A   PURE MULTIPLICATIVE SENSITIVITY IN STRICT TURNSTILE MODEL

In this section, we derive the pure multiplicative sensitivity of F_p in the strict turnstile model. In the strict turnstile model, for a key-value stream $\mathcal{S} = \{(k_1, v_1), \cdots, (k_n, v_n)\}$ $(n \geq 1)$ where $k_i \in [m]$ $(m \geq 2), v_i \in \{-M, \cdots, M\}$ $(M \geq 1)$, the sum of vs of the same key should always be non-negative:

$$\sum_{i=1}^{n} \mathbb{I}(k_i = k)v_i \geq 0$$

Besides, for the utility of the result, we need to assume that $M < n - 1$.

Theorem 6 (Multiplicative sensitivity of F_p in strict turnstile model). *A mechanism $\mathcal{M} < n - 1$ which calculates $F_p, p \in (0, 1]$ in the strict turnstile model when has pure multiplicative sensitivity upper bounded by*

$$\rho_p^{st} \leq 2^{2-2p}(1 + \frac{2M}{n - 1 - M})^p$$

Proof for Theorem 6. An upper bound for the sensitivity of F_p in the strict turnstile model can be derived by taking the division of the upper and lower bound in the incremental setting following the same logic as the proof for Theorem 1. The upper bound is the same as in the proof of Theorem 6 so we only need to calculate the lower bound of the following expression.

First, we observe the following two inequalities.

$$\forall a, b, d > 0, c \geq 0, a \leq b, c \leq d, \frac{a + c}{a + d} \leq \frac{b + c}{b + d}. \tag{7}$$

$$\frac{\sum_{i=2}^{m} u_i^p + (u_1 - \Delta)^p}{\sum_{i=2}^{m} u_i^p + u_1^p} \overset{(3)+(7)}{\geq} \frac{(\sum_{i=2}^{m} u_i)^p + (u_1 - \Delta)^p}{(\sum_{i=2}^{m} u_i)^p + u_1^p}$$

$$= \frac{(s - u_1)^p + (u_1 - \Delta)^p}{(s - u_1)^p + u_1^p}$$

$$\overset{(3)}{\geq} 2^{p-1}(1 - \frac{\Delta}{s})^p$$

Taking the division between the supremum and the infimum, we get

$$\rho_p^{st} \leq 2^{2-2p}(1 + \frac{2\Delta}{s - \Delta})^p \leq 2^{2-2p}(1 + \frac{2M}{n - 1 - M})^p \qquad \square$$

As shown in Figure 7, when m is the same, the sensitivity is very close to the sensitivity in the cash register model if $M \ll n$.
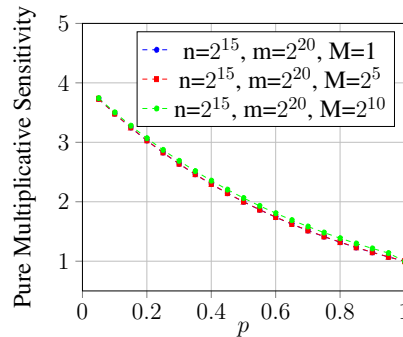

Figure 7: Pure multiplicative sensitivity in the Strict Turnstile Model.