# Compressed Sensing with Invertible Generative Models and Dependent Noise

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NEURAL INFORMATION PROCESSING SYSTEMS

## Compressed Sensing

### Motivation

- Inverse problems seek to recover an unknown signal from the given observations produced by a noisy process that transforms the original signal:

$$\boldsymbol{y} = f(\boldsymbol{x}) + \boldsymbol{\delta}$$

- A wide range of applications can be posed under this formulation with an appropriate choice of $f(\boldsymbol{x})$ and $\boldsymbol{\delta}$, such as compressed sensing, computed tomography, magnetic resonance imaging, and image super-resolution.
- Since $f(\boldsymbol{x})$ is not invertible in general, this problem is ill-posed even in the noiseless case. Classically, this is dealt by assuming a **sparsity prior** over the signal in some basis.

### Compressed Sensing with a GAN

- Bora et al. [1] proposed to replace the sparsity prior with a **GAN prior** trained on the domain of interest. Since GANs are capable of modeling complex high-dimensional signals, they can be an effective learned prior for inverse problems.
- Thus the signal recovery is done by finding a point in the range of GAN that minimizes reconstruction loss:

$$L_{\text{Bora}}(\boldsymbol{z};\boldsymbol{y}) = \|\boldsymbol{y} - AG(\boldsymbol{z})\|^2 + \lambda\|z\|^2$$

where $G$ is the generator from a pre-trained GAN and $\lambda$ is a regularization coefficient.

### Compressed Sensing with a Flow Model

- Asim et al. [2] pointed out that GAN prior does not generalize well to out-of-distribution samples, since it cannot represent any signal that's outside its range. Thus they proposed to use a **flow prior** instead where the generator $G$ is invertible:

$$L(\boldsymbol{z};\boldsymbol{y}) = \|\boldsymbol{y} - AG(\boldsymbol{z})\|^2 + \gamma\|\boldsymbol{z}\|$$

- Since $G$ is invertible, it can represent any signal in its range.
- We show that this loss is related to a specific instance of our general formulation for Gaussian noise $\boldsymbol{\delta} \sim \mathcal{N}(\mathbf{0}, \gamma I)$.

## MAP Formulation

### Maximum a Posteriori Objective

- We generalize Asim et al. [2] to the case of nonlinear forward operator $f(\boldsymbol{x})$ and general noise distribution $p_\Delta$.

$$L_{\text{G}}(\boldsymbol{z};\boldsymbol{y},\beta) = -\log p_\Delta(\boldsymbol{y} - f(G(\boldsymbol{z}))) - \beta \log p(G(\boldsymbol{z}))$$

