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SCAN ME

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Covering Relations in the Poset of Combinatorial Neural Codes

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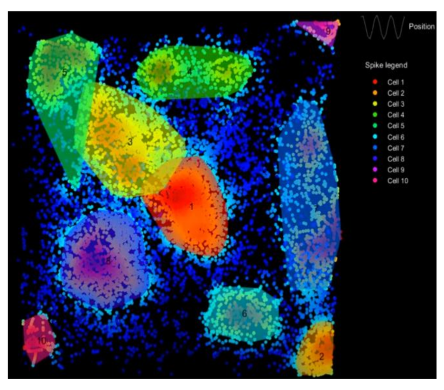
NEURAL INFORMATION PROCESSING SYSTEMS

MOTIVATION



Place fields of 10 recorded place cells in a rat [1] are approximately convex.

- **Place cells** fire in specific regions called **place fields**.
- Place fields are allocentric and believed to contribute to the **neural maps** of the environment.
- Place fields away from the walls are approximately open **convex** (e.g., circular or elliptical [2], [3]) for dorsal hippocampal place cells.

Collective cell co-firings can tell us about the convexity of their place fields [5].

PROBLEM

😳 **How can we determine if a neural code admits an arrangement of open convex place fields?**

🚨 **Complexity[7]: NP-hard and ∃ℝ-hard.**

PRELIMINARIES

✅ A **(combinatorial) neural code** is a collection of some subsets of $[n] = \{1, \ldots, n\}$ and $\emptyset$.

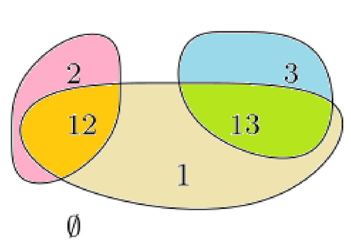
✅ Elements are **codewords**, written as strings (e.g., 12 instead of {1,2}).

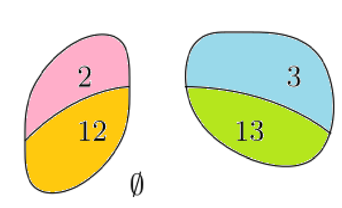
✅ **The code** of an open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ is

$$\mathcal{C}(\mathcal{U}) = \left\{\sigma \subseteq [n] : \bigcap_{i\in\sigma} U_i \setminus \bigcup_{j\in[n]\setminus\sigma} U_j \neq \emptyset\right\}$$

The code $\mathcal{C}_1 = \{\emptyset, \mathbf{1}, 2, 3, 12, 13\}$ is convex because it is the code $\mathcal{C}(\mathcal{U})$ of the below open convex cover $\mathcal{U}$.

The code $\mathcal{C}_2 = \{\emptyset, 2, 3, 12, 13\}$ cannot admit any arrangement of open convex cover.



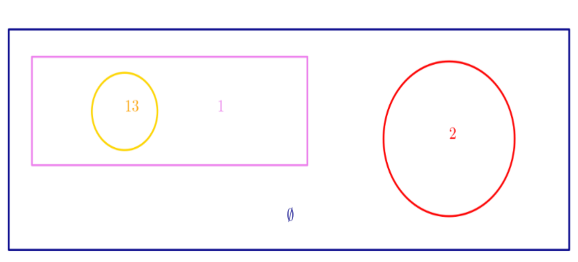


✅ A **trunk**[6] in a neural code $\mathcal{C}$ is either $\emptyset$ or $\text{Tk}_\mathcal{C}(\sigma)$ for some $\sigma \subseteq [n]$ where

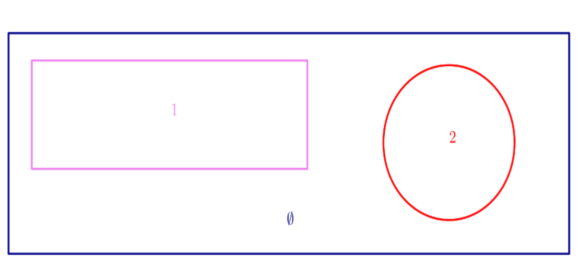
$$\text{Tk}_\mathcal{C}(\sigma) = \{\tau \in \mathcal{C} : \sigma \subseteq \tau\}$$

E.g.: $\mathcal{D} = \{\emptyset, 1, 2, 13\}$

- $\text{Tk}_\mathcal{D}(\emptyset) = \{\emptyset, 1, 2, 13\} = \mathcal{D}$
- $\text{Tk}_\mathcal{D}(1) = \{1, 13\}$
- $\text{Tk}_\mathcal{D}(2) = \{2\}$
- $\text{Tk}_\mathcal{D}(3) = \{13\}$



Trunks of $\emptyset$ (dark blue), 1 (pink), 2 (red), and 3 (orange) in $\mathcal{D}$.



