Appendix A
Under LS reconstruction, $\Delta_1 \leq 0$

For LS we have:

$$\boldsymbol{R}_{\mathcal{S}} = \boldsymbol{U}_k(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^{\dagger}.$$

Lemma 1. For any matrix $\boldsymbol{A}$, $||\boldsymbol{U}_k\boldsymbol{A}||_F^2 = ||\boldsymbol{A}||_F^2$

Proof.

$$\begin{aligned}||\boldsymbol{U}_k\boldsymbol{A}||_F^2 &= \mathrm{tr}(\boldsymbol{U}_k\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{U}_k^T) = \mathrm{tr}(\boldsymbol{U}_k^T\boldsymbol{U}_k\boldsymbol{A}\boldsymbol{A}^T)\\ &= \mathrm{tr}(\boldsymbol{A}\boldsymbol{A}^T) = ||\boldsymbol{A}||_F^2.\end{aligned}$$

□

Lemma 2. For LS, $\xi_1(\mathcal{S}) = k - \mathrm{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)$.

Proof. Using Lemma 1,

$$\begin{aligned}\xi_1(\mathcal{S}) &= ||\boldsymbol{U}_k - \boldsymbol{R}_{\mathcal{S}}\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k||_F^2\\ &= ||\boldsymbol{U}_k - \boldsymbol{U}_k(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^{\dagger}\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k||_F^2\\ &= ||\boldsymbol{I}_k - (\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^{\dagger}\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k||_F^2\end{aligned}$$

Let $\boldsymbol{\Pi} = (\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^{\dagger}\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k$. $\boldsymbol{\Pi}$ is of the form $\boldsymbol{A}^{\dagger}\boldsymbol{A}$, so is a symmetric orthogonal projection onto the range of $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$ [23, p. 258]. Orthogonal projections are idempotent ($\boldsymbol{\Pi} = \boldsymbol{\Pi}^2$) hence have eigenvalues which are 0 or 1, and therefore $\mathrm{tr}(\boldsymbol{\Pi}) = \mathrm{rank}((\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T) = \mathrm{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)$. We then have:

$$\begin{aligned}\xi_1(\mathcal{S}) &= ||\boldsymbol{I}_k - \boldsymbol{\Pi}||_F^2\\ &= \mathrm{tr}((\boldsymbol{I}_k - \boldsymbol{\Pi})(\boldsymbol{I}_k - \boldsymbol{\Pi})^T)\\ &= \mathrm{tr}((\boldsymbol{I}_k - \boldsymbol{\Pi})(\boldsymbol{I}_k - \boldsymbol{\Pi}))\\ &= \mathrm{tr}(\boldsymbol{I}_k - 2\boldsymbol{\Pi} + \boldsymbol{\Pi}^2)\\ &= \mathrm{tr}(\boldsymbol{I}_k - \boldsymbol{\Pi})\\ &= \mathrm{tr}(\boldsymbol{I}_k) - \mathrm{tr}(\boldsymbol{\Pi})\\ &= k - \mathrm{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k).\end{aligned}$$

□

Lemma 3. For LS, $\Delta_1(\mathcal{S}, v) \in \{0, -1\}$.

Proof. Removing a vertex from $\mathcal{S}$ removes a row from $\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k$, reducing the rank by 0 or 1.

$$\begin{aligned}\Delta_1(\mathcal{S}, v) &= \xi_1(\mathcal{S}) - \xi_1(\mathcal{S}\backslash\{v\})\\ &= -\mathrm{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k) + \mathrm{rank}(\boldsymbol{M}_{\mathcal{S}\backslash\{v\}}\boldsymbol{U}_k)\\ &\in \{0, -1\}.\end{aligned}$$

□

Non-positivity of Δ_1 immediately follows from Lemma 3.

Appendix B
Under LS reconstruction, $\Delta_1 < 0 \iff \Delta_2 > 0$

We first need the following lemmas.

Lemma 4.

$$\xi_2(\mathcal{S}) = \sum_{\lambda_i^{\mathcal{S}} \neq 0} \frac{1}{\lambda_i^{\mathcal{S}}} \tag{20}$$

where $\lambda_i^{\mathcal{S}}$ is the i^{th} eigenvalue of $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$.

