A Proving Property Equivalence Between LFPS and FPS

Theorem For a task in which n points are to be selected from a bounded $\mathcal{R}^2$ region defined as $\{(x, y) \in \mathcal{R}^2 : a \leq x \leq b, c \leq y \leq d\}$, let S denote the set of selected points. Given that the maximum attainable first nearest neighbor distance is R , and with an estimated optimal distance $R_E = R + \varepsilon$, there exists $\varepsilon > 0$ such that the distribution of points in S that minimizes the loss function $\mathcal{L}_{LFPS}$ exhibits the same distance properties as FPS in the two-dimensional continuous case.

Proof If the sampled points are distributed in such a manner that the distance from any point to one of its neighbors is equal to R, it is possible to draw circles with a radius of $R/2$ around each sampled point and the circles around neighboring points will touch each other. As the radius of the circle increases with the area covered by them, the configuration where each point maintains a distance of R to its neighbors must be a solution with the densest arrangement of circles, i.e., a solution to the circle packing problem. The authors in Thue (1892) established that hexagonal lattice packing (see Fig. 8) represents the densest achievable circle packing. In this configuration, the fraction of the total area covered by all circles is approximately $\pi(2\sqrt{3})^{-1} \approx 0.9069$. Consequently, for an area of size $A = (b - a) \cdot (d - c)$ the maximum radius R can be calculated as follows:

$$R = \sqrt{\frac{4A}{2\sqrt{3}n}}. \tag{6}$$

Incorporating the estimated maximal distance $R_E < R$ in the loss computation results in optimal values of the loss function, achieving a perfect value of 0. This outcome arises from the ability to position all neighboring sampled points outside the circumcircle with radius $R_E/2$ for every selected point. However, these configurations may not uniformly cover the entire area, as smaller circles can cluster only in specific regions. If $R_E = R$, there is only one optimal solution when all points are in the lattice form, as there is no denser packing available. Only then, the loss function is 0, as alternative configurations would result in selected points within the $R_E/2$ radius of other selected points. Consequently, the minimum correspond to a distribution of points where all neighboring points maintain an equal distance to each other, satisfying the distance bounds for FPS with $R_m = R_M$.

If $R_E = R + \varepsilon$ with $\varepsilon > 0$, it follows that $\mathcal{L}_{\text{LFPS}}(S) > 0$. In this case, an optimal solution may involve an uneven distribution of distances to minimize the sum of squared bounded similarities (see Fig. 8). This optimization strategy might involve increasing the similarity of one pair of points to decrease the similarity of other pairs.

Let $l_{\text{hex}} = l(x_i, N_i^S)/6$ represent the loss per point neighbor pair in the case of lattice packing. If the loss function attains a minimum for a sample configuration different from lattice packing, there must exist point pairs $(x_i^-, x_j^-) \in S^-$ for which $\hat{l}(x_i^-, x_j^-) < l_{\text{hex}}$, and point pairs $(x_i^+, x_i^+) \in S^+$ for which $\hat{l}(x_i^+, x_j^+) > l_{\text{hex}}$, where $\hat{l}$ is the squared bounded similarity from Eq. (2). Consequently, the following condition must hold:

$$l_{\text{hex}}|S^-| - \sum_{i,j} \hat{l}(x_i^-, x_j^-) > -l_{\text{hex}}|S^+| + \sum_{i,j} \hat{l}(x_i^+, x_j^+) \tag{7}$$

Since $\hat{l}$ is squared, squeezing a distance between two points by factor $(1 - p)$ to expand the distance between two other points cannot satisfy the upper inequality. Hence, multiple point pairs must be expanded for one squeezed neighbor distance, and for every squeezed point pair, there must be multiple point pairs that are expanded. This implies the existence of a substructure with $n > 3$ points for a squeezed point pair (x_i^+, x_j^+) such that a subset of expanded point pairs outweigh $2 \cdot (\hat{l}(x_i^+, x_j^+) - l_{\text{hex}})$ for x_i and x_j with distance $(1 - p)R$ to each other. Assuming that R_E is set to