### Theoretical Analysis

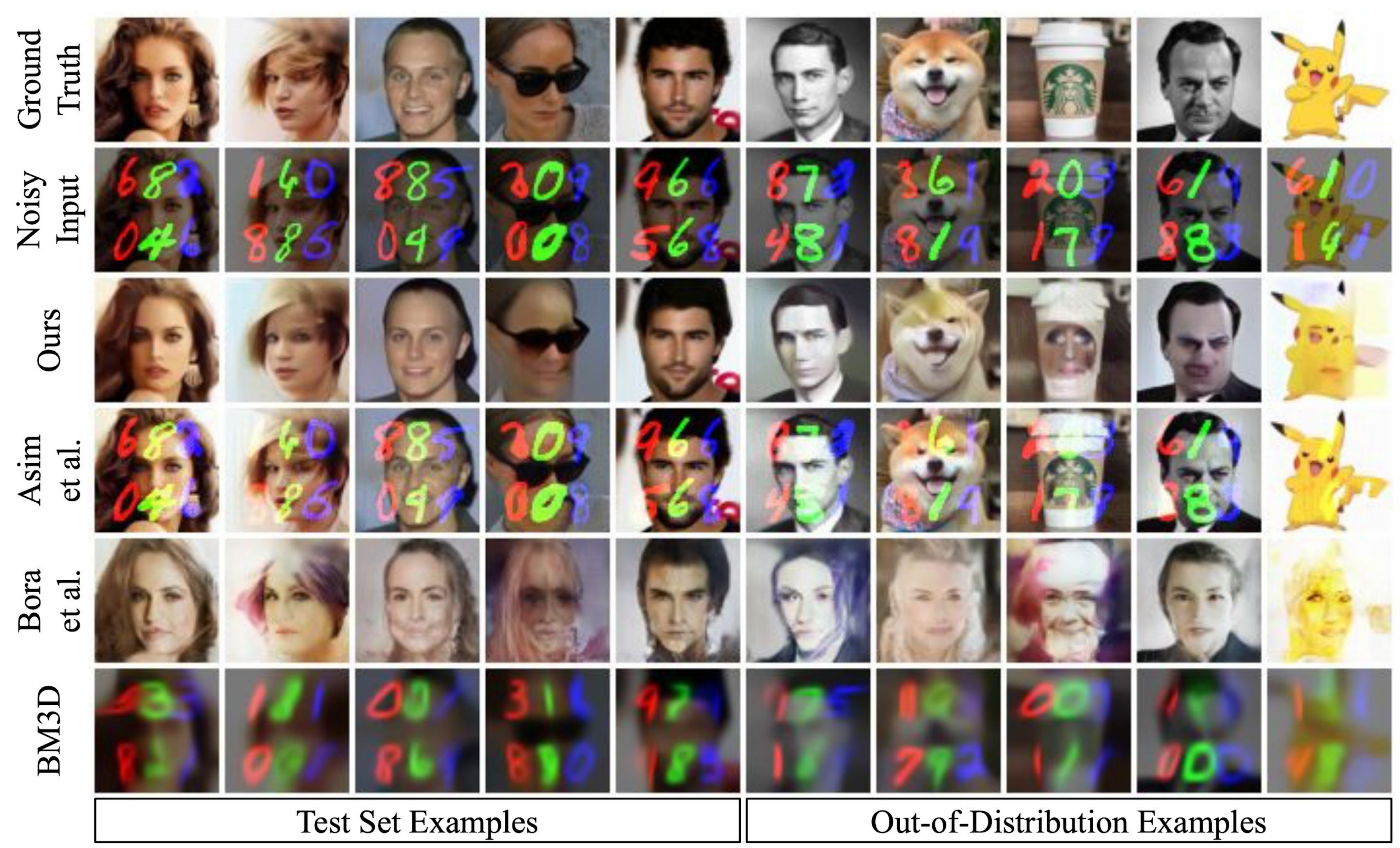
- (Gaussian Denoising) We show that when the log likelihood of the flow model is $\mu$ locally-concave around the ground truth signal, the L2 reconstruction error is bounded as

$$\|\bar{\boldsymbol{x}} - \boldsymbol{x}^*\| \le \frac{1}{\mu\sigma^2+1}\|\boldsymbol{\delta}\|$$

- (Compressed Sensing) We show that the worst-case reconstruction error for compressed sensing with Gaussian noise is bounded as $\mathbb{E}\|\bar{\boldsymbol{x}} - \boldsymbol{x}^*\| \le \sqrt{8\pi}\left(\frac{w(S(\rho))}{\sqrt{m}} + \epsilon\right)$, where $w(\cdot)$ is the Gaussian mean-width and $S(\rho) = \{\boldsymbol{x} | \log p(\boldsymbol{x}) \ge \rho\}$.

