Conditional Causal Effect for Individual Attribution (Supplementary Material)

A PROOF OF LEMMA 1

We can write the conditional probability as

$$\mathrm{P}\left(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1 \mid \mathbf{X} = \mathbf{x}\right) = \frac{\mathrm{P}\left(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{X} = \mathbf{x}\right)}{\mathrm{P}\left(\mathbf{X} = \mathbf{x}\right)}.$$

We first show the identifiability of the numerator.

$$\begin{aligned}
&\mathrm{P}\left(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{X} = \mathbf{x}\right) \\
=&\mathrm{P}\left(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{A}_k = \mathbf{a}_k, X_k = x_k, \mathbf{D}_{k+1} = \mathbf{d}_{k+1}\right) \\
=&\sum_{c_k \leq x_k} \mathrm{P}\left(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{A}_k = \mathbf{a}_k, (X_k)_{\mathbf{a}_k} = x_k, (X_k)_{\mathbf{a}_k, \mathbf{x}^0_S} = c_k, \mathbf{D}_{k+1} = \mathbf{d}_{k+1}\right) \\
=&\sum_{c_k \leq x_k} \mathrm{P}\left(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{A}_k = \mathbf{a}_k, C_k = c_k, \mathbf{D}_{k+1} = \mathbf{d}_{k+1}\right) \\
=&\sum_{(c_k, c_{k+1}) \preceq (x_k, x_{k+1})} \mathrm{P}\left(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{A}_k = \mathbf{a}_k, C_k = c_k, (X_{k+1})_{\mathbf{a}_{k+1}} = x_{k+1},\right. \\
&\qquad\qquad \left.(X_{k+1})_{\mathbf{a}_k, c_k, \mathbf{x}^0_S} = c_{k+1}, \mathbf{D}_{k+2} = \mathbf{d}_{k+2}\right) \\
=&\sum_{\mathbf{c}_{k:k+1} \preceq \mathbf{x}_{k:k+1}} \mathrm{P}\left(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{A}_k = \mathbf{a}_k, C_k = c_k, C_{k+1} = c_{k+1}, \mathbf{D}_{k+2} = \mathbf{d}_{k+2}\right),
\end{aligned}$$

where for ease of presentation we use $C_l = c_l$ to denote $((X_l)_{\mathbf{a}_l}, (X_l)_{\mathbf{a}_k, \mathbf{c}_{k:l-1}, \mathbf{x}^0_{\mathbf{S}}}) = (x_l, c_l)$ for $k \leq l \leq p$ and $x_l \geq c_l$, and $c_l = x^0_l$ if $l \in \mathbf{S}$. The second equality holds because of the consistency and the monotonicity assumptions.

Recursively, by the consistency and the composition, we have

$$\begin{aligned}
&\mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^0}=1,\mathbf{X}=\mathbf{x}\right)\\
=&\mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^0}=1,\mathbf{A}_k=\mathbf{a}_k,\mathbf{D}_k=\mathbf{d}_k\right)\\
=&\sum_{\mathbf{c}_{k:p}\preceq\mathbf{d}_k}\mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^0}=1,\mathbf{A}_k=\mathbf{a}_k,C_k=c_k,\cdots,C_p=c_p\right)\\
=&\sum_{\mathbf{c}_{k:p}\preceq\mathbf{d}_k}\mathrm{P}\left(Y_{\mathbf{a}_k,\mathbf{c}_{k:p}}=1,\mathbf{A}_k=\mathbf{a}_k,C_k=c_k,\cdots,C_p=c_p\right)\\
=&\sum_{\mathbf{c}_{k:p}\preceq\mathbf{d}_k}\mathrm{P}\left(Y_{\mathbf{a}_k,\mathbf{c}_{k:p}}=1,C_k=c_k,\cdots,C_p=c_p\mid\mathbf{A}_k=\mathbf{a}_k\right)\times\mathrm{P}(\mathbf{A}_k=\mathbf{a}_k),\\
=&\sum_{\mathbf{c}_{k:p}\preceq\mathbf{d}_k}\mathrm{P}\left(Y_{\mathbf{a}_k,\mathbf{c}_{k:p}}=1\mid\mathbf{A}_k=\mathbf{a}_k\right)\times\prod_{l=k}^{p}\mathrm{P}\left(C_l=c_l\mid\mathbf{A}_k=\mathbf{a}_k\right)\times\mathrm{P}(\mathbf{A}_k=\mathbf{a}_k),
\end{aligned}$$

where the last equality holds as the potential outcomes $\mathbf{C}_{k:p}=(C_k,\cdots,C_p)$ are conditionally independent given $\mathbf{A}_k$. By the no confounding assumption, the first factor can be identified by

$$\mathrm{P}\left(Y_{\mathbf{a}_k,\mathbf{c}_{k:p}}=1\mid\mathbf{A}_k=\mathbf{a}_k\right)=\mathrm{P}(Y=1\mid\mathbf{A}_k=\mathbf{a}_k,X_k=c_k,\cdots,X_p=c_p).$$

Next, we consider the identifiability of $\mathrm{P}(C_l=c_l\mid\mathbf{A}_k=\mathbf{a}_k)$ for $l=k+1,\ldots,p$. For $l\in\mathbf{S}$, we have

$$\begin{aligned}
&\mathrm{P}\left(C_l=c_l\mid\mathbf{A}_k=\mathbf{a}_k\right)\\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_l}=x_l,(X_l)_{\mathbf{a}_k,\mathbf{c}_{k:l-1},\mathbf{x}_{\mathbf{S}}^1}=c_l\mid\mathbf{A}_k=\mathbf{a}_k\right)\\
=&1_{c_l=x_l}\cdot\mathrm{P}\left((X_l)_{\mathbf{a}_l}=x_l,(X_l)_{\mathbf{a}_k,\mathbf{c}_{k:l-1},x_l^1}=x_l^1\mid\mathbf{A}_k=\mathbf{a}_k\right)\\
=&1_{c_l=x_l}\cdot\mathrm{P}\left((X_l)_{\mathbf{a}_l}=x_l\mid\mathbf{A}_k=\mathbf{a}_k\right)\\
=&1_{c_l=x_l}\cdot\mathrm{P}\left(X_l=x_l\mid\mathbf{A}_k=\mathbf{a}_k\right),
\end{aligned}$$

where the second equality holds by the definition of c_l and the third equality holds by the consistency.

