Supplement to “SA-Learner: Surrogate-Assisted Meta-Learner with Missing Outcomes”

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A Demonstrate Meta-Learner with Surrogate Outcomes

For illustration, we focus on four specified meta learners that are used in our experiments for Algorithm 1: S-learner (Wager and Athey, 2018), T-learner (Wager and Athey, 2018), X-learner (Wager and Athey, 2018), and DR-learner (Kennedy, 2023).

S-learner. We train $\hat{\mu}(x, a, s)$ on the labeled data $\mathbf{L}$ and regress $\hat{\mu}(X, A, S)$ on (X, A) for the entire data $\mathbf{S}$ to get $\hat{\nu}(x, a)$ as the estimator of $\nu(x, a)$. The CATE estimator of the S-learner is obtained by

$$\hat{\tau}(x) = \hat{\nu}(x, 1) - \hat{\nu}(x, 0).$$

T-learner. We train $\hat{\mu}(x, a, s)$ on the labeled data $\mathbf{L}$ and regress $\hat{\mu}(X, A, S)$ on X for both the treated and control groups to obtain $\hat{\nu}_0(x)$ and $\hat{\nu}_1(x)$ for each group. The CATE estimator of the T-learner is obtained by

$$\hat{\tau}(x) = \hat{\nu}_1(x) - \hat{\nu}_0(x).$$

X-learner. X-learner and DR-learner are slightly complicated. The outcome models $\mu(x, a, s)$ and $\nu(x, a)$ are identical to those used in T-learner. In addition to the outcome models, we train the propensity score (PS) $\pi(x)$ on the entire data $\mathbf{S}$ and obtain the estimator $\hat{\pi}(x)$. The PS estimator $\hat{\pi}(x)$ will be used for both X-learner and DR-learner.

For the X-learner, we need to train two models for the imputed treatment effects of the treated and control groups. We regress $\hat{\nu}_1(X) - \hat{\mu}(X, A, S)$ on X from the control group, i.e., $\{(X_i, \hat{\nu}_1(X_i) - \hat{\mu}(X_i, A_i, S_i) : A_i = 0, Z_i \in \mathbf{S}\}$ to obtain $\hat{D}_0(x)$. Analogously, we regress $\hat{\mu}(X, A, S) - \hat{\nu}_0(X)$ on X from the treated group, i.e., $\{(X_i, \hat{\mu}(X_i, A_i, S_i) - \hat{\nu}_0(X_i)) : A_i = 1, Z_i \in \mathbf{S}\}$ to obtain $\hat{D}_1(x)$. The CATE estimator of the X-learner is then obtained by

$$\hat{\tau}(x) = \hat{D}_0(x)\hat{\pi}(x) + \hat{D}_1(x)(1 - \hat{\pi}(x)).$$

DR-learner. We regress the pseudo-outcome

$$\frac{A - \hat{\pi}(X)}{\hat{\pi}(X)(1 - \hat{\pi}(X))}(\hat{\mu}(X, A, S) - \hat{\nu}_A(X))$$

on X from the entire data $\mathbf{S}$ to obtain the CATE estimator of the DR-learner $\hat{\tau}(x)$. Note that because we replace Y_i by the proxy $\mu(X_i, A_i, S_i)$ the DR-learner is no longer enjoy the semi-parametric efficiency.

B CATE Estimation of the Targeted Population

We show that the empirical loss of the SA-learner can be used for learning other estimates of interest.