Trunks of $\emptyset$ (dark blue), 1 (pink), and 2 (red) in $\mathcal{C}$.

E.g.: $\mathcal{C} = \{\emptyset, 1, 2\}$

- $\text{Tk}_\mathcal{C}(\emptyset) = \{\emptyset, 1, 2\} = \mathcal{C}$
- $\text{Tk}_\mathcal{C}(1) = \{1\}$
- $\text{Tk}_\mathcal{C}(2) = \{2\}$

✅ A **code morphism**[7] is a function $f: \mathcal{D} \to \mathcal{C}$ of neural codes such that preimage of every proper trunk $T \subseteq \mathcal{C}$ is a proper trunk in $\mathcal{D}$.

E.g.: $\mathcal{D} = \{\emptyset, 1, 2, 13\}$ and $\mathcal{C} = \{\emptyset, 1, 2\}$

$$f: \{\emptyset, 1, 2, 13\} \to \{\emptyset, 1, 2\}$$

$$\emptyset \mapsto \emptyset; \quad 1, 13 \mapsto 1; \quad 2 \mapsto 2$$

is a code surjective morphism since f is onto and

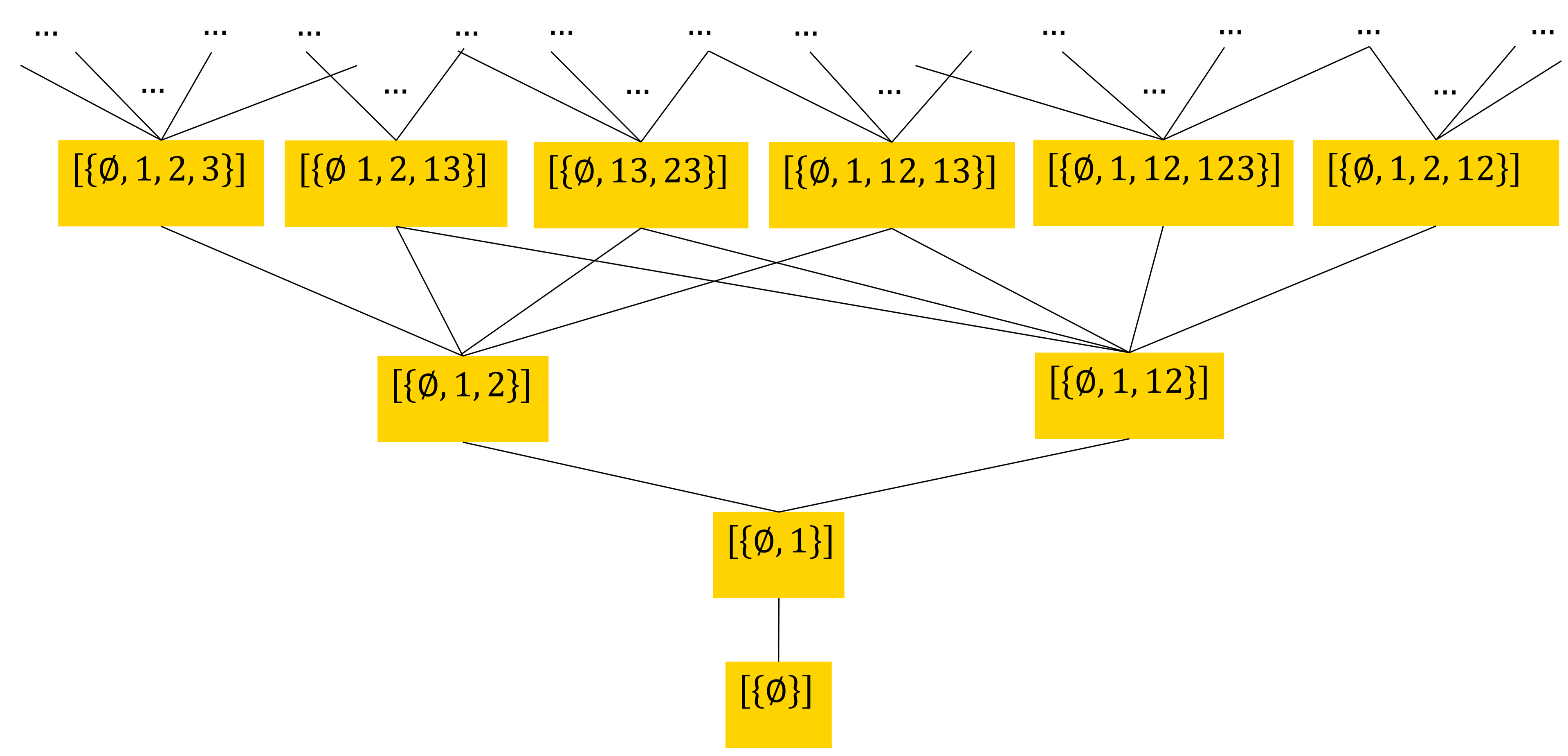
$$f^{-1}(\text{Tk}_\mathcal{C}(\emptyset)) = f^{-1}(\mathcal{C}) = \mathcal{D} = \text{Tk}_\mathcal{D}(\emptyset)$$

