A Additional Background Work

Developing efficient algorithms for decentralized systems has been a popular research area in recent years. Among them, gossiping algorithms have been proven to be successful [Scaman et al., 2017, Duchi et al., 2011, Nedic and Ozdaglar, 2009]. In this approach, each client computes iteratively a weighted average of local estimators and network-wide estimators obtained from neighbors. The goal is to derive an estimator that converges to the average of the true values across the entire system. The weights are represented by a matrix that respects the graph structure under certain conditions. The gossiping-based averaging approach enables the incorporation of MAB methods in decentralized settings. In particular, motivated by the success of the UCB algorithm [Auer et al., 2002a] in stochastic MAB, [Landgren et al., 2016a,b, 2021, Zhu et al., 2020, 2021a,b, Martínez-Rubio et al., 2019, Chawla et al., 2020, Wang et al., 2021] import it to various decentralized settings under the assumption of sub-Gaussianity, including homogeneous or heterogeneous rewards, different graph assumptions, and various levels of global information. The regret bounds obtained are typically of order $\log T$. However, most existing works assume that the graph is time-invariant under further conditions, which is often not the case. For example, [Wang et al., 2021] provide a optimal regret guarantee for complete graphs which are essentially a centralized batched bandit problem [Perchet et al., 2016]. Connected graphs are also considered, but [Zhu et al., 2020] assume that the rewards are homogeneous and graphs are time-invariant related to doubly stochastic matrices. In addition, [Martínez-Rubio et al., 2019] propose the DDUCB algorithm for settings with time-invariant graphs and homogeneous rewards, dealing with deterministically delayed feedback and assuming knowing the number of vertices and the spectral gap of the given graph. Meanwhile, [Jiang and Cheng, 2023] propose an algorithm C-CBGE that is robust to Gaussian noises and deals with client-dependent MAB, but requires time-invariant regular graphs. [Zhu et al., 2021b] propose a gossiping-based UCB-variant algorithm for time-invariant graphs. In this approach, each client maintains a weighted averaged estimator by gossiping, uses doubly stochastic weight matrices depending on global information of the graph, and adopts a UCB-based decision rule by constructing upper confidence bounds. Recently, [Zhu and Liu, 2023], revisit the algorithm and add an additional term to the UCB rule for time-varying repeatedly strongly connected graphs, assuming no global information. However, the doubly stochasticity assumption excludes many graphs from consideration. Our algorithm builds on the approach proposed by [Zhu et al., 2021b] with new weight matrices that do not require the doubly stochasticity assumption. Our weight matrices leverage more local graph information, rather than just the size of the vertex set as in [Zhu and Liu, 2023, Zhu et al., 2021b]. We introduce the terminology of the stopping time for randomly delayed feedback, along with new upper confidence bounds that consider random graphs and sub-exponentiality. This leads to smaller high probability regret bounds, and the algorithm only requires knowledge of the number of vertices that can be obtained at initialization or estimated as in [Martínez-Rubio et al., 2019].

In the context of bandits with heavy-tailed distributed rewards, the UCB algorithm continues to play a significant role. [Dubey et al., 2020] are the first to consider the multi-agent MAB setting with homogeneous heavy-tailed rewards. They develop a UCB-based algorithm with an instance-dependent regret bound of order $\log T$. They achieve this by adopting larger upper confidence bounds and finding cliques of vertices, even though the graphs are time-invariant and known to clients. In a separate line of work, [Jia et al., 2021] consider the single-agent MAB setting with sub-exponential rewards, and propose a UCB-based algorithm that enlarges or pretrains the upper confidence bounds, achieving a mean-gap independent regret bound of order $\sqrt{T \log T}$. We extend this technique to the decentralized multi-agent MAB setting with heterogeneous sub-exponential rewards, using a gossiping approach, and establish both an optimal instance-dependent regret bound of $O(\log T)$ and a nearly optimal mean-gap independent regret bound of $O(\sqrt{T} \log T)$, up to a $\log T$ factor.

Our work draws on the classical literature on random graphs. From the perspective of generating random connected graphs, we build upon a numerically efficient algorithm introduced in [Gray et al., 2019], which is based on the Metropolis-Hastings algorithm [Chib and Greenberg, 1995], despite its lack of finite convergence rate for non-sparse graphs. We follow their algorithm and, in addition, provide a new analysis on the convergence rate and mixing time of the underlying Markov chain. In terms of the E-R model, it has been thoroughly examined in various areas, such as mean-field games [Delarue, 2017] and majority vote settings [Lima et al., 2008]. However, these random graphs have not yet been applied to the decentralized multi-agent MAB setting that is partially determined by the underlying graphs. Our formulation and analyses bridge this gap, providing insights into the dynamics of decentralized multi-agent MAB in the context of random graphs.

B Future work

Recent advancements have been made in reducing communication costs with respect to the dependency in multi-agent MAB with homogeneous rewards (in the generalized linear bandit setting [Li and Wang, 2022], the ground truth of the unknown parameter is the same for all clients), such as achieving $O(\sqrt{T}M^2)$ in [Li and Wang, 2022] for centralized settings or $O(M^3 \log T)$ through Global Information Synchronization (GIS) communication protocols assuming time-invariant graphs in [Li and Song, 2022]. Likewise, [Sankararaman et al., 2019, Chawla et al., 2020] improve the communication cost of order $\log T$ or $o(T)$ through asynchronous communication protocols and balancing the trade-off between regret and communication cost. More recently, [Wang et al., 2022] establish a novel communication protocol, TCOM, which is of order $\log \log T$ by means of concentrating communication around sub-optimal arms and performing aggregation of estimators across time steps. Furthermore, [Wang et al., 2020] develops a new leader-follower communication protocol, which selects a leader that communicates to the followers. Here the communication cost is independent of T which is much smaller. The incorporation of random graph structures and heterogeneous rewards introduces its own complexities, which poses challenges to reductions in communication costs. These great advancements introduce a promising direction for communication efficiency as a next step within the context herein.

C Details on numerical experiments in Section 4

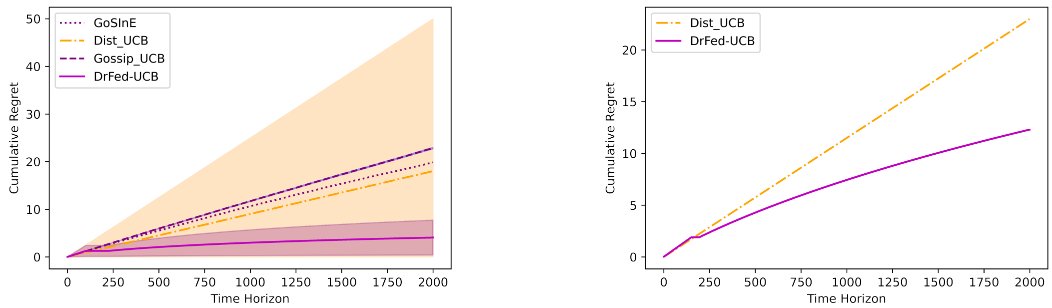
We report the experimental details in Section 4, including both benchmarking and regret properties of the algorithms. The implementation details of the experiments are as follows, including the data generation, benchmarks, and the evaluation metrics.

The process of data generation involves both reward generation and graph generation. First we generate different numbers of arms and clients, denoted as K and M, respectively. Specifically, we generate rewards using the Bernoulli distribution in the sub-Gaussian distribution family, varying the mean values μ_i^m by introducing multiple levels of heterogeneity denoted as $h = \max_{i,j,m} |\mu_i^m - \mu_j|$ and then for each arm k, partitioning the range $[0.1, 0.1 + (k+1) \cdot h/K]$ into M intervals. In terms of graph generation, we generate E-R models with varying values of c, to capture graph complexity. Specifically, for the benchmarking experiment with time-invariant graphs, we set $K = 2, M = 5, h = 0.1, c = 1$, i.e. complete graphs. For the benchmarking experiment with time-varying graphs, we set $K = 2, M = 5, h = 0.1, c = 0.9$. For the regret experiments, the parameters are $h \in \{0.1, 0.2, 0.3\}$, $M \in \{5, 8, 12\}$, $c \in \{0.2, 0.5, 0.9, 1\}$, and $K \in \{2, 3, 4\}$. We selected the least positive number of arms $K = 2$ to keep computational times low and $M = 5$ to have small graphs but still a variety of them.

We compare the new method DrFed-UCB with the classical methods, such as the Gossiping Insert-Eliminate algorithm (GoSInE) in [Chawla et al., 2020] which focuses on deterministic graphs and sub-Gaussian rewards and motivated our work. We also include the Gossip UCB algorithm (Gossip_UCB) [Zhu et al., 2021b] as a benchmark. Meanwhile, in terms of time-varying graphs, we implement the algorithm, Distributed UCB (Dist_UCB) in [Zhu and Liu, 2023] that has been developed for time-varying graphs, and compare our algorithm to this benchmark.

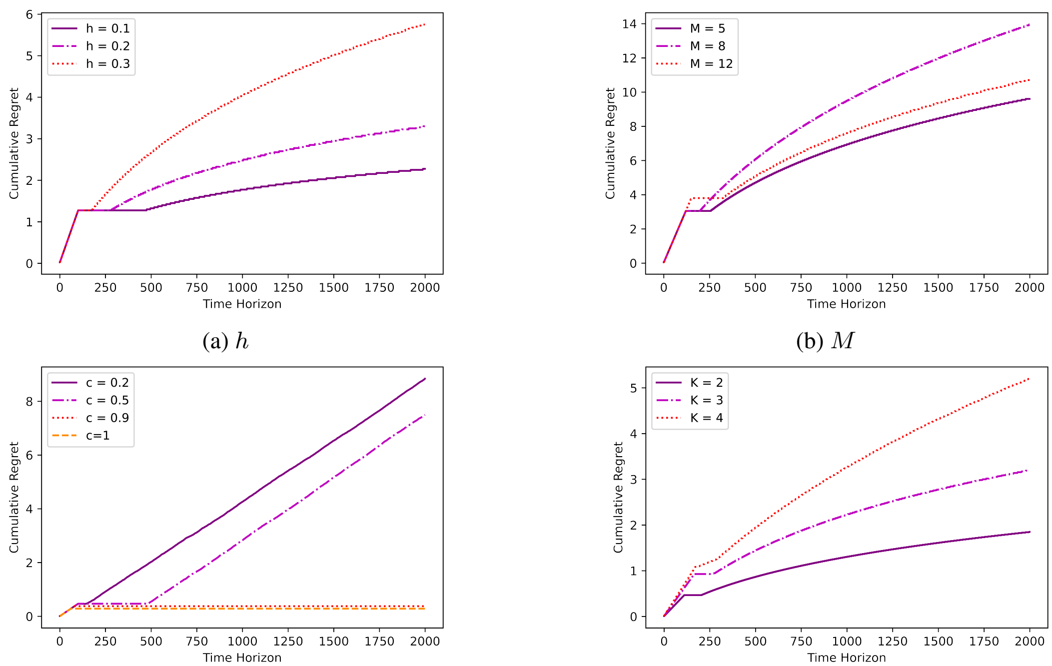
Evaluation The evaluation metric is the regret measure as defined in Section 4. More specifically, for the experiments, we use the average regret over 50 runs for each benchmark and also report the 95% confidence intervals across the 50 runs. With respect to the communication cost as another performance measure, it is computed explicitly. Additionally, the runtime can provide insights into the time complexity of the models.

Benchmark comparison results The results for time-invariant and time-varying graphs are presented in Figure 1 (a) and Figure 1 (b), respectively. The x-axis represents time steps, while the y-axis shows the average regret up to that time step. Figure 1 (a) demonstrates that DrFed-UCB exhibits the smallest average regret among all methods in time-invariant graphs, with significant improvements. More precisely, with respect to the Area Under the Curve (AUC) of the regret curve, the improvements of DrFed-UCB over GoSInE, Gossip_UCB, and Dist_UCB are 132%, 158%, and 128%, respectively, showcasing the regret improvement of the newly proposed algorithm compared to the benchmarks. Notably, both Dist_UCB and DrFed-UCB result in larger variances, primarily observed in Dist_UCB. This phenomenon may be attributed to the communication mechanisms designed for time-varying graphs. In Figure 1 (b), we observe that our regret is notably smaller compared to Dist_UCB in settings with time-varying graphs. Specificaly, the AUC of Dist_UCB is



(a) time-invariant graphs (b) time-varying graphs

Figure 1: The regret of different methods in settings with both time-invariant and time-varying graphs



(a) h (b) M

(c) c (d) K

Figure 2: The regret of the proposed algorithm in problem settings with different parameters

96.6% larger than that of our regret curve, which implies the significant improvement in this setting with time-varying graphs. Furthermore, we perform a time complexity comparison, revealing that DrFed-UCB and GoSInE are approximately six times faster than Dist_UCB. Lastly, communication cost is directly computed by the total number of communication rounds and follows an explicit formula. Specifically, the communication costs of DrFed-UCB, Gossip_UCB, and Dist_UCB are of order T, whereas GoSInE exhibits only $o(T)$, suggesting a potential direction for optimizing communication costs.

Regret dependency results Meanwhile, we illustrate how DrFed-UCB's regret depends on several factors: the number of clients (M), the number of arms (K), the Bernoulli parameter (c) for the E-R model, and heterogeneity measured by h. The regret metrics are presented as (a), (b), (c), and (d) in Figure 2, respectively. We observe that regret monotonically increases with the level of heterogeneity and the number of arms, while decreasing with connectivity, which is equivalent to an increase in graph complexity. However, this monotonic trend does not apply to the number of clients. This is due to the following considerations. On one hand, a large M implies a greater number of incident edges

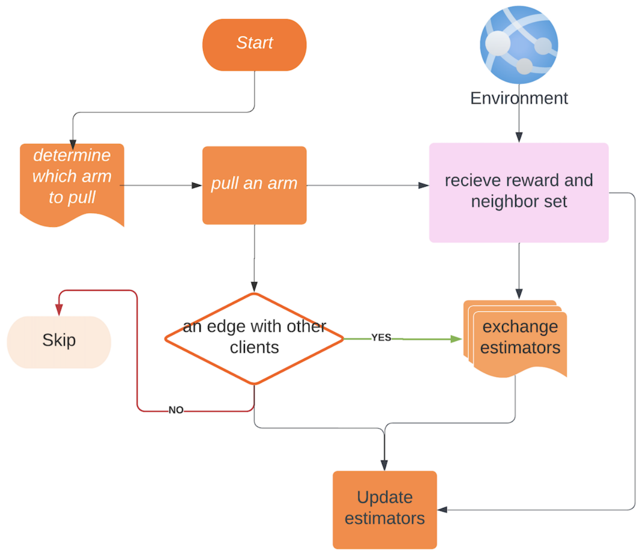


Figure 3: Flowchart of Algorithm 2

of each client, providing more global information access and potentially leading to smaller regret. On the other hand, a large M also weakens the Chernoff-Hoeffding inequality for clients transmitting information, which might result in larger regret.