B.1 CATE on the Treated Population

The CATE on the Treated (CATT) is defined as

$$\tau_{CATT}(x) = \mathbb{E}[Y(1) - Y(0)|A = 1, X = x].$$

It measures the heterogeneous treatment effect among those who are already treated (Heckman et al., 1997). Assumption 2(b) in the main paper can be weakened by Ignoriability of the control: $(S(0), Y(0)) \perp A|X$. Under the refined assumptions, we define the doubly robust score for CATT as

$$\zeta_{CATT}(z; \overline{\mu}, \overline{\rho}, \overline{\nu}, \overline{\pi}) = \frac{a}{\mathbb{P}(A = 1)}(\overline{\nu}(x, 1) - \overline{\nu}(x, 0)) + \varphi_{CATT}(z; \overline{\mu}, \overline{\rho}, \overline{\nu}, \overline{\pi}),$$

where

$$\varphi_{CATT}(z; \overline{\mu}, \overline{\rho}, \overline{\nu}, \overline{\pi}) = \frac{a - \overline{\pi}(x)}{\mathbb{P}(A = a)(1 - \overline{\pi}(x))}\left(\frac{r(y - \overline{\mu}(x, a, s))}{\overline{\rho}(x, a, s)} + \overline{\mu}(x, a, s) - \overline{\nu}(x, a)\right).$$

The loss function is obtained by replacing the probability measure $\mathbb{P}$ by the empirical measure $\mathbb{P}_n$, and the nuisance function (μ, ρ, ν, π) by their estimation in $\zeta_{CATT}(z; \mu, \rho, \nu, \pi)$.

Let $\hat{\zeta}^{(-c(i))}_{CATT}(z) = \zeta_{CATT}(z; \hat{\mu}^{(-c(i))}, \hat{\rho}^{(-c(i))}, \hat{\nu}^{(-c(i))}, \hat{\pi}^{(-c(i))})$. The loss function is written by

$$\hat{L}_{CATT}(\tau) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\zeta}^{(-c(i))}_{CATT}(Z_i) - \tau(X_i))^2.$$

The rest of the learning procedure follows Algorithm 2. The theory for CATT can be derived analogously.

B.2 CATE on the Control Population

The CATE on the Control (CATC) is defined as

$$\tau_{CATC}(x) = \mathbb{E}[Y(1) - Y(0)|A = 0, X = x].$$

It measures the heterogeneous treatment effect among those who are not treated yet. Assumption 2(b) in the main paper can be weakened by Ignoriability of the treated: $(S(1), Y(1)) \perp A|X$. Under the refined assumptions, we define the doubly robust score for CATC as

$$\zeta_{CATC}(z; \overline{\mu}, \overline{\rho}, \overline{\nu}, \overline{\pi}) = \frac{1 - a}{\mathbb{P}(A = 0)}(\overline{\nu}(x, 1) - \overline{\nu}(x, 0)) + \varphi_{CATC}(z; \overline{\mu}, \overline{\rho}, \overline{\nu}, \overline{\pi}),$$

where

$$\varphi_{CATC}(z; \overline{\mu}, \overline{\rho}, \overline{\nu}, \overline{\pi}) = \frac{a - \overline{\pi}(x)}{\mathbb{P}(A = 1 - a)(1 - \overline{\pi}(x))}\left(\frac{r(y - \overline{\mu}(x, a, s))}{\overline{\rho}(x, a, s)} + \overline{\mu}(x, a, s) - \overline{\nu}(x, a)\right).$$

The loss function is obtained by replacing the probability measure $\mathbb{P}$ by the empirical measure $\mathbb{P}_n$, and the nuisance function (μ, ρ, ν, π) by their estimation.

Let $\hat{\zeta}^{(-c(i))}_{CATC}(z) = \zeta_{CATC}(z; \hat{\mu}^{(-c(i))}, \hat{\rho}^{(-c(i))}, \hat{\nu}^{(-c(i))}, \hat{\pi}^{(-c(i))})$. The loss function is written by

$$\hat{L}_{CATC}(\tau) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\zeta}^{(-c(i))}_{CATC}(Z_i) - \tau(X_i))^2.$$

The rest of the learning procedure follows Algorithm 2. The theory for CATC can be derived analogously.

B.3 CATE on the Labeled Population

The CATE on the Labeled (CATL) is defined as

$$\tau_{CATL}(x) = \mathbb{E}[Y(1) - Y(0)|R = 1, X = x].$$

It measures the heterogeneous treatment effect among the labeled subjects. In this case, there is no need of surrogate outcomes since we already fully observe the primary outcome in the labeled population of interest. The semiparametric efficiency analysis of the CATL estimator immediately follows from restricting the analysis to the labeled subpopulation $\mathbf{L}$.