$$f^{-1}(\text{Tk}_\mathcal{C}(1)) = \{1,13\} = \text{Tk}_\mathcal{D}(1)$$

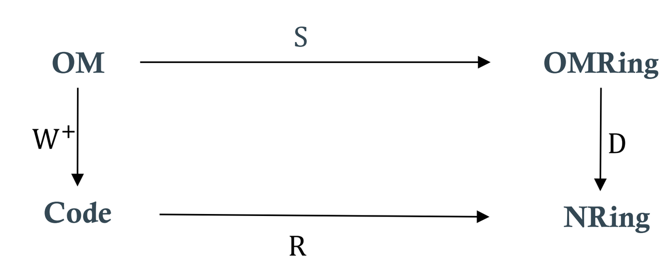
$$f^{-1}(\text{Tk}_\mathcal{C}(2)) = \{2\} = \text{Tk}_\mathcal{D}(2)$$

THE POSET $\mathbf{P}_{\textbf{Code}}$

✅ Code surjective morphisms induce a partial order named "**is a minor of**" on the set of isomorphism classes of neural codes, resulting in a poset called $\mathbf{P}_{\textbf{Code}}$[7].

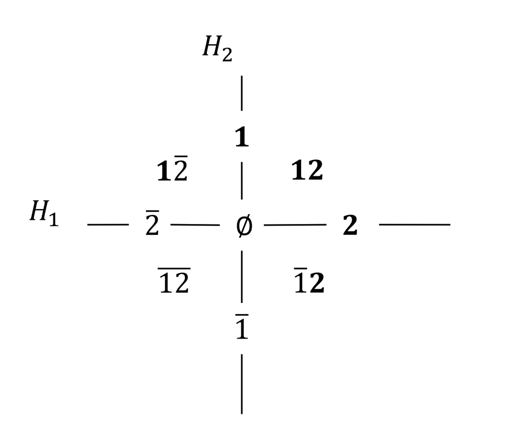


✅ The category of acyclic oriented matroids **OM**, that of oriented matroid rings **OMRing**, that of neural codes **Code**, and that of neural rings **NRing** commute in the following diagram[8]:

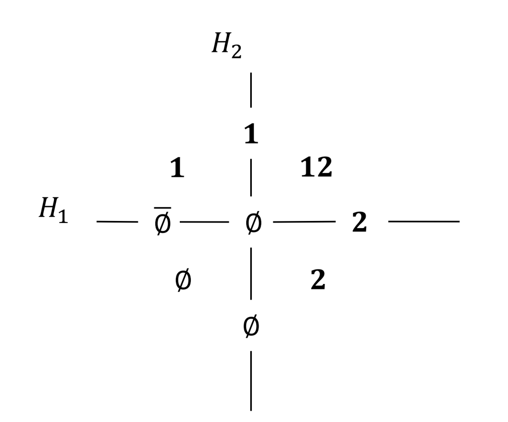


😳 CONJECTURE 😳

Every open convex neural code is the image of some representable oriented matroid under some code surjective morphism[8].



