A Omitted Proofs

Theorem (7)**.** *Let the crossover rate $p_c \in [0,1)$, μ be the maximum size of parent population P with $\mu > n/2$, and λ the size of offspring population Q with $\lambda = O(\mu)$. Consider using Algorithm 1 with random selection, one-bit mutation or bit-wise mutation to generate P', and survival selection with Property $\mathcal{A}$, to optimize LF'_ε. Then the expected number of fitness evaluations for achieving an additive ε-approximation w.r.t. LF'_ε is $O(\mu n \log n)$.*

Proof of Theorem 7. From Lemma 4, we know that an additive ε-approximation of LF'_ε is reached if and only if, for each $k \in [0..n/2]$, the population includes a solution x such that $|x'|_1 = k$. As Property $\mathcal{A}$ guarantees that once a k value is reached it will never be lost, we only need to consider the time to generate an individual with k 1-bits in its first half bitstring. Consider a solution x such that $\big||x'|_1 - |y'|_1\big| \neq 1$ for all $y \in P$. From Lemma 4, such a solution x always exists if and only if an additive ε-approximation has not been obtained. The probability of selecting x in single generation is at least

$$1-\left(1-\frac{1}{N}\right)^{\lambda} \geq 1-e^{\frac{-\lambda}{N}} \geq \frac{\lambda}{N+\lambda},$$

where the last inequality uses $e^a \geq 1+a$, for any $a \in \mathbb{R}$. Conditioned on selecting x as a parent, the probability to generate z with $|z'|_1 = |x'|_1 + 1$ or $|z'|_1 = |x'|_1 - 1$ is at least

$$\left(1-\frac{1}{n}\right)^{n-1}(1-p_c)\frac{\min\{|x'|_1, n/2-|x'|_1\}}{n} \geq \frac{(1-p_c)\min\{|x'|_1, n/2-|x'|_1\}}{en}$$

for the bit-wise mutation, and is at least

$$(1-p_c)\frac{\min\{|x'|_1, n/2-|x'|_1\}}{n} > \frac{(1-p_c)\min\{|x'|_1, n/2-|x'|_1\}}{en}$$

for the one-bit mutation. Note that an individual with $|x'|_1$ 1-bits in its first half bitstring will exist in all future generations from Property $\mathcal{A}$. Then the expected number of iterations to generate such a z is at most

$$\left(\frac{\lambda}{N+\lambda}\frac{(1-p_c)\min\{|x'|_1, n/2-|x'|_1\}}{en}\right)^{-1} = \frac{en(N+\lambda)}{(1-p_c)\lambda\min\{|x'|_1, n/2-|x'|_1\}}.$$

Hence, to reach a population P with $\{|y'|_1 \mid y \in P\} = [0..n/2]$, it requires at most

$$en\frac{N+\lambda}{(1-p_c)\lambda}\sum_{|x'|_1=0}^{n/2}\frac{1}{\min\{|x'|_1, n/2-|x'|_1\}} \leq \frac{en(N+\lambda)}{(1-p_c)\lambda}\sum_{|x'|_1=1}^{n/4+1}\frac{1}{|x'|_1} < \frac{en(N+\lambda)}{(1-p_c)\lambda}\ln n$$

expected number of iterations, that is, at most $\frac{en(N+\lambda)}{(1-p_c)}\ln n$ expected number of fitness evaluations. □

Lemma (8)**.** *Let $N \geq \frac{2n}{3}+3$. Consider using the NSGA-II with the survival selection based on the current crowding distance to optimize LF'_ε with problem size n. Assume that at some iteration t, the combined parent and offspring population $R_t = P_t \bigcup Q_t$ contains an individual x with $|x'|_1 = k$, then the next parent population P_{t+1} also contains an individual y with $|y'|_1 = k$.*

Proof of Lemma 8. Since all individuals in R_t are Pareto optimal from Lemma 3, we know that all individuals are in F_1, and then the selection of the next parent population P_{t+1} from R_t consists of N times removing an individual with the smallest current crowding distance, with ties broken randomly. Let R denote the set of individuals remaining from R_t at an arbitrary removal step and write $r := |R| > N$. We call a solution $s \in R$ *sole* if the size of the multiset $\{u \in R \mid |u'|_1 = |s'|_1\}$ is 1. We will prove that whenever such a sole solution s exists, there is always another solution $z \in R$ whose crowding distance satisfies $\mathrm{cDis}(z) < \mathrm{cDis}(s)$ and therefore s will be kept in the next removal. To this end, we first calculate a lower bound for $\mathrm{cDis}(s)$ and then upper bound $\mathrm{cDis}(z)$.