For $l\notin\mathbf{S}$, we have the following three cases according to the values of (x_l,c_l):

- $(x_l,c_l)=(0,0)$: for this case, we have

$$\begin{aligned}
&\mathrm{P}\left(C_l=c_l\mid\mathbf{A}_k=\mathbf{a}_k\right)\\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_l}=0,(X_l)_{\mathbf{a}_k,\mathbf{c}_{k:l-1},\mathbf{x}_{\mathbf{S}}^0}=0\mid\mathbf{A}_k=\mathbf{a}_k\right)\\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_l}=0\mid\mathbf{A}_k=\mathbf{a}_k\right)\\
=&\mathrm{P}\left(X_l=0\mid\mathbf{A}_l=\mathbf{a}_l\right),
\end{aligned}$$

  where the second and the third equalities hold because of the monotonicity and no confounding assumptions, respectively;

- For the case of $(x_l,c_l)=(1,1)$, we have

$$\begin{aligned}
&\mathrm{P}\left(C_l=c_l\mid\mathbf{A}_k=\mathbf{a}_k\right)\\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_l}=1,(X_l)_{\mathbf{a}_k,\mathbf{c}_{k:l-1},\mathbf{x}_{\mathbf{S}}^0}=1\mid\mathbf{A}_k=\mathbf{a}_k\right)\\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_k,\mathbf{c}_{k:l-1},\mathbf{x}_{\mathbf{S}}^0}=1\mid\mathbf{A}_k=\mathbf{a}_k\right)\\
=&\mathrm{P}\left(X_l=1\mid\mathbf{A}_k=\mathbf{a}_k,\mathbf{X}_{k:l-1}=\mathbf{c}_{k+l-1}\right);
\end{aligned}$$

- For the case of $(x_l, c_l) = (1, 0)$, we have

$$
\begin{aligned}
&\mathrm{P}\left(C_l = c_l \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_l} = 1, (X_l)_{\mathbf{a}_k, \mathbf{c}_{k:l-1}, \mathbf{x}^0_{\mathbf{S}}} = 0 \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_l} = 1 \mid \mathbf{A}_k = \mathbf{a}_k\right) - \mathrm{P}\left((X_l)_{\mathbf{a}_l} = 1, (X_l)_{\mathbf{a}_k, \mathbf{c}_{k:l-1}, \mathbf{x}^0_{\mathbf{S}}} = 1 \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&\mathrm{P}(X_l = 1 \mid \mathbf{A}_l = \mathbf{a}_l) - \mathrm{P}\left(X_l = 1 \mid \mathbf{A}_k = \mathbf{a}_k, \mathbf{X}_{k:l-1} = \mathbf{c}_{k+l-1}\right).
\end{aligned}
$$

Summarizing the identification equations for the three cases, we get

$$
\begin{aligned}
&\prod_{l=k}^{p} \mathrm{P}\left(C_l = c_l \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
= &\prod_{i \in \{k,\ldots,p\} \setminus \mathbf{S}} \Big\{(1 - x_i) \times \mathrm{P}(X_i = 0 \mid \mathbf{A}_i = \mathbf{a}_i) + x_i(1 - c_i) \times \mathrm{P}(X_i = 1 \mid \mathbf{A}_i = \mathbf{a}_i) \\
&+ x_i(-1)^{1-c_i} \times \mathrm{P}\left(X_i = 1 \mid \mathbf{A}_k = \mathbf{a}_k, \mathbf{X}_{k:i-1} = \mathbf{c}_{k:i-1}\right)\Big\} \times \prod_{i \in \mathbf{S}} \mathrm{P}(X_i = x_i \mid \mathbf{A}_i = \mathbf{a}_i) \\
&\times 1_{\mathbf{x}_S = \mathbf{c}_S}.
\end{aligned}
$$

From the above results, the identification formula of $\mathrm{P}\left(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1 \mid \mathbf{X} = \mathbf{x}\right)$ can be derived as follows