C Proofs

C.1 Proof for Proposition 1

Without loss of generality, we show that $\mathbb{E}[Y(1) \mid X] = \mathbb{E}_S[\mu(X, A, S) \mid A = 1, X]$. Under Assumption 1, we have

$$\begin{aligned}
\mathbb{E}_S[\mu(X, 1, S) \mid A = 1, X] &= \mathbb{E}_S[\mathbb{E}[Y \mid X, A = 1, S, R = 1] \mid A = 1, X] \\
&= \mathbb{E}_S[\mathbb{E}[Y(1) \mid X, A = 1, S(1), R = 1] \mid A = 1, X] \\
&= \mathbb{E}_S[\mathbb{E}[Y(1) \mid X, A = 1, S(1)] \mid A = 1, X] \\
&= \mathbb{E}[Y(1) \mid X, A = 1] \\
&= \mathbb{E}[Y(1) \mid X].
\end{aligned}$$

The second equation follows from Assumption 1(a), the third from Assumption 1(d), and the fourth from the law of iterated expectations. The final equation is justified by Assumption 1(b), while Assumption 1(c) ensures the existence of the relevant conditional expectations.

We remark that our assumptions are weaker than the literature (Kallus and Mao, 2024; Zeng et al., 2024; Gao et al., 2025) as we do not require the covariates X to be the confounders between the surrogate outcomes $S(a)$ and the treatment A. Because we are only interested in the causal effects of the treatment A on the primary outcome Y, impose such restriction on the covariates X is irrelevant to the problem of interest.

C.2 Proof for Proposition 2

Without loss of generality, we show that

$$\mathbb{E}\left[\overline{\nu}(x, 1) + \frac{a}{\overline{\pi}(x)}\left(\frac{r(y - \overline{\mu}(x, 1, s))}{\overline{\rho}(x, 1, s)} + \overline{\mu}(x, 1, s) - \overline{\nu}(x, 1)\right) - Y(1)\right] = 0$$

if $(\overline{\mu}, \overline{\nu}) = (\mu, \nu)$ or $(\overline{\rho}, \overline{\pi}) = (\rho, \pi)$.

By the law of the total expectation, the left-hand side can be simplify by

$$
\begin{aligned}
&\mathbb{E}\left[\overline{\nu}(x,1)+\frac{\pi(x)}{\overline{\pi}(x)}\left(\frac{\rho(x,1,s)}{\overline{\rho}(x,1,s)}(\mu(x,1,s)-\overline{\mu}(x,1,s))+\overline{\mu}(x,1,s)-\overline{\nu}(x,1)\right)-\nu(x,1)\right]\\
&=\mathbb{E}\left[\left(1-\frac{\pi(x)}{\overline{\pi}(x)}\right)(\overline{\nu}(x,1)-\nu(x,1))\right]+\mathbb{E}\left[\frac{\pi(x)}{\overline{\pi}(x)}\left(\frac{\rho(x,1,s)}{\overline{\rho}(x,1,s)}-1\right)(\mu(x,1,s)-\overline{\mu}(x,1,s))\right]\\
&+\mathbb{E}\left[\frac{\pi(x)}{\overline{\pi}(x)}(\mu(x,1,s)-\nu(x,1))\right]\\
&:=I+II+III.
\end{aligned}
$$

We show that $III=0$. Consider

$$
|III|\leq\sup_{x\in\mathcal{X}}\left|\frac{\pi(x)}{\overline{\pi}(x)}\right||\mathbb{E}[\mu(x,1,s)-\nu(x,1)]|=0
$$

The last equation follows from $\mathbb{E}[\mu(x,1,s)\mid X,A=1]=\nu(x,1)$, and both $\pi(x)$ and $\overline{\pi}(x)$ are bounded in $[c,1-c]\subset(0,1)$ for some constant $c>0$ and for all $x\in\mathcal{X}$. Therefore, if $(\overline{\mu},\overline{\nu})=(\mu,\nu)$ or $(\overline{\rho},\overline{\pi})=(\rho,\pi)$, $I=II=0$, where $II=0$ is obtained by applying an analogous argument for $III=0$.