Sort the individuals in R by increasing f_1 values as $x_1, \ldots, x_r$ and by increasing f_2 values as $y_1, \ldots, y_r$. Since s is a sole solution, we further know

$$\{f_1(x_r) - f_1(x_1), f_2(y_r) - f_2(y_1)\} > 0. \quad (1)$$

Let $i,j \in [1..r]$ such that $s = x_i = y_j$. If $\{i,j\} \bigcap \{1,r\} \neq \emptyset$, then $\mathrm{cDis}(s) = +\infty$. Otherwise $1 < i,j < r$, and we compute

$$\mathrm{cDis}(s) = \frac{f_1(x_{i+1}) - f_1(x_{i-1})}{f_1(x_r) - f_1(x_1)} + \frac{f_2(y_{j+1}) - f_2(y_{j-1})}{f_2(y_r) - f_2(y_1)}.$$

Since $f_1(x) - f_1(y) \leq n\varepsilon$, and $f_2(x) - f_2(y) \leq n\varepsilon$ for any $x, y \in \{0,1\}^n$, together with (1), we have

$$\mathrm{cDis}(s) \geq \frac{f_1(x_{i+1}) - f_1(x_{i-1}) + f_2(y_{j+1}) - f_2(y_{j-1})}{n\varepsilon}. \tag{2}$$

If $\min\{|s'|_0, |s'|_1\} < \sqrt{n}$, for the worst case where $\min\{|x'_{i+1}|_0, |x'_{i+1}|_1\} \geq \sqrt{n}$, $\min\{|x'_{i-1}|_0, |x'_{i-1}|_1\} < \sqrt{n}$, $\min\{|y'_{j+1}|_0, |y'_{j+1}|_1\} < \sqrt{n}$, and $\min\{|y'_{j-1}|_0, |y'_{j-1}|_1\} \geq \sqrt{n}$, we have

$$f_1(x_{i+1}) - f_1(x_{i-1}) + f_2(y_{j+1}) - f_2(y_{j-1}) \geq (4 + 4 - 1 + 2^{-n/2}) \cdot \varepsilon.$$

Then (2) gives

$$\mathrm{cDis}(s) \geq \frac{(4 + 4 - 1 + 2^{-n/2}) \cdot \varepsilon}{n\varepsilon} > \frac{7}{n}.$$

Similarly, if $\min\{|s'|_0, |s'|_1\} \geq \sqrt{n}$, we pessimistically consider the case where $\min\{|x'_{i+1}|_0, |x'_{i+1}|_1\} \geq \sqrt{n}$, $\min\{|x'_{i-1}|_0, |x'_{i-1}|_1\} \geq \sqrt{n}$, $\min\{|y'_{j+1}|_0, |y'_{j+1}|_1\} \geq \sqrt{n}$, and $\min\{|y'_{j-1}|_0, |y'_{j-1}|_1\} \geq \sqrt{n}$, then we have

$$f_1(x_{i+1}) - f_1(x_{i-1}) + f_2(y_{j+1}) - f_2(y_{j-1}) \geq (3 + 2^{-n/2} + 3 + 2^{-n/2}) \cdot \varepsilon.$$

Hence, (2) gives

$$\mathrm{cDis}(s) \geq \frac{(3 + 2^{-n/2} + 3 + 2^{-n/2}) \cdot \varepsilon}{n\varepsilon} > \frac{6}{n}$$

Therefore, for a sole solution s, we have

$$\mathrm{cDis}(s) > \frac{6}{n}. \tag{3}$$

Next, we estimate the sum of the finite crowding distances over the individuals in R to prove the existence of a solution $z \in R$ with $\mathrm{cDis}(z) < \mathrm{cDis}(s)$. We compute

$$\begin{aligned}\sum_{i=2}^{r-1}(f_1(x_{i+1}) - f_1(x_{i-1})) &= f_1(x_r) + f_1(x_{r-1}) - f_1(x_2) - f_1(x_1)\\ &\leq 2(f_1(x_r) - f_1(x_1)).\end{aligned}$$

An analogous estimate holds for f_2. Hence, we have

$$\begin{aligned}\sum_{x \in R^*} \mathrm{cDis}(x) &\leq \frac{\sum_{i=2}^{r-1}(f_1(x_{i+1}) - f_1(x_{i-1}))}{f_1(x_r) - f_1(x_1)} + \frac{\sum_{j=2}^{r-1}(f_2(y_{j+1}) - f_2(y_{j-1}))}{f_2(y_r) - f_2(y_1)}\\ &\leq \frac{2(f_1(x_r) - f_1(x_1))}{f_1(x_r) - f_1(x_1)} + \frac{2(f_2(y_r) - f_2(y_1))}{f_2(y_r) - f_2(y_1)} = 4,\end{aligned} \tag{4}$$