$$R_E = \sqrt{\frac{4A}{2\sqrt{3}(n-1)}} = \sqrt{\frac{n}{n-1}}R \tag{8}$$

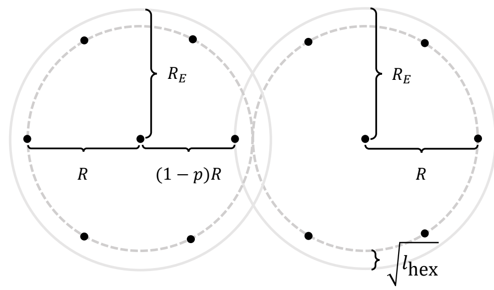

Figure 8: If $R_E > R$, placing a point (middle) farther away from another point (right) causes a smaller loss value for both, but also pushes two other points (middle and left) closer together.

such that if x_i^+ and x_j^+ are completely squeezed together, the configuration yielding the minimum loss for all other points would again be lattice packing. In this substructure,

$$\sum_{i,j} l_{\text{hex}} - s(x_i^-, x_j^-) \leq \sum_{i,j} l_{\text{hex}}. \tag{9}$$

Furthermore, each point has a maximum of six neighbor relationships that can be expanded, and for the two completely squeezed points, there are five points that could potentially be expanded. With this, and utilizing Eqs. (2) and (8), the upper bound can be calculated as

$$\sum_{i,j} l_{\text{hex}} < (6n-2)\left(\frac{R_E - R}{R_E}\right)^2 = (6n-2)\left(1 - \underbrace{\sqrt{\frac{n-1}{n}}}_{\delta}\right)^2. \tag{10}$$

Inserting this into Eq. (7), we obtain

$$(6n-2)(1-\delta)^2 > 2 \cdot \left((1-(1-p)\cdot\delta)^2 - (1-\delta)^2\right) \tag{11}$$

$$\Leftrightarrow \qquad \frac{\left(\sqrt{3n}-1\right)\cdot(1-\delta)}{\delta} > p. \tag{12}$$

The left-hand side approaches 0 for $n \to \infty$, and since at least $n = 4$ was required, we obtain $R_E/R \approx 1.15$. For values up to $1 - p \approx 0.62$, the inequality does not hold, and therefore $R_E < 2 \cdot (1-p)$. It is important to note that the upper bound for

$$l_{\text{hex}}|S^-| - \sum_{i,j} \hat{l}(x_i^-, x_j^-) \tag{13}$$

was intentionally set very high. This implies that there exists a range of values for R_E for which a minimum of the loss function results in a point sampling, ensuring the same properties as a sampling from FPS.

B K-Nearest Neighbor Experiments Across Various Datasets

We conduct experiments to assess the performance of LFPS on diverse datasets, aiming to demonstrate its broad applicability for point clouds. For the ModelNet dataset (Wu et al., 2015), where 256 points are selected per model from a pool of 1 024 FPS pre-sampled points, a setup akin to the S3DIS experiment yields comparable results (compare Fig. 9 and Fig. 5). Notably, LFPS exhibits a higher variance in sampling compared to FPS; however, both mean nearest neighbor distances are similar, with LFPS outperforming other sampling methods.

In a subsequent experiment, we generate an artificial dataset by sampling 4 000 points within a cube and drawing coordinates from a random uniform distribution. The results (Fig. 10) resemble those observed in the original experiment (Fig. 5).

Furthermore, we repeat the ModelNet experiment with a variation, setting $d_f = 8$ to investigate the impact of the decrease factor on selection performance. The results, depicted in Fig. 11, indicate that the network requires slightly more k-nearest neighbors, as expected, while 32 remains a reasonable default choice.