$$
\begin{aligned}
&\mathrm{P}\left(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1 \mid \mathbf{X} = \mathbf{x}\right) = \frac{\mathrm{P}(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{X} = \mathbf{x})}{\mathrm{P}(\mathbf{X} = \mathbf{x})} \\
= &\sum_{\mathbf{c}_{k:p} \preceq \mathbf{d}_k} \left[\frac{\mathrm{P}\left(Y_{\mathbf{a}_k, \mathbf{c}_{k:p}} = 1 \mid \mathbf{A}_k = \mathbf{a}_k\right)}{\mathrm{P}(\mathbf{D}_k = \mathbf{d}_k \mid \mathbf{A}_k = \mathbf{a}_k)} \times \prod_{l=k}^{p} \mathrm{P}\left(C_l = c_l \mid \mathbf{A}_k = \mathbf{a}_k\right)\right] \\
= &\sum_{\mathbf{c}_{k:p} \preceq \mathbf{d}_k} \Big\{1_{\mathbf{x}_S = \mathbf{c}_S} \times \frac{\mathrm{P}\left(Y = 1 \mid \mathbf{A}_k = \mathbf{a}_k, \mathbf{D}_k = \mathbf{c}_{k:p}\right)}{\mathrm{P}(\mathbf{D}_k = \mathbf{d}_k \mid \mathbf{A}_k = \mathbf{a}_k)} \times \prod_{i \in \mathbf{S}} \mathrm{P}(X_i = x_i \mid \mathbf{A}_i = \mathbf{a}_i) \\
&\times \prod_{i \in \{k,\ldots,p\} \setminus \mathbf{S}} \Big[(1 - x_i) \times \mathrm{P}(X_i = 0 \mid \mathbf{A}_i = \mathbf{a}_i) + x_i(1 - c_i) \times \mathrm{P}(X_i = 1 \mid \mathbf{A}_i = \mathbf{a}_i) \\
&+ x_i(-1)^{1-c_i} \times \mathrm{P}\left(X_i = 1 \mid \mathbf{A}_k = \mathbf{a}_k, \mathbf{X}_{k:i-1} = \mathbf{c}_{k:i-1}\right)\Big]\Big\} \\
= &\sum_{\mathbf{c}_{k:p} \preceq \mathbf{d}_k} \Big\{1_{\mathbf{x}_S = \mathbf{c}_S} \times \mathrm{P}(Y = 1 \mid \mathbf{A}_k = \mathbf{a}_k, \mathbf{D}_k = \mathbf{c}_{k:p}) \\
&\times \prod_{i \in \{k,\ldots,p\} \setminus \mathbf{S}} \Big[1 - x_i c_i + x_i(-1)^{1-c_i} \times \frac{\mathrm{P}(X_i = 1 \mid \mathbf{A}_k = \mathbf{a}_k, \mathbf{X}_{k:i-1} = \mathbf{c}_{k:i-1})}{\mathrm{P}(X_i = x_i \mid \mathbf{A}_i = \mathbf{a}_i)}\Big]\Big\},
\end{aligned}
$$

where the last equality holds because

$$
(1 - x_i) \times \frac{\mathrm{P}(X_i = 0 \mid \mathbf{A}_i = \mathbf{a}_i)}{\mathrm{P}(X_i = x_i \mid \mathbf{A}_i = \mathbf{a}_i)} = \begin{cases} 0, & \text{if } x_i = 1; \\ 1 - x_i, & \text{if } x_i = 0; \end{cases}
$$

and

$$
x_i(1 - c_i) \times \frac{\mathrm{P}(X_i = 1 \mid \mathbf{A}_i = \mathbf{a}_i)}{\mathrm{P}(X_i = x_i \mid \mathbf{A}_i = \mathbf{a}_i)} = \begin{cases} x_i(1 - c_i), & \text{if } x_i = 1; \\ 0, & \text{if } x_i = 0. \end{cases}
$$

B PROOF OF LEMMA 2

We write the conditional probability as

$$
\mathrm{P}\left(Y_{\mathbf{x}^1_{\mathbf{S}}} = 1 \mid \mathbf{X} = \mathbf{x}\right) = \frac{\mathrm{P}\left(Y_{\mathbf{x}^1_{\mathbf{S}}} = 1, \mathbf{X} = \mathbf{x}\right)}{\mathrm{P}\left(\mathbf{X} = \mathbf{x}\right)},
$$

and we first show the identifiability of the numerator above.

$$
\begin{aligned}
& \mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^1}=1, \mathbf{X}=\mathbf{x}\right) \\
= & \mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^1}=1, \mathbf{A}_k=\mathbf{a}_k, X_k=x_k, \mathbf{D}_{k+1}=\mathbf{d}_{k+1}\right) \\
= & \sum_{c_k \geq x_k} \mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^1}=1, \mathbf{A}_k=\mathbf{a}_k,(X_k)_{\mathbf{a}_k}=x_k,(X_k)_{\mathbf{a}_k, \mathbf{x}_S^1}=c_k, \mathbf{D}_{k+1}=\mathbf{d}_{k+1}\right) \\
= & \sum_{c_k \geq x_k} \mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^1}=1, \mathbf{A}_k=\mathbf{a}_k, C_k=c_k, \mathbf{D}_{k+1}=\mathbf{d}_{k+1}\right) \\
= & \sum_{(c_k, c_{k+1}) \succeq (x_k, x_{k+1})} \mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^1}=1, \mathbf{A}_k=\mathbf{a}_k, C_k=c_k,(X_{k+1})_{\mathbf{a}_{k+1}}=x_{k+1},\right. \\
& \left.(X_{k+1})_{\mathbf{a}_k, c_k, \mathbf{x}_S^1}=c_{k+1}, \mathbf{D}_{k+2}=\mathbf{d}_{k+2}\right) \\
= & \sum_{\mathbf{c}_{k:k+1} \succeq \mathbf{x}_{k:k+1}} \mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^1}=1, \mathbf{A}_k=\mathbf{a}_k, C_k=c_k, C_{k+1}=c_{k+1}, \mathbf{D}_{k+2}=\mathbf{d}_{k+2}\right),
\end{aligned}
$$

where $C_l=c_l$ denotes $((X_l)_{\mathbf{a}_l},(X_l)_{\mathbf{a}_k,\mathbf{c}_{k:l-1},\mathbf{x}_{\mathbf{S}}^1})=(x_l,c_l)$ for any $k \leq l \leq p$ satisfying $x_l \leq c_l$ and $c_l=x_l^1$ if $l \in \mathbf{S}$. The second equality holds because of the consistency and Assumption 2(a).