where R^* denotes the individuals in R with finite crowding distances. By definition of crowding distance, at most 4 individuals have an infinite value of crowding distance, implying $|R^*| \geq r - 4$. Then with (4), an averaging argument implies that there exists at least one solution $z \in R^*$ with

$$\mathrm{cDis}(z) \leq \frac{4}{|R^*|} \leq \frac{4}{r-4} \leq \frac{4}{N-3} \leq \frac{6}{n},$$

where the third inequality uses $r = |R| \geq N + 1$, and the last inequality uses $N \geq \frac{2}{3}n + 3$. Together with (3), we now proved the existence of $z \in R$ with $\mathrm{cDis}(z) < \mathrm{cDis}(s)$, if a sole solution s exists. Hence, among all removal steps in the survival selection, we will never see a sole solution be removed, thus we will never see $\{u \in R \mid |u'|_1 = |x'|_1\}$ be removed to empty. Then this lemma is proved. $\square$

Lemma (10)**.** *Let $N = N_r \geq 2n+3$ and a given positive threshold $\epsilon_{nad} \geq n\varepsilon$. Consider using the NSGA-III to optimize LF'_ε with problem size n. Define $z_j^{\min} := \min\{f_j(x) \mid x \in R_t\}$ and $z_j^{\max} := \max\{f_j(x) \mid x \in R_t\}$, $j = 1,2$. Then the next parent population P_{t+1} will preserve two individuals x, y such that $f_1(x) = z_1^{\min}$ and $f_1(y) = z_1^{\max}$.*

Proof of Lemma 10. Similar to [DZD25, Lemma 4], we establish our proof based on the following facts.

- Fact I: Any two different function values are incomparable w.r.t. LF'_ε.
- Fact II: Any individual z with $f_1(z) = z_1^{\min}$ will have $f_2(z) = z_2^{\max}$, and z with $f_1(z) = z_1^{\max}$ will have $f_2(z) = z_2^{\min}$.
- Fact III: The ideal point estimate $\hat{z}^* = (z_1^{\min}, z_2^{\min})$, and the Nadir point estimate $\hat{z}^{\mathrm{nad}} = (z_1^{\max}, z_2^{\max})$.
- Fact IV: In each generation, for any reference point with associated individual(s) in the critical front F_{i^*}, one of its individual(s) in F_{i^*} with minimal distance will survive to the next generation.

Fact III results in that the normalized function value $f^n(x) = (f_1^n(x), f_2^n(x))$ of an individual x is $f_j^n(x) = \frac{f_j(x) - z_j^{\min}}{z_j^{\max} - z_j^{\min}}$, $j = 1,2$. Then, with Fact II, we know that any individual z with $f(z) = (z_1^{\min}, z_2^{\max})$ or $f(z) = (z_1^{\max}, z_2^{\min})$ will be normalized to $(0,1)$ or $(1,0)$ respectively, and then will be associated to the reference point $(0,1)$ or $(1,0)$ respectively. Together with $i^* = 1$, that is, all individuals are in F_1, from Fact I, Fact IV shows that the reference points $(0,1)$ and $(1,0)$ will have their associated x and y into P_{t+1}. Then this lemma is proved.

Note that Fact IV has already been proved in [DZD25, Lemma 4], stemming from the survival selection mechanism of the NSGA-III. Also note that Fact II is a direct corollary of the easy-to-obtain Fact I. $\hat{z}^*$ in Fact III is obvious from the definitions of $z_1^{\min}$, $z_2^{\min}$, and $\hat{z}^*$. For $\hat{z}^{\mathrm{nad}}$, from the procedure of the normalization, we know that either $\hat{z}_j^{\mathrm{nad}} = \max_{x \in F_1} f_j(x)$ or $\hat{z}_j^{\mathrm{nad}} = I_j$. For the first case, $\hat{z}_j^{\mathrm{nad}} = z_j^{\max}$, $j = 1,2$, is obtained. For the second case, $I_j \geq \epsilon_{\mathrm{nad}}$ and $I_j \leq z_j^{\max}$. Since $\epsilon_{\mathrm{nad}} \geq n\varepsilon$ from our assumption and $z_j^{\max} \leq n\varepsilon$ from the definition of LF'_ε, we know that $I_j = \epsilon_{\mathrm{nad}} = z_j^{\max} = n\varepsilon$. Thus, $\hat{z}_j^{\mathrm{nad}} = I_j = z_j^{\max}$. $\square$

