcal{F}_t\right) \mid \mathcal{F}_{t-1}\right) = \mathbb{P}\left(S_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) \leq b_t \mid \mathcal{F}_{t-1}\right) \leq \Delta_t < 1$$

by construction, which implies that

$$\mathbb{P}\left(S_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) > B \mid \mathcal{F}_t\right) \geq \mathbb{P}\left(S_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) > b_t \mid \mathcal{F}_t\right) > 0$$

almost surely. If $m_t = 0$, we have that

$$\mathbb{P}\left(S_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) > B \mid \mathcal{F}_t\right) = \mathbb{P}\left(S_{t-1}^{\mathcal{T}}(1) - S_{t-1}^{\mathcal{T}}(0) > B \mid \mathcal{F}_{t-1}\right) > 0$$

from inductive hypothesis. This shows (16a).

To show (16b), note that under Definition 3.4,

$$\begin{aligned} &\left[s_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) \mid \mathcal{F}_{t-1}, S_{t-1}^{\mathcal{T}}(0) < S_{t-1}^{\mathcal{T}}(1) - B\right] \\ &\quad \overset{d}{=} s_t^{\mathcal{T}}(1) - s_t^{\mathcal{T}}(0) - \left[S_{t-1}^{\mathcal{T}}(0) \mid \mathcal{F}_{t-1}, S_{t-1}^{\mathcal{T}}(0) < S_{t-1}^{\mathcal{T}}(1) - B\right] \end{aligned}$$

On the rhs, $s_t^{\mathcal{T}}(1) - s_t^{\mathcal{T}}(0)$ is independent of

$$\left[S_{t-1}^{\mathcal{T}}(0) \mid \mathcal{F}_{t-1}, S_{t-1}^{\mathcal{T}}(0) < S_{t-1}^{\mathcal{T}}(1) - B\right]$$

and that $S_{t-1}^{\mathcal{T}}(1) - B \in \mathcal{F}_{t-1}$. It follows from these, (16a) and Lemma B.1 that

$$\begin{aligned} &\mathbb{P}\left(s_t^{\mathcal{T}}(1) - s_t^{\mathcal{T}}(0) - S_{t-1}^{\mathcal{T}}(0) \leq b_t \middle| \mathcal{F}_{t-1}, S_{t-1}^{\mathcal{T}}(0) < S_{t-1}^{\mathcal{T}}(1) - B\right) \\ &\quad \leq \mathbb{P}\left(s_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) \leq b_t \mid \mathcal{F}_{t-1}\right) \end{aligned}$$

Therefore, for each $t \geq 1$, if $m_t \neq 0$,

$$\mathbb{P}\left(S_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) \leq b_t \mid \mathcal{F}_{t-1}, S_{t-1}^{\mathcal{T}}(1) - S_{t-1}^{\mathcal{T}}(0) > B\right) \leq \Delta_t$$

as required. This concludes the proof. □

When are (13) **satisfied** Fix any $t \geq 1$ where $m_t \neq 0$. Note that

$$\left[s_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) \mid \mathcal{F}_{t-1}\right] = \sum_{i \in \mathcal{T}_t} \left(Y_{i,t}(1) - Y_{i,t}(0)\right) - \sum_{r \in [t-1]} \sum_{i \in \mathcal{T}_r} \left[Y_{i,r}(0) \mid Y_{i,r}(1)\right]$$

The summands on the rhs are independent random variables under Definition 3.4. We thus expect that when m_t or M_{t-1} are sufficiently large,

$$\frac{\left[s_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) \mid \mathcal{F}_{t-1}\right] - \mathbb{E}\left[s_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) \mid \mathcal{F}_{t-1}\right]}{\sqrt{\mathbb{V}\left[s_t^{\mathcal{T}}(1) - S_t^{\mathcal{T}}(0) \mid \mathcal{F}_{t-1}\right]}} \approx N(0,1)$$

by central limit theorem under mild moment-growth conditions (e.g. Lyapunov's conditions). We thus expect that first condition in (13) holds when m_t or $M_{t-1}^{(1)}$ are sufficiently large for each $t \geq 1$.