D Algorithms and Tables in Section 3

The algorithm for generating random connected graphs is presented in Algorithm 3 as follows.

Algorithm 3: Generate a uniformly distributed connected graph

---

Initialization: Let τ_1 be given; Generate a random graph G^{init} by selecting each edge with probability $\frac{1}{2}$;
Connectivity: make G^{init} connected by adding the least many edges to get G_0 ;
for $t = 0, 1, 2, \ldots, \tau_1$ **do**
  Randomly sample an edge pair $e = (i, j)$;
  Denote the edge set of G_s as E_s;
  if $e \in E_s$ **then**
    Remove e from E_s to get $G'_s = (V, E_s \backslash \{e\})$;
    if G'_s *is connected* **then**
      $G_{s+1} = G'_s$;
    else
      reject G'_s and set $G_{s+1} = G_s$;
    end
  else
    $G_{s+1} = (V, E_s \cup \{e\})$;
  end
end

The flowchart of Algorithm 2 is presented in Figure 3 to illustrate the information flow in the algorithm. The table below displays the various settings we consider for the regret analysis.

Table 1: Settings

	E-R	uniform	M	reward
s_1	✓		any	sub-g
s_2		✓	$[1, 10]$	sub-g
s_3		✓	$[11, \infty)$	sub-g
S_1	✓		any	sub-e
S_2		✓	$[1, 10]$	sub-e
S_3		✓	$[11, \infty)$	sub-e

E Remarks on the theoretical results in Section 3.2

E.1 Remarks on Theorem 2

Remark (The condition on the time horizon). *Although the above regret bound holds for any $T > L$, the same bound applies to $T \leq L$ as follows. Assuming $T \leq L$, we obtain $E[R_T|A_{\epsilon,\delta}] \leq T \leq L$ where the first inequality is by noting that the rewards are within the range of $[0, 1]$.*

Remark (The upper bound on the expected regret). *Theorem 2 states a high probability regret bound, while the expected regret is often considered in the existing literature. As a corollary of Theorem 2, we establish the upper bound on $E[R_T]$ if $\epsilon = \frac{\log T}{MT}$ as follows. Note that*

$$E[R_T] = E[R_T|A_{\epsilon,\delta}]P(A_{\epsilon,\delta}) + E[R_T|A^c_{\epsilon,\delta}]P(A^c_{\epsilon,\delta}) \leq P(A_{\epsilon,\delta}) \cdot E[R_T|A_{\epsilon,\delta}] + T \cdot (1 - P(A_{\epsilon,\delta}))$$
$$\leq (1-7\epsilon)(L + \sum_{i \neq i^*}(\max\{[\frac{4C_1 \log T}{\Delta_i^2}], 2(K^2 + MK)\} + \frac{2\pi^2}{3P(A_{\epsilon,\delta})} + K^2 + (2M-1)K)) + 7\epsilon T$$
$$\leq l_1 + l_2 \log T + \sum_{i \neq i^*}(\max\{[\frac{4C_1 \log T}{\Delta_i^2}], 2(K^2 + MK)\} + \frac{2\pi^2}{3(1-7\epsilon)} + K^2 + (2M-1)K) + 7\frac{\log T}{M}$$

where the first inequality uses $E[R_T|A_{\epsilon,\delta}] \leq T$ and the second inequality follows by Theorem 1. Here l_1 and l_2 are constants depending on $K, M, \delta, \min_{i \neq i^*} \Delta_i$, and λ.

Remark (Comparison with previous work). *A comparison to the regret bounds in the existing literature considering sub-Gaussian rewards is as follows. Our regret bounds are consistent with the prior works where the expected regret bounds are of order $\log T$. Note that the regret bounds in [Zhu and Liu, 2023] cannot be used here since the update rule and the settings are different. Their update rule and analyses cannot carry over to our settings, which explains why we invent these modifications and proofs. On the one hand, the time-varying graphs they consider do not include the E-R model, and we can find counter-examples where their doubly stochastic weight matrices W_t result in the divergence of $W_1 \cdot W_2 \ldots W_T$. This makes the key proof step invalid in our framework. On the other hand, their time-varying graphs include the connected graphs when $l = 1$, but they also make an additional assumption of doubly stochastic weight matrices, which is not applicable to regular graphs. Furthermore, they study an expected regret upper bound, while we prove a high probability regret bound that captures the dynamics in the random graphs. The graph assumptions in other works, however, are stronger, such as [Zhu et al., 2021b] consider time-invariant graphs and [Wang et al., 2021] assume graphs are complete [Perchet et al., 2016]. In contrast to some work that focuses on homogeneous rewards in decentralized multi-agent MAB, we derive regret bounds of the same order $\log T$ in a heterogeneous setting. If we take a closer look at the coefficients in terms of K, M, λ, Δ_i, our regret bound is determined by $O(\max(K, \frac{1+\lambda}{1-\lambda}, \frac{1}{M^2 \Delta_i}) \log T)$. The work of [Zhu and Liu, 2023] arrives at $O(\max\{\frac{\log T}{\Delta_i}, K_1, K_2\})$ where K_1, K_2 are related to T without explicit formulas. Our regret is smaller when $K\Delta_i \leq 1$ and $\frac{1+\lambda}{1-\lambda}\Delta_i \leq 1$, which can always hold by rescaling Δ_i, i.e. for many cases we get substantial improvement.*

E.2 Remarks on Theorem 4

Remark. *Based on the expression of L_1, we obtain that L_1 is independent of the sub-optimality gap Δ_i. Meanwhile, we have $C_1 = 8\sigma^2 \cdot 12\frac{M(M+2)}{M^4}$ and $C_2 = \frac{3}{2}C_1 = 12\sigma^2 \cdot 12\frac{M(M+2)}{M^4}$. This implies that the established regret bound in Theorem 4 does not rely on Δ_i but does depend on σ^2. To this end, we use the terminology, mean-gap independent bounds, to only represent bounds having no dependency on Δ_i, rather than instance independent that seems to be an overclaim in this case.*

Remark (**Comparison with previous work**). *For decentralized multi-agent MAB with homogeneous heavy-tailed rewards and time-invariant graphs, [Dubey et al., 2020] provide an instance-dependent regret bound of order $\log T$. In contrast, our regret bound has the same order for heterogeneous settings with random graphs, as shown in Theorem 3. Additionally, we provide a mean-gap independent regret bound as in Theorem 4. In the single-agent MAB setting, [Jia et al., 2021] consider sub-exponential rewards and derive a mean-gap independent regret upper bound of order $\sqrt{T \log T}$. Our regret bound of $\sqrt{T} \log T$ is consistent with theirs, up to a logarithmic factor. Furthermore, our result is consistent with the regret lower bound as proposed in [Slivkins et al., 2019], up to a $\log T$ factor, indicating the tightness of our regret bound.*

Remark. *The discussion regarding the conditions on T, the expected regret $E[R_T]$, and the parameter specifications follow the same logic as those in Theorem 2. We omit the details here.*

F Proof of results in Section 3.2

F.1 Lemmas and Propositions

Lemma 1. *For any $m, i, t > L$, we have*

$$n_{m,i}(t) \geq N_{m,i}(t) - K(K + 2M)$$

Proof of Lemma 1. The proof is referred to [Zhu and Liu, 2023]. □

Lemma 2. *For any $m, i, t > L$, if $n_{m,i}(t) \geq 2(K^2 + KM + M)$ and graph G_t is connected, then we have*

$$n_{m,i}(t) \leq 2 \min_j n_{j,i}(t).$$

where the min is taken over all clients, not just the neighbors.

Proof of Lemma 2. The proof is referred to [Zhu and Liu, 2023]. □

Lemma 3 (Generalized Holder's inequality)**.** *For any $r > 0$ and measurable functions h_i for $i = 1, \dots, n$, if $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{r}$, then*

$$||\Pi_{i=1}^n h_i||_r \leq \Pi_{i=1}^n ||h_i||_{p_k}.$$

The proof follows from the Young's inequality for products.

Lemma 4. *Suppose that random variables $X_1, X_2, \dots, X_n$ are such that $Y_i = E[(X_1, \dots, X_n)|(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)]$ are sub-Gaussian distributed with variance proxy $\sigma_1, \sigma_2, \dots, \sigma_n$, respectively. Then the sum of these sub-Gaussian random variables, $\sum_{i=1}^n X_i$, is again sub-Gaussian with variance proxy $(\sum_{i=1}^n \sigma_i)^2$.*

Proof. First, without loss of generality, let us assume $E[X_i] = 0$. Otherwise, we can always construct a random variable $X_i - E[X_i]$ which has the same variance proxy with a difference up to a constant.

Defining $p_i = \frac{\sum_{k=1}^n \sigma_k}{\sigma_i}$ gives $\sum_{i=1}^n \frac{1}{p_i} = 1$. Let μ be the distribution function of random vector $(X_1, \ldots, X_n)$. By specifying $h_i(x) = \exp(\lambda x)$ and $r = 1$, we obtain that for any $\lambda > 0$ we have

$$
\begin{aligned}
&E[\exp\{\lambda(\sum_{i=1}^n X_i)\}] \\
&= E[\Pi_{i=1}^n \exp\{\lambda X_i\}] \\
&= \int_0^\infty \Pi_{i=1}^n \exp\{\lambda X_i\} d\mu \\
&\leq \Pi_{i=1}^n || \exp\{\lambda X_i\}||_{\frac{\sum_{k=1}^n \sigma_k}{\sigma_i}} \\
&= \Pi_{i=1}^n (\int_0^\infty \exp\{\lambda X_i\}^{\frac{\sum_{k=1}^n \sigma_k}{\sigma_i}} d\mu)^{\frac{\sigma_i}{\sum_{k=1}^n \sigma_k}} \\
&= \Pi_{i=1}^n (E_{Y_i}[\exp\{\lambda X_i \frac{\sum_{k=1}^n \sigma_k}{\sigma_i}\}]^{\frac{\sigma_i}{\sum_{k=1}^n \sigma_k}} \\
&\leq \Pi_{i=1}^n [\exp\{\frac{1}{2}\sigma_i^2 \lambda^2 \frac{(\sum_{k=1}^n \sigma_k)^2}{\sigma_i^2}\}]^{\frac{\sigma_i}{\sum_{k=1}^n \sigma_k}} \\
&= [\exp\{\frac{1}{2}\lambda^2 (\sum_{k=1}^n \sigma_k)^2\}]^{\sum_{i=1}^n \frac{\sigma_i}{\sum_{k=1}^n \sigma_k}} \\
&= \exp\{\frac{1}{2}\lambda^2 (\sum_{k=1}^n \sigma_k)^2\}
\end{aligned}
$$

where the first inequality is by Lemma 3 and the second inequality follows the definition of sub-Gaussian random variables.

□

Lemma 5. *Suppose that random variables $X_1, X_2, \ldots, X_n$ are independent sub-Gaussian distributed with variance proxy $\sigma_1, \sigma_2, \ldots, \sigma_n$, respectively. Then we have that the sum of these sub-Gaussian random variables, $\sum_{i=1}^n X_i$, is again sub-Gaussian with variance proxy $\sum_{i=1}^n \sigma_i^2$.*

Proof. For any $\lambda > 0$ note that

$$
\begin{aligned}
&E[\exp\{\lambda(\sum_{i=1}^n X_i)\}] \\
&= E[\Pi_{i=1}^n \exp\{\lambda X_i\}].
\end{aligned}
$$

Since $X_1, X_2, \ldots, X_n$ are independent random variables, we further have

$$
\begin{aligned}
&E[\exp\{\lambda(\sum_{i=1}^n X_i)\}] \\
&= \Pi_{i=1}^n E[\exp \lambda X_i] \\
&\leq \Pi_{i=1}^n \exp\{\frac{1}{2}\lambda^2 \sigma_i^2\} \\
&= \exp\{\frac{1}{2}\lambda^2 \sum_{i=1} \sigma_i^2\}
\end{aligned}
$$

where the inequality is by the definition of sub-Gaussian random variables.

This concludes the proof.

□

Proposition 1. *Under E-R, for any $1 > \delta, \epsilon > 0$, and any fixed t, $t \geq L_{s1}$, the maintained matrix P_t satisfies*

$$||P_t - cE||_\infty \leq \delta$$

with probability $1 - \frac{\epsilon}{T}$*. This implies that with probability at least* $1 - \epsilon$ *for any* $t \geq L_{s_1}$*, we have*

$$||P_t - cE||_\infty \leq \delta.$$

Proof. We start with the convergence rate of matrix P_t for fixed t.

We recall that in E-R, the indicator function $X_{i,j}^s$ for edge (i,j) at time step s follows a Bernoulli distribution with mean value c. This implies that $\{X_{i,j}^s\}_s$ are i.i.d. random variables which allows us to use the Chernoff-Hoeffding inequality

$$P(|\frac{\sum_{s=1}^t X_{i,j}^s}{t} - c| > \delta) \leq 2\exp\{-2t\delta^2\}.$$

For the probability term, we note that for any $t \geq L_{s_1}$,

$$2\exp\{-2t\delta^2\} \leq \frac{\epsilon}{T}$$

since $t \geq L_{s_1} \geq \frac{\ln \frac{T}{2\epsilon}}{2\delta^2}$ by the choice L_{s_1} of the burn-in period in setting 1.

As a result, the maintained matrix P_t satisfies with probability at least $1 - \frac{\epsilon}{T}$ that

$$\begin{aligned} &||P_t - cE||_\infty \\ &= \max_{i,j} |\frac{\sum_{s=1}^t X_{i,j}^s}{t} - c| \\ &\leq \delta \end{aligned}$$

which concludes the first part of the statement.

Subsequently, consider the probability $P(||P_t - cE||_\infty < \delta, \forall t > L_{s_1})$. We obtain

$$\begin{aligned} &P(||P_t - cE||_\infty < \delta, \forall t > L_{s_1}) \\ &= 1 - P(\cup_{t \geq L_{s_1}} ||P_t - cE||_\infty < \delta) \\ &\geq 1 - \sum_{t \geq L_{s_1}} P(||P_t - cE||_\infty < \delta) \\ &\geq 1 - (T - L_{s_1})\frac{\epsilon}{T} \geq 1 - \epsilon \end{aligned}$$

where the first inequality uses the Bonferroni's inequality.