Recursively, by the consistency and the composition, we have

$$
\begin{aligned}
& \mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^1}=1, \mathbf{X}=\mathbf{x}\right) \\
= & \mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^1}=1, \mathbf{A}_k=\mathbf{a}_k, \mathbf{D}_k=\mathbf{d}_k\right) \\
= & \sum_{\mathbf{c}_{k:p} \succeq \mathbf{d}_k} \mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^1}=1, \mathbf{A}_k=\mathbf{a}_k, C_k=c_k, \cdots, C_p=c_p\right) \\
= & \sum_{\mathbf{c}_{k:p} \succeq \mathbf{d}_k} \mathrm{P}\left(Y_{\mathbf{a}_k, \mathbf{c}_{k:p}}=1, \mathbf{A}_k=\mathbf{a}_k, C_k=c_k, \cdots, C_p=c_p\right) \\
= & \sum_{\mathbf{c}_{k:p} \succeq \mathbf{d}_k} \mathrm{P}\left(Y_{\mathbf{a}_k, \mathbf{c}_{k:p}}=1, C_k=c_k, \cdots, C_p=c_p \mid \mathbf{A}_k=\mathbf{a}_k\right) \times \mathrm{P}(\mathbf{A}_k=\mathbf{a}_k), \\
= & \sum_{\mathbf{c}_{k:p} \succeq \mathbf{d}_k} \mathrm{P}\left(Y_{\mathbf{a}_k, \mathbf{c}_{k:p}}=1 \mid \mathbf{A}_k=\mathbf{a}_k\right) \times \prod_{l=k}^{p} \mathrm{P}\left(C_l=c_l \mid \mathbf{A}_k=\mathbf{a}_k\right) \times \mathrm{P}(\mathbf{A}_k=\mathbf{a}_k),
\end{aligned}
$$

where the last equality holds because of the conditional independencies between the potential outcomes $\mathbf{C}_{k:p}=(C_k,\cdots,C_p)$ given $\mathbf{A}_k$. By the no confounding assumption, the first factor above can be identified by

$$
\begin{aligned}
& \mathrm{P}\left(Y_{\mathbf{a}_k, \mathbf{c}_{k:p}}=1 \mid \mathbf{A}_k=\mathbf{a}_k\right) \\
= & \mathrm{P}(Y=1 \mid \mathbf{A}_k=\mathbf{a}_k, X_k=c_k, \cdots, X_p=c_p).
\end{aligned}
$$

Next, we consider the identifiability of $\mathrm{P}(C_l=c_l \mid \mathbf{A}_k=\mathbf{a}_k)$ for $l=k+1,\ldots,p$.

For $l \in \mathbf{S}$, we have

$$
\begin{aligned}
& \mathrm{P}\left(C_l=c_l \mid \mathbf{A}_k=\mathbf{a}_k\right) \\
= & \mathrm{P}\left((X_l)_{\mathbf{a}_l}=x_l,(X_l)_{\mathbf{a}_k, \mathbf{c}_{k:l-1}, \mathbf{x}_{\mathbf{S}}^1}=c_l \mid \mathbf{A}_k=\mathbf{a}_k\right) \\
= & 1_{c_l=x_l} \cdot \mathrm{P}\left((X_l)_{\mathbf{a}_l}=x_l,(X_l)_{\mathbf{a}_k, \mathbf{c}_{k:l-1}, x_l^1}=x_l^1 \mid \mathbf{A}_k=\mathbf{a}_k\right) \\
= & 1_{c_l=x_l} \cdot \mathrm{P}\left((X_l)_{\mathbf{a}_l}=x_l \mid \mathbf{A}_k=\mathbf{a}_k\right) \\
= & 1_{c_l=x_l} \cdot \mathrm{P}\left(X_l=x_l \mid \mathbf{A}_k=\mathbf{a}_k\right),
\end{aligned}
$$

where the second equality holds by the definition of c_l and the third equality holds by the consistency.

For $l \notin \mathbf{S}$, according to the value of (x_l, c_l) we discuss it for three cases.

- For the case of $(x_l, c_l) = (0, 0)$, we have

$$
\begin{aligned}
&\mathrm{P}\left(C_l = c_l \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_l} = 0, (X_l)_{\mathbf{a}_k, \mathbf{c}_{k:l-1}, \mathbf{x}_{\mathbf{S}}^1} = 0 \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_k, \mathbf{c}_{k:l-1}, \mathbf{x}_{\mathbf{S}}^1} = 0 \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&\mathrm{P}\left(X_l = 0 \mid \mathbf{A}_k = \mathbf{a}_k, \mathbf{X}_{k:l-1} = \mathbf{c}_{k:l-1}\right),
\end{aligned}
$$

  where the second and the third equalities hold bacause of the monotonicity and no confounding assumptions, respectively;

- For the case of $(x_l, c_l) = (1, 1)$, we have

$$
\begin{aligned}
&\mathrm{P}\left(C_l = c_l \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_l} = 1, (X_l)_{\mathbf{a}_k, \mathbf{c}_{k:l-1}, \mathbf{x}_{\mathbf{S}}^1} = 1 \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_l} = 1 \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&\mathrm{P}\left(X_l = 1 \mid \mathbf{A}_l = \mathbf{a}_l\right);
\end{aligned}
$$