We now focus on the second condition in (13). Suppose that $\Delta_t \leq 0.5$, which implies $z_t \leq 0$ by (11). Note that we can write

$$\begin{aligned}
\breve{\mu}_t &= \sum_{i \in \mathcal{T}_t} \mathbb{E}\left(Y_{i,t}(1) - Y_{i,t}(0)\right) - \sum_{r \in [t-1]} \sum_{i \in \mathcal{T}_r} \mathbb{E}\left[Y_{i,t}(0) \mid Y_{i,t}(1)\right] \\
\breve{\sigma}_t^2 &= \sum_{r \in [t-1]} \mathbb{V}\left(Y_{i,t}(1) - Y_{i,t}(0)\right) + \sum_{i \in \mathcal{T}_r} \mathbb{V}\left[Y_{i,t}(0) \mid \sigma\left(Y_{i,t}(1)\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mu}_t &= m_t\left(\mu_{p,t}(1) - \mu_{p,t}(0)\right) - \mu_{p,t}(0) M_{t-1}^{(1)} \\
\tilde{\sigma}_t^2 &= m_t \cdot \left(\sigma(1)^2 + \sigma(0)^2\right) + M_{t-1}^{(1)} \cdot \sigma(0)^2 + m_t^2 \cdot \sigma_{p,t}(1)^2 + \left(m_t + M_{t-1}^{(1)}\right)^2 \cdot \sigma_{p,t}(0)^2.
\end{aligned}$$

For $t = 1$,

$$\breve{\mu}_t = \sum_{i \in \mathcal{T}_t} \mathbb{E}\left(Y_{i,t}(1) - Y_{i,t}(0)\right), \quad \breve{\sigma}_t^2 = \sum_{i \in \mathcal{T}_t} \mathbb{V}\left(Y_{i,t}(1) - Y_{i,t}(0)\right)$$

and

$$\begin{aligned}
\tilde{\mu}_t &= m_t\left(\mu_0(1) - \mu_0(0)\right), \\
\tilde{\sigma}_t^2 &= m_t \cdot \left(\sigma(0)^2 + \sigma(1)^2\right) + m_t^2 \cdot \left(\sigma_0(1)^2 + \sigma_0(0)^2\right).
\end{aligned}$$

So, second condition in (13) holds for $t = 1$ if we have chosen prior and model parameters such that

$$\begin{aligned}
\mu_0(1) - \mu_0(0) &\leq \frac{1}{m_t} \sum_{i \in \mathcal{T}_1} \mathbb{E}\left(Y_{i,1}(1) - Y_{i,1}(0)\right) \\
\sigma(0)^2 + \sigma(1)^2 + m_t \cdot \left(\sigma_0(1)^2 + \sigma_0(0)^2\right) &\geq \frac{1}{m_t} \sum_{i \in \mathcal{T}_t} \mathbb{V}\left(Y_{i,t}(1) - Y_{i,t}(0)\right)
\end{aligned}$$

This corresponds to that we choose prior and model parameters conservatively in the sense that we do not overestimate treatment effect or underestimate its variability. Now fix any $t \geq 2$. From the law of large number, we expect that for $M_{t-1}^{(1)}$ sufficiently large

$$\begin{aligned}
&\mu_{p,t}(0) \approx \frac{1}{M_{t-1}^{(1)}} \sum_{r \in [t-1]} \sum_{i \in \mathcal{T}_r} \mathbb{E}\left[Y_{i,t}(0)\right] \\
&\frac{1}{M_{t-1}^{(1)}} \sum_{r \in [t-1]} \sum_{i \in \mathcal{T}_r} \mathbb{E}\left[Y_{i,t}(0) \mid Y_{i,t}(1)\right] \approx \frac{1}{M_{t-1}^{(1)}} \sum_{r \in [t-1]} \sum_{i \in \mathcal{T}_r} \mathbb{E}\left[Y_{i,t}(0)\right] \\
&\mu_{p,t}(1) - \mu_{p,t}(0) \approx \frac{1}{M_{t-1}^{(1)}} \sum_{r \in [t-1]} \sum_{i \in \mathcal{T}_r} \mathbb{E}\left[Y_{i,t}(1) - Y_{i,t}(0)\right] \\
&\frac{1}{M_{t-1}^{(1)}} \sum_{r \in [t-1]} \sum_{i \in \mathcal{T}_r} \mathbb{V}\left[Y_{i,t}(0) \mid Y_{i,t}(1)\right] \approx \frac{1}{M_{t-1}^{(1)}} \sum_{r \in [t-1]} \sum_{i \in \mathcal{T}_r} \mathbb{E}\mathbb{V}\left[Y_{i,t}(0) \mid Y_{i,t}(1)\right] \\
&\quad \leq \frac{1}{M_{t-1}^{(1)}} \sum_{r \in [t-1]} \sum_{i \in \mathcal{T}_r} \mathbb{V}\left[Y_{i,t}(0)\right]
\end{aligned}$$