Proof. By definition and Appendix A, Lemma 1

$$\begin{aligned}\xi_2(\mathcal{S}) &= ||\boldsymbol{R}_{\mathcal{S}}||_F^2\\ &= ||\boldsymbol{U}_k(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^{\dagger}||_F^2\\ &= ||(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^{\dagger}||_F^2\end{aligned}$$

which is the sum of the squares of the singular values of $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^{\dagger}$ [23, Corollary 2.4.3]. The pseudoinverse maps the singular values of $\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k$ onto the singular values of $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^{\dagger}$ in the following way [23, Section 5.5.2]:

$$\sigma_i((\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^{\dagger}) = \begin{cases}0 & \text{if } \sigma_i(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k) = 0\\ \sigma_i(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^{-1} & \text{otherwise}\end{cases} \tag{21}$$

and the squares of the singular values of $\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k$ are λ_i [23, Eq. (8.6.1)]. Summing them gives the result. □

Lemma 5.

$$\mathrm{rank}((\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T) = \mathrm{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k) \leq k.$$

Proof. For the equality: $\mathrm{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)$ is the number of strictly positive singular values it has [23, Corollary 2.4.6]. By [23, Eq. (8.6.2)], this is the same as the number of strictly positive eigenvalues of $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T)$, which is $\mathrm{rank}((\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T)$.

For the inequality: $\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k$ has k columns and so must have column rank less than or equal to k. Row rank being equal to column rank gives the result. □

Lemma 6. For LS, $\Delta_1 = 0 \iff \Delta_2 \leq 0$.

Proof. Note that $(\boldsymbol{M}_{\mathcal{S}\backslash\{v\}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}\backslash\{v\}}\boldsymbol{U}_k)^T$ is a principal submatrix of $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$. Write the eigenvalues of $(\boldsymbol{M}_{\mathcal{S}\backslash\{v\}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}\backslash\{v\}}\boldsymbol{U}_k)^T$ as $\lambda_1, \ldots, \lambda_n$ and the eigenvalues of $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$ as $\mu_1, \ldots \mu_{n+1}$. Then by Cauchy's Interlacing Theorem [24, p. 59],

$$0 \leq \mu_1 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \mu_{n+1} \leq 1 \tag{22}$$

where the outer bounds come from the fact that both matrices are principal submatrices of $\boldsymbol{U}_k\boldsymbol{U}_k^T$, an orthogonal projection matrix.

1) $\Delta_1 = 0 \implies \Delta_2 \leq 0$: $\Delta_1 = 0$ implies the rank of $\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k$ does not change with the removal of v, so neither does the rank of $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$. As the rank is unchanged, $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$ has one more zero-eigenvalue than $(\boldsymbol{M}_{\mathcal{S}\backslash\{v\}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}\backslash\{v\}}\boldsymbol{U}_k)^T$. This means:

$$\mu_1 = 0 \tag{23}$$

$$\lambda_i = 0 \iff \mu_{i+1} = 0 \tag{24}$$

By Cauchy's Interlacing Theorem, $\lambda_i \leq \mu_{i+1}$ and so

$$\frac{1}{\lambda_i} \geq \frac{1}{\mu_{i+1}} \text{ if } \lambda_i \neq 0 \text{ and } \mu_{i+1} \neq 0. \tag{25}$$

Therefore

$$\sum_{\lambda_i^{\mathcal{S}} \neq 0} \frac{1}{\lambda_i^{\mathcal{S}}} \geq \sum_{\mu_i^{\mathcal{S}} \neq 0} \frac{1}{\mu_i^{\mathcal{S}}} \tag{26}$$

as we have the same number of non-zero terms in each of these terms by (23) and (24), and the inequality is proved by summing over the non-zero terms using (25). Equation (26) is exactly

$$\xi_2(\mathcal{S}\backslash\{v\}) \geq \xi_2(\mathcal{S}). \tag{27}$$

Rearranging gives $\Delta_2 \leq 0$.