Lemma (11)**.** *Let $N = N_r \geq 2n+3$ and $\epsilon_{nad} \geq n\varepsilon$. Consider using the NSGA-III to optimize LF'_ε with problem size n. Assume that at some iteration t, the two extreme points $(0, n\varepsilon)$ and $(n\varepsilon, 0)$ are covered by the combined parent and offspring population $R_t = P_t \bigcup Q_t$. If R_t contains an individual x with $|x'|_1 = k$, then the next parent population P_{t+1} also contains an individual $\tilde{x}$ with $|\tilde{x}'|_1 = k$, and covers $(0, n\varepsilon)$ and $(n\varepsilon, 0)$ as well.*

Proof of Lemma 11. We prove this lemma by showing that all individuals associated with the same reference point will have a same number of 1-bits in the first half of their bit-strings. That is, the individual x with $|x'|_1 = k$ will be associated to a reference point any of whose associated individuals, say z, has $|z'|_1 = k$. From Fact IV in the proof of Lemma 10, we know that this reference point will contribute a satisfying $\tilde{x}$ to P_{t+1}.

To show the above statement, we will prove that any two individuals x and y with $|x'|_1 \neq |y'|_1$ will be associated to different reference points. Since $(0, n\varepsilon)$ and $(n\varepsilon, 0)$ are covered by R_t, we know the normalized $f_j^n(x) = \frac{f_j(x) - z_j^{\min}}{z_j^{\max} - z_j^{\min}} = \frac{f_j(x) - 0}{n\varepsilon - 0} = \frac{f_j(x)}{n\varepsilon}$ and $f_j^n(y) = \frac{f_j(x)}{n\varepsilon}$, $j = 1,2$, from the procedure of the normalization and Fact III in the proof of Lemma 10. Say $(\frac{m}{N_r - 1}, 1 - \frac{m}{N_r - 1})$, $m \in [0..N_r - 1]$ the m-th reference point. Then $(0, n\varepsilon)$ and $(n\varepsilon, 0)$ are associated to 0-th and $(N_r - 1)$-th reference points respectively. From

$$\frac{1 - \frac{m_x}{N_r - 1}}{\frac{m_x}{N_r - 1}} = \frac{f_2^n(x)}{f_1^n(x)}$$

with $f_1(x) \neq 0$, we have

$$m_x = \frac{N_r - 1}{1 + \frac{f_2^n(x)}{f_1^n(x)}} = \frac{N_r - 1}{1 + \frac{f_2(x)}{f_1(x)}}.$$

Then x will be associated to $\lfloor m_x \rfloor$-th or $\lceil m_x \rceil$-th reference point. Analogously, y will be associated to $\lfloor m_y \rfloor$-th or $\lceil m_y \rceil$-th reference point, where $m_y = \frac{N_r - 1}{1 + \frac{f_2(y)}{f_1(y)}}$. W.l.o.g. let $|x'|_1 > |y'|_1$. If $|y'|_1 > 0$, then $f_1(y) \neq 0$ and $f_1(x) \neq 0$. From an easy-to-obtain fact that $f_1(x) > f_1(y) > 0$ and $f_2(y) > f_2(x) \geq 0$ for $|x'|_1 > |y'|_1 > 0$, we know that $\frac{f_2(x)}{f_1(x)} < \frac{f_2(y)}{f_1(y)}$. Then

$$\begin{aligned}
|m_x - m_y| &= \left| \frac{N_r - 1}{1 + \frac{f_2(x)}{f_1(x)}} - \frac{N_r - 1}{1 + \frac{f_2(y)}{f_1(y)}} \right| = \frac{N_r - 1}{1 + \frac{f_2(x)}{f_1(x)}} - \frac{N_r - 1}{1 + \frac{f_2(y)}{f_1(y)}} \\
&\geq \frac{N_r - 1}{1 + \frac{(2|x'|_0 + 2^{-n/2} BV(\overline{x''}))\varepsilon}{2|x'|_1 \varepsilon}} - \frac{N_r - 1}{1 + \frac{2|y'|_0 \varepsilon}{(2|y'|_1 + 2^{-n/2} BV(y''))\varepsilon}} \\
&> \frac{N_r - 1}{1 + \frac{2|x'|_0 + 1}{2|x'|_1}} - \frac{N_r - 1}{1 + \frac{2|y'|_0}{2|y'|_1 + 1}} \\
&= \frac{N_r - 1}{n + 1}(2|x'|_1 - 2|y'|_1 - 1) \\
&\geq \frac{2n + 3 - 1}{n + 1} = 2,
\end{aligned}$$

where the first inequality uses $f_2(x) \leq (2|x'|_0 + 2^{-n/2} BV(\overline{x''}))\varepsilon$, $f_1(x) \geq 2|x'|_1 \varepsilon$, $f_2(y) \geq 2|y'|_0 \varepsilon$, and $f_1(y) \leq (2|y'|_1 + 2^{-n/2} BV(y''))\varepsilon$ from the definitions of f_1 and f_2, the second inequality uses $\{BV(\overline{x''}), BV(y'')\} < 2^{-n/2}$, the third equality uses $|x'|_1 + |x'|_0 = \frac{n}{2} = |y'|_1 + |y'|_0$, and the last inequality uses $N_r \geq 2n + 3$ and $|x'|_1 - |y'|_1 \geq 1$. If $|y'|_1 = 0$, then analogously, we have

$$|m_x - 0| = m_x = \frac{Nr - 1}{1 + \frac{f_2(x)}{f_1(x)}} > 2.$$

Then we know that at least two reference points lie between x and y. Thus x and y will be associated to different reference points, and this lemma is proved. □