So if the treatment effects increase or stay roughly constant throughout the experiments

$$\frac{1}{m_t}\sum_{i\in\mathcal{T}_t}\mathbb{E}\left(Y_{i,t}(1)-Y_{i,t}(0)\right)\geq\frac{1}{M_{t-1}^{(1)}}\sum_{r\in[t-1]}\sum_{i\in\mathcal{T}_r}\mathbb{E}\left[Y_{i,t}(1)-Y_{i,t}(0)\right]$$

and our variance estimates $\sigma(0)^2, \sigma(1)^2$ are accurate or conservative in the sense that

$$\sigma(0)^2\geq\frac{1}{M_{t-1}^{(1)}}\sum_{r\in[t-1]}\sum_{i\in\mathcal{T}_r}\mathbb{V}\left[Y_{i,t}(0)\right],\quad\sigma(0)^2+\sigma(1)^2\geq\frac{1}{m_t}\sum_{i\in\mathcal{T}_t}\mathbb{V}\left(Y_{i,t}(1)-Y_{i,t}(0)\right)$$

the second condition in (13) holds for each $t\geq 2$ and the experiment produced by Algorithm 1 is (δ, B)-RRC.

E Algorithm for general Bayesian models and costs

The following outcome model is a generalization of Definition 3.1. Here, experiment outcomes are allowed to be multivariate with each coordinate corresponds a different business metric.

Definition E.1 (General Bayesian model)**.** Fix $p, q\geq 1$. The model parameter $\theta_{\mathrm{true}}\in\mathbb{R}^q$ is generated from certain prior π_0. The experiment outcome of unit i at stage t are distributed independently and identically as

$$\Big(Y_{i,t}(0), Y_{i,t}(0)\Big)\overset{\mathrm{iid}}{\sim}p(\theta_{\mathrm{true}})$$

where $Y_{i,t}(0), Y_{i,t}(0)\in\mathbb{R}^q$ and $p(\theta_{\mathrm{true}})$ is a probability distribution on $\mathbb{R}^{p\times p}$.

The following is a generalization of Definition 2.1. It allows for general experiment cost beyond treatment effect. The cost of treating unit i is now $h_{it}=h_t(Y_{i,t}(1), Y_{i,t}(0))$ for some function $h_t:\mathbb{R}^{p\times p}\mapsto\mathbb{R}$ chosen by the user. For instance, h_t can be chosen to compute the worst treatment effect across multiple business metrics.

Definition E.2 (General experiment cost)**.** For each $t\geq 1$, let the experiment cost from stage-t and treated unit i be $h_{it}=h_t(Y_{i,t}(1), Y_{i,t}(0))$ where $h_t:\mathbb{R}^{p\times p}\mapsto\mathbb{R}$ is any user-chosen function. Then define $r_t:=\sum_{i\in\mathcal{T}_t}h_{i,t}$. We let $r_t=0$ if $\mathcal{T}_t=\emptyset$. Define the cumulative experiment cost up to stage t as $R_t:=\sum_{k\in[t]}r_k$.

We now move to derive an explicit algorithm Algorithm 1 from Theorem 3.1 that output $(m_t)_{t\geq 1}$ such that the experiment is (δ, B)-RRC. Compared to Algorithm 1, the algorithm developed in this section will require Monte-Carlo simulations and generally gives more conservative ramp schedule.

We first review the Cantelli's inequality, which is an improved version of the well-known Chebyshev's inequality for one-sided tail bounds.