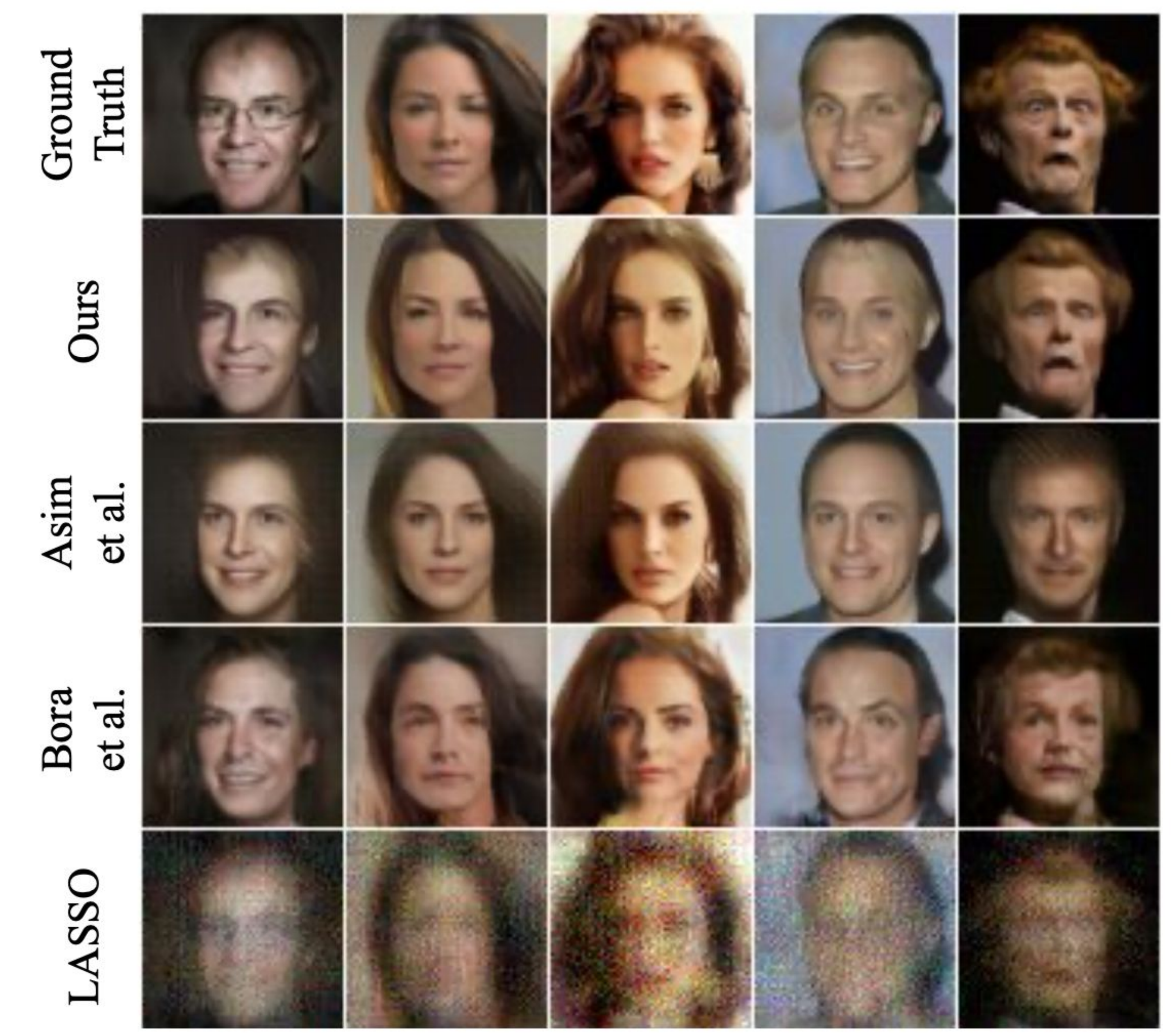
## Experimental Results

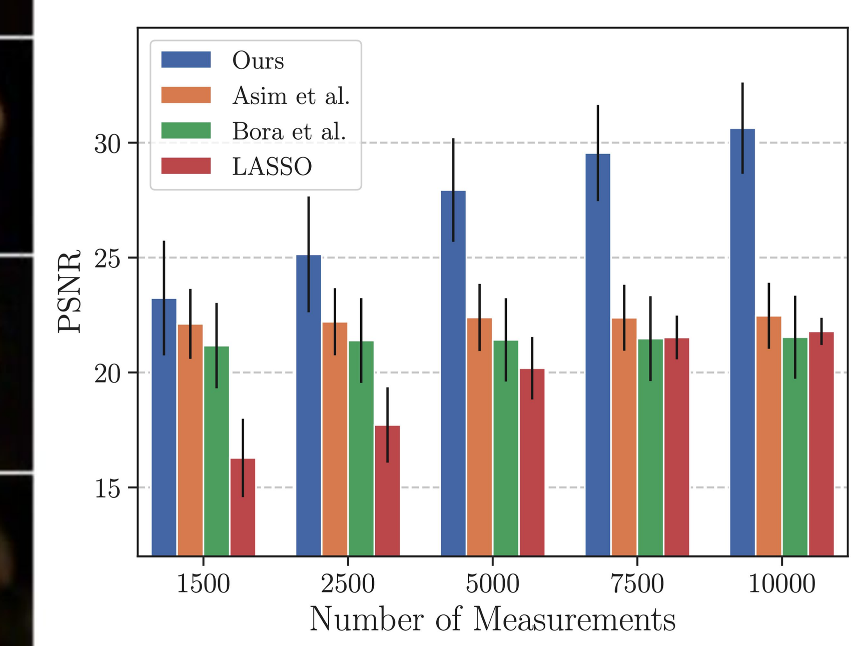
### Denoising MNIST digits from CelebA-HQ faces and OOD images