Theorem (12). *Let $N = N_r \geq 2n + 3$, $\epsilon_{nad} \geq n\varepsilon$ and $p_c \in [0, 1)$. Consider using the NSGA-III with uniform selection and one-bit mutation or bit-wise mutation to optimize LF'_ε with problem size n. Then after an expected number of $O(Nn \log n)$ fitness evaluations, the population achieves an additive ε-approximation of LF'_ε.*

Proof of Theorem 12. Assume that at some time t_0, the population contains two solutions x, y such that $|x'|_1 = n/2$ and $|y'|_1 = 0$, this is $(0, n\varepsilon)$ and $(n\varepsilon, 0)$ are covered. Then from Lemma 11, we know that Property $\mathcal{A}$ will be satisfied in current and all future generations. Thus Theorem 7 will result in additional $O(Nn \log n)$ expected number of fitness evaluations to reach an additive ε-approximation of LF'_ε from time t_0.

In the following, we only need to analyze the time t_0 required to generate such solutions x and y. Let $x_{\max}$ denote an individual in the population with the maximum number of 1-bits in its first half bitstring. Then the probability to generate an offspring in $S_{\max} := \{z \mid |z'|_1 > |x'_{\max}|_1\}$ is at least

$$(1 - p_c)\left(1 - \left(1 - \frac{1}{N}\right)^N\right) \frac{\frac{n}{2} - |x'_{\max}|_1}{n} \left(1 - \frac{1}{n}\right)^{n-1} \geq (1 - p_c)\left(1 - \frac{1}{e}\right) \frac{n - 2|x'_{\max}|_1}{2en}.$$

As $f_1(z)$ increases as $|z'|_1$ increases, from Lemma 10, we know that one offspring in $S_{\max}$ (the one with maximal number of 1-bits in the first half bitstring) will survive to the next generation. As $z_1^{\min}$ will not decrease from Lemma 10, $|x'_{\max}|_1$ will not decrease. Hence, the expected number of iterations to generate an offspring in $S_{\max}$ is at most

$$\frac{1}{(1 - p_c)(1 - \frac{1}{e}) \frac{n - 2|x'_{\max}|_1}{2en}} = \frac{2e^2 n}{(1 - p_c)(e - 1)(n - 2|x'_{\max}|)}.$$

Therefore, to reach an x with $|x'|_1 = n/2$, we need at most

$$\sum_{k=0}^{\frac{n}{2}-1} \frac{2e^2 n}{(1-p_c)(e-1)(n-2k)} = O(n \log n)$$

expected number of iterations. Analogously, the same bound of $O(n\log n)$ holds for generating a y with $|y'|_1 = 0$. Therefore, $E[t_0] = O(n\log n)$, and the expected number of fitness evaluations to cover $(0, n\varepsilon)$ and $(n\varepsilon, 0)$ is $O(Nn\log n)$. Together with the additional $O(Nn\log n)$ expected number of fitness evaluations for an additive ε-approximation, this lemma is proved. $\square$

Lemma (13). *Let $N \geq \frac{n}{2}+3$ and $r = (r_1, r_2)$ with $r_1 \leq -\varepsilon$, $r_2 \leq -\varepsilon$. Consider using the SMS-EMOA to optimize LF'_ε with problem size n. Assume that at some iteration t the combined parent and offspring population R_t contains an individual x with $|x'|_1 = k$, then the next parent population P_{t+1} contains an individual y that $|y'|_1 = k$.*

Proof of Lemma 13. Recall the definition of a sole solution s in the proof of Lemma 8, that is, the size of the multiset $\{z \in R_t \mid |z'|_1 = |s'|_1\}$ is 1. Since the SMS-EMOA only deletes one individual in R_t for the survival selection, we easily know that this lemma holds if x is not a sole solution. Hence, we only consider the case when x is a sole solution. As the survival selection removes one individual with smallest hypervolume contribution, we only need to prove that there is another solution $z \in R_t$ with $\Delta_r(z, R_t) < \Delta_r(x, R_t)$.