Lemma E.3 (Cantelli's inequality)**.** *For any $\lambda\geq 0$, and real-valued random variable X with finite variance,*

$$\mathbb{P}(X-\mathbb{E}(X)\geq\lambda)\leq\frac{1}{1+\lambda^2/\mathbb{V}(X)}$$

Given that (i) $\mathbb{P}\left(R_{t-1}\geq B\mid\mathcal{F}_{t-1}\right)>0$ and that (ii) $\mathbb{E}\left[R_t\mid R_{t-1}\geq B,\mathcal{F}_{t-1}\right]\geq b_t$, a direct application of Cantelli's inequality shows that

$$\begin{aligned}&\mathbb{P}\left(R_t\leq b_t\mid R_{t-1}>B,\mathcal{F}_{t-1}\right)\\&=\mathbb{P}\Big(\mathbb{E}\left[R_t\mid R_{t-1}>B,\mathcal{F}_{t-1}\right]-R_t\geq\mathbb{E}\left[R_t\mid R_{t-1}>B,\mathcal{F}_{t-1}\right]-b_t\Big|R_{t-1}>B,\mathcal{F}_{t-1}\Big)\\&\leq\left(1+\frac{\left(\mathbb{E}\left[R_t\mid R_{t-1}>B,\mathcal{F}_{t-1}\right]-b_t\right)^2}{\mathbb{V}\left(R_t\mid R_{t-1}>B,\mathcal{F}_{t-1}\right)}\right)^{-1}\end{aligned}$$

where $\mathcal{F}_0$ denotes trivial σ-algebra.

Our strategy to construct an algorithm that selects ramp size m_t such that (1), (2) hold is as follows: we first verify that condition (i) holds; if not, set $m_t = 0$ and otherwise find m_t such that the following two inequalities hold

$$\mathbb{E}\left[R_t \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right] \geq b_t \tag{17a}$$

$$\frac{1}{1 + \frac{(\mathbb{E}[R_t|R_{t-1}>B,\mathcal{F}_{t-1}]-b_t)^2}{\mathbb{V}(R_t|R_{t-1}>B,\mathcal{F}_{t-1})}} \leq \Delta_t \tag{17b}$$

To accomplish this, note that by exchangeability of the outcomes under Definition E.1,

$$\begin{aligned}\mathbb{E}\left[R_t \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right] &= \mathbb{E}\left[r_t \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right] + \mathbb{E}\left[R_{t-1} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right] \\ &= m_t \mathbb{E}\left[h_{i=1,t} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right] + \mathbb{E}\left[R_{t-1} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right]\end{aligned} \tag{18}$$

and

$$\begin{aligned}\mathbb{V}\left(R_t \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) &= \mathbb{V}\left(r_t \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) + \mathbb{V}\left(R_{t-1} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \\ &\quad + \mathrm{Cov}\left(r_t, R_{t-1} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \\ &= m_t \mathbb{V}\left(h_{i=1,t} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \\ &\quad + m_t\left(m_t - 1\right)\mathrm{Cov}\left(h_{i=1,t}, h_{i=2,t} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \\ &\quad + \mathbb{V}\left(R_{t-1} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) + \mathrm{Cov}\left(r_t, R_{t-1} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right)\end{aligned} \tag{19}$$

We thus require a Monte-Carlo procedure to output estimates $\hat{\varphi}_t(0), \ldots, \hat{\varphi}_t^{(6)}$ for the following posterior quantities on the rhs of (18), (19)

$$\begin{aligned}&\mathbb{P}\left(R_{t-1} \geq B \mid \mathcal{F}_{t-1}\right) \leftarrow \hat{\varphi}_t(0) \\ &\mathbb{E}\left(h_{i=1,t} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \leftarrow \hat{\varphi}_t(1) \\ &\mathbb{E}\left(R_{t-1} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \leftarrow \hat{\varphi}_t^{(2)} \\ &\mathbb{V}\left(h_{i=1,t} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \leftarrow \hat{\varphi}_t^{(3)} \\ &\mathrm{Cov}\left(h_{i=1,t}, h_{i=2,t} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \leftarrow \hat{\varphi}_t^{(4)} \\ &\mathbb{V}\left[R_{t-1} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right] \leftarrow \hat{\varphi}_t^{(5)} \\ &\mathrm{Cov}\left(h_{i=1,t}, R_{t-1} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \leftarrow \hat{\varphi}_t^{(6)}\end{aligned}$$

where $h_{i=1,t}, h_{i=2,t}$ denote costs from treating two units $i = 1, 2$ at stage t. Recall that under (...), the outcome of the units are exchangeable. So $i = 1, 2$ simply refers to any two distinct units. These quantities will be used to construct estimates of $\mathbb{E}\left[R_t \mid R_{t-1} > B, \mathcal{F}_{t-1}\right]$ and $\mathbb{V}\left(R_t \mid R_{t-1} > B, \mathcal{F}_{t-1}\right)$ as functions of m_t chosen.