- For the case of $(x_l, c_l) = (0, 1)$, we have

$$
\begin{aligned}
&\mathrm{P}\left(C_l = c_l \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_l} = 0, (X_l)_{\mathbf{a}_k, \mathbf{c}_{k:l-1}, \mathbf{x}_{\mathbf{S}}^1} = 1 \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&\mathrm{P}\left((X_l)_{\mathbf{a}_l} = 0 \mid \mathbf{A}_k = \mathbf{a}_k\right) - \mathrm{P}\left((X_l)_{\mathbf{a}_l} = 0, (X_l)_{\mathbf{a}_k, \mathbf{c}_{k:l-1}, \mathbf{x}_{\mathbf{S}}^1} = 0 \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&\mathrm{P}(X_l = 0 \mid \mathbf{A}_l = \mathbf{a}_l) - \mathrm{P}\left(X_l = 0 \mid \mathbf{A}_k = \mathbf{a}_k, \mathbf{X}_{k:l-1} = \mathbf{c}_{k+l-1}\right).
\end{aligned}
$$

Summarizing the identification equations for the three cases, we get

$$
\begin{aligned}
&\prod_{l=k}^{p} \mathrm{P}\left(C_l = c_l \mid \mathbf{A}_k = \mathbf{a}_k\right) \\
=&1_{\mathbf{x}_S = \mathbf{c}_S} \times \prod_{i \in \{k, \ldots, p\} \setminus \mathbf{S}} \Big\{(1 - x_i) c_i \times \mathrm{P}(X_i = 0 \mid \mathbf{A}_i = \mathbf{a}_i) + x_i \times \mathrm{P}(X_i = 1 \mid \mathbf{A}_i = \mathbf{a}_i) \\
&+ (1 - x_i)(-1)^{c_i} \times \mathrm{P}\left(X_i = 0 \mid \mathbf{A}_k = \mathbf{a}_k, \mathbf{X}_{k:i-1} = \mathbf{c}_{k:i-1}\right)\Big\} \times \prod_{i \in \mathbf{S}} \mathrm{P}(X_i = x_i \mid \mathbf{A}_i = \mathbf{a}_i).
\end{aligned}
$$

From the above results, the identification formula of $\mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^{1}}=1 \mid \mathbf{X}=\mathbf{x}\right)$ can be derived as follows

$$
\begin{aligned}
&\mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^{1}}=1 \mid \mathbf{X}=\mathbf{x}\right)=\frac{\mathrm{P}(Y_{\mathbf{x}_{\mathbf{S}}^{1}}=1, \mathbf{X}=\mathbf{x})}{\mathrm{P}(\mathbf{X}=\mathbf{x})} \\
=&\sum_{\mathbf{c}_{k:p} \succeq \mathbf{d}_k}\left[\frac{\mathrm{P}\left(Y_{\mathbf{a}_k, \mathbf{c}_{k:p}}=1 \mid \mathbf{A}_k=\mathbf{a}_k\right)}{\mathrm{P}(\mathbf{D}_k=\mathbf{d}_k \mid \mathbf{A}_k=\mathbf{a}_k)} \times \prod_{l=k}^{p} \mathrm{P}\left(C_l=c_l \mid \mathbf{A}_k=\mathbf{a}_k\right)\right] \\
=&\sum_{\mathbf{c}_{k:p} \succeq \mathbf{d}_k}\left\{1_{\mathbf{x}_S=\mathbf{c}_S} \times \frac{\mathrm{P}\left(Y=1 \mid \mathbf{A}_k=\mathbf{a}_k, \mathbf{D}_k=\mathbf{c}_{k:p}\right)}{\mathrm{P}(\mathbf{D}_k=\mathbf{d}_k \mid \mathbf{A}_k=\mathbf{a}_k)} \times \prod_{i \in \mathbf{S}} \mathrm{P}(X_i=x_i \mid \mathbf{A}_i=\mathbf{a}_i)\right. \\
&\times \prod_{i \in\{k, \ldots, p\} \backslash \mathbf{S}}\Big[(1-x_i) c_i \times \mathrm{P}(X_i=0 \mid \mathbf{A}_i=\mathbf{a}_i)+x_i \times \mathrm{P}(X_i=1 \mid \mathbf{A}_i=\mathbf{a}_i) \\
&\left.+(1-x_i)(-1)^{c_i} \times \mathrm{P}\left(X_i=0 \mid \mathbf{A}_k=\mathbf{a}_k, \mathbf{X}_{k:i-1}=\mathbf{c}_{k:i-1}\right)\Big]\right\} \\
=&\sum_{\mathbf{c}_{k:p} \succeq \mathbf{d}_k}\left\{1_{\mathbf{x}_S=\mathbf{c}_S} \times \mathrm{P}(Y=1 \mid \mathbf{A}_k=\mathbf{a}_k, \mathbf{D}_k=\mathbf{c}_{k:p})\right. \\
&\left.\times \prod_{i \in\{k, \ldots, p\} \backslash \mathbf{S}}\left[x_i+c_i-x_i c_i+(1-x_i)(-1)^{c_i} \times \frac{\mathrm{P}(X_i=0 \mid \mathbf{A}_k=\mathbf{a}_k, \mathbf{X}_{k:i-1}=\mathbf{c}_{k:i-1})}{\mathrm{P}(X_i=x_i \mid \mathbf{A}_i=\mathbf{a}_i)}\right]\right\},
\end{aligned}
$$

where the last equality holds because

$$
(1-x_i) c_i \times \frac{\mathrm{P}(X_i=0 \mid \mathbf{A}_i=\mathbf{a}_i)}{\mathrm{P}(X_i=x_i \mid \mathbf{A}_i=\mathbf{a}_i)}= \begin{cases}0, & \text{if } x_i=1; \\ (1-x_i) c_i, & \text{if } x_i=0;\end{cases}
$$

and

$$
x_i \times \frac{\mathrm{P}(X_i=1 \mid \mathbf{A}_i=\mathbf{a}_i)}{\mathrm{P}(X_i=x_i \mid \mathbf{A}_i=\mathbf{a}_i)}= \begin{cases}x_i, & \text{if } x_i=1; \\ 0, & \text{if } x_i=0.\end{cases}
$$

C PROOF OF THEOREM 1

The conclusion follows directly from Lemma 1, Lemma 2 and the definition of CCE.