Let $s_1, \ldots, s_{N+1}$ be the solutions in R_t sorted by increasing f_1 values (and hence by decreasing f_2 values). Then

$$\Delta_r(s_i, R_t) = \begin{cases} \big(f_1(s_1) - r_1\big)\big(f_2(s_1) - f_2(s_2)\big), & i = 1, \\ \big(f_1(s_{i+1}) - f_1(s_i)\big)\big(f_2(s_{i-1}) - f_2(s_i)\big), & i \in [2..N], \\ \big(f_1(s_{N+1}) - f_1(s_N)\big)\big(f_2(s_{N+1}) - r_2\big), & i = N+1. \end{cases} \tag{5}$$

If $x = s_1$, together with $r_1 \leq -\varepsilon$, then

$$\Delta_r(x, R_t) = \big(\mathrm{LF}'_{\varepsilon,1}(x) - r_1\big)\big(\mathrm{LF}'_{\varepsilon,2}(x) - \mathrm{LF}'_{\varepsilon,2}(s_2)\big) \geq (0 - r_1)\cdot 2\varepsilon \geq 2\varepsilon^2. \tag{6}$$

Similarly, if $x = s_{N+1}$, together with $r_2 \leq -\varepsilon$, then

$$\Delta_r(x, R_t) = \big(\mathrm{LF}'_{\varepsilon,1}(s_{N+1}) - \mathrm{LF}'_{\varepsilon,1}(s_N)\big)\big(\mathrm{LF}'_{\varepsilon,2}(s_{N+1}) - r_2\big) \geq 2\varepsilon\cdot(0 - r_2) \geq 2\varepsilon^2. \tag{7}$$

Otherwise, let $\tilde{i} \in [2..N]$ such that $x = s_{\tilde{i}}$. If $\min\{|x'|_0, |x'|_1\} < \sqrt{n}$, from (5) and Definition 2, we pessimistically consider that $\min\{|s'_{\tilde{i}-1}|_0, |s'_{\tilde{i}-1}|_1\} < \sqrt{n}$ and $\min\{|s'_{\tilde{i}+1}|_0, |s'_{\tilde{i}+1}|_1\} \geq \sqrt{n}$, and have

$$\begin{aligned} \Delta_r(x, R_t) &\geq \Big(\Big(2|s'_{\tilde{i}+1}|_1 + 2^{-n/2}BV(s''_{\tilde{i}+1})\Big)\varepsilon - 2|x'|_1\varepsilon\Big)\cdot\Big(2|s'_{\tilde{i}-1}|_0\varepsilon - 2|x'|_0\varepsilon\Big) \\ &\geq (1 + 2^{-n/2})\varepsilon\cdot 2\varepsilon > 2\varepsilon^2. \end{aligned} \tag{8}$$

Analogously, for $\min\{|x'|_0, |x'|_1\} \geq \sqrt{n}$, we pessimistically consider that $\min\{|s'_{\tilde{i}-1}|_0, |s'_{\tilde{i}-1}|_1\} \geq \sqrt{n}$ and $\min\{|s'_{\tilde{i}+1}|_0, |s'_{\tilde{i}+1}|_1\} \geq \sqrt{n}$, and obtain

$$\begin{aligned} \Delta_r(x, R_t) &\geq \Big(\Big(2|s'_{\tilde{i}+1}|_1 + 2^{-n/2}BV(s''_{\tilde{i}+1})\Big)\varepsilon - \Big(2|x'|_1 + 2^{-n/2}BV(x'')\Big)\varepsilon\Big) \\ &\quad\cdot\Big(\Big(2|s'_{\tilde{i}-1}|_0 + 2^{-n/2}BV(\overline{s''_{\tilde{i}-1}})\Big)\varepsilon - \Big(2|x'|_0 + 2^{-n/2}BV(\overline{x''})\Big)\varepsilon\Big) \\ &\geq (2 - 2^{-n/2}BV(x''))\varepsilon\cdot(2 - 2^{-n/2}BV(\overline{x''}))\varepsilon > 2\varepsilon^2. \end{aligned} \tag{9}$$

With (6)-(9), we know the hypervolume contribution of a sole solution s

$$\Delta_r(x, R_t) \geq 2\varepsilon^2.$$

Now we will show that R_t contains a solution z with $\Delta_r(z, R_t) < 2\varepsilon^2 < \Delta_r(x, R_t)$. Note that for any $u \in \{0,1\}^n$, $|u'|_1$ takes its value from $[0..n/2]$, a set with size $n/2+1$. Since R_t has $N+1 \geq n/2+4$ (indeed $n/2+2$ is enough) from our assumption $N \geq n/2+3$ ($N \geq n/2+1$ is sufficient as well), the pigeonhole principle shows that there is a solution $s \in R_t$ such that the multiset $S := \{a \in R_t \mid |a'|_1 = |s'|_1\}$ has its size at least 2. If $\min\{|s'|_0, |s'|_1\} < \sqrt{n}$, from Definition 2, we know that for any $u, v \in S$, $f(u) = f(v)$. Then setting $z = s$, we have $\Delta_r(z, R_t) = 0$.