We now outline a procedure to construct $\hat{\varphi}_t(0), \ldots, \hat{\varphi}_t^{(6)}$. Firstly, suppose that we can obtain K samples from the posterior distribution

$$\left[\left(Y_{i,r}(0)\right)_{i \in \mathcal{T}_r, r \in [t-1]}, Y_{i=1,t}(0), Y_{i=1,t}(1), Y_{i=2,t}(0), Y_{i=2,t}(1) \middle| \mathcal{F}_{t-1}\right], \tag{20}$$

from certain MCMC algorithms. The specific details of the MCMC algorithm will depend on the Bayesian model used, but generating posterior-predictive samples while imputing unobserved data, as required in (20), is a common objective of such algorithms (see e.g. [15, Chapter 18]). Let us denote the K samples as

$$\left(Y_{i,r}^{\{k\}}(0)\right)_{i \in \mathcal{T}_r, r \in [t-1]}, \left(Y_{i,t}^{\{k\}}(0)\right), Y_{i=1,t}^{\{k\}}(1), Y_{i=2,t}^{\{k\}}(0), Y_{i=2,t}^{\{k\}}(1), \quad k = 1, \ldots, K \tag{21}$$

These will give us K samples from $\left[h_{i=1,t}, h_{i=2,t}, R_{t-1} \mid \mathcal{F}_{t-1}\right]$ as follows:

$$\begin{aligned}\left(\hat{h}_{i=1,t}^{\{k\}}, \hat{h}_{i=2,t}^{\{k\}}, \hat{R}_{t-1}^{\{k\}}\right) = &\Bigg(h_t\left(Y_{i=1,t}^{\{k\}}(1) - Y_{i=1,t}^{\{k\}}(0)\right), h_t\left(Y_{i=2,t}^{\{k\}}(1) - Y_{i=2,t}^{\{k\}}(0)\right), \\ &\sum_{r=1}^{t-1}\sum_{i \in \mathcal{T}_r} h_r\left(Y_{i,r}^{\{k\}}(1) - Y_{i,r}^{\{k\}}(0)\right)\Bigg), \quad k = 1, \ldots, K\end{aligned}$$

Then we can estimate $\mathbb{P}\left(R_{t-1} \geq B \mid \mathcal{F}_{t-1}\right)$ by

$$\mathbb{P}\left(R_{t-1} \geq B \mid \mathcal{F}_{t-1}\right) \leftarrow \hat{\varphi}_t(0) = \frac{1}{K}\sum_{k=1}^{K} \mathbb{I}\left(\hat{R}_{t-1}^{\{k\}} \geq B\right)$$

Let

$$\mathcal{L}_t := \left\{k \in [K] : \widehat{R}_{t-1}^{\{k\}} \geq B\right\} \subset [K]$$

which denotes the subset of the K Monte-Calor samples for which the budgets are not depleted.

If $\hat{\varphi}_t(0) = 0 \iff \mathcal{L}_t = \emptyset$, we can simply out $m_t = 0$ since this corresponds to the case that the condition (i) does not hold, i.e. $\mathbb{P}\left(R_t \leq b_t \mid R_{t-1} > B, \mathcal{F}_{t-1}\right) \approx 0$. Otherwise, we continue to construct $\hat{\varphi}_t(1), \ldots, \hat{\varphi}_t^{(6)}$ as follows:

$$\begin{aligned}
&\mathbb{E}\left(h_{i=1,t} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \leftarrow \hat{\varphi}_t(1) = \frac{1}{|\mathcal{L}_t|}\sum_{k\in\mathcal{L}_t} \hat{h}_{i=1,t}^{\{k\}} \\
&\mathbb{E}\left(R_{t-1} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \leftarrow \hat{\varphi}_t^{(2)} = \frac{1}{|\mathcal{L}_t|}\sum_{k\in\mathcal{L}_t} \hat{R}_{t-1}^{\{k\}} \\
&\mathbb{V}\left(h_{i=1,t} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \leftarrow \hat{\varphi}_t^{(3)} = \frac{1}{|\mathcal{L}_t|}\sum_{k\in\mathcal{L}_t} \left(\hat{h}_{i=1,t}^{kk\}}\right)^2 - \left(\hat{\varphi}_t(1)\right)^2 \\
&\operatorname{Cov}\left(h_{i=1,t}, h_{i=2,t} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \\
&\qquad\qquad \leftarrow \hat{\varphi}_t^{(4)} = \frac{1}{|\mathcal{L}_t|}\sum_{k\in\mathcal{L}_t} \hat{h}_{i=1,t}^{\{k\}}\hat{h}_{i=2,t}^{\{k\}} - \hat{\varphi}_t(1)\left(\frac{1}{|\mathcal{L}_t|}\sum_{k\in\mathcal{L}_t} \hat{h}_{i=2,t}^{\{k\}}\right) \\
&\mathbb{V}\left[R_{t-1} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right] \leftarrow \hat{\varphi}_t^{(5)} = \frac{1}{|\mathcal{L}_t|}\sum_{k\in\mathcal{L}_t} \left(\hat{R}_{t-1}^{\{k\}}\right)^2 - \left(\hat{\varphi}_t^{(2)}\right)^2 \\
&\operatorname{Cov}\left(h_{i=1,t}, R_{t-1} \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right) \leftarrow \hat{\varphi}_t^{(6)} = \frac{1}{|\mathcal{L}_t|}\sum_{k\in\mathcal{L}_t} \hat{h}_{i=1,t}^{\{k\}}\hat{h}_{i=2,t}^{\{k\}} - \hat{\varphi}_t(1)\hat{\varphi}_t^{(2)}
\end{aligned} \tag{22}$$

From (18), (19) and the Monte-Carlo estimates above, we then have estimators for $\mathbb{E}\left[R_t \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right], \mathbb{V}\left[R_t \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right]$ in terms of $\hat{\varphi}_t(1), \ldots, \hat{\varphi}_t^{(6)}$ as follows

$$\begin{aligned}
&\mathbb{E}\left[R_t \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right] \leftarrow m_t \cdot \hat{\varphi}_t(1) + \hat{\varphi}_t^{(2)} \\
&\mathbb{V}\left[R_t \mid R_{t-1} \geq B, \mathcal{F}_{t-1}\right] \leftarrow \left(m_t \cdot \hat{\varphi}_t^{(3)} + m_t\left(m_t - 1\right) \cdot \hat{\varphi}_t^{(4)}\right) + \hat{\varphi}_t^{(5)} + m_t \cdot \hat{\varphi}_t^{(6)}
\end{aligned}$$

The two inequalities in (17) then become

$$m_t \cdot \hat{\varphi}_t(1) + \hat{\varphi}_t^{(2)} \geq b_t \tag{23a}$$

$$\frac{1}{1 + \frac{\left(m_t \cdot \hat{\varphi}_t(1) + \hat{\varphi}_t^{(2)} - b_t\right)^2}{\left(m_t \cdot \hat{\varphi}_t^{(3)} + m_t(m_t-1)\cdot\hat{\varphi}_t^{(4)}\right) + \hat{\varphi}_t^{(5)} + m_t \cdot \hat{\varphi}_t^{(6)}}} \leq \Delta_t \tag{23b}$$

respectively. Assume that $\Delta_t > 0$ or else set $m_t = 0$ directly. Observe that (23b) can be written as, with $q_t := \Delta_t^{-1} - 1$,

$$A_t m_t^2 + B_t m_t + C_t \geq 0$$

where

$$\begin{aligned}
A_t &:= \left(\hat{\varphi}_t(1)\right)^2 - q_t\hat{\varphi}_t^{(4)} \\
B_t &:= 2\widehat{\varphi}_t(1)\left(\hat{\varphi}_t^{(2)} - b_t\right) - q_t\hat{\varphi}_t^{(3)} + q_t\hat{\varphi}_t^{(4)} - q_t\hat{\varphi}_t^{(6)} \\
C_t &:= \left(\hat{\varphi}_t^{(2)} - b_t\right)^2 - q_t\hat{\varphi}_t^{(5)}
\end{aligned} \tag{24}$$