D PROOF OF COROLLARY 1

For any subset $\mathbf{X}' \subset \mathbf{X}$, we have

$$
\begin{aligned}
&\mathrm{CCE}\left(\mathbf{X}_{\mathbf{S}} \Rightarrow Y \mid \mathbf{X}'=\mathbf{x}'\right) \\
=&\mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^{1}}=1 \mid \mathbf{X}'=\mathbf{x}'\right)-\mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^{0}}=1 \mid \mathbf{X}'=\mathbf{x}'\right) \\
=&\sum_{\mathbf{x}: \mathbf{x} \supset \mathbf{x}'}\left[\mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^{1}}=1 \mid \mathbf{X}=\mathbf{x}\right)-\mathrm{P}\left(Y_{\mathbf{x}_{\mathbf{S}}^{0}}=1 \mid \mathbf{X}=\mathbf{x}\right)\right] \times \mathrm{P}(\mathbf{X}=\mathbf{x} \mid \mathbf{X}'=\mathbf{x}') \\
=&\sum_{\mathbf{x}: \mathbf{x} \supset \mathbf{x}'} \mathrm{CCE}\left(\mathbf{X}_{\mathbf{S}} \Rightarrow Y \mid \mathbf{X}=\mathbf{x}\right) \times \mathrm{P}(\mathbf{X}=\mathbf{x} \mid \mathbf{X}'=\mathbf{x}').
\end{aligned}
$$

Hence, $\mathrm{CCE}\left(\mathbf{X}_{\mathbf{S}} \Rightarrow Y \mid \mathbf{X}'=\mathbf{x}'\right)$ is identifiable if and only if $\mathrm{CCE}\left(\mathbf{X}_{\mathbf{S}} \Rightarrow Y \mid \mathbf{X}=\mathbf{x}\right)$ is identifiable, and its identification formula can be obtained by Theorem 1.

E PROOF OF THEOREM 2

E.1 $\mathbf{CCE}(\mathbf{X_S} \Rightarrow Y \mid \mathbf{X} = \mathbf{x}, Y = 1)$

For $Y = 1$, we have

$$\begin{aligned}&\mathrm{CCE}(\mathbf{X_S} \Rightarrow Y \mid \mathbf{X} = \mathbf{x}, Y = 1) = \mathrm{E}(Y_{\mathbf{x}^1_{\mathbf{S}}} - Y_{\mathbf{x}^0_{\mathbf{S}}} \mid \mathbf{X} = \mathbf{x}, Y = 1)\\ =&1 - \mathrm{P}(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1 \mid \mathbf{X} = \mathbf{x}, Y = 1) = 1 - \frac{\mathrm{P}(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{X} = \mathbf{x}, Y = 1)}{\mathrm{P}(\mathbf{X} = \mathbf{x}, Y = 1)}.\end{aligned}$$

By consistency, composition and Assumption 2(a), we have

$$\begin{aligned}&\mathrm{P}\left(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{X} = \mathbf{x}, Y = 1\right)\\ =&\sum_{\mathbf{c}_k \preceq \mathbf{d}_k} \mathrm{P}\Big(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, Y_{\mathbf{x}} = 1, (\mathbf{A}_k)_{\mathbf{x}^0_{\mathbf{S}}} = \mathbf{a}_k, (\mathbf{D}_k)_{\mathbf{a}_k, \mathbf{x}^0_{\mathbf{S}}} = \mathbf{c}_k, \mathbf{X} = \mathbf{x}\Big)\\ =&\sum_{\mathbf{c}_k \preceq \mathbf{d}_k} \mathrm{P}\Big(Y_{\mathbf{a}_k, \mathbf{x}^0_{\mathbf{S}}, \mathbf{c}_k} = 1, Y_{\mathbf{x}} = 1, (\mathbf{A}_k)_{\mathbf{x}^0_{\mathbf{S}}} = \mathbf{a}_k, (\mathbf{D}_k)_{\mathbf{a}_k, \mathbf{x}^0_{\mathbf{S}}} = \mathbf{c}_k, \mathbf{X} = \mathbf{x}\Big)\\ =&\sum_{\mathbf{c}_k \preceq \mathbf{d}_k} \mathrm{P}\Big(Y_{\mathbf{a}_k, \mathbf{x}^0_{\mathbf{S}}, \mathbf{c}_k} = 1, (\mathbf{A}_k)_{\mathbf{x}^0_{\mathbf{S}}} = \mathbf{a}_k, (\mathbf{D}_k)_{\mathbf{a}_k, \mathbf{x}^0_{\mathbf{S}}} = \mathbf{c}_k, \mathbf{X} = \mathbf{x}\Big)\\ =&\sum_{\mathbf{c}_k \preceq \mathbf{d}_k} \mathrm{P}\Big(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, (\mathbf{D}_k)_{\mathbf{a}_k, \mathbf{x}^0_{\mathbf{S}}} = \mathbf{c}_k, \mathbf{X} = \mathbf{x}\Big)\\ =&\mathrm{P}\left(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{X} = \mathbf{x}\right),\end{aligned}$$