2) $\Delta_1 = 0 \iff \Delta_2 \leq 0$: We prove the equivalent statement

$$\Delta_1 \neq 0 \implies \Delta_2 > 0. \tag{28}$$

By Lemma 3, if $\Delta_1 \neq 0$ then $\Delta_1 = -1$. This means that the rank of $\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k$ is reduced by 1 by the removal of v, therefore $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$ has one more non-zero eigenvalue than $(\boldsymbol{M}_{\mathcal{S}\backslash\{v\}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}\backslash\{v\}}\boldsymbol{U}_k)^T$. This means:

$$\mu_{n+1} > 0 \tag{29}$$

$$\lambda_i \neq 0 \iff \mu_i \neq 0 \tag{30}$$

By Cauchy's interlacing theorem, $\lambda_i \geq \mu_i$ and so

$$\frac{1}{\lambda_i} \leq \frac{1}{\mu_i} \text{ if } \lambda_i \neq 0 \text{ and } \mu_i \neq 0. \tag{31}$$

Let I be the number of zero eigenvalues of $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$. Then

$$\sum_{I\leq i\leq n} \frac{1}{\lambda_i^{\mathcal{S}}} \leq \sum_{I\leq i\leq n} \frac{1}{\mu_i^{\mathcal{S}}} < \sum_{I\leq i\leq n+1} \frac{1}{\mu_i^{\mathcal{S}}}. \tag{32}$$

With the left inequality by matching terms via (30) and then summing over (31), and the right inequality because (29) means $\frac{1}{\mu_{n+1}} > 0$. We then note the left and the right terms in this equality say:

$$\sum_{\lambda_i^{\mathcal{S}}\neq 0} \frac{1}{\lambda_i^{\mathcal{S}}} < \sum_{\mu_i^{\mathcal{S}}\neq 0} \frac{1}{\mu_i^{\mathcal{S}}} \tag{33}$$

or equivalently,

$$\xi_2(\mathcal{S}\backslash\{v\}) < \xi_2(\mathcal{S}). \tag{34}$$

Rearranging gives $\Delta_2 > 0$. □

We finally have the following:

Lemma 7. For LS, $\Delta_1 < 0 \iff \Delta_2 > 0$.

Proof. By Lemma 3 and Lemma 6. □

Appendix C
Proof of Theorem 1

Proof. For brevity, we fix $\mathcal{S}$ and v and write $\Delta_1 = \Delta_1(\mathcal{S}, v)$ and $\Delta_2 = \Delta_2(\mathcal{S}, v)$.

Rearranging (14) gives us that v improves $\mathcal{S}$ if and only if

$$\Delta_1 + \sigma^2 \cdot \Delta_2 > 0 \tag{35}$$

or equivalently if and only if

$$\Delta_1 > -\sigma^2 \cdot \Delta_2. \tag{36}$$

By definition, $\sigma^2 = \frac{k}{N\cdot\text{SNR}}$, so this condition is equivalent to

$$\Delta_1 > -\frac{k}{N\cdot\text{SNR}}\Delta_2 \tag{37}$$

and as SNR is strictly positive, this is equivalent to

$$\text{SNR}\cdot\Delta_1 > -\frac{k}{N}\Delta_2. \tag{38}$$

We can now use the major lemmas from the previous appendices. By Lemma 3, we have two possible values of $\Delta_1(\mathcal{S}, v)$:

$\Delta_1 = 0$:

Lemma 6 means $\Delta_2 < 0$, so

$$\Delta_1 + \sigma^2 \cdot \Delta_2 = \sigma^2 \cdot \Delta_2 < 0 \tag{39}$$

and so v does not improve $\mathcal{S}$.

$\Delta_1 = -1$:

Eq. (38) simplifies to:

$$-\text{SNR} > -\frac{k}{N}\Delta_2 \tag{40}$$

which is equivalent to

$$\text{SNR} < \frac{k}{N}\Delta_2. \tag{41}$$

On the one hand, v improves $\mathcal{S}$ implies $\Delta_1 = -1$, which implies (41). On the other hand, (41) implies $\Delta_2 > 0$ which in turn implies $\Delta_1 = -1$, which means (41) implies (38), which implies v improves $\mathcal{S}$.