Then one can choose m_t to be the largest, positive integer in the range defined by

$$m_t \cdot \hat{\varphi}_t(1) + \hat{\varphi}_t^{(2)} \geq b_t, \quad A_t m_t^2 + B_t m_t + C_t \geq 0$$

If the range does not contain any positive integer, we set $m_t = 0$. Note that the range can be easily identified after solving the quadratic equation $A_t m_t^2 + B_t m_t + C_t = 0$. Algorithm 2 gives the algorithm that outputs ramp sizes adaptively. Note that by construction, it gives a (δ, B)-RRC experiments if the Monte-Carlo estimators are sufficiently accurate.

Algorithm 2 Output ramp size adaptively

Input: $B < 0, \delta \in [0, 1)$

```
Initialize t ← 1, ∏_{r=1}^{0} (1 − Δ_r) ← 1
while ∏_{r=1}^{t−1} (1 − Δ_r) > 1 − δ do
    choose Δ_t ∈ [0, (1−δ)/∏_{r=1}^{t−1}(1−Δ_r) − 1], b_t ≥ B
    run MCMC to obtain posterior samples in (21) and computes φ̂_t(0)
    if φ̂_t(0) ← 0 then m_t ← 0
    else
        compute φ̂_t(1), ..., φ̂_t^(6) using (22) and then A_t, B_t, C_t by (24)
        find V_t ← {m ∈ N_+ ∩ [0, N_t/2] : m · φ̂_t(1) + φ̂_t^(2) ≥ b_t, A_t m^2 + B_t m + C_t ≥ 0}
        if V_t ≠ ∅ then
            m_t ← max V_t
        else
            m_t ← 0
        end if
    end if
    Output m_t and then conduct stage t-experiment and observe the outcomes
    update t ← t + 1
end while
```

We have conducted preliminary simulations of the proposed procedure for a multivariate Gaussian outcome model with Gaussian-inverse-Wishart prior, and observed satisfactory results. However, we defer presenting numerical results until future work when a more systematic investigation of Monte-Carlo based procedures can be conducted.

F Linkedin experiment data

In Table 1 below, $\mu_{\text{true}}(w), \sigma(w)^2, w = 0, 1$ are sample statistics from the actual LinkedIn experiment. N_t are incoming population size reduced by 10^4 factor for tractability on a personal computer.

Stages t	1	2	3	4	5	6
$\mu_{\text{true}}(0)$	0.3648	0.3780	0.3752	0.2317	0.4009	0.3930
$\mu_{\text{true}}(1)$	0.3659	0.3788	0.3754	0.2317	0.4010	0.3941
$\sigma(0)^2$	2.0993	2.2769	2.0909	1.1165	2.2705	2.3982
$\sigma(1)^2$	2.0923	2.2248	2.0135	1.0526	2.2476	2.4430
N_t	10,756	10,460	10,598	7,580	10,550	10,688

Table 1: Linkedin experiment data

G Thompson-sampling based Bayesian bandit

This algorithm is developed in [27, Section 4] for clinical trials. The algorithm assigns a user i at stage $t \geq 1$ to treatment with probability

$$\mathbb{P}(i \in \mathcal{T}_t) = \frac{\mathbb{P}(\mu_{\text{true}}(1) > \mu_{\text{true}}(0) \mid \mathcal{F}_{t-1})^c}{\mathbb{P}(\mu_{\text{true}}(1) > \mu_{\text{true}}(0) \mid \mathcal{F}_{t-1})^c + \mathbb{P}(\mu_{\text{true}}(1) \leq \mu_{\text{true}}(0) \mid \mathcal{F}_{t-1})^c}$$

for tuning parameter $c > 0$. Under Definition 3.1, by (15d), we have that

$$\mathbb{P}\left(\mu_{\text{true}}\left(1\right) > \mu_{\text{true}}\left(0\right) \mid \mathcal{F}_{t-1}\right) = \Phi\left(\frac{\mu_{p,t}(1) - \mu_{p,t}(0)}{\sqrt{\sigma_{p,t}(0)^2 + \sigma_{p,t}(1)^2}}\right).$$