This completes the second part of the statement.

□

We next pin down the Markov chain governing Algorithm 1. Its states compound to all connected graphs if G and G' are connected, then the transition probability is defined by

$$\pi(G'|G) = \begin{cases} 0 \text{ if } |E(G')\Delta E(G)| > 1 \\ \frac{2}{M(M-1)} \text{ if } |E(G')\Delta E(G)| = 1 \\ 1 - \frac{2\alpha(G)}{M(M-1)} \text{ if } G' = G. \end{cases}$$

Here Δ denotes the symmetric difference and $\alpha(G)$ is the number of all connected graph that differ with G by at most one edge. Algorithm 1 is a random walk in the Markov chain denoted as $CG-MC$. The intriguing question is if the stationary distribution corresponds to the uniform distribution on all connected graphs on M nodes and if it is rapidly mixing. The next paragraph gives affirmative answers.

Proposition 2. *In* $CG - MC$*, for any time step* $n \geq 1$ *and initial connected graph* G^{init}*, we have*

$$||\pi^n(\cdot|G^{init}) - \pi^*(\cdot)||_{TV} \leq 2(p^*)^n$$

where $p^* = p^*(M) < 1$ *and* π^* *is the uniform distribution on all connected graphs.*

Proof. Based on the definition of π^*, we have

$$\pi^* = \frac{1}{\#\{connected \quad graphs\}}.$$

Therefore, there exists a constant $0 < C_f < 1$ such that for any two connected graphs G, G' with $|E(G)\Delta E(G')| = 1$ we have

$$\pi(G|G') \geq C_f \pi^*.$$

In essence $C_f = \frac{1}{\pi^*} \min_{G,G'} \pi(G|G') < 1$.

If $G = G'$, then there are two possible cases. First, if $\alpha(G) < \frac{(M(M-1)}{2}$, then $\pi(G|G) > \frac{2}{M(M-1)} > \pi^*$ and $\pi(G|G) > 0$. Otherwise, we have $\pi(G|G) = 0$. In other words, the set $G \notin \{G' : \pi(G'|G) \leq \pi^*(G'), \pi(G'|G) > 0\}$.

This implies that for $G' \in \{G' : \pi(G'|G) \leq \pi^*(G'), \pi(G'|G) > 0\}$, we have $|E(G)\Delta E(G')| = 1$ and subsequently $\pi(G|G') \geq C_f \pi^*$.

We start with the one-step transition and obtain

$$\begin{aligned}
&||\pi(\cdot|G) - \pi^*(\cdot)||_{TV} \\
&= 2 \sup_A |\int_A (\pi(G'|G) - \pi^*(G'))d_{G'}| \\
&\leq 2 \int_{\{G':\pi(G'|G)-\pi^*(G')\leq 0\}} (-\pi(G'|G) + \pi^*(G'))d_{G'} \\
&\leq 2 \int_{\{G':\pi(G'|G)=0\}} (-\pi(G'|G) + \pi^*(G'))d_{G'} + \\
&\qquad 2 \int_{\{G':\pi(G'|G)>0, \pi(G'|G)-\pi^*(G')\leq 0\}} (-\pi(G'|G) + \pi^*(G'))d_{G'} \\
&= 2 \int_{\{G':\pi(G'|G)=0\}} (\pi^*(G'))d_{G'} + 2(1 - C_f) \int_{\{G':\pi(G'|G)>0, \pi(G'|G)-\pi^*(G')\leq 0\}} (\pi^*(G'))d_{G'} \\
&\leq 2P(\{G' : \pi(G'|G) = 0\}) + 2(1 - C_f)(1 - P(\{G' : \pi(G'|G) = 0\})) \\
&\leq 2(1 - \frac{1}{\#\{connected \quad graphs\}}) + 2(1 - C_f)(1 - (1 - \frac{1}{\#\{connected \quad graphs\}})) \\
&\doteq 2p' + 2(1 - C_f)(1 - p') = 2(p' + (1 - C_f)(1 - p')) \\
&\doteq 2p^*
\end{aligned}$$

where we denote the term $1 - \frac{1}{\#\{connected \quad graphs\}}$ and the term $p' + (1 - C_f)(1 - p')$ by p' and p^*, respectively. It is worth noting that $p^* = p^*(M)$ and $p^* < 1$ since $p', C_f < 1$. Here the third inequality uses the above argument on the graphs in the set $\{G' : \pi(G'|G) \leq \pi^*(G'), \pi(G'|G) > 0\}$, and the last inequality uses the following result. By definition,

$$\begin{aligned}
&P(\{G' : \pi(G'|G) = 0\}) \\
&= 1 - P(\{G' : \pi(G'|G) > 0\}) \\
&\leq 1 - P(\{G' : |E(G)\Delta E(G')| = 1\}) \\
&= 1 - \frac{\alpha(G)}{\#\{connected \quad graphs\}} \leq 1 - \frac{1}{\#\{connected \quad graphs\}}
\end{aligned}$$

where the last inequality uses $\alpha(G) \geq 1$ by the definition of $\alpha(G)$.

Suppose at time step n, the result holds, i.e. for any G

$$||\pi^n(\cdot|G) - \pi^*(\cdot)||_{TV} \leq 2(p^*)^n.$$

Then we consider the transition kernel at the $n+1$ step. Note that

$$\begin{aligned}
&||\pi^{n+1}(\cdot|G) - \pi^*(\cdot)||_{TV} \\
&= 2\sup_A |\int_A (\pi^{n+1}(G'|G) - \pi^*(G'))d_{G'}| \\
&\leq 2\sup_A |\int_S \int_A (\pi^n(G'|S) - \pi^*(G'))(\pi(S|G) - \pi^*(S))d_{G'}d_S| \\
&= 2\sup_A |\int_S (\pi(S|G) - \pi^*(S))(\int_A (\pi^n(G'|S) - \pi^*(G'))d_{G'})d_S| \\
&\leq 2 \cdot \frac{1}{2}||\pi^n(\cdot|S) - \pi^*(\cdot)||_{TV} \cdot \frac{1}{2}||\pi(\cdot|G) - \pi^*(\cdot)||_{TV} \\
&= \frac{1}{2}||\pi^n(\cdot|S) - \pi^*(\cdot)||_{TV}||\pi(\cdot|G) - \pi^*(\cdot)||_{TV} \\
&\leq \frac{1}{2} \cdot 2(p^*)^n \cdot 2p^* = 2(p^*)^{n+1}
\end{aligned}$$

where the second inequality is by using the definition of $||\cdot||_{TV}$ and the last inequality holds by the results in the basis step and the induction step, respectively. The first inequality requires more arguments as follows. Consider the integral

$$\begin{aligned}
&\int_S \int_A (\pi^n(G'|S) - \pi^*(G'))(\pi(S|G) - \pi^*(S))d_{G'}d_S \\
&= \int_S \int_A (\pi^n(G'|S) - \pi^*(G'))\pi(S|G)d_{G'}d_S - (\pi^n(G'|S) - \pi^*(G'))\pi^*(S)d_{G'}d_S \\
&= \int_S \int_A \pi^n(G'|S)\pi(S|G)d_{G'}d_S - \int_S \int_A \pi^*(G')\pi^n(S|G)d_{G'}d_S - \\
&\qquad \int_S \int_A \pi^n(G'|S)\pi^*(S)d_{G'}d_S + \int_S \int_A \pi^*(G')\pi^*(S)d_{G'}d_S \\
&\geq \int_A \pi^{n+1}(G'|G)d_{G'} - \int_A \pi^*(G')d_{G'} - \\
&\qquad \int_S \pi^*(S)d_S + \int_S \pi^*(G')d_{G'} \\
&= \int_A \pi^{n+1}(G'|G)d_{G'} - \int_A \pi^*(S)d_S = \int_A (\pi^{n+1}(G'|G) - \pi^*(G'))d_{G'}
\end{aligned}$$

where the results hold by exchanging the orders of the integrals as a result of Funibi's Theorem and the inequality uses the fact that $\int_A \pi^n(G'|S)d_{G'} \leq 1$.

This completes the proof by concluding the mathematical induction.

□

Proposition 3. *For any $1 > \delta, \epsilon > 0$, we obtain that for setting 2, for any fixed $t \geq L_{s_2}$, the maintained matrix P_t satisfies with probability $1 - 2\frac{\epsilon}{T}$*

$$||P_t - cE||_\infty \leq \delta.$$

Meanwhile, the graph generated by Algorithm 3 converges to the stationary distribution with

$$||\pi^t(\cdot|G) - \pi^*(\cdot)||_{TV} \leq \frac{1\delta}{5},$$

where π^* is the uniform distribution on all connected graphs.

Proof. Suppose we run the rapidly mixing markov chain for a time period of length $\tau_1 = \frac{\ln \frac{\zeta}{2}}{\ln p^*}$ where $\zeta < \frac{\delta}{5}$. By applying Proposition 2, we obtain that for any time step $t > \tau_1$,

$$||\pi^t(\cdot|G) - f(\cdot)||_{TV} \leq 2(p^*)^{\tau_1}.$$

For the second phase, we reset the counter of time step and denote the starting point $t = \tau_1 + 1$ as $t = 1$. Based on the definition of P_t, we have $P_t = (\frac{\sum_{s=1}^t X_{i,j}^s}{t})_{1 \le i \ne j \le M}$ where $X_{i,j}^s$ is the indicator function for edge (i, j) on Graph G_s that follows the distribution $\pi^{\tau_1+s}(G, \cdot)$. Let us denote $Y_{i,j}^s$ the indicator function for edge (i, j) on Graph G_s^{obj} following the distribution $\pi^*(\cdot)$ and $P_t^{obj} = (\frac{\sum_{s=1}^t Y_{i,j}^s}{t})_{1 \le i \ne j \le M}$.

By the Chernoff-Hoeffding inequality and specifying $\zeta = 2(p^*)^{\tau_1}$, we derive

$$P(|E[Y_{i,j}^1] - \frac{\sum_{s=1}^t Y_{i,j}^s}{t}| \ge \zeta) \le 2\exp\{-2t\zeta^2\}, \tag{3}$$

i.e.

$$||P_t^{obj} - cE||_\infty \le \zeta$$

holds with probability $1 - 2\exp\{-2t\zeta^2\}$.

Consider the difference between P_t and P_t^{obj} and we obtain that

$$\begin{aligned}
&P_t - P_t^{obj} \\
&= (\frac{\sum_{s=1}^t X_{i,j}^s - \sum_{s=1}^t Y_{i,j}^s}{t})_{1 \le i \ne j \le M} \\
&= (\frac{\sum_{s=1}^t X_{i,j}^s}{t} - E[X_{i,j}^1] + E[X_{i,j}^1 - Y_{i,j}^1] + E[Y_{i,j}^1] - \frac{\sum_{s=1}^t Y_{i,j}^s}{t})_{1 \le i \ne j \le M}
\end{aligned}$$

where the last term is bounded by (3).

For the second quantity $E[(X_{i,j}^s - Y_{i,j}^s)]$, we have that for any s, i, j

$$\begin{aligned}
&E[(X_{i,j}^s - Y_{i,j}^s)|G_s, G_s^{obj}] \\
&= 1_{G_s \text{ contains edge } (i,j) \& G_s^{obj} \text{ does not contain edge } (i,j)} - 1_{(G_s \text{ does not contain edge } (i,j) \& G_s^{obj} \text{ contains edge } (i,j))}
\end{aligned}$$

and subsequently by the law of total expectation, we further obtain

$$
\begin{aligned}
&E[(X_{i,j}^s - Y_{i,j}^s)] \\
&= E[E[(X_{i,j}^s - Y_{i,j}^s)|G_s, G_s^{obj}]] \\
&= E[1_{G_s \text{ contains edge } (i,j) \& G_s^{obj} \text{ does not contain edge } (i,j)} - 1_{(G_s \text{ does not contain edge } (i,j) \& G_s^{obj} \text{ contains edge } (i,j))}] \\
&= E[1_{G_s \text{ contains edge } (i,j)}] \cdot E[1_{G_s^{obj} \text{ does not contain edge } (i,j)}] - \\
&\qquad E[1_{(G_s \text{ does not contain edge } (i,j)}] \cdot E[1_{G_s^{obj} \text{ contains edge } (i,j))}] \text{ since } G_s \text{ and } G_s^{obj} \text{ are independent} \\
&= \int_A 1_{S \text{ contains edge } (i,j)} \pi^{\tau_1+s}(S|G) dS \cdot \int_A 1_{S \text{ does not contain edge } (i,j)} \pi^*(S) dS - \\
&\qquad \int_A 1_{S \text{ does not contain edge } (i,j)} \pi^{\tau_1+s}(S|G) dS \cdot \int_A 1_{S \text{ contains edge } (i,j)} \pi^*(S) dS \\
&= \int_A 1_{S \text{ contains edge } (i,j)} \pi^{\tau_1+s}(S|G) dS \cdot \int_A 1_{S \text{ does not contain edge } (i,j)} \pi^*(S) dS - \\
&\qquad \int_A 1_{S \text{ contains edge } (i,j)} \pi^*(S) dS \cdot \int_A 1_{S \text{ does not contain edge } (i,j)} \pi^*(S) dS + \\
&\qquad \int_A 1_{S \text{ contains edge } (i,j)} \pi^*(S) dS \cdot \int_A 1_{S \text{ does not contain edge } (i,j)} \pi^*(S) dS - \\
&\qquad \int_A 1_{S \text{ does not contain edge } (i,j)} \pi^{\tau_1+s}(S|G) dS \cdot \int_A 1_{S \text{ contains edge } (i,j)} \pi^*(S) dS \\
&= \int_A 1_{S \text{ does not contain edge } (i,j)} \pi^*(S) dS (\int_A 1_{S \text{ contains edge } (i,j)} (\pi^{\tau_1+s}(S|G) - \pi^*(S)) dS) + \\
&\qquad \int_A 1_{S \text{ contains edge } (i,j)} \pi^*(S) dS (\int_A 1_{S \text{ contains edge } (i,j)} (-\pi^{\tau_1+s}(S|G) + \pi^*(S)) dS) \\
&\leq \int_A 1_{S \text{ does not contain edge } (i,j)} \pi^*(S) dS \cdot ||\pi^{\tau_1+s}(\cdot|G) - \pi^*(\cdot)||_{TV} + \\
&\qquad \int_A 1_{S \text{ contains edge } (i,j)} \pi^*(S) dS \cdot ||\pi^{\tau_1+s}(\cdot|G) - \pi^*(\cdot)||_{TV} \\
&= ||\pi^{\tau_1+s}(\cdot|G) - \pi^*(\cdot)||_{TV} \\
&\leq 2(p^*)^{\tau_1}.
\end{aligned}
$$

In like manner, we achieve

$$
\begin{aligned}
&E[(-X_{i,j}^s + Y_{i,j}^s)] \\
&\leq 2(p^*)^{\tau_1}.
\end{aligned} \tag{4}
$$

We now proceed to the analysis on the first term. Though $\{X_{i,j}^s\}_s$ are neither independent or identically distributed random variables, the difference $\frac{\sum_{s=1}^t X_{i,j}^s}{t} - E[X_{i,j}^1]$ can be upper bounded by the convergence property of π^n. Note that $X_{i,j}^s$ is only different from $X_{i,j}^{s+1}$ when edge (i,j) is sampled at time step s and the generated graph is accepted.