If $\min\{|s'|_0, |s'|_1\} \geq \sqrt{n}$, let $j \in [1..N+1]$ such that $|s'|_1 = |s'_j|_1$, and rewrite $S = \{s_{j_a}, s_{j_{a+1}}, \ldots, s_{j_b}\}$ with $j_a \leq j \leq j_b$ and $j_b - j_a \geq 2$. For the case when $j_b - j_a \geq 3$, we set $z = s_{j_{a+1}}$, and we have

$$\begin{aligned}\Delta_r(z, R_t) &\leq \left(\left(2|s'_{j_b}|_1 + 2^{-n/2}BV(s''_{j_b})\right)\varepsilon - \left(2|z'|_1 + 2^{-n/2}BV(z'')\right)\varepsilon\right)\\
&\quad\cdot \left(\left(2|s'_{j_a}|_0 + 2^{-n/2}BV(\overline{s''_{j_a}})\right)\varepsilon - \left(2|z'|_0 + 2^{-n/2}BV(\overline{z''})\right)\varepsilon\right)\\
&\leq (1 - 2^{-n/2} - 2^{-n/2}BV(z''))\varepsilon \cdot (1 - 2^{-n/2} - 2^{-n/2}BV(\overline{z''}))\varepsilon\\
&\leq \left(\frac{2 - 1 - 2^{-n/2}}{2}\right)^2 \varepsilon^2 \leq \frac{\varepsilon^2}{4}.\end{aligned}$$

For the case when $j_b - j_a = 2$, we set $z = s_{j_b}$ if $j_b \in [2..N]$ and $z = s_{j_a}$ if $j_a \in [2..N]$, and we simply denote $z = s_j$ here. We pessimistically consider the case where both neighbors of z satisfy $\min\{|z'_{j+1}|_0, |z'_{j+1}|_1\} \geq \sqrt{n}$ and $\min\{|z'_{j-1}|_0, |z'_{j-1}|_1\} \geq \sqrt{n}$, then we have

$$\begin{aligned}\Delta_r(z, R_t) &= \left(\left(2|s'_{j+1}|_1 + 2^{-n/2}BV(s''_{j+1})\right)\varepsilon - \left(2|z'|_1 + 2^{-n/2}BV(z'')\right)\varepsilon\right)\\
&\quad\cdot \left(\left(2|s'_{j-1}|_0 + 2^{-n/2}BV(\overline{s''_{j-1}})\right)\varepsilon - \left(2|z'|_0 + 2^{-n/2}BV(\overline{z''})\right)\varepsilon\right)\\
&\leq (3 - 2^{-n/2} - 2^{-n/2}BV(z''_i))\varepsilon \cdot (1 - 2^{-n/2} - 2^{-n/2}BV(\overline{z''}))\varepsilon\\
&\leq 2\varepsilon \cdot (1 - 2^{-n/2})\varepsilon < 2\varepsilon^2.\end{aligned}$$

□

Lemma (15). *Let $N \geq n/2 + 2$. Consider using the SPEA2 to optimize LF'_ε with problem size n. If at some iteration, the combined population R_t contains an individual x with $|x'|_1 = k$, then the next population P_{t+1} will also include an individual y with $|y'|_1 = k$.*

Proof of Lemma 15. Since all solutions in R_t are mutually non-dominated, we know that $|R_t| = N + \lambda > N$, and the truncation operator will iteratively remove solutions in R_t for λ times (See Step 5 in Algorithm 8). Let R denote the set of remaining individuals during this process until λ solutions are removed, and let $S = \{s \in R \mid |s'|_1 = |x'|_1\}$. Then we only need to prove that S will not be empty after the survival selection. Thus we only focus on the case when S decreases to 1, that is, $x \in S$ is a sole solution. We will prove that there always exists a $z \in R$ such that $\sigma_z^1 < \sigma_x^1$. Then by the truncation operator, we know an individual with smallest value of σ^1 will be removed (thus x will be kept) in the survival selection.

