Multidimensional Persistence Module Classification via Lattice-Theoretic Convolutions

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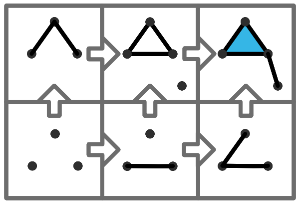

$$\begin{array}{ccccc} H_*(\mathbb{X}_{12}) & \longrightarrow & H_*(\mathbb{X}_{22}) & \longrightarrow & H_*(\mathbb{X}_{23}) \\ \uparrow & & \uparrow & & \uparrow \\ H_*(\mathbb{X}_{11}) & \longrightarrow & H_*(\mathbb{X}_{21}) & \longrightarrow & H_*(\mathbb{X}_{31}) \end{array}$$

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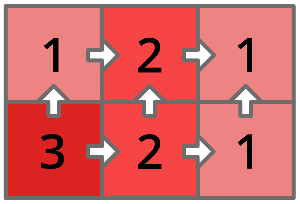


Multidimensional Persistent Homology

Rips bifiltration

$\mathbb{X}_{r,t} = \mathrm{Rips}_r\{x \in \mathcal{M} \mid \rho(x) \leqslant t\}$.

$PH_i(r,t) = H_i(\mathbb{X}_{r,t}, \mathbf{k})$

Hilbert function

$\mathrm{Hilb}_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{Z}$

$(r,t) \mapsto \dim(PH_i(r,t))$

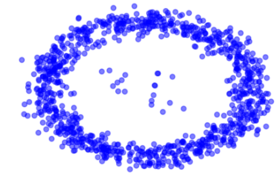
Multigraded Betti numbers

$F_2 \to F_1 \to F_0 \to PH_i$

$\xi_j : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{Z}$

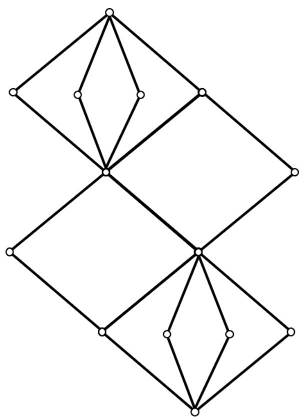
dimension of resolution at (r,t)

Example: Density



Lattice Signal Processing

Lattice: a poset where every pair of elements has a supremum (join) and an infimum (meet). These two operations give an **algebraic description** of the lattice. This algebra suggests a **convolution** operation for **signals** indexed by the elements of the lattice.



Group convolutions

$$(f * g)(x) = \sum_{a \in \mathbb{R}^n} f(x-a)g(x)$$

Lattice convolutions

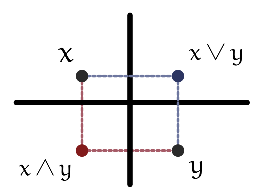
$$(f *_\vee g)(x) = \sum_{a \in \mathbb{R}^n} f(x \vee a)g(x)$$

$$(f *_\wedge g)(x) = \sum_{a \in \mathbb{R}^n} f(x \wedge a)g(x)$$

$\mathbb{R}^n$ is a lattice

$$(x_1,x_2) \wedge (y_1,y_2) = (\min(x_1,y_1), \min(x_2,y_2))$$

$$(x_1,x_2) \vee (y_1,y_2) = (\max(x_1,y_1), \max(x_2,y_2))$$



Lattice convolutions combine information from regions which are algebraically close, but not necessarily geometrically nearby.

Lattice Neural Networks

Persistent homology is indexed by a poset/lattice. Does using this poset structure in a CNN make sense?

$$\mathrm{MeetConv}(f)(x,y)^j = \sum_i (f_i *_\wedge g_j^i)(x,y) = \sum_i \sum_{(a,b)\in[m]\times[n]} f_i(x \wedge a, y \wedge b) g_j^i(a,b)$$

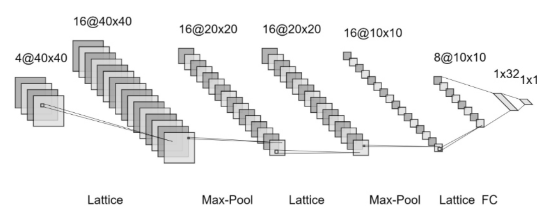
$$\mathrm{JoinConv}(f)(x,y)^j = \sum_i (f_i *_\vee g_j^i)(x,y) = \sum_i \sum_{(a,b)\in[m]\times[n]} f_i(x \vee a, y \vee b) g_j^i(a,b)$$

Lattice convolutions might capture structure which is preserved over meets and joins: quasi-persistence?

Receptive fields for a neuron are strangely shaped: make sure to choose convolution kernels g supported in appropriate regions (the neutral element of the operation).

Architecture

Compare classification accuracy for two networks, one with standard CNN and one with Lattice CNN layers.



Point cloud data from ModelNet-10, processed with RIVET to produce Hilbert function and multigraded Betti numbers.