Noting that the right-hand side of (41) is $\tau(\mathcal{S}, v)$, which completes the proof. □

Appendix D
Proof of Theorem 2

We restate the theorem:

Theorem 4. Consider any sequence of vertices $v_1, \ldots, v_N$ with no repeated vertices, and let $\mathcal{S}_i = \{v_1, \ldots, v_i\}$. Then there are exactly k indices $I_1, \ldots, I_k$ such that under LS reconstruction of a noisy k-bandlimited signal,

$$\forall 1 \leq j \leq k : \tau(\mathcal{S}_{I_j}, v_{I_j}) > 0 \tag{42}$$

and so for some SNR > 0 removing v_{I_j} would improve $\mathcal{S}_{I_j}$.

Proof. By Appendix C, Lemma 2:

$$\xi_1(\mathcal{S}_i) = k - \text{rank}(\boldsymbol{M}_{\mathcal{S}_i}\boldsymbol{U}_k). \tag{43}$$

By Appendix C, Lemma 3, $\Delta_1 \in \{0, -1\}$ and as $\text{rank}(\boldsymbol{U}_k) = k$, $\xi_1(\mathcal{S}_N) = 0$. As $\xi_1(\mathcal{S}_0) = k$, we must have exactly k indices for which $\Delta_1(\mathcal{S}_i, v_i) = -1$, and by Appendix C, Lemma 6 we have exactly k indices for which $\Delta_2(\mathcal{S}_i, v_i) > 0$. As $\tau(\mathcal{S}_i, v_i) = \frac{k}{N}\Delta_2(\mathcal{S}_i, v_i)$, we're done. □

APPENDIX E
PROOF OF THEOREM 3

Proof. By Appendix C, Lemma 2, the noiseless error

$$\xi_1(\mathcal{S}) = k - \text{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k) \tag{44}$$

must be 0, as we can perfectly reconstruct any k-bandlimited signal. Therefore, $\text{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k) = k$.

$\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k$ is a $k \times k$ matrix of full rank, so its rows must be linearly independent. Any subset of linearly independent rows is linearly independent, so for any non-empty $\mathcal{R} \subset \mathcal{S}$, $\boldsymbol{M}_{\mathcal{R}}\boldsymbol{U}_k$ has linearly independent rows.

Greedy schemes pick increasing sample sets: that is, if asked to pick a vertex sample set $\mathcal{S}_m$ of size m for $m < k$ and a sample set $\mathcal{S}$ of size k, $\mathcal{S}_m \subset \mathcal{S}$. Therefore for any sample set $\mathcal{S}_m$ of size $m \le k$ picked by the scheme, $\boldsymbol{M}_{\mathcal{S}_m}\boldsymbol{U}_k$ has independent rows.

If $\boldsymbol{M}_{\mathcal{S}_m}\boldsymbol{U}_k$ has independent rows, then removal of any row (corresponding to removing any vertex) reduces its rank by 1; that is,

$$\forall m \le k: \ \forall v \in \mathcal{S}_m: \ \Delta_1(\mathcal{S}_m, v) = -1 \tag{45}$$

Then, by Appendix C, Lemma 7,

$$\forall m \le k: \ \forall v \in \mathcal{S}_m: \ \Delta_2(\mathcal{S}_m, v) > 0 \tag{46}$$

and as $\tau(\mathcal{S}_m, v) = \frac{k}{N}\Delta_2(\mathcal{S}_m, v)$ and $\frac{k}{N} > 0$,

$$\forall m \le k: \ \forall v \in \mathcal{S}_m: \ \tau(\mathcal{S}_m, v) > 0. \tag{47}$$

This proves (18).

As $\boldsymbol{M}_{\mathcal{S}_k}\boldsymbol{U}_k$ has k independent rows, it is of rank k. Adding further rows can't decrease its rank, so for $m' > k$, $\text{rank}(\boldsymbol{M}_{\mathcal{S}_{m'}}\boldsymbol{U}_k) \ge k$. As $\boldsymbol{U}_k$ is of rank k, $\text{rank}(\boldsymbol{M}_{\mathcal{S}_{m'}}\boldsymbol{U}_k) \le k$. This means for all samples sizes $m' > k$, $\text{rank}(\boldsymbol{M}_{\mathcal{S}_{m'}}\boldsymbol{U}_k) = k$. This says that further additions of rows do not change rank; that is:

$$\forall m' > k: \ \forall v \in \mathcal{S}_{m'} \backslash \mathcal{S}_k: \ \Delta_1(\mathcal{S}_{m'}, v) = 0 \tag{48}$$