The covectors of an oriented matroid arising from an arrangement of H_1 and H_2.



The convex code $\mathcal{C} = \{\emptyset, 1, 2, 12\}$ given by the positive open half-spaces of H_1 and H_2.

OUR CONTRIBUTIONS

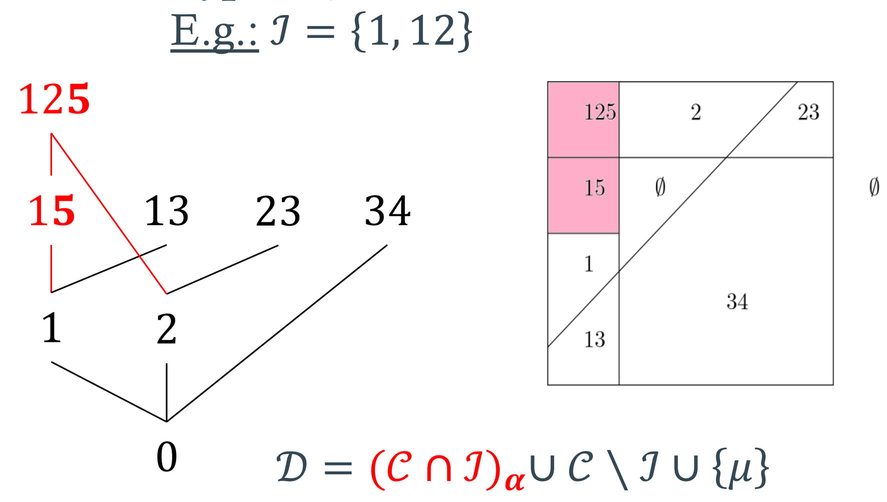
💡 ISOLATED SUBSETS 💡

Given an ∩-complete neural code $\mathcal{C}$. An **isolated subset** $\mathcal{I}$ of $\mathcal{C}$ is an ∩-complete subset that satisfies the condition that $\sigma \not\supseteq \tau, \forall \sigma \in \mathcal{C} \setminus \mathcal{I}, \forall \tau \in \mathcal{I} \setminus \{\mu\}$ where μ is the minimal codeword in $\mathcal{I}$.

💡 COVERING CODES 💡

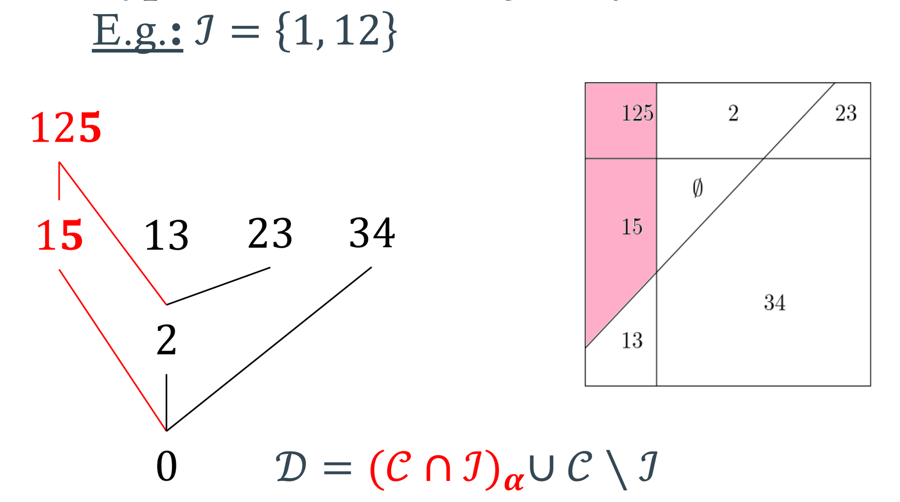
For every isolated subset $\mathcal{I}$ of the ∩-completion $\hat{\mathcal{C}}$ of $\mathcal{C}$, depending on the conditions on $\mathcal{I}$, one can construct a neural code $\mathcal{D}$ covering $\mathcal{C}$ in **Pcode** by one of the following ways (with examples).