We observe that

$$
\begin{aligned}
&P(G_{s+1}|G_s, G_{s-1}, \ldots, G_1) \\
&= P(G_{s+1}|G_s).
\end{aligned}
$$

Meanwhile, we can write $X_{i,j}^{s+1} = 1_{X_{i,j}^{s+1}=1} = 1_{G_{s+1} \text{ contains edge } (i,j)}$ and similarly, $X_{i,j}^s = 1_{X_{i,j}^s=1} = 1_{G_s \text{ contains edge } (i,j)}$. Denote event E as $E = \{\text{connected graph G contains edge } (i,j)\}$. This gives us

$$
\begin{aligned}
X_{i,j}^{s+1} &= 1_{G_{s+1} \in E}, \\
X_{i,j}^s &= 1_{G_s \in E}.
\end{aligned}
$$

Furthermore, the quantity $\int_A 1_{S\in E}\pi^*(S)dS$ can be simplified as

$$E[Y^1_{i,j}] = E[1_{G^{obj}_s \in E}] \\ = \int_A 1_{S\in E}\pi^*(S)dS$$

since G^{obj}_s follows a distribution with density $\pi^*(\cdot)$.

A new Hoeffding lemma for markov chains has been recently shown as follows in [Fan et al., 2021]. Let $a(\lambda) = \frac{1+\lambda}{1-\lambda}$ where λ is the spectrum of the Markov chain $CG - MC$ and by the Theorem 2.1 in [Fan et al., 2021], we obtain that

$$P(|\sum_{s=1}^{t} 1_{G_s\in E} - tE[Y^1_{i,j}]| > t\zeta) \leq 2\exp\{-2a(\lambda)^{-1}t\zeta^2\}$$

$$i.e. P(|\frac{\sum_{s=1}^{t} X^s_{i,j}}{t} - E[Y^1_{i,j}]| > \zeta) \leq 2\exp\{-2a(\lambda)^{-1}t\zeta^2\} \tag{5}$$

since $X^s_{i,j} = 1_{G_s\in E}$ satisfies $0 \leq 1_{G_s\in E} \leq 1$, i.e. the values are within the range of $[0, 1]$.

By the result $E[X^1_{i,j}] - \zeta \leq E[Y^1_{i,j}] \leq E[X^1_{i,j}] + \zeta$, we obtain

$$P(|\frac{\sum_{s=1}^{t} X^s_{i,j}}{t} - E[X^1_{i,j}]| > 2\zeta) \leq 2\exp\{-2a(\lambda)^{-1}t\zeta^2\}. \tag{6}$$

Putting the results (3), (4) and (6) together, we derive

$$\begin{aligned} &||P_t - P^{obj}_t||_\infty \\ &= \max_{i,j} |\frac{\sum_{s=1}^{t} X^s_{i,j}}{t} - E[X^1_{i,j}] + E[X^1_{i,j} - Y^1_{i,j}] + E[Y^1_{i,j}] - \frac{\sum_{s=1}^{t} Y^s_{i,j}}{t}| \\ &\leq \max_{i,j}(|\frac{\sum_{s=1}^{t} X^s_{i,j}}{t} - E[X^1_{i,j}]| + |E[X^1_{i,j} - Y^1_{i,j}]| + |E[Y^1_{i,j}] - \frac{\sum_{s=1}^{t} Y^s_{i,j}}{t}| \\ &\leq 2\zeta + \zeta + \zeta = 4\zeta \end{aligned}$$

which holds with probability at least $1 - 2\exp\{-2a(\lambda)^{-1}t\zeta^2\} - 2\exp\{-2t\zeta^2\}$.

For the probability term $1 - 2\exp\{-2a(\lambda)^{-1}t\zeta^2\} - 2\exp\{-2t\zeta^2\}$, we have

$$2\exp\{-2a(\lambda)^{-1}t\zeta^2\} \leq \frac{\epsilon}{T}, \\ 2\exp\{-2t\zeta^2\} \leq \frac{\epsilon}{T}$$

which holds by

$$t \geq L_{s_2} - \tau_1 = a(\lambda)\frac{\ln\frac{T}{2\epsilon}}{2\zeta^2}.$$

Therefore, the distance between the empirical matrix and the constant matrix reads as with probability at least $1 - 2\frac{\epsilon}{T}$,

$$\begin{aligned} &||P_t - cE||_\infty \\ &\leq ||P_t - P^{obj}_t||_\infty + ||P^{obj}_t - cE||_\infty \\ &\leq 4\zeta + \zeta = 5\zeta < \delta \end{aligned}$$

where $\zeta = 2(p^*)^{\tau_1} < \frac{1}{5}\delta$ by the choice of parameter δ and ζ. This completes the proof of Proposition 3.

□

Next, we proceed to explicitly characterize the spectrum of $CG - MC$ which plays a role in the length of burning period L_{s_2} and L_{s_3}.

Proposition 4. *In setting 2, the spectral gap $1-\lambda$ of $CG-MC$ satisfies that for $\theta > 1$,*

$$1-\lambda \geq \frac{1}{2\frac{\ln\theta}{\ln 2p^*}\ln 2\theta + 1}.$$

Proof. It is worth noting that for $G \neq G'$ and $|E(G)\Delta E(G')| = 1$, we have

$$\pi^*(G)\pi(G'|G) = \pi^*(G')\pi(G|G')$$

by the fact that $\pi^*(G) = \pi^*(G')$ and $\pi(G'|G) = \pi(G|G') = \frac{2}{M(M-1)}$.

For $G \neq G'$ and $|E(G)\Delta E(G')| > 1$ or $|E(G)\Delta E(G')| = 0$, we have $\pi^*(G)\pi(G'|G) = \pi^*(G')\pi(G|G')$ since $\pi(G'|G) = \pi(G|G') = 0$.

For $G = G'$, we have $\pi^*(G)\pi(G'|G) = \pi^*(G')\pi(G|G')$ by the expression.

As a result, $CG-MC$ is reversible. Meanwhile, it is ergodic since it has a stationary distribution π^* as stated in Proposition 2.

Henceforth, by the result of Theorem 1 in [McNew, 2011] that holds for any ergodic and reversible Markov chain, we have

$$\frac{1}{2\ln 2e}\frac{\lambda_2}{1-\lambda_2} \leq \tau(e)$$

where $\tau(e)$ is the mixing time for an error tolerance e and λ_2 is the second largest eigenvalue of $CG-MC$. Choosing $e > \frac{1}{2}$ immediately gives us

$$\lambda_2 \leq \frac{2\tau(e)\ln 2\frac{1}{5}\delta}{2\tau(e)\ln 2e + 1}. \tag{7}$$

Again by Proposition 2, we have

$$||\pi^{\tau(e)}(\cdot|G) - \pi^*(\cdot)||_{TV} \leq 2(p^*)^{\tau(e)} = e.$$

Consequently, we arrive at

$$\tau(e) = \frac{\ln e}{\ln 2p^*}$$

and subsequently

$$\lambda_2 \leq \frac{2\frac{\ln e}{\ln 2p^*}\ln 2e}{2\frac{\ln e}{\ln 2p^*}\ln 2e + 1}.$$

by plugging $\tau(e)$ into (7).

This completes the proof of the lower bound on the spectral gap $1-\lambda_2$.

$\square$

In the following proposition, we show the sufficient condition for graphs generated by the E-R model being connected.

Proposition 5. *Assume c in setting 1 meets the condition*

$$1 \geq c \geq \frac{1}{2} + \frac{1}{2}\sqrt{1-(\frac{\epsilon}{MT})^{\frac{2}{M-1}}},$$

where $0 < \epsilon < 1$. Then, with probability $1-\epsilon$, for any $t > 0$, G_t following the E-R model is connected.

Proof. For $1 \leq j \leq M$, we denote the degree of client j as d_j.

It is straightforward to have 1) $\sum_{j=1}^{M} d_j = 2 \cdot$ total number of edegs, 2) $E[\text{total number of edges}] = c \cdot \frac{M(M-1)}{2}$ and 3) random variables $d_1, d_2, \ldots, d_M$ are dependent but follow the same distribution. Note that d_j follows a binomial distribution with $E[d_j] = c \cdot (M-1)$ where c is the probability of an edge. Then by the Chernoff Bound inequality, we have

$$P(d_j < \frac{M-1}{2}) \leq \exp\{-(M-1) \cdot KL(0.5||c)\}$$

where $KL(0.5||c)$ denotes the KL divergence between Bernoulli(0.5) and Bernoulli(c).

For the term $KL(0.5||c)$, we can further show that

$$KL(0.5||c) = \frac{1}{2} \log \frac{\frac{1}{2}}{c} + \frac{1}{2} \log \frac{\frac{1}{2}}{1-c} = \frac{1}{2} \log \frac{1}{4c(1-c)}$$

which leads to $P(d_j < \frac{M-1}{2}) \leq \exp\{(M-1) \cdot \frac{1}{2} \log 4c(1-c)\}$.

Meanwhile, we have specified the choice of c as

$$\frac{1}{2} + \frac{1}{2}\sqrt{1 - (\frac{\epsilon}{MT})^{\frac{2}{M-1}}\}} \leq c < 1$$

which guarantees $\exp\{(M-1) \cdot \frac{1}{2} \log 4c(1-c)\} \leq \frac{\epsilon}{MT}$ as follow. We observe that

$$\begin{aligned}
& c \geq \frac{1}{2} + \frac{1}{2}\sqrt{1 - (\frac{\epsilon}{MT})^{\frac{2}{M-1}}} \\
& \Longrightarrow 4c(1-c) \leq (\frac{\epsilon}{MT})^{\frac{2}{M-1}} \\
& \Longrightarrow \log 4c(1-c) \leq \frac{2 \log \frac{\epsilon}{MT}}{M-1} \\
& \Longrightarrow (M-1) \cdot \frac{1}{2} \log 4c(1-c) \leq \log \frac{\epsilon}{MT} \\
& \Longrightarrow \exp\{(M-1) \cdot \frac{1}{2} \log 4c(1-c)\} \leq \frac{\epsilon}{MT}.
\end{aligned}$$

This is summarized as for any j

$$P(d_j < \frac{M-1}{2}) \leq \exp\{(M-1) \cdot \frac{1}{2} \log 4c(1-c)\} \leq \frac{\epsilon}{MT}. \tag{8}$$

Meanwhile, it is known as if $\delta(G_t) \geq \frac{M-1}{2}$, then we have that graph G_t is connected where $\delta(G_t) = \min_m d_m$.

As a result, consider the probability and we obtain that

$$\begin{aligned}
& P(\text{graph } G_t \text{ is connected}) \\
& \geq P(\min_j d_j \geq \frac{M-1}{2}) \\
& = P(\bigcap_j \{d_j \geq \frac{M-1}{2}\}) \\
& = 1 - P(\bigcup_j \{d_j < \frac{M-1}{2}\}) \\
& \geq 1 - \sum_j P(d_j < \frac{M-1}{2}) \\
& = 1 - MP(d_j < \frac{M-1}{2}) \\
& \geq 1 - M\frac{\epsilon}{MT} = 1 - \frac{\epsilon}{T}
\end{aligned}$$

where the second inequality holds by the Bonferroni's inequality and the third inequality uses (8).

Consequently, we obtain

$$
\begin{aligned}
& P(\text{graph } G_t \text{ is connected}) \\
& = P(\cap_t \{G_t \text{ is connected}\}) \\
& \geq 1 - \sum_t P(G_t \text{ is not connected}) \\
& = 1 - \sum_t (1 - P(G_t \text{ is connected})) \\
& \geq 1 - \sum_t (1 - (1 - \frac{\epsilon}{T}) = 1 - \epsilon
\end{aligned}
$$

where the first inequality holds again by the Bonferroni's inequality and the second inequality results from the above derivation.

This completes the proof.

□

On graphs with the established properties, we next show the results on the transmission gap between two consecutive rounds of communication for any two clients and the number of arm pulls for all clients.

Proposition 6. *We have that with probability* $1 - \epsilon$*, for any* $t > L$ *and any* m*, there exists* t_0 *such that*

$$
t + 1 - \min_j t_{m,j} \leq t_0, t_0 \leq c_0 \min_l n_{l,i}(t+1)
$$

where $c_0 = c_0(K, \min_{i \neq i^*} \Delta_i, M, \epsilon, \delta)$.

Proof. The edges in setting 1 follow a Bernoulli distribution with a given parameter c by definition. Though setting 2 does not explicitly define the edge distribution, the probability of an edge existing in a connected graph, denoted as c, is deterministic, independent of time since graphs are i.i.d. over time and homogeneous among edges.

Henceforth, it is straightforward that c satisfies

$$
\frac{M(M-1)}{2} c = E(N)
$$

and equivalently $c = \frac{2E(N)}{M(M-1)}$ where N denotes the number of edges in a random connected graph.

We observe that $0 \leq N \leq \frac{M(M-1)}{2}$. Furthermore, the existing result in [Trevisan] yields

$$
E[N] = M \log M.
$$

Consequently, the probability term c has an explicit expressions $c = \frac{2E[N]}{M(M-1)} = \frac{2 \log M}{M-1}$.