Let s be the nearest neighbor of a sole solution x. We have

$$\sigma_x^1 = \sqrt{\left(\mathrm{LF}'_{\varepsilon,1}(s) - \mathrm{LF}'_{\varepsilon,1}(x)\right)^2 + \left(\mathrm{LF}'_{\varepsilon,2}(s) - \mathrm{LF}'_{\varepsilon,2}(x)\right)^2}.$$

If $\min\{|s'|_0, |s'|_1\} < \sqrt{n}$ and $\min\{|x'|_0, |x'|_1\} < \sqrt{n}$, then

$$\begin{aligned}\sigma_x^1 &= \sqrt{(2|s'|_1\varepsilon - 2|x'|_1\varepsilon)^2 + (2|s'|_0\varepsilon - 2|x'|_0\varepsilon)^2}\\
&\geq \sqrt{(2\varepsilon)^2 + (2\varepsilon)^2} \geq 2\sqrt{2}\varepsilon.\end{aligned}$$

If $\min\{|s'|_0, |s'|_1\} < \sqrt{n}$ and $\min\{|x'|_0, |x'|_1\} \geq \sqrt{n}$, then

$$\begin{aligned}\sigma_x^1 &= \sqrt{\begin{aligned}&\left(2|s'|_1\,\varepsilon - \left(2|x'|_1 - 2^{-n/2}BV(x'')\right)\varepsilon\right)^2\\ &\quad + \left(2|s'|_0\,\varepsilon - \left(2|x'|_0 - 2^{-n/2}BV(\overline{x''})\right)\varepsilon\right)^2\end{aligned}}\\
&\geq \sqrt{(2\varepsilon)^2 + \left((2 - 1 + 2^{-n/2})\varepsilon\right)^2} > \sqrt{5}\,\varepsilon.\end{aligned}$$

Analogously, $\sigma_x^1 > \sqrt{5}\varepsilon$ also holds if $\min\{|s'|_0, |s'|_1\} \geq \sqrt{n}$ and $\min\{|x'|_0, |x'|_1\} < \sqrt{n}$. If $\min\{|s'|_0, |s'|_1\} \geq \sqrt{n}$ and $\min\{|x'|_0, |x'|_1\} \geq \sqrt{n}$, then

$$\begin{aligned}\sigma_s^1 &= \sqrt{\begin{aligned}&\big((2|s'|_1 - 2^{-n/2}BV(s''))\,\varepsilon - (2|x'|_1 - 2^{-n/2}BV(x''))\,\varepsilon\big)^2 \\ &\quad + \big((2|s'|_0 - 2^{-n/2}BV(s''))\,\varepsilon - (2|x'|_0 - 2^{-n/2}BV(\overline{x''}))\,\varepsilon\big)^2\end{aligned}} \\ &\geq \sqrt{\big((2-1+2^{-n/2})\varepsilon\big)^2 + \big((2-1+2^{-n/2})\varepsilon\big)^2} > \sqrt{2}\,\varepsilon.\end{aligned}$$

Therefore, we have

$$\sigma_s^1 > \sqrt{2}\varepsilon.$$

Now, we will show the existence of a solution $z \in R$ with $\sigma_z^1 < \sqrt{2}\varepsilon$. Since the population size $N \geq \frac{n}{2} + 1$, there are two solutions u and v with $|u'|_1 = |v'|_1$ (and hence $|u'|_0 = |v'|_0$). Let $\sigma_{u,v}$ denote the distance between $LF'_\varepsilon(u)$ and $LF'_\varepsilon(v)$. If $\min\{|u'|_0, |u'|_1\} < \sqrt{n}$, we know that $(\mathrm{LF}'_{\varepsilon,1}(u), \mathrm{LF}'_{\varepsilon,2}(u)) = (\mathrm{LF}'_{\varepsilon,1}(v), \mathrm{LF}'_{\varepsilon,2}(v))$, then $\sigma_{u,v}$ is 0. If $\min\{|u'|_0, |u'|_1\} \geq \sqrt{n}$, we have

$$\begin{aligned}\sigma_{u,v} &= \sqrt{\begin{aligned}&\big((2|u'|_1 - 2^{-n/2}BV(u''))\,\varepsilon - (2|v'|_1 - 2^{-n/2}BV(v''))\,\varepsilon\big)^2 \\ &\quad + \big((2|u'|_0 - 2^{-n/2}BV(u''))\,\varepsilon - (2|v'|_0 - 2^{-n/2}BV(\overline{v''}))\,\varepsilon\big)^2\end{aligned}} \\ &\leq \sqrt{\big((1-2^{-n/2})\varepsilon\big)^2 + \big((1-2^{-n/2})\varepsilon\big)^2} < \sqrt{2}\varepsilon.\end{aligned}$$

Recalling the definitions of σ_u^1, σ_v^1, and $\sigma_{u,v}$, we have $\sigma_u^1 \leq \sigma_{u,v}$ and $\sigma_v^1 \leq \sigma_{u,v}$. With z being one of the two solutions, we proved this lemma. □