C.3 Proof for Proposition 3

Continue the proof of Proposition 2. We replace the nuisance function $(\overline{\mu},\overline{\rho},\overline{\nu},\overline{\pi})$ in the expression by their estimator $(\hat{\mu}^{(-c(i))},\hat{\rho}^{(-c(i))},\hat{\nu}^{(-c(i))},\hat{\pi}^{(-c(i))})$. According to the expression of I and II, we have $|I|=O_p(r_\nu(n)r_\pi(n))$ and $|II|=O_p(r_\mu(n)r_\rho(n))$, where $|II|$ follows again by the boundedness of both PS $\pi(x)$ and its estimator $\hat{\pi}(x)$. Therefore, the error term is of the order

$$
\begin{aligned}
|\mathbb{E}[\hat{\zeta}^{(-c(i))}(Z)]-\psi|&\leq|I|+|II|\\
&=O_p(\max(r_\nu(n)r_\pi(n),r_\mu(n)r_\rho(n))).
\end{aligned}
$$

C.4 Useful Definitions and Lemmas for Theorem 1

We define the directional derivative of a functional $F:\mathcal{F}\to\mathbb{R}$ by

$$
D_fF(f)[h]=\frac{\partial}{\partial\epsilon}F(f+\epsilon h)\Big|_{\epsilon=0}
$$

for functions $f,h\in\mathcal{F}$. Analogously, we define the kth-order directional derivative of a functional $F:\mathcal{F}\to\mathbb{R}$ by

$$
D_f^kF(f)[h_1,\dots,h_k]=\frac{\partial^k}{\partial\epsilon_1\dots\partial\epsilon_k}F(f+\epsilon_1h_1+\cdots+\epsilon_kh_k)\Big|_{\epsilon_1=\cdots=\epsilon_k=0}
$$

for functions $f,h_1,\dots,h_k\in\mathcal{F}$. In addition to the directional derivative, we also define the gradient of a functional $F:\mathcal{F}^k\to\mathbb{R}$ by

$$
\nabla_fF(f)=\left(\frac{\partial}{\partial f_1}F(f),\dots,\frac{\partial}{\partial f_k}F(f)\right)
$$

for functions $f = (f_1, \dots, f_k) \in \mathcal{F}^k$. A key connection between the directional derivative and the gradient is

$$D_f F(f)[h] = \langle \nabla_f F(f), h \rangle.$$

Let $\ell(\tau, \zeta(g)) = (\tau - \zeta(g))^2$ and $L(\tau, g) = \mathbb{E}[\ell(\tau, \zeta(g))]$ to highlight the inputs of the CATE function τ and nuisance function $g = (\mu_1, \mu_0, \rho_1, \rho_0, \nu_1, \nu_0, \pi)$, where $f_a(x, s) = f(x, a, s)$ for f being μ, ρ, and ν; let τ_0 and g_0 represent the underlying CATE function and nuisance functions, respectively. The following lemmas are used in the proof of Theorem 1.

Lemma 1 (First-order Optimality). *We have*

$$D_\tau L(\tau, g_0)[\tau - \tau_0] \geq 0$$

for all $\tau \in \Gamma$.

Lemma 2 (Orthogonality). *We have*

$$D_g D_\tau L(\tau, g)[\tau - \tau_0, g - g_0] = 0$$

for all $\tau \in \Gamma$ and $g \in \mathcal{G}$.