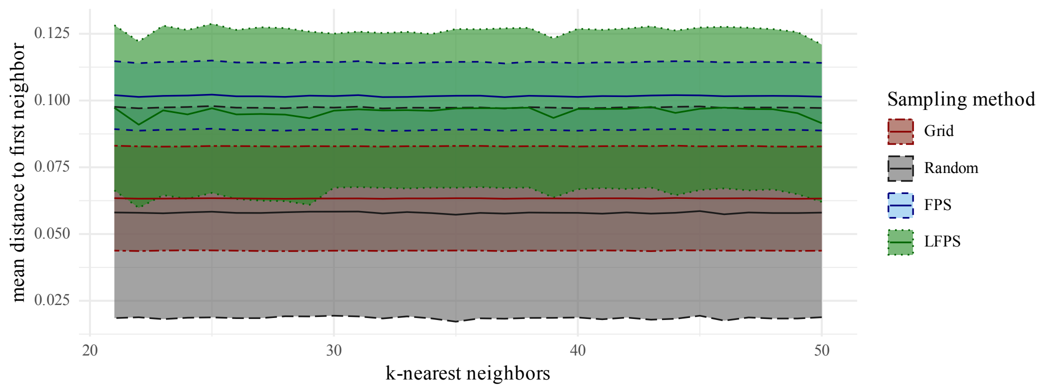

Figure 9: Development of the mean and variance of the first nearest neighbor distance across various neighborhood sizes k, on the ModelNet (Wu et al., 2015) dataset.

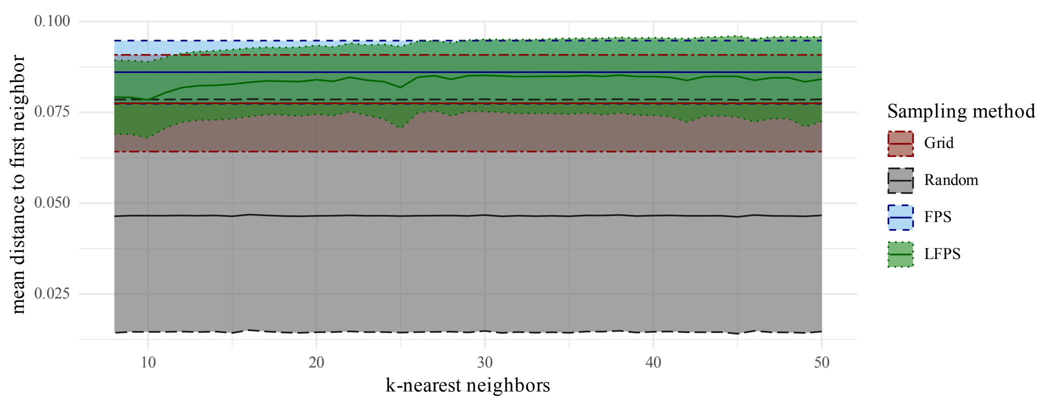

Figure 10: Development of the mean and variance of the first nearest neighbor distance across various neighborhood sizes k, on randomly placed points.

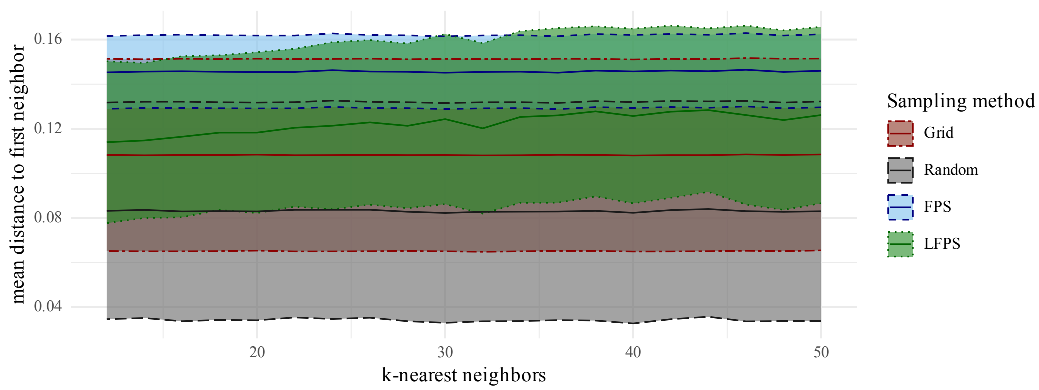

Figure 11: Development of the mean and variance of the first nearest neighbor distance across various neighborhood sizes k, on the ModelNet dataset with $f_d = 8$.