Then, by Appendix C, Lemma 6,

$$\forall m' > k: \ \forall v \in \mathcal{S}_{m'} \backslash \mathcal{S}_k: \ \Delta_2(\mathcal{S}_{m'}, v) \le 0 \tag{49}$$

and, like for (18, as $\tau(\mathcal{S}_m, v) = \frac{k}{N}\Delta_2(\mathcal{S}_m, v)$ and $\frac{k}{N} > 0$,

$$\forall m' > k: \ \forall v \in \mathcal{S}_{m'} \backslash \mathcal{S}_k: \ \tau(\mathcal{S}_{m'}, v) \le 0. \tag{50}$$

This proves (19). □

APPENDIX F
PROOF OF REMARK 4

A-Optimality

A-optimality depends on the existence of the inverse of $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$ existing, which requires it to be of full rank. By Appendix C, Lemma 5, if an A-optimal scheme picks a set $\mathcal{S}$ of size k, then $\text{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k) = k$. Therefore, $\mathcal{S}$ is a uniqueness set [17] and can perfectly reconstruct any k-bandlimited signal.

D- and E-optimality

We show that for sample sizes less than k we can always pick a row which keeps $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$ full rank (of rank $|\mathcal{S}|$), and that D- and E-optimal schemes do so.

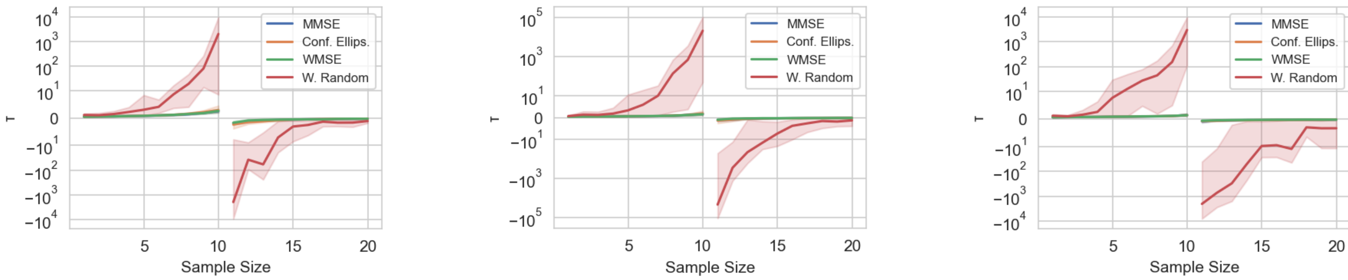
By Appendix C, Lemma 5, $\text{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T = \text{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)$, so we only need to ensure $\text{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k) = |\mathcal{S}|$.

We proceed by induction: given $\mathcal{S}_1$ with $|\mathcal{S}_1| = 1$, $\text{rank}(\boldsymbol{M}_{\mathcal{S}_1}\boldsymbol{U}_k) = 1$. Assume that for $\mathcal{S}_i$ with $|\mathcal{S}_i| = i < k$, $\text{rank}(\boldsymbol{M}_{\mathcal{S}_i}\boldsymbol{U}_k) = i$. As $\text{rank}(\boldsymbol{U}_k) = k$ and $i < k$, we can find a row to add to $\boldsymbol{M}_{\mathcal{S}_i}\boldsymbol{U}_k$ which will increase its rank (else all other rows would lie in the i-dimensional space spanned by the rows of $\boldsymbol{M}_{\mathcal{S}_i}\boldsymbol{U}_k$, which would imply $\text{rank}(\boldsymbol{U}_k) = i$, which is a contradiction as $i < k$). Adding the vertex which corresponds to the row to $\mathcal{S}_i$ gives $\mathcal{S}_{i+1}$ with $\text{rank}(\boldsymbol{M}_{\mathcal{S}_{i+1}}\boldsymbol{U}_k) = i + 1$.