where $k = \min \mathbf{S}$. Hence, we have

$$\begin{aligned}&\mathrm{CCE}(\mathbf{X_S} \Rightarrow Y \mid \mathbf{X} = \mathbf{x}, Y = 1) = 1 - \frac{\mathrm{P}(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{X} = \mathbf{x}, Y = 1)}{\mathrm{P}(\mathbf{X} = \mathbf{x}, Y = 1)}\\ =&1 - \frac{\mathrm{P}(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1, \mathbf{X} = \mathbf{x})}{\mathrm{P}(\mathbf{X} = \mathbf{x}, Y = 1)} = 1 - \frac{\mathrm{P}(Y_{\mathbf{x}^0_{\mathbf{S}}} = 1 \mid \mathbf{X} = \mathbf{x})}{\mathrm{P}(Y = 1 \mid \mathbf{X} = \mathbf{x})}.\end{aligned}$$

E.2 $\mathbf{CCE}(\mathbf{X_S} \Rightarrow Y \mid \mathbf{X} = \mathbf{x}, Y = 0)$

For $Y = 0$, we have

$$\begin{aligned}&\mathrm{CCE}(\mathbf{X_S} \Rightarrow Y \mid \mathbf{X} = \mathbf{x}, Y = 0) = \mathrm{E}(Y_{\mathbf{x}^1_{\mathbf{S}}} - Y_{\mathbf{x}^0_{\mathbf{S}}} \mid \mathbf{X} = \mathbf{x}, Y = 0)\\ =&\mathrm{P}(Y_{\mathbf{x}^1_{\mathbf{S}}} = 1 \mid \mathbf{X} = \mathbf{x}, Y = 0) - 0 = 1 - \mathrm{P}(Y_{\mathbf{x}^1_{\mathbf{S}}} = 0 \mid \mathbf{X} = \mathbf{x}, Y = 0)\\ =&1 - \frac{\mathrm{P}(Y_{\mathbf{x}^1_{\mathbf{S}}} = 0, \mathbf{X} = \mathbf{x}, Y = 0)}{\mathrm{P}(\mathbf{X} = \mathbf{x}, Y = 0)}.\end{aligned}$$

By consistency, composition and Assumption 2(a), we have

$$\begin{aligned}&\mathrm{P}\left(Y_{\mathbf{x}^1_{\mathbf{S}}} = 0, \mathbf{X} = \mathbf{x}, Y = 0\right)\\ =&\sum_{\mathbf{c}_k \succeq \mathbf{d}_k} \mathrm{P}\Big(Y_{\mathbf{x}^1_{\mathbf{S}}} = 0, Y_{\mathbf{x}} = 0, (\mathbf{A}_k)_{\mathbf{x}^1_{\mathbf{S}}} = \mathbf{a}_k, (\mathbf{D}_k)_{\mathbf{a}_k, \mathbf{x}^1_{\mathbf{S}}} = \mathbf{c}_k, \mathbf{X} = \mathbf{x}\Big)\\ =&\sum_{\mathbf{c}_k \succeq \mathbf{d}_k} \mathrm{P}\Big(Y_{\mathbf{a}_k, \mathbf{x}^1_{\mathbf{S}}, \mathbf{c}_k} = 0, Y_{\mathbf{x}} = 0, (\mathbf{A}_k)_{\mathbf{x}^1_{\mathbf{S}}} = \mathbf{a}_k, (\mathbf{D}_k)_{\mathbf{a}_k, \mathbf{x}^1_{\mathbf{S}}} = \mathbf{c}_k, \mathbf{X} = \mathbf{x}\Big)\\ =&\sum_{\mathbf{c}_k \succeq \mathbf{d}_k} \mathrm{P}\Big(Y_{\mathbf{a}_k, \mathbf{x}^1_{\mathbf{S}}, \mathbf{c}_k} = 0, (\mathbf{A}_k)_{\mathbf{x}^1_{\mathbf{S}}} = \mathbf{a}_k, (\mathbf{D}_k)_{\mathbf{a}_k, \mathbf{x}^1_{\mathbf{S}}} = \mathbf{c}_k, \mathbf{X} = \mathbf{x}\Big)\\ =&\sum_{\mathbf{c}_k \succeq \mathbf{d}_k} \mathrm{P}\Big(Y_{\mathbf{x}^1_{\mathbf{S}}} = 0, (\mathbf{D}_k)_{\mathbf{a}_k, \mathbf{x}^1_{\mathbf{S}}} = \mathbf{c}_k, \mathbf{X} = \mathbf{x}\Big)\\ =&\mathrm{P}\left(Y_{\mathbf{x}^1_{\mathbf{S}}} = 0, \mathbf{X} = \mathbf{x}\right),\end{aligned}$$

where $k = \min \mathbf{S}$. Hence, we have

$$\mathrm{CCE}(\mathbf{X_S} \Rightarrow Y \mid \mathbf{X}=\mathbf{x}, Y=0) = 1 - \frac{\mathrm{P}(Y_{\mathbf{x}_\mathbf{S}^1}=0, \mathbf{X}=\mathbf{x}, Y=0)}{\mathrm{P}(\mathbf{X}=\mathbf{x}, Y=0)}$$

$$= 1 - \frac{\mathrm{P}(Y_{\mathbf{x}_\mathbf{S}^1}=0, \mathbf{X}=\mathbf{x})}{\mathrm{P}(\mathbf{X}=\mathbf{x}, Y=0)} = 1 - \frac{\mathrm{P}(Y_{\mathbf{x}_\mathbf{S}^1}=0 \mid \mathbf{X}=\mathbf{x})}{\mathrm{P}(Y=0 \mid \mathbf{X}=\mathbf{x})}.$$