Lemma 3 (Convexity). *Let*

$$r_g = \|\nu_1 - \nu_1^*\|_\infty \left\| 1 - \frac{\pi_1}{\pi_1^*} \right\|_\infty + \|\mu_1 - \mu_1^*\|_\infty \left\| 1 - \frac{\rho_1}{\rho_1^*} \right\|_\infty, \tag{1}$$

where f^* indicates the underlying functions of f. Then,

$$|D_g^2 D_\tau L(\tau, g_0)[\tau - \tau_0, g - g_0, g - g_0]| \leq 4 r_g \|\tau - \tau_0\|_2$$

for all $\tau \in \Gamma$ and $g \in \mathcal{G}$.

Lemma 4 (Excess Risk). *We have*

$$L(\hat{\tau}, \hat{g}) - L(\tau_0, \hat{g}) \asymp \|\tilde{\tau} - \tau_0\|_2^2.$$

C.5 Proof for Theorem 1

We follow Foster and Syrgkanis (2023) to apply orthogonal statistical learning theory. By a second-order Taylor expansion, there exists $\overline{\tau} = \lambda \hat{\tau} + (1 - \lambda)\tau_0$ for some $\lambda \in [0, 1]$ such that

$$\frac{1}{2} D_\tau^2 L(\overline{\tau}, \hat{g})[\hat{\tau} - \tau_0, \hat{\tau} - \tau_0] = L(\hat{\tau}, \hat{g}) - L(\tau_0, \hat{g}) - D_\tau L(\hat{\tau}_0, \hat{g})[\hat{\tau} - \tau_0]. \tag{2}$$

Besides, a direct calculation yields

$$D_\tau^2 L(\overline{\tau}, g_0)[\hat{\tau} - \tau_0, \hat{\tau} - \tau_0] = 2\|\hat{\tau} - \tau_0\|_2^2. \tag{3}$$

Combine Equation (3) and (2), we have

$$\|\hat{\tau} - \tau_0\|_2^2 = L(\hat{\tau}, \hat{g}) - L(\tau_0, \hat{g}) - D_\tau L(\tau_0, \hat{g})[\hat{\tau} - \tau_0]. \tag{4}$$

Apply a second-order Taylor expansion again, there exists $\bar{g} = \lambda\hat{g} + (1-\lambda)g_0$ for some $\lambda \in [0,1]$ such that

$$\begin{aligned}
-D_\tau L(\tau_0, \hat{g})[\hat{\tau}-\tau_0] =& - D_\tau L(\tau_0, g_0)[\hat{\tau}-\tau_0] - D_g D_\tau L(\tau_0, g_0)[\hat{\tau}-\tau_0, \hat{g}-g_0] \\
&- \frac{1}{2} D_g^2 D_\tau L(\tau_0, g_0)[\hat{\tau}-\tau_0, \hat{g}-g_0, \hat{g}-g_0] \\
=& -D_\tau L(\tau_0, g_0)[\hat{\tau}-\tau_0] - \frac{1}{2} D_g^2 D_\tau L(\tau_0, g_0)[\hat{\tau}-\tau_0, \hat{g}-g_0, \hat{g}-g_0]. \qquad (5)
\end{aligned}$$

The second equation follows from Lemma 2. We plug in Equation (5) into Equation (4) and apply Lemma 3 to obtain

$$\begin{aligned}
\|\hat{\tau}-\tau_0\|_2^2 &= L(\hat{\tau}, \hat{g}) - L(\tau_0, \hat{g}) - D_\tau L(\tau_0, g_0)[\hat{\tau}-\tau_0] - \frac{1}{2} D_g^2 D_\tau L(\tau_0, g_0)[\hat{\tau}-\tau_0, \hat{g}-g_0, \hat{g}-g_0]. \\
&\leq L(\hat{\tau}, \hat{g}) - L(\tau_0, \hat{g}) - D_\tau L(\tau_0, g_0)[\hat{\tau}-\tau_0] + 2r_g\|\tau-\tau_0\|_2 \\
&\leq L(\hat{\tau}, \hat{g}) - L(\tau_0, \hat{g}) - D_\tau L(\tau_0, g_0)[\hat{\tau}-\tau_0] + \eta\|\tau-\tau_0\|_2^2 + \frac{2}{\eta} r_g^2. \qquad (6)
\end{aligned}$$