We have shown that we can greedily choose to keep $\text{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k) = |\mathcal{S}|$. We now show that D- and E-optimal schemes do so. The eigenvalues of $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$ are non-negative (see Appendix C, Eq. (22)), so any invertible $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$ will have a strictly positive determinant and minimum eigenvalue, which are preferable under the D- and E- optimality criterion respectively to a non-invertible $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$, which has a determinant and minimum eigenvalue of 0. Therefore, greedy D- and E- optimal sampling schemes will make sure $(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k)^T$ is invertible, and thus keep $\text{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k) = |\mathcal{S}|$ for $|\mathcal{S}| \le k$. Therefore when D- and E- optimal schemes pick a set $\mathcal{S}$ of size k, $\text{rank}(\boldsymbol{M}_{\mathcal{S}}\boldsymbol{U}_k) = k$. Therefore, $\mathcal{S}$ is a uniqueness set [17] and can perfectly reconstruct any k-bandlimited signal.

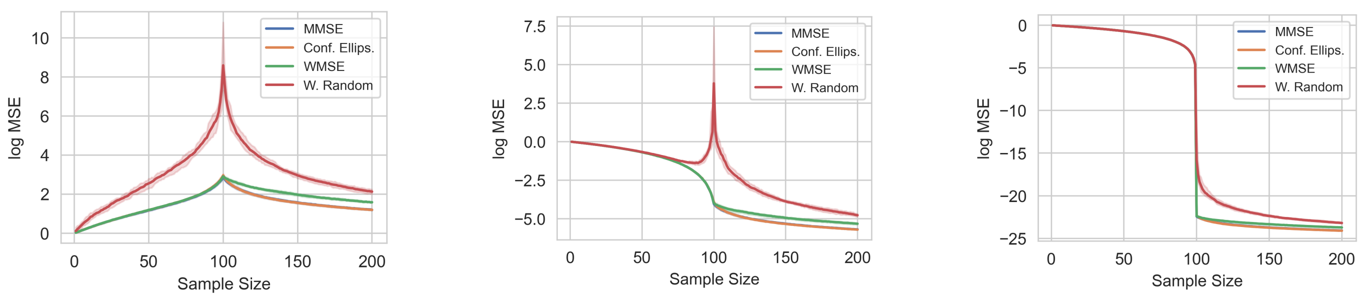
APPENDIX G
ADDITIONAL RESULTS

We show thresholds for the ER, BA and SBM graphs with 100 vertices (Fig. 3). We also present MSE plots for the larger BA (Fig 4) and SBM (Fig 5) graphs.



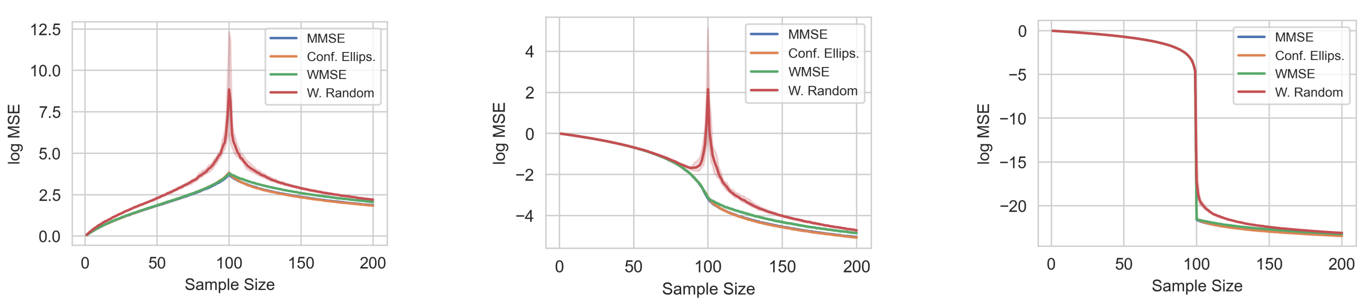
(a) Erdos-Renyi (b) Barabasi-Albert (c) SBM

Fig. 3: τ for different random graph models under LS reconstruction (#vertices = 100, bandwidth = 10)



(a) SNR = 10^{-1} (b) SNR = 10^{2} (c) SNR = 10^{10}

Fig. 4: Average MSE for LS reconstruction on BA Graphs (#vertices=1000, bandwidth = 100) with different SNRs



(a) SNR = 10^{-1} (b) SNR = 10^{2} (c) SNR = 10^{10}

Fig. 5: Average MSE for LS reconstruction on SBM Graphs (#vertices=1000, bandwidth = 100) with different SNRs