F PROOF OF LEMMA 3

Using the notations in this lemma, we have

$$\begin{aligned}
&\mathrm{P}\left(Y_{\mathbf{x}_\mathbf{S}^*}=1, \mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z}\right)\\
=&\sum_{\mathbf{x}^*} \mathrm{P}\left(Y_{\mathbf{x}_\mathbf{S}^*}=1, \mathbf{X}_{\mathbf{x}_\mathbf{S}^*}=\mathbf{x}^*, \mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z}\right)\\
=&\sum_{\mathbf{x}^*} \mathrm{P}\left(Y_{\mathbf{x}^*}=1, \mathbf{X}_{\mathbf{x}_\mathbf{S}^*}=\mathbf{x}^*, \mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z}\right)\\
=&\sum_{\mathbf{x}^*} \mathrm{P}\left(Y_{\mathbf{x}^*}=1, \mathbf{X}_{\mathbf{x}_\mathbf{S}^*}=\mathbf{x}^*, \mathbf{Z}=\mathbf{z} \mid \mathbf{X}=\mathbf{x}, Y=y\right) \times \mathrm{P}(\mathbf{X}=\mathbf{x}, Y=y)\\
=&\sum_{\mathbf{x}^*} \mathrm{P}\left(Y_{\mathbf{x}^*}=1, \mathbf{X}_{\mathbf{x}_\mathbf{S}^*}=\mathbf{x}^* \mid \mathbf{X}=\mathbf{x}, Y=y\right) \times \mathrm{P}(\mathbf{Z}=\mathbf{z} \mid \mathbf{X}=\mathbf{x}, Y=y) \times \mathrm{P}(\mathbf{X}=\mathbf{x}, Y=y)\\
=&\sum_{\mathbf{x}^*} \mathrm{P}\left(Y_{\mathbf{x}^*}=1, \mathbf{X}_{\mathbf{x}_\mathbf{S}^*}=\mathbf{x}^* \mid \mathbf{X}=\mathbf{x}, Y=y\right) \times \mathrm{P}(\mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z}),
\end{aligned}$$

where the second and the fourth equalities hold because of the composition and Assumption 1(c), respectively. Hence, we have

$$\begin{aligned}
&\mathrm{P}\left(Y_{\mathbf{x}_\mathbf{S}^*}=1 \mid \mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z}\right) = \frac{\mathrm{P}\left(Y_{\mathbf{x}_\mathbf{S}^*}=1, \mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z}\right)}{\mathrm{P}(\mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z})}\\
=&\sum_{\mathbf{x}^*} \mathrm{P}\left(Y_{\mathbf{x}^*}=1, \mathbf{X}_{\mathbf{x}_\mathbf{S}^*}=\mathbf{x}^* \mid \mathbf{X}=\mathbf{x}, Y=y\right)\\
=&\mathrm{P}\left(Y_{\mathbf{x}_\mathbf{S}^*}=1 \mid \mathbf{X}=\mathbf{x}, Y=y\right).
\end{aligned}$$

G PROOF OF COROLLARY 3

The conclusion follows directly from Lemma 3 and the definition of CCE.

H PROOF OF THEOREM 3

For any subset $\mathbf{W} \subset (\mathbf{X}, Y, \mathbf{Z})$, we have

$$\begin{aligned}
&\mathrm{CCE}(\mathbf{X_S} \Rightarrow Y \mid \mathbf{W}=\mathbf{w})\\
=&\mathrm{P}\left(Y_{\mathbf{x}_\mathbf{S}^1}=1 \mid \mathbf{W}=\mathbf{w}\right) - \mathrm{P}\left(Y_{\mathbf{x}_\mathbf{S}^0}=1 \mid \mathbf{W}=\mathbf{w}\right)\\
=&\sum_{(\mathbf{x},y,\mathbf{z}):(\mathbf{x},y,\mathbf{z})\supset\mathbf{w}} \mathrm{P}(\mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z} \mid \mathbf{W}) \times \Big[\mathrm{P}\left(Y_{\mathbf{x}_\mathbf{S}^1}=1 \mid \mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z}\right)\\
&\qquad - \mathrm{P}\left(Y_{\mathbf{x}_\mathbf{S}^0}=1 \mid \mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z}\right)\Big]\\
=&\sum_{(\mathbf{x},y,\mathbf{z}):(\mathbf{x},y,\mathbf{z})\supset\mathbf{w}} \mathrm{CCE}(\mathbf{X_S} \Rightarrow Y \mid \mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z}) \times \mathrm{P}(\mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z} \mid \mathbf{W})\\
=&\sum_{(\mathbf{x},y,\mathbf{z}):(\mathbf{x},y,\mathbf{z})\supset\mathbf{w}} \mathrm{CCE}(\mathbf{X_S} \Rightarrow Y \mid \mathbf{X}=\mathbf{x}, Y=y) \times \mathrm{P}(\mathbf{X}=\mathbf{x}, Y=y, \mathbf{Z}=\mathbf{z} \mid \mathbf{W}),
\end{aligned}$$

where the last equality holds because of Corollary 3. Hence, $\mathrm{CCE}(\mathbf{X_S} \Rightarrow Y \mid \mathbf{W}=\mathbf{w})$ is identifiable if and only if $\mathrm{CCE}(\mathbf{X_S} \Rightarrow Y \mid \mathbf{X}=\mathbf{x}, Y=y)$ is identifiable for any $(\mathbf{x}, y, \mathbf{z}) \supset \mathbf{w}$, and under Assumption 1 and Assumption 2, the identification equations are given by Theorem 2.