For setting s_2, S_2 we have $c = \frac{2 \log M}{M-1} \geq \frac{1}{2}$ since $M < 10$, while in setting s_1, S_1, the condition on c guarantees $c > \frac{1}{2}$. Note that $t + 1 - t_{m,j}$ follows a geometric distribution since each edge follows a Bernoulli distribution, which holds by

$$
\begin{aligned}
& P(t + 1 - t_{m,j} = 1 | t_{m,j}) \\
& = \frac{P(\text{there is an edge between m and j at time step t+1 and } t_{m,j})}{P(\text{there is an edge between m and j at time step } t_{m,j})} \\
& = \frac{(c)^2}{c} = c
\end{aligned}
$$

and

$$
\begin{aligned}
&P(t+1-t_{m,j}=k|t_{m,j})\\
&=\frac{P(\text{there is an edge between m and j at time step t+k and } t_{m,j},\ \text{no edge at time step} t+1,\ldots,t+k-1)}{P(\text{there is an edge between m and j at time step } t_{m,j})}\\
&=\frac{(1-c)^{k-1}c^2}{c}=c(1-c)^{k-1}.
\end{aligned}
$$

Note that $P(t+1-\min_j t_{m,j}\geq t_0)$ which denotes the tail of a geometric distribution depends on the choice of c. More precisely, the tail probability P_0 is monotone decreasing in c.

When $c=\frac{1}{2}$, we obtain that

$$
P_0=P(t+1-\min_j t_{m,j}>t_0)\leq\sum_{s>t_0}(\frac{1}{2})^s\leq(\frac{1}{2})^{t_0}. \tag{9}
$$

Choosing $t_0=\frac{\ln\frac{M^2T}{\epsilon}}{\ln 2}$ leads to $P_0=1-(\frac{1}{2})^{t_0}=1-\frac{\epsilon}{M^2T}$ and

$$
\begin{aligned}
&c_0\min_l n_{l,i}(t+1)\geq c_0\min_l n_{l,i}(L)\\
&\geq c_0\frac{L}{K}\geq c_0\frac{\ln\frac{M^2T}{\epsilon}}{c_0\ln 2}=\frac{\ln\frac{M^2T}{\epsilon}}{\ln 2}=t_0
\end{aligned}
$$

where the last inequality holds by the choice of L. This implies $\min_l n_{l,i}(t+1)-t_0\geq(1-c_0)\min_l n_{l,i}(t+1)$, i.e. $t_0\leq c_0\min_l n_{l,i}(t+1)$.

Therefore, with probability $1-\frac{\epsilon}{M^2T}$,

$$
\begin{aligned}
&\min_l n_{l,i}(t_{m,l})\\
&\geq\min_l n_{l,i}(\min_j t_{m,j})\\
&\geq\min_l n_{l,i}(t+1-t_0)\\
&\geq\min_l n_{l,i}(t+1)-t_0\\
&\geq(1-c_0)\cdot\min_l n_{l,i}(t+1)
\end{aligned}
$$

where the first inequality results from the fact $t_{m,l}\geq\min_j t_{m,j}$, the second inequality uses the fact from (9), the third inequality applies the definition of n, and the last inequality holds by the choice of t_0.

Consider setting s_3,S_3 where $M>10$. Generally, for a given parameter c, we obtain

$$
\begin{aligned}
&P(t+1-\min_j t_{m,j}=1)=c,\\
&P(t+1-\min_j t_{m,j}=2)=c(1-c),\\
&\ldots,\\
&P(t+1-\min_j t_{m,j}=n)=c(1-c)^{n-1}
\end{aligned}
$$

and subsequently

$$
P_0=P(t+1-\min_j t_{m,j}>t_0)\leq\sum_{s>t_0}c(1-c)^{s-1}\leq c(\frac{1}{c}-\frac{1-(1-c)^{t_0}}{c})=(1-c)^{t_0}.
$$

For the probability term P_0, we further arrive at

$$
P_0\geq 1-\frac{\epsilon}{M^2T}
$$

by the choice of $t_0\geq\frac{\ln(\frac{\epsilon}{M^2T})}{\ln(1-c)}$.

Meanwhile, we claim that the choice of t_0 satisfies

$$t_0 \leq c_0 \min_l n_{l,i}(t+1)$$

since $\frac{\ln(\frac{\epsilon}{M^2T})}{\ln(1-c)} \leq c_0 \min_l n_{l,i}(t+1)$ holds by noting $n_{l,i}(t+1) \geq n_{l,i}(L) \geq \frac{L}{K}$ and

$$L \geq \frac{K \ln(\frac{\epsilon}{M^2T})}{c_0 \ln(1-c)} = \frac{K \ln(\frac{M^2T}{\epsilon})}{c_0 \ln(\frac{1}{1-c})}.$$

To summarize, in all the settings, we have that with probability at least $1 - \frac{\epsilon}{M^2T}$,

$$\begin{aligned} & t + 1 - \min_j t_{m,j} \leq t_0, \\ & t_0 \leq c_0 \min_l n_{l,i}(t+1). \end{aligned} \tag{10}$$

Therefore, we obtain that in setting s_1, S_1

$$\begin{aligned} & P(\forall m, t+1-\min_j t_{m,j} \leq t_0 \leq c_0 \min_l n_{l,i}(t+1)) \\ & \doteq (1-P_0)^M \geq 1 - MP_0 = 1 - \frac{\epsilon}{MT} \end{aligned} \tag{11}$$

where the inequality is a result of the Bernoulli's inequality.

In setting s_2, S_2, s_3, S_3, $\{t+1-\min_j t_{1,j}, \ldots, t+1-\min_j t_{M,j}\}$ follow the same distribution, but are dependent since they construct a connected graph. However, we have the following result

$$\begin{aligned} & P(\forall m, t+1-\min_j t_{m,j} \leq t_0 \leq c_0 \min_l n_{l,i}(t+1)) \\ & = 1 - P(\cup_m \{t+1-\min_j t_{m,j} \geq t_0\}) \\ & \geq 1 - \sum_m P(t+1-\min_j t_{m,j} \geq t_0) \\ & = 1 - MP_0 = 1 - \frac{\epsilon}{MT} \end{aligned} \tag{12}$$

by the Bonferroni's inequality.

As a consequence, we arrive at that in setting $s_1, S_1, s_2, S_2, s_3, S_3$,

$$\begin{aligned} & P(\forall t, \forall m, t+1-\min_j t_{m,j} \leq t_0 \leq c_0 \min_l n_{l,i}(t+1)) \\ & \geq 1 - \sum_t \sum_m P(t+1-\min_j t_{m,j} \leq t_0 \leq c_0 \min_l n_{l,i}(t+1)) \\ & \geq 1 - MT(1-(1-\frac{\epsilon}{MT})) = 1 - \epsilon \end{aligned}$$

where the first inequality again uses the Bonferroni's inequality and the second inequality holds by applying (11) and (12).

□

After establishing the transmissions among clients, we next proceed to show the concentration properties of the network-wide estimators maintained by the clients.

The first is to demonstrate the unbiasedness of these estimators with respect to the global expected rewards.

Proposition 7. *Assume the parameter δ satisfies that $0 < \delta < c = f(\epsilon, M, T)$. For any arm i and any client m, at every time step t, we have*

$$E[\tilde{\mu}_i^m(t)|A_{\epsilon,\delta}] = \mu_i.$$

Proof. The result can be shown by induction as follows. We start with the basis step by considering any time step $t \leq L+1$. By the definition of $\tilde{\mu}_i^m(t) = \tilde{\mu}_i^m(L+1)$, we arrive at

$$\begin{aligned}
&E[\tilde{\mu}_i^m(t)|A_{\epsilon,\delta}] \\
&= E[\tilde{\mu}_i^m(L+1)|A_{\epsilon,\delta}] \\
&= E[\sum_{j=1}^{M} P'_{m,j}(L)\hat{\tilde{\mu}}_{i,j}^m(h_{m,j}^L)|A_{\epsilon,\delta}]
\end{aligned} \tag{13}$$

where $P'_{m,j}(L) = \begin{cases} \frac{1}{M}, & \text{if } P_L(m,j) > 0 \\ 0, & \text{else} \end{cases}$. The definition of $A_{\epsilon,\delta}$ and the choice of δ guarantee that $|P_L - cE| < \delta < c$ on event $A_{\epsilon,\delta}$, i.e. we have for any $t \geq L$, $P_t > 0$ and thereby obtaining

$$P'_{m,j}(L) = \frac{1}{M}. \tag{14}$$

Therefore, we continue with (13) and have

$$\begin{aligned}
(13) &= E[\sum_{j=1}^{M} \frac{1}{M}\hat{\tilde{\mu}}_{i,j}^m(h_{m,j}^L)|A_{\epsilon,\delta}] \\
&= \frac{1}{M}\sum_{j=1}^{M} E[\bar{\mu}_i^j(h_{m,j}^L)|A_{\epsilon,\delta}] \\
&= \frac{1}{M}\sum_{j=1}^{M} E[\frac{\sum_s r_i^j(s)}{n_{j,i}(h_{m,j}^L)}|A_{\epsilon,\delta}] \\
&= \frac{1}{M}\sum_{j=1}^{M} E[E[\frac{\sum_s r_i^j(s)}{n_{j,i}(h_{m,j}^L)}|\sigma(n_{j,i}(l))_{l\leq h_{m,j}^L}, A_{\epsilon,\delta}|A_{\epsilon,\delta}]
\end{aligned}$$

where the last equality uses the law of total expectation.

With the derivations, we further have

$$\begin{aligned}
(13) &= \frac{1}{M}\sum_{j=1}^{M} E[\frac{1}{n_{j,i}(h_{m,j}^L)} E[\sum_{s:n_{j,i}(s)-n_{j,i}(s-1)=1} r_i^j(s)|\sigma(n_{j,i}(l))_{l\leq h_{m,j}^L}, A_{\epsilon,\delta}|A_{\epsilon,\delta}] \\
&= \frac{1}{M}\sum_{j=1}^{M} E[\frac{1}{n_{j,i}(h_{m,j}^L)} \sum_{s:n_{j,i}(s)-n_{j,i}(s-1)=1} E[r_i^j(s)|\sigma(n_{j,i}(l))_{l\leq h_{m,j}^L}, A_{\epsilon,\delta}|A_{\epsilon,\delta}] && (15) \\
&= \frac{1}{M}\sum_{j=1}^{M} E[\frac{1}{n_{j,i}(h_{m,j}^L)} \sum_{s:n_{j,i}(s)-n_{j,i}(s-1)=1} \mu_i^j|A_{\epsilon,\delta}] && (16) \\
&= \frac{1}{M}\sum_{j=1}^{M} E[\mu_i^j|A_{\epsilon,\delta}] = \mu_i && (17)
\end{aligned}$$

where the second equality (15) uses the fact that $\{s : n_{j,i}(s) - n_{j,i}(s-1) = 1\}$ is contained in $\sigma(n_{j,i}(l))_{l\leq L_{m,j}}$ and the third equality (16) results from that $r_i^j(s)$ is independent of everything else given s and $E[r_i^j(s)] = \mu_i^j$.

The induction step follows a similar analysis as follows. Suppose that for any $s \leq t$ we have $E[\tilde{\mu}_i^m(s)|A_{\epsilon,\delta}] = \mu_i$.

For time step $t+1$, we first write it as

$$
\begin{aligned}
&E[\tilde{\mu}_i^m(t+1)|A_{\epsilon,\delta}] \\
&= E[\sum_{j=1}^{M} P_t'(m,j)\hat{\tilde{\mu}}_{i,j}^m(t_{m,j}) + d_{m,t}\sum_{j\in N_m(t)}\hat{\bar{\mu}}_{i,j}^m(t) + d_{m,t}\sum_{j\notin N_m(t)}\hat{\bar{\mu}}_{i,j}^m(t_{m,j})|A_{\epsilon,\delta}] \\
&= E[E[\sum_{j=1}^{M} P_t'(m,j)\tilde{\mu}_i^j(t_{m,j}) + d_{m,t}\sum_{j\in N_m(t)}\bar{\mu}_i^j(t) + d_{m,t}\sum_{j\notin N_m(t)}\bar{\mu}_i^j(t_{m,j})|\sigma(n_{j,i}(t))_{j,i,t}, A_{\epsilon,\delta}|A_{\epsilon,\delta}]
\end{aligned} \tag{18}
$$

where $P_t'(m,j)$ and d are constants since $P_t(m,j)>0$ for $t\geq L$ on event $A_{\epsilon,\delta}$ and the last equality is again by the law of total expectation.

This gives us that by the law of total expectation

$$
\begin{aligned}
(18) = E[&\sum_{j=1}^{M} P_t'(m,j)E[\tilde{\mu}_i^j(t_{m,j})|\sigma(n_{j,i}(t))_{j,i,t}, A_{\epsilon,\delta}+ \\
&d_{m,t}\sum_{j\in N_m(t)} E[\bar{\mu}_i^j(t)|\sigma(n_{j,i}(t))_{j,i,t}, A_{\epsilon,\delta}+ \\
&d_{m,t}\sum_{j\notin N_m(t)} E[\bar{\mu}_i^j(t_{m,j})|\sigma(n_{j,i}(t))_{j,i,t}, A_{\epsilon,\delta}|A_{\epsilon,\delta}] \\
= \sum_{j=1}^{M} & P'(m,j)E[E[\tilde{\mu}_i^j(t_{m,j})|\sigma(n_{j,i}(t))_{j,i,t}, A_{\epsilon,\delta}|A_{\epsilon,\delta}]+ \\
&E[d_{m,t}\sum_{j\in N_m(t)} E[\bar{\mu}_i^j(t)|\sigma(n_{j,i}(t))_{j,i,t}, A_{\epsilon,\delta}+ \\
&d_{m,t}\sum_{j\notin N_m(t)} E[\bar{\mu}_i^j(t_{m,j})|\sigma(n_{j,i}(t))_{j,i,t}, A_{\epsilon,\delta}|A_{\epsilon,\delta}] \\
= \sum_{j=1}^{M} & P'(m,j)E[\tilde{\mu}_i^j(t_{m,j})|A_{\epsilon,\delta}] + E[d_{m,t}\sum_{j} E[\bar{\mu}_i^j(t_{m,j})|\sigma(n_{j,i}(t))_{j,i,t}, A_{\epsilon,\delta}|A_{\epsilon,\delta}] \\
= \sum_{j=1}^{M} & P'(m,j)E[\tilde{\mu}_i^j(t_{m,j})|A_{\epsilon,\delta}]+ \\
&d_{m,t}\sum_{j} E[E[\frac{1}{n_{j,i}(t_{m,j})}\sum_{s:n_{j,i}(s)-n_{j,i}(s-1)=1} E[r_i^j(s)|\sigma(n_{j,i}(t))_{j,i,t}, A_{\epsilon,\delta}|A_{\epsilon,\delta}] \\
= \sum_{j=1}^{M} & P'(m,j)\mu_i + d_{m,t}M\mu_i = (\sum_{j=1}^{M} P'(m,j) + Md_{m,t})\mu_i = \mu_i
\end{aligned}
$$

where the first equality uses $P_t'(m,j)$ and d are constants on event $A_{\epsilon,\delta}$, the second equality is derived by re-organizing the terms, the third equality again uses the law of total expectation and integrates the second term by $t_{m,j}$, the fourth equality elaborate the second term and the equality in the last line follows from the induction and (15, 16 17).