Type 1: $\mu \in \mathcal{C}$.
E.g.: $\mathcal{I} = \{1, 12\}$



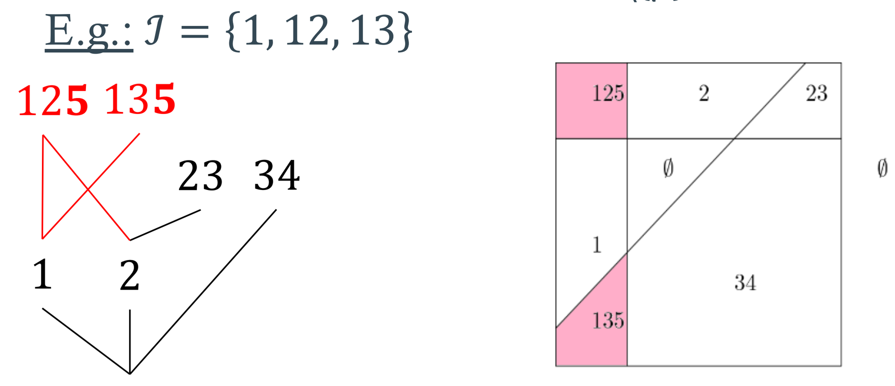
$$\mathcal{D} = (\mathcal{C} \cap \mathcal{I})_\alpha \cup \mathcal{C} \setminus \mathcal{I} \cup \{\mu\}$$

Type 2: $\mu \in \mathcal{C}$ and $\text{Tk}_\mathcal{C}(\mu) \setminus \mathcal{I} \neq \emptyset$.
E.g.: $\mathcal{I} = \{1, 12\}$



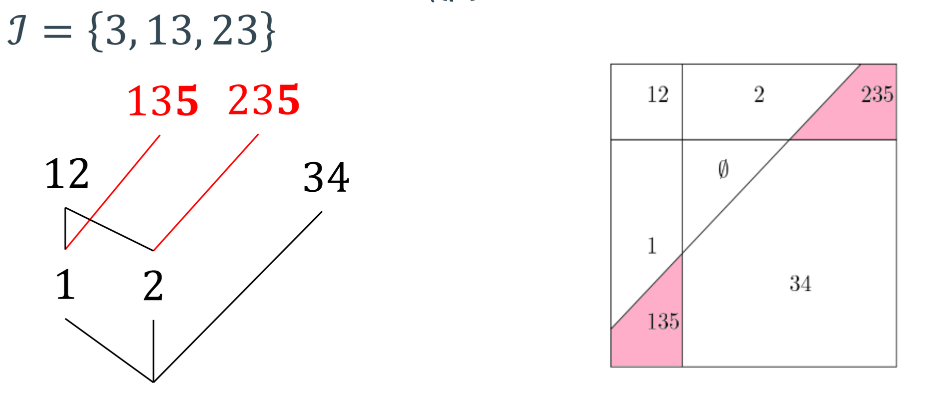
$$\mathcal{D} = (\mathcal{C} \cap \mathcal{I})_\alpha \cup \mathcal{C} \setminus \mathcal{I}$$

Type 3: $\mu \in \mathcal{C}$ and $\mu = \bigcap_{\sigma \in \mathcal{C} \cap \mathcal{I} \setminus \{\mu\}} \sigma$.
E.g.: $\mathcal{I} = \{1, 12, 13\}$



$$\mathcal{D} = (\mathcal{C} \cap \mathcal{I} \setminus \{\mu\})_\alpha \cup \mathcal{C} \setminus \mathcal{I} \cup \{\mu\}$$

Type 4: $\mu \notin \mathcal{C}$, $\mu = \bigcap_{\sigma \in \mathcal{C} \cap \mathcal{I} \setminus \{\mu\}} \sigma$, and $\text{Tk}_\mathcal{C}(\mu) \setminus \mathcal{I} \neq \emptyset$.
E.g.: $\mathcal{I} = \{3, 13, 23\}$



$$\mathcal{D} = (\mathcal{C} \cap \mathcal{I})_\alpha \cup \mathcal{C} \setminus \mathcal{I}$$

*Note: $(S)_\alpha$ denotes the operation of adding a new neuron α to all codewords in S.

References

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