The last inequality follows from Young's inequality for any $\eta > 0$. Rearranging Equation (6) and applying Lemma 1, we select $\eta \in (0,1)$ and obtain

$$\begin{aligned}
\|\hat{\tau}-\tau_0\|_2^2 &\leq \frac{1}{1-\eta}(L(\hat{\tau}, \hat{g}) - L(\tau_0, \hat{g}) - D_\tau L(\tau_0, g_0)[\hat{\tau}-\tau_0]) + \frac{2}{(1-\eta)\eta} r_g^2 \\
&\leq \frac{1}{1-\eta}(L(\hat{\tau}, \hat{g}) - L(\tau_0, \hat{g})) + \frac{2}{(1-\eta)\eta} r_g^2. \qquad (7)
\end{aligned}$$

We apply Lemma 4 and a basic inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ to Equation (7); then

$$\begin{aligned}
\|\hat{\tau}-\tau_0\|_2 &\lesssim \|\tilde{\tau}-\tau_0\|_2 + r_g \\
&= O_p(n^{-\gamma}) + O_p(r(n)).
\end{aligned}$$

The last equation follows by Assumption 2 and the expression of r_g in Equation (1). The rate of convergence for r_g is obtained by an analogous argument of Proposition 2.

C.6 Proof for Lemma 1

According to the variance-bias decomposition, we have $L(\tau, g_0) = \mathbb{E}[(\tau-\tau_0)^2] + \mathrm{Var}(\zeta(g_0))$ as $\mathbb{E}[\zeta(g_0) \mid X] = \tau(X_0)$; then a direct calculation yields

$$D_\tau L(\tau, g_0)[\tau-\tau_0] = 2\mathbb{E}[(\tau(X)-\tau_0(X))^2] \geq 0$$

for all $\tau \in \Gamma$.

C.7 Proof for Lemma 2

Let $V = (X, A, S)$. Notice

$$\begin{aligned}
D_g D_\tau L(\tau, g)[\tau-\tau_0, g-g_0] &= -2\mathbb{E}[(\tau(X)-\tau_0(X))\langle\nabla_g\zeta(g), g(V)-g_0(V)\rangle] \\
&= -2\mathbb{E}[(\tau(X)-\tau_0(X))\mathbb{E}[\langle\mathbb{E}[\nabla_g\zeta(g) \mid V], g(V)-g_0(V)\rangle \mid X]], \qquad (8)
\end{aligned}$$

where

$$\nabla_g \zeta(g) = \Bigg(\frac{a}{\pi}\left(1-\frac{r}{\rho_1}\right), \frac{1-a}{1-\pi}\left(1-\frac{r}{\rho_0}\right), \frac{-ra}{\rho_1^2 \pi}(y-\mu_1), \frac{r(1-a)}{\rho_0^2(1-\pi)}(y-\mu_0),$$

$$1-\frac{a}{\pi}, 1-\frac{1-a}{1-\pi}, \frac{-a}{\pi^2}\left(\frac{r}{\rho_1}(y-\mu_1)+\mu_1-\nu_1\right) - \frac{1-a}{(1-\pi)^2}\left(\frac{r}{\rho_0}(y-\mu_0)+\mu_0-\nu_0\right)\Bigg).$$

Equation (8) holds by the law of iterated expectations. For convenience, let $\tilde{g} = (0,0,0,0,\nu_1,\nu_0,\pi)$. A straightforward calculation yields

$$\mathbb{E}[\nabla_g \zeta(g) \mid V] = \left(0,0,0,0,1-\frac{a}{\pi}, 1-\frac{1-a}{1-\pi}, \frac{-a}{\pi^2}(\mu_1-\nu_1) - \frac{1-a}{(1-\pi)^2}(\mu_0-\nu_0)\right)$$

and thus

$$\begin{aligned} \mathbb{E}[\langle \mathbb{E}[\nabla_g \zeta(g) \mid V], g(V)-g_0(V)\rangle \mid X] &= \langle \mathbb{E}[\nabla_g \zeta(g) \mid X], \tilde{g}(X)-\tilde{g}_0(X)\rangle \\ &= 0 \end{aligned} \tag{9}$$