This completes the induction step and thus shows the unbiasedness of the network-wide estimators conditional on event $A_{\epsilon,\delta}$.

□

Then we characterize the moment generating functions of the network-wide estimators and conclude that they have similar properties as their local rewards.

Proposition 8. *Assume the parameter δ satisfies that $0<\delta<c=f(\epsilon,M,T)$. In setting s_1,s_2,s_3 where rewards follow sub-gaussian distributions, for any m,i,λ and $t>L$ where L is the length of* *the burn-in period, the global estimator $\tilde{\mu}_i^m(t)$ is sub-Gaussian distributed. Moreover, the conditional moment generating function satisfies that with $P(A_{\epsilon,\delta}) = 1 - 7\epsilon$,*

$$E[\exp\{\lambda(\tilde{\mu}_i^m(t) - \mu_i)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})]$$
$$\leq \exp\{\frac{\lambda^2}{2}\frac{C\sigma^2}{\min_j n_{j,i}(t)}\}$$

where $\sigma^2 = \max_{j,i}(\tilde{\sigma}_i^j)^2$ and $C = \max\{\frac{4(M+2)(1-\frac{1-c_0}{2(M+2)})^2}{3M(1-c_0)}, (M+2)(1+4Md_{m,t}^2)\}$.

Proof. We prove the statement on the conditional moment generating functions by induction. Let us start with the basis step.

Note that the definition of $A_{\epsilon,\delta}$ and the choice of δ again guarantee that for $t \geq L$, $|P_t - cE| < \delta < c$ on event $A_{\epsilon,\delta}$. This implies that for any $t \geq L$, m and j, $P_t(m,j) > 0$, and if $t = L$

$$P_t'(m,j) = \frac{1}{M} \tag{19}$$

and if $t > L$

$$P_t'(m,j) = \frac{M-1}{M^2}. \tag{20}$$

Consider the time step $t \leq L+1$. The quantity satisfies that

$$\begin{aligned}
&E[\exp\{\lambda(\tilde{\mu}_i^m(t) - \mu_i)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})\\
&= E[\exp\{\lambda(\tilde{\mu}_i^m(L+1) - \mu_i)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})\\
&= E[\exp\{\lambda(\sum_{j=1}^M P'_{m,j}(L)\hat{\bar{\mu}}_{i,j}^m(h_{m,j}^L) - \mu_i)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})]\\
&= E[\exp\{\lambda(\sum_{j=1}^M \frac{1}{M}\hat{\bar{\mu}}_{i,j}^m(h_{m,j}^L) - \mu_i)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})]\\
&= E[\exp\{\lambda\sum_{j=1}^M \frac{1}{M}(\hat{\bar{\mu}}_{i,j}^m(h_{m,j}^L) - \mu_i^j)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})]\\
&\leq \Pi_{j=1}^M(E[(\exp\{(\lambda\frac{1}{M}(\bar{\mu}_i^j(h_{m,j}^L) - \mu_i^j)\}1_{A_{\epsilon,\delta}})^M|\sigma(\{n_{m,i}(t)\}_{t,i,m}))])^{\frac{1}{M}}
\end{aligned} \tag{21}$$

where the third equality holds by (19), the fourth equality uses the definition $\mu_i = \frac{1}{M}\sum_{i=1}^M \mu_i^j$, and the last inequality results from the generalized hoeffding inequality as in Lemma 3 and the fact that $\hat{\bar{\mu}}_{i,j}^m(h_{m,j}^L) = \bar{\mu}_i^j(h_{m,j}^L)$.

Note that for any client j, we have

$$\begin{aligned}
&E[(\exp\{(\lambda\frac{1}{M}(\bar{\mu}_i^j(h_{m,j}^L) - \mu_i^j)\}1_{A_{\epsilon,\delta}})^M|\sigma(\{n_{m,i}(t)\}_{t,i,m}))]\\
&= E[\exp\{(\lambda(\bar{\mu}_i^j(h_{m,j}^L) - \mu_i^j)\}1_{A_{\epsilon,\delta}})|\sigma(\{n_{m,i}(t)\}_{t,i,m})]\\
&= E[\exp\{(\lambda\frac{\sum_s(r_i^j(s) - \mu_i^j)}{n_{j,i}(h_{m,j}^L)}\}1_{A_{\epsilon,\delta}})|\sigma(\{n_{m,i}(t)\}_{t,i,m})]\\
&= E[\exp\{\sum_s(\lambda\frac{(r_i^j(s) - \mu_i^j)}{n_{j,i}(h_{m,j}^L)}\}1_{A_{\epsilon,\delta}})|\sigma(\{n_{m,i}(t)\}_{t,i,m})].
\end{aligned} \tag{22}$$

It is worth noting that given s, $r_i^j(s)$ is independent of everything else, which gives us

$$
\begin{aligned}
(22) &= \Pi_s E[\exp\{\lambda \frac{(r_i^j(s)-\mu_i^j)}{n_{j,i}(h_{m,j}^L)}\} 1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&= \Pi_s E[\exp\{\lambda \frac{(r_i^j(s)-\mu_i^j)}{n_{j,i}(h_{m,j}^L)}\} | \sigma(\{n_{m,i}(t)\}_{t,i,m})] \cdot E[1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&= \Pi_s E_r[\exp\{\lambda \frac{(r_i^j(s)-\mu_i^j)}{n_{j,i}(h_{m,j}^L)}\}] \cdot E[1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&\leq \Pi_s \exp\{\frac{(\frac{\lambda}{n_{j,i}(h_{m,j}^L)})^2\sigma^2}{2}\} \cdot E[1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&\leq (\exp\{\frac{(\frac{\lambda}{n_{j,i}(h_{m,j}^L)})^2\sigma^2}{2}\})^{n_{j,i}(h_{m,j}^L)} \\
&= \exp\{\frac{\frac{\lambda^2}{n_{j,i}(h_{m,j}^L)}\sigma^2}{2}\} \\
&\leq \exp\{\frac{\lambda^2\sigma^2}{2\min_j n_{j,i}(h_{m,j}^L)}\}
\end{aligned} \tag{23}
$$

where the first inequality holds by the definition of sub-Gaussian random variables $r_i^j(s) - \mu_i^j$ with an mean value 0, the second inequality results from $1_{A_{\epsilon,\delta}} \leq 1$, and the last inequality uses $n_{j,i}(h_{m,j}^L) \geq \min_j n_{j,i}(h_{m,j}^L)$ for any j.

Therefore, we obtain that by plugging (23) into (21)

$$
\begin{aligned}
(21) &\leq \Pi_{j=1}^M (\exp\{\frac{\lambda^2\sigma^2}{2\min_j n_{j,i}(h_{m,j}^L)}\})^{\frac{1}{M}} \\
&= ((\exp\{\frac{\lambda^2\sigma^2}{2\min_j n_{j,i}(h_{m,j}^L)}\})^{\frac{1}{M}})^M \\
&= \exp\{\frac{\lambda^2\sigma^2}{2\min_j n_{j,i}(h_{m,j}^L)}\}
\end{aligned}
$$

which completes the basis step.

Now we proceed to the induction step. Suppose that for any $s < t+1$ where $t \geq L$, we have

$$
\begin{aligned}
&E[\exp\{\lambda(\tilde{\mu}_i^m(s) - \mu_i)\} 1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(s)\}_{s,i,m})] \\
&\leq \exp\{\frac{\lambda^2}{2} \frac{C\sigma^2}{\min_j n_{j,i}(s)}\}.
\end{aligned} \tag{24}
$$

The update rule of $\tilde{\mu}_i^m$ implies that

$$
\begin{aligned}
&E[\exp\{\lambda(\tilde{\mu}_i^m(t+1)-\mu_i)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(s)\}_{s,i,m})]\\
&= E[\exp\{\lambda(\sum_{j=1}^{M} P_t'(m,j)(\hat{\tilde{\mu}}_{i,j}^m(t_{m,j})-\mu_i) + d_{m,t}\sum_{j\in N_m(t)}(\hat{\bar{\mu}}_{i,j}^m(t)-\mu_i^j)\\
&\qquad + d_{m,t}\sum_{j\notin N_m(t)}(\hat{\bar{\mu}}_{i,j}^m(t_{m,j})-\mu_i^j))\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(s)\}_{s,i,m})]\\
&= E[\exp\{\lambda(\sum_{j=1}^{M} P_t'(m,j)(\tilde{\mu}_i^j(t_{m,j})-\mu_i) + d_{m,t}\sum_{j\in N_m(t)}(\bar{\mu}_i^j(t)-\mu_i^j)\\
&\qquad + d_{m,t}\sum_{j\notin N_m(t)}(\bar{\mu}_i^j(t_{m,j})-\mu_i^j))\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(s)\}_{s,i,m})]\\
&= E[\Pi_{j=1}^M \exp\{\lambda P_t'(m,j)(\tilde{\mu}_i^j(t_{m,j})-\mu_i)\}1_{A_{\epsilon,\delta}}\cdot\Pi_{j\in N_m(t)}\exp\{\lambda d_{m,t}(\bar{\mu}_i^j(t)-\mu_i^j)\}1_{A_{\epsilon,\delta}}\\
&\qquad \cdot\Pi_{j\notin N_m(t)}\exp\{\lambda d_{m,t}(\bar{\mu}_i^j(t_{m,j})-\mu_i^j)1_{A_{\epsilon,\delta}}\}|\sigma(\{n_{m,i}(s)\}_{s,i,m})]\\
&\le \Pi_{j=1}^M(E[(\exp\{\lambda P_t'(m,j)(\tilde{\mu}_i^j(t_{m,j})-\mu_i)\})^{M+2}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(s)\}_{s,i,m})])^{\frac{1}{M+2}}\cdot\\
&\qquad E[\Pi_{j\in N_m(t)}(\exp\{\lambda d_{m,t}(\bar{\mu}_i^j(t)-\mu_i^j)\})^{M+2}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(s)\}_{s,i,m})]^{\frac{1}{M+2}}\cdot\\
&\qquad E[\Pi_{j\notin N_m(t)}(\exp\{\lambda d_{m,t}(\bar{\mu}_i^j(t_{m,j})-\mu_i^j)\})^{M+2}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(s)\}_{s,i,m})]^{\frac{1}{M+2}}\\
&= \Pi_{j=1}^M(E[(\exp\{\lambda P_t'(m,j)(M+2)(\tilde{\mu}_i^j(t_{m,j})-\mu_i)\})1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(s)\}_{s,i,m})])^{\frac{1}{M+2}}\cdot\\
&\qquad E[\Pi_{j\in N_m(t)}(\exp\{\lambda d_{m,t}(M+2)(\bar{\mu}_i^j(t)-\mu_i^j)\})1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(s)\}_{s,i,m})]^{\frac{1}{M+2}}\cdot\\
&\qquad E[\Pi_{j\notin N_m(t)}(\exp\{\lambda d_{m,t}(M+2)(\bar{\mu}_i^j(t_{m,j})-\mu_i^j)\})1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(s)\}_{s,i,m})]^{\frac{1}{M+2}}\\
&\le \Pi_{j=1}^M(\exp\{\frac{\lambda^2(P_t'(m,j))^2(M+2)^2}{2}\frac{C\sigma^2}{\min_j n_{j,i}(t_{m,j})}\})^{\frac{1}{M+2}}\cdot\\
&\qquad \Pi_{j\in N_m(t)}\Pi_s(E_r[\exp\{\lambda d_{m,t}(M+2)\frac{(r_i^j(s)-\mu_i^j)}{n_{j,i}(t)}\}]\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})])^{\frac{1}{M+2}}\cdot\\
&\qquad \Pi_{j\notin N_m(t)}\Pi_s(E_r[\exp\{\lambda d_{m,t}(M+2)\frac{(r_i^j(s)-\mu_i^j)}{n_{j,i}(t_{m,j})}\}]\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})])^{\frac{1}{M+2}}
\end{aligned}
\tag{25}
$$

where the first inequality uses Lemma 3 and the second inequality applies (24) as the assumption for the induction step and holds by exchanging the expectations with the multiplication since again given s the reward $(r_i^j(s)-\mu_i^j)$ is independent of other random variables.