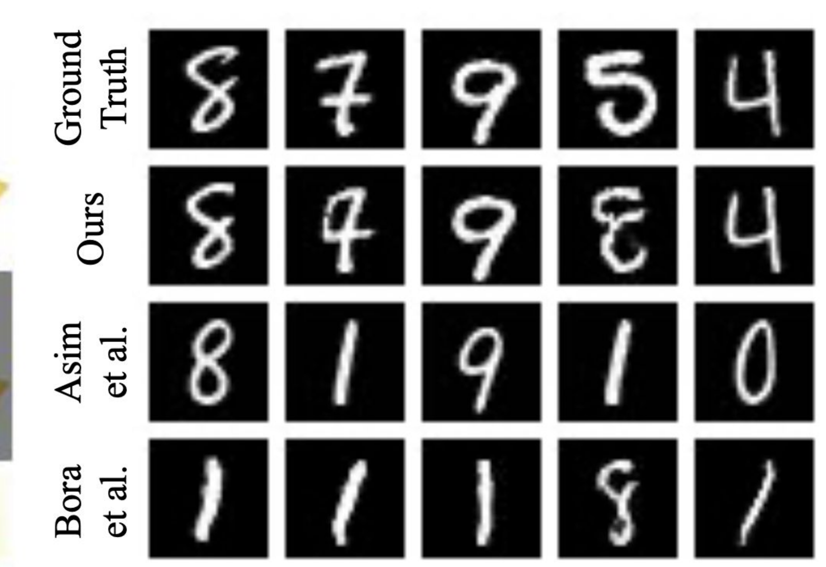

## Experimental Results

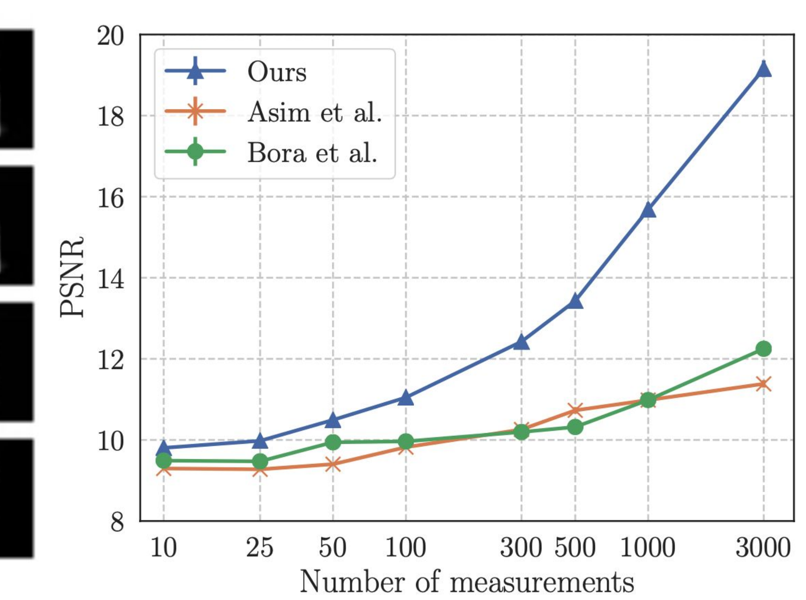
### Noisy Compressed Sensing




### 1-bit Compressed Sensing




## Conclusion

We propose a novel method that generalizes [1,2] to solve inverse problems for general differentiable forward operators and structured noise. The power of our approach stems from the flexiblity of flow models which can be combined in a modular way to solve inverse problems via MAP inference.

**References**

[1] Ashish Bora, Ajil Jalal, Eric Price, and Alexandros G Dimakis. Compressed sensing using generative models. In Proceedings of the 34th International Conference on Machine Learning- Volume 70, pages 537–546. JMLR. org, 2017.

[2] Muhammad Asim, Ali Ahmed, and Paul Hand. Invertible generative models for inverse problems: mitigating representation error and dataset bias. CoRR, abs/1905.11672, 2019. URL http://arxiv.org/abs/1905.11672.