The last equation follows by $\mathbb{E}[\nabla_g \zeta(g) \mid X] = \mathbb{E}[\mathbb{E}[\nabla_g \zeta(g) \mid V] \mid X] = (0,\ldots,0)$. Putting Equation (9) back into Equation (8), we conclude that $D_g D_\tau L(\tau,g)[\tau-\tau_0, g-g_0] = 0$ for all $\tau \in \Gamma$ and $g \in \mathcal{G}$.

C.8 Proof for Lemma 3

A direct calculation yields

$$\begin{aligned} D_g^2 D_\tau L(\tau,g_0)[\tau-\tau_0, g-g_0, g-g_0] &= -2\mathbb{E}[(\tau(X)-\tau_0(X))\langle g(V)-g_0(V), \nabla_g^2 \zeta(g)(g(V)-g_0(V))\rangle] \\ &= -2\mathbb{E}[(\tau(X)-\tau_0(X))\mathbb{E}[\langle g(V)-g_0(V), \mathbb{E}[\nabla_g^2 \zeta(g) \mid V](g(V)-g_0(V))\rangle \mid X]], \end{aligned}$$

where the last equation follows from the law of iterated expectations. Analogous to the proof of Lemma 2, a direct calculation yields

$$\begin{aligned} \mathbb{E}[\langle g(V)-g_0(V), \mathbb{E}[\nabla_g^2 \zeta(g) \mid V](g(V)-g_0(V))\rangle \mid X] &= 2(\nu_1-\nu_1^*)\left(1-\frac{\pi_1}{\pi_1^*}\right) - 2\mathbb{E}\left[(\mu_1-\mu_1^*)\left(1-\frac{\rho_1}{\rho_1^*}\right)\middle| X\right] \\ &\leq 2r_g, \end{aligned}$$

Then, we have

$$|D_g^2 D_\tau L(\tau,g_0)[\tau-\tau_0, g-g_0, g-g_0]| \leq 4r_g\|\tau-\tau_0\|_2$$

for all $\tau \in \Gamma$.

C.9 Proof for Lemma 4

Because $\hat{\tau}$ is the minimizer of $L(\tau,\hat{g})$, we have $L(\hat{\tau},\hat{g}) \leq L(\tilde{\tau},\hat{g})$. Notice that

$$\begin{aligned} L(\tilde{\tau},\hat{g}) - L(\tau_0,\hat{g}) &= \|\tilde{\tau}-\tau_0\|_2^2 + 2\mathbb{E}[(\tilde{\tau}-\tau_0)(\tau_0-\zeta(\hat{g}))] \\ &= \|\tilde{\tau}-\tau_0\|_2^2 + 2\mathbb{E}[(\tilde{\tau}(X)-\tau_0(X))\mathbb{E}[\zeta(g_0)-\zeta(\hat{g}) \mid V]] \\ &\leq \|\tilde{\tau}-\tau_0\|_2^2 + 2\|\tilde{\tau}-\tau_0\|_2^2 \sup_{v\in\mathcal{V}} |\mathbb{E}[\zeta(g_0)-\zeta(\hat{g}) \mid V=v]| \end{aligned}$$

The second equation follows from the law of iterated expectations. Furthermore, by Proposition 3,

$$\sup_{v \in \mathcal{V}} |\mathbb{E}[\zeta(g_0) - \zeta(\hat{g}) \mid V = v]| = O_p(r(n)).$$

Hence, under Assumption 2,

$$L(\hat{\tau}, \hat{g}) - L(\tau_0, \hat{g}) = O_p(\max(n^{-2\gamma}, n^{-2\gamma} r(n))) = O_p(n^{-2\gamma}) \asymp \|\tilde{\tau} - \tau_0\|_2^2.$$

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