We continue bounding the last two terms by using the definition of sub-Gaussian random variables $(r_i^j(s)-\mu_i^j)$ and obtain

$$
\begin{aligned}
(25) &\le (\exp\{\frac{\lambda^2(P_t'(m,j))^2(M+2)^2}{2}\frac{C\sigma^2}{\min_j n_{j,i}(t_{m,j})}\})^{\frac{M}{M+2}}\cdot\\
&\qquad \Pi_{j\in N_m(t)}\Pi_s(\exp\frac{\lambda^2 d_{m,t}^2(M+2)^2\sigma^2}{2n_{j,i}^2(t)}\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})])^{\frac{1}{M+2}}\cdot\\
&\qquad \Pi_{j\notin N_m(t)}\Pi_s(\exp\frac{\lambda^2 d_{m,t}^2(M+2)^2\sigma^2}{2n_{j,i}^2(t_{m,j})}\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})])^{\frac{1}{M+2}}\\
&= (\exp\{\frac{\lambda^2(P_t'(m,j))^2(M+2)^2}{2}\frac{C\sigma^2}{\min_j n_{j,i}(t_{m,j})}\})^{\frac{M}{M+2}}\cdot\\
&\qquad \Pi_{j\in N_m(t)}\exp\{\frac{n_{j,i}(t)}{M+2}\frac{\lambda^2 d_{m,t}^2(M+2)^2\sigma^2}{2n_{j,i}^2(t)}\}\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})]\cdot\\
&\qquad \Pi_{j\notin N_m(t)}\exp\{\frac{n_{j,i}(t_{m,j})}{M+2}\frac{\lambda^2 d_{m,t}^2(M+2)^2\sigma^2}{2n_{j,i}^2(t_{m,j})}\}\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})]
\end{aligned}
$$

Building on that, we establish

$$
\begin{aligned}
(25) \leq & (\exp\{\frac{\lambda^2(P_t'(m,j))^2(M+2)^2}{2}\frac{C\sigma^2}{\min_j n_{j,i}(t_{m,j})}\})^{\frac{M}{M+2}} \cdot \\
& (\exp\{\frac{\lambda^2 d_{m,t}^2(M+2)\sigma^2}{2\min_j n_{j,i}(t)}\})^{|N_m(t)|} \cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \cdot \\
& (\exp\{\frac{\lambda^2 d_{m,t}^2(M+2)\sigma^2}{2\min_j n_{j,i}(t_{m,j})}\})^{|M-N_m(t)|} \cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
= & E[(\exp\{\frac{\lambda^2(P_t'(m,j))^2 M(M+2)}{2}\frac{C\sigma^2}{\min_j n_{j,i}(t_{m,j})}\}) \cdot (\exp\{\frac{\lambda^2 d_{m,t}^2(M+2)|N_m(t)|}{2\min_j n_{j,i}(t)}\}) \\
& \cdot (\exp\{\frac{\lambda^2 d_{m,t}^2(M+2)\sigma^2|M-N_m(t)|}{2\min_j n_{j,i}(t_{m,j})}\})1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
\leq & E[(\exp\{\frac{\lambda^2(P_t'(m,j))^2 M(M+2)}{2(1-c_0)}\frac{C\sigma^2}{\min_j n_{j,i}(t+1)}\}) \cdot (\exp\{\frac{\lambda^2 d_{m,t}^2(M+2)|N_m(t)|\sigma^2}{2\frac{L/K}{L/K+1}\min_j n_{j,i}(t+1)}\}) \\
& \cdot (\exp\{\frac{\lambda^2 d_{m,t}^2(M+2)|M-N_m(t)|\sigma^2}{2(1-c_0)\min_j n_{j,i}(t+1)}\})1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})]
\end{aligned}
$$

where the first inequality uses the fact that for any j, $n_{j,i}(t) \geq \min_j n_{j,i}(t)$ and $n_{j,i}(t_{m,j}) \geq \min_j n_{j,i}(t_{m,j})$. For the second inequality, the first term is a result of $\frac{\min_j n_{j,i}(t)}{\min_j n_{j,i}(t+1)} \geq \frac{\min_j n_{j,i}(t)}{\min_j n_{j,i}(t)+1} \geq \frac{L/K}{L/K+1}$ since $n_{j,i}(t) > n_{j,i}(L) = L/K$ and the ratio is monotone increasing in n, and the second term is bounded based on the following derivations

$$
\begin{aligned}
\min_j n_{j,i}(t_{m,j}) &\geq \min_j n_{j,i}(t+1-t_0) \\
&\geq \min_j n_{j,i}(t+1) - t_0 \\
&\geq \min_j n_{j,i}(t+1) - c_0 \min_j n_{j,i}(t+1) \\
&= (1-c_0)\min_j n_{j,i}(t+1)
\end{aligned}
$$

where the last inequality holds by applying Proposition 6 that holds on event $A_{\epsilon,\delta}$.

Therefore, we can rewrite the above expression as

$$
\begin{aligned}
(25) = & E[(\exp\{\frac{\lambda^2\sigma^2}{2\min_j n_{j,i}(t+1)} \cdot (\frac{C(P_t'(m,j))^2 M(M+2)}{2(1-c_0)} + \\
& \frac{d_{m,t}^2(M+2)|N_m(t)|}{\frac{L/K}{L/K+1}} + \frac{d_{m,t}^2(M+2)|M-N_m(t)|}{(1-c_0)})\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
\leq & E[\exp\{\frac{C\lambda^2\sigma^2}{2\min_j n_{j,i}(t+1)}\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
\leq & \exp\{\frac{C\lambda^2\sigma^2}{2\min_j n_{j,i}(t+1)}\}
\end{aligned}
$$

where the first inequality holds by the choice of $P_t'(m,j), d_{m,t}, L, c_0$ and C and the second inequality uses the fact that $1_{A_{\epsilon,\delta}} \leq 1$ and $\min_j n_{j,i}(t+1) \in \sigma(\{n_{m,i}(t)\}_{t,i,m})$.

This completes the induction step and subsequently concludes the proof.

□

Proposition 9. *Assume the parameter δ satisfies that $0 < \delta < c = f(\epsilon, M, T)$. In setting s_1, s_2, and s_3, for any m, i and $t > L$ where L is the length of the burn-in period, $\tilde{\mu}_{m,i}(t)$ satisfies that if if*

$n_{m,i}(t) \geq 2(K^2 + KM + M)$, *then with* $P(A_{\epsilon,\delta}) = 1 - 7\epsilon$,

$$P(\tilde{\mu}_{m,i}(t) - \mu_i \geq \sqrt{\frac{C_1 \log t}{n_{m,i}(t)}} | A_{\epsilon,\delta}) \leq \frac{1}{P(A_{\epsilon,\delta})} \frac{1}{t^2},$$
$$P(\mu_i - \tilde{\mu}_{m,i}(t) \geq \sqrt{\frac{C_1 \log t}{n_{m,i}(t)}} | A_{\epsilon,\delta}) \leq \frac{1}{P(A_{\epsilon,\delta}) t^2}.$$

Proof. By Proposition 7, we have $E[\tilde{\mu}_{m,i}(t) - \mu_i | A_{\epsilon,\delta}] = 0$, which allows us to consider the tail bound of the global estimator $\tilde{\mu}_i^m(t)$ conditional on event $A_{\epsilon,\delta}$ as follows.

Note that

$$\begin{aligned}
&P(\tilde{\mu}_{m,i}(t) - \mu_i \geq \sqrt{\frac{C_1 \log t}{n_{m,i}(t)}} | A_{\epsilon,\delta}) \\
&= E[1_{\tilde{\mu}_{m,i}(t) - \mu_i \geq \sqrt{\frac{C_1 \log t}{n_{m,i}(t)}}} | A_{\epsilon,\delta}] \\
&= \frac{1}{P(A_{\epsilon,\delta})} E[1_{\tilde{\mu}_{m,i}(t) - \mu_i \geq \sqrt{\frac{C_1 \log t}{n_{m,i}(t)}}} 1_{A_{\epsilon,\delta}}] \\
&= \frac{1}{P(A_{\epsilon,\delta})} E[1_{\exp\{\lambda(n)(\tilde{\mu}_{m,i}(t) - \mu_i)\} \geq \exp\{\lambda(n)\sqrt{\frac{C_1 \log t}{n_{m,i}(t)}}\}} 1_{A_{\epsilon,\delta}}] \\
&\leq \frac{1}{P(A_{\epsilon,\delta})} E[\frac{\exp\{\lambda(n)(\tilde{\mu}_{m,i}(t) - \mu_i)\}}{\exp\{\lambda(n)\sqrt{\frac{C_1 \log t}{n_{m,i}(t)}}\}} 1_{A_{\epsilon,\delta}}]
\end{aligned} \tag{26}$$

where the last inequality is by the fact that $1_{\exp\{\lambda(n)(\tilde{\mu}_{m,i}(t) - \mu_i)\} \geq \exp\{\lambda(n)\sqrt{\frac{C_1 \log t}{n_{m,i}(t)}}\}} \leq \frac{\exp\{\lambda(n)(\tilde{\mu}_{m,i}(t) - \mu_i)\}}{\exp\{\lambda(n)\sqrt{\frac{C_1 \log t}{n_{m,i}(t)}}\}}$.

By the assumption that $\delta < c$, we have Proposition 8 holds. Subsequently, by Proposition 8 and Lemma 2 which holds since $n_{m,i}(t) \geq 2(K^2 + KM + M)$, we have for any λ

$$\begin{aligned}
E[\exp\{\lambda(\tilde{\mu}_i^m(t) - \mu_i)\} 1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(t)\}_{t,i,m})] &\leq \exp\{\frac{\lambda^2}{2} \frac{C\sigma^2}{\min_j n_{j,i}(t)}\} \\
&\leq \exp\{\frac{\lambda^2}{1} \frac{C\sigma^2}{n_{m,i}(t)}\}.
\end{aligned} \tag{27}$$

Again, we utilize the law of total expectation and further obtain

$$\begin{aligned}
(26) &= \frac{1}{P(A_{\epsilon,\delta})} E[E[\frac{\exp\{\lambda(n)(\tilde{\mu}_{m,i}(t) - \mu_i)\}}{\exp\{\lambda(n)\sqrt{\frac{C_1 \log t}{n_{m,i}(t)}}\}} 1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(t)\}_{m,i,t})]] \\
&= \frac{1}{P(A_{\epsilon,\delta})} E[E[\frac{\exp\{\lambda(n)(\tilde{\mu}_{m,i}(t) - \mu_i)\}}{\exp\{\lambda(n)\sqrt{\frac{C_1 \log t}{n_{m,i}(t)}}\}} 1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(t)\}_{m,i,t})]] \\
&= \frac{1}{P(A_{\epsilon,\delta})} E[\frac{1}{\exp\{\lambda(n)\sqrt{\frac{C_1 \log t}{n_{m,i}(t)}}\}} E[\exp\{\lambda(n)(\tilde{\mu}_{m,i}(t) - \mu_i)\} 1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(t)\}_{m,i,t})]] \\
&\leq \frac{1}{P(A_{\epsilon,\delta})} E[\frac{1}{\exp\{\lambda(n)\sqrt{\frac{C_1 \log t}{n_{m,i}(t)}}\}} \cdot \exp\{\frac{\lambda^2(n)}{1} \frac{C\sigma^2}{n_{m,i}(t)}\}] \\
&\leq \frac{1}{P(A_{\epsilon,\delta})} \exp\{-2 \log t\} = \frac{1}{P(A_{\epsilon,\delta}) t^2}
\end{aligned} \tag{28}$$

where the first inequality holds by (44) and the second inequality holds by choosing $\lambda(n) = \frac{\sqrt{\frac{C_1 \log t}{n_{m,i}(t)}}}{2\frac{C\sigma^2}{n_{m,i}(t)}}$ and by the choice of parameter C_1 such that $\frac{C_1}{4C\sigma^2} \geq 2$ or equivalently $C_1 \geq 8C\sigma^2$.

In like manner, we obtain that by repeating the above steps with $\mu_i - \tilde{\mu}_{m,i}(t)$, we have

$$P(\mu_i - \tilde{\mu}_{m,i}(t) \geq \sqrt{\frac{C_1 \log t}{n_{m,i}(t)}} | A_{\epsilon,\delta}) \leq \frac{1}{P(A_{\epsilon,\delta})t^2} \tag{29}$$

which complete the proof.

□

Proposition 10. *Assume the parameter δ satisfies that $0 < \delta < c = f(\epsilon, M, T)$. An arm k is said to be sub-optimal if $k \neq i^*$ where i^* is the unique optimal arm in terms of the global reward, i.e. $i^* = \arg\max \frac{1}{M}\sum_{j=1}^{M} \mu_i^j$. Then in setting s_1, s_2 and s_3, when the game ends, for every client m, $0 < \epsilon < 1$ and $T > L$, the expected numbers of pulling sub-optimal arm k after the burn-in period satisfies with $P(A_{\epsilon,\delta}) = 1 - 7\epsilon$*

$$\begin{aligned}
&E[n_{m,k}(T)|A_{\epsilon,\delta}] \\
&\leq \max\{[\frac{4C_1 \log T}{\Delta_i^2}], 2(K^2 + MK + M)\} + \frac{2\pi^2}{3P(A_{\epsilon,\delta})} + K^2 + (2M-1)K \\
&\leq O(\log T).
\end{aligned}$$

Proof of Proposition 10. We claim that what lead to pulling an sub-optimal arm i are explicit by the decision rule of Algorithm 2, meaning that the result $a_t^m = i$ holds when any of the following conditions is met:

- Case 1: $n_{m,i}(t) \leq \mathcal{N}_{m,i}(t) - K$,
- Case 2: $\tilde{\mu}_{m,i} - \mu_i > \sqrt{\frac{C_1 \log t}{n_{m,i}(t-1)}}$,
- Case 3: $-\tilde{\mu}_{m,i^*} + \mu_{i^*} > \sqrt{\frac{C_1 \log t}{n_{m,i^*}(t-1)}}$,
- Case 4: $\mu_{i^*} - \mu_i < 2\sqrt{\frac{C_1 \log t}{n_{m,i}(t-1)}}$.

Then we formally consider the number of pulling arms $n_{m,i}(T)$ starting from $L+1$. For any $l > 1$, we have that based on the above listed conditions

$$\begin{aligned}
n_{m,i}(T) \leq l + &\sum_{t=L+1}^{T} 1_{\{a_t^m = i, n_{m,i}(t) > l\}} \\
\leq l + &\sum_{t=L+1}^{T} 1_{\{\tilde{\mu}_i^m - \sqrt{\frac{C_1 \log t}{n_{m,i}(t-1)}} > \mu_i, n_{m,i}(t-1) \geq l\}} \\
&+ \sum_{t=L+1}^{T} 1_{\{\tilde{\mu}_{i^*}^m + \sqrt{\frac{C_1 \log t}{n_{m,i^*}(t-1)}} < \mu_{i^*}, n_{m,i}(t-1) \geq l\}} \\
&+ \sum_{t=L+1}^{T} 1_{\{n_{m,i}(t) < \mathcal{N}_{m,i}(t) - K, a_t^m = i, n_{m,i}(t-1) \geq l\}} \\
&+ \sum_{t=L+1}^{T} 1_{\{\mu_i + 2\sqrt{\frac{C_1 \log t}{n_{m,i}(t-1)}} > \mu_{i^*}, n_{m,i}(t-1) \geq l\}}.
\end{aligned}$$

Consequently, the expected value of $n_{m,i}(t)$ conditional on $A_{\epsilon,\delta}$ reads as

$$
\begin{aligned}
&E[n_{m,i}(T)|A_{\epsilon,\delta}] \\
&= l + \sum_{t=L+1}^{T} P(\tilde{\mu}_i^m - \sqrt{\frac{C_1 \log t}{n_{m,i}(t-1)}} > \mu_i, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta}) \\
&\qquad + \sum_{t=L+1}^{T} P(\tilde{\mu}_{i^*}^m + \sqrt{\frac{C_1 \log t}{n_{m,i^*}(t-1)}} < \mu_{i^*}, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta}) \\
&\qquad + \sum_{t=L+1}^{T} P(n_{m,i}(t) < \mathcal{N}_{m,i}(t) - K, a_t^m = i, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta}) \\
&\qquad + \sum_{t=L+1}^{T} P(\mu_i + 2\sqrt{\frac{C_1 \log t}{n_{m,i}(t-1)}} > \mu_{i^*}, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta}) \\
&= l + \sum_{t=L+1}^{T} P(Case2, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta}) + \sum_{t=L+1}^{T} P(Case3, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta}) \\
&\quad + \sum_{t=L+1}^{T} P(Case1, a_t^m = i, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta}) + \sum_{t=L+1}^{T} P(Case4, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta})
\end{aligned}
\tag{30}
$$

where $l = \max\{[\frac{4C_1 \log T}{\Delta_i^2}], 2(K^2 + MK + M)\}$.

For the last term in (30), we have

$$
\sum_{t=L+1}^{T} P(Case4 : \mu_i + 2\sqrt{\frac{C_1 \log t}{n_{m,i}(t-1)}} > \mu_{i^*}, n_{m,i}(t-1) \geq l) = 0 \tag{31}
$$

since the choice of l satisfies $l \geq [\frac{4C_1 \log T}{\Delta_i^2}]$ with $\Delta_i = \mu_{i^*} - \mu_i$.

For the first two terms, we have on event $A_{\epsilon,\delta}$

$$
\begin{aligned}
&\sum_{t=L+1}^{T} P(Case2, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta}) + \sum_{t=1}^{T} P(Case3, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta}) \\
&\leq \sum_{t=L+1}^{T} P(\tilde{\mu}_{m,i} - \mu_i > \sqrt{\frac{C_1 \log t}{n_{m,i}(t-1)}} | A_{\epsilon,\delta}) + \sum_{t=1}^{T} P(-\tilde{\mu}_{m,i^*} + \mu_{i^*} > \sqrt{\frac{C_1 \log t}{n_{m,i^*}(t-1)}} | A_{\epsilon,\delta}) \\
&\leq \sum_{t=1}^{T} (\frac{1}{t^2}) + \sum_{t=1}^{T} (\frac{1}{t^2}) \leq \frac{\pi^2}{3}
\end{aligned}
\tag{32}
$$

where the first inequality holds by the property of the probability measure when removing the event $n_{m,i}(t-1) \geq l$ and the second inequality holds by (47) and (29) as stated in Proposition 9, which holds by the assumption that $\delta < c$.

For Case 1, we note that Lemma 1 implies that

$$
n_{m,i}(t) > \mathcal{N}_{m,i}(t) - K(K + 2M)
$$

with the definition of $N_{m,i}(t+1) = \max\{n_{m,i}(t+1), N_{j,i}(t), j \in \mathcal{N}_m(t)\}$.

Departing from the result that the difference between $N_{m,i}(t)$ and $n_{m,i}(t)$ is at most $K(K+2M)$, we then present the following analysis on how long it takes for the value $-n_{m,i}(t) + N_{m,i}(t)$ to be smaller than K.

At time step t, if Case 1 holds for client m, then $n_{m,i}(t+1)$ is increasing by 1 on the basis of $n_{m,i}(t)$. What follows characterizes the change of $N_{m,i}(t+1)$. Client m satisfying $n_{m,i}(t) \leq \mathcal{N}_{m,i}(t) - K$ will not change the value of $N_{m,i}(t+1)$ by the definition $N_{m,i}(t+1) = \max\{n_{m,i}(t+1), N_{j,i}(t), j \in$

$\mathcal{N}_m(t)\}$. Moreover, for client $j \in \mathcal{N}_m(t)$ with $n_{j,i}(t) < \mathcal{N}_{j,i}(t) - K$, i.e. $\mathcal{N}_{j,i}(t+1)$ will not be affected by $n_{j,i}(t+1) \leq n_{j,i}(t) + 1$. Thus, the value of $N_{m,i}(t+1) = \max\{n_{m,i}(t+1), N_{j,i}(t), j \in \mathcal{N}_m(t)\}$ is independent of such clients. We observe that for client $j \in \mathcal{N}_m(t)$ with $n_{j,i}(t) > \mathcal{N}_{j,i}(t) - K$, the value $N_{j,i}(t)$ will be the same if the client does not sample arm i, which leads to a decrease of 1 in the difference $-n_{m,i}(t) + N_{m,i}(t)$. Otherwise, if such a client samples arm i which brings an increment of 1 to $N_{m,i}(t)$, the difference between $n_{m,i}(t)$ and $N_{m,i}(t)$ will remain the same. However, the latter has just been discussed and must be the cases as in Case 2 and Case 3, the total length of which has already been upper bounded by $\frac{\pi^2}{3}$ as shown in (32).

Therefore, the gap is at most $K(K+2M) - K + \frac{\pi^2}{3}$, i.e.

$$\sum_{t=1}^{T} P(Case1, a_t^m = i, n_{m,i}(t-1) \geq l | A) \leq K(K+2M) - K + \frac{\pi^2}{3}. \tag{33}$$

Subsequently, we derive that

$$\begin{aligned}
E[n_{m,i}(T)|A_{\epsilon,\delta}] &\leq l + \frac{\pi^2}{3} + K(K+2M) - K + \frac{\pi^2}{3} + 0 \\
&= l + \frac{2\pi^2}{3} + K^2 + (2M-1)K \\
&= \max\{[\frac{4C_1 \log T}{\Delta_i^2}], 2(K^2 + MK + M)\} + \frac{2\pi^2}{3} + K^2 + (2M-1)K
\end{aligned}$$

where the inequality results from (30), (31), (32), and (33).

This completes the proof steps.

□

Next, we establish the concentration inequalities of the network-wide estimators when the rewards follow sub-exponential distributions, i.e. in setting S_1, S_2, and S_3.

Proposition 11. *Assume the parameter δ satisfies that $0 < \delta < c = f(\epsilon, M, T)$. In setting S_1, S_2, and S_3, for any m, i, λ and $t > L$ where L is the length of the burn-in period, the global estimator $\tilde{\mu}_i^m(t)$ is sub-exponentially distributed. Moreover, the conditional moment generating function satisfies that with $P(A_{\epsilon,\delta}) = 1 - 7\epsilon$, for $|\lambda| < \frac{1}{\alpha}$*

$$\begin{aligned}
&E[\exp\{\lambda(\tilde{\mu}_i^m(t) - \mu_i)\} 1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&\leq \exp\{\frac{\lambda^2}{2} \frac{C\sigma^2}{\min_j n_{j,i}(t)}\}
\end{aligned}$$

where $\sigma^2 = \max_{j,i}(\tilde{\sigma}_i^j)^2$ and $C = \max\{\frac{4(M+2)(1-\frac{1-c_0}{2(M+2)})^2}{3M(1-c_0)}, (M+2)(1+4Md_{m,t}^2)\}$.

Proof. Assume that parameter $|\lambda| < \frac{1}{\alpha}$. We prove the statement on the conditional moment generating function by induction. Let us start with the basis step.

Note that the definition of A and the choice of δ again guarantee that for $t \geq L$, $|P_t - cE| < \delta < c$ on event A. This implies that for any $t \geq L$, $P_t > 0$ and thereby obtaining that if $t = L$

$$P'_{m,j}(t) = \frac{1}{M} \tag{34}$$

and if $t > L$

$$P'_{m,j}(t) = \frac{M-1}{M^2}. \tag{35}$$

Consider the time step $t \le L+1$. The quantity

$$
\begin{aligned}
&E[\exp\{\lambda(\tilde{\mu}_i^m(t)-\mu_i)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&= E[\exp\{\lambda(\tilde{\mu}_i^m(L+1)-\mu_i)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&= E[\exp\{\lambda(\sum_{j=1}^{M} P'_{m,j}(L)\hat{\tilde{\mu}}_{i,j}^m(h_{m,j}^L)-\mu_i)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&= E[\exp\{\lambda(\sum_{j=1}^{M}\frac{1}{M}\bar{\mu}_i^j(h_{m,j}^L)-\mu_i)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&= E[\exp\{\lambda\sum_{j=1}^{M}\frac{1}{M}(\bar{\mu}_i^j(h_{m,j}^L)-\mu_i^j)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&\le \Pi_{j=1}^M(E[(\exp\{(\lambda\frac{1}{M}(\bar{\mu}_i^j(h_{m,j}^L)-\mu_i^j)\}1_{A_{\epsilon,\delta}})^M|\sigma(\{n_{m,i}(t)\}_{t,i,m}))])^{\frac{1}{M}} \qquad (36)
\end{aligned}
$$

where the third equality holds by (34), the fourth equality uses the definition $\mu_i = \frac{1}{M}\sum_{i=1}^M \mu_i^j$ and the last inequality results from the generalized hoeffding inequality as in Lemma 3.

Note that for any client j, by the definition of $\bar{\mu}_i^j(h_{m,j}^L)$ we have

$$
\begin{aligned}
&E[(\exp\{(\lambda\frac{1}{M}(\bar{\mu}_i^j(h_{m,j}^L)-\mu_i^j)\}1_{A_{\epsilon,\delta}})^M|\sigma(\{n_{m,i}(t)\}_{t,i,m}))] \\
&= E[\exp\{(\lambda(\bar{\mu}_i^j(h_{m,j}^L)-\mu_i^j)\}1_{A_{\epsilon,\delta}})|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&= E[\exp\{(\lambda\frac{\sum_s(r_i^j(s)-\mu_i^j)}{n_{j,i}(h_{m,j}^L)}\}1_{A_{\epsilon,\delta}})|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&= E[\exp\{\sum_s(\lambda\frac{(r_i^j(s)-\mu_i^j)}{n_{j,i}(h_{m,j}^L)}\}1_{A_{\epsilon,\delta}})|\sigma(\{n_{m,i}(t)\}_{t,i,m})]. \qquad (37)
\end{aligned}
$$

It is worth noting that given s, $r_i^j(s)$ is independent of everything else, which gives us

$$
\begin{aligned}
(37) &= \Pi_s E[\exp\{\lambda\frac{(r_i^j(s)-\mu_i^j)}{n_{j,i}(h_{m,j}^L)}\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&= \Pi_s E[\exp\{\lambda\frac{(r_i^j(s)-\mu_i^j)}{n_{j,i}(h_{m,j}^L)}\}|\sigma(\{n_{m,i}(t)\}_{t,i,m})]\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&= \Pi_s E_r[\exp\{\lambda\frac{(r_i^j(s)-\mu_i^j)}{n_{j,i}(h_{m,j}^L)}\}]\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&\le \Pi_s \exp\{\frac{(\frac{\lambda}{n_{j,i}(h_{m,j}^L)})^2\sigma^2}{2}\}\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
&\le (\exp\{\frac{(\frac{\lambda}{n_{j,i}(h_{m,j}^L)})^2\sigma^2}{2}\})^{n_{j,i}(h_{m,j}^L)} \\
&= \exp\{\frac{\frac{\lambda^2}{n_{j,i}(h_{m,j}^L)}\sigma^2}{2}\} \le \exp\{\frac{\lambda^2\sigma^2}{2\min_j n_{j,i}(h_{m,j}^L)}\} \qquad (38)
\end{aligned}
$$

where the first inequality holds by the definition of sub-exponential random variables $r_i^j(s)-\mu_i^j$ with mean 0, the second inequality again uses $1_{A_{\epsilon,\delta}} \le 1$, and the last inequality is by the fact that $n_{j,i}(h_{m,j}^L) \ge \min_j n_{j,i}(h_{m,j}^L)$.

Therefore, we obtain that by plugging (38) into (36)

$$(36) \leq \Pi_{j=1}^{M}(\exp\{\frac{\lambda^2\sigma^2}{2\min_j n_{j,i}(h_{m,j}^L)}\})^{\frac{1}{M}}$$
$$= ((\exp\{\frac{\lambda^2\sigma^2}{2\min_j n_{j,i}(h_{m,j}^L)}\})^{\frac{1}{M}})^M = \exp\{\frac{\lambda^2\sigma^2}{2\min_j n_{j,i}(h_{m,j}^L)}\}$$

which completes the basis step.

Now we proceed to the induction step. Suppose that for any $s < t+1$ where $t+1 > L+1$, we have

$$E[\exp\{\lambda(\tilde{\mu}_i^m(s) - \mu_i)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(s)\}_{s,i,m})]$$
$$\leq \exp\{\frac{\lambda^2}{2}\frac{C\sigma^2}{\min_j n_{j,i}(s)}\} \tag{39}$$

The update rule of $\tilde{\mu}_i^m$ again and (25) implies that

$$E[\exp\{\lambda(\tilde{\mu}_i^m(t+1) - \mu_i)\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(s)\}_{s,i,m})]$$
$$\leq \Pi_{j=1}^{M}(\exp\{\frac{\lambda^2(P_t'(m,j))^2(M+2)^2}{2}\frac{C\sigma^2}{\min_j n_{j,i}(t_{m,j})}\})^{\frac{1}{M+2}}\cdot$$
$$\Pi_{j\in N_m(t)}\Pi_s(E_r[\exp\{\lambda d_{m,t}(M+2)\frac{(r_i^j(s)-\mu_i^j)}{n_{j,i}(t)}\}]\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})])^{\frac{1}{M+2}}\cdot$$
$$\Pi_{j\notin N_m(t)}\Pi_s(E_r[\exp\{\lambda d_{m,t}(M+2)\frac{(r_i^j(s)-\mu_i^j)}{n_{j,i}(t_{m,j})}\}]\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})])^{\frac{1}{M+2}}. \tag{40}$$

We continue bounding the last two terms by using the definition of sub-exponential random variables $(r_i^j(s) - \mu_i^j)$ and obtain

$$(40) \leq (\exp\{\frac{\lambda^2(P_t'(m,j))^2(M+2)^2}{2}\frac{C\sigma^2}{\min_j n_{j,i}(t_{m,j})}\})^{\frac{M}{M+2}}\cdot$$
$$\Pi_{j\in N_m(t)}\Pi_s(\exp\frac{\lambda^2 d_{m,t}^2(M+2)^2\sigma^2}{2n_{j,i}^2(t)}\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})])^{\frac{1}{M+2}}\cdot$$
$$\Pi_{j\notin N_m(t)}\Pi_s(\exp\frac{\lambda^2 d_{m,t}^2(M+2)^2\sigma^2}{2n_{j,i}^2(t_{m,j})}\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})])^{\frac{1}{M+2}}$$
$$= (\exp\{\frac{\lambda^2(P_t'(m,j))^2(M+2)^2}{2}\frac{C\sigma^2}{\min_j n_{j,i}(t_{m,j})}\})^{\frac{M}{M+2}}\cdot$$
$$\Pi_{j\in N_m(t)}\exp\{\frac{n_{j,i}(t)}{M+2}\frac{\lambda^2 d_{m,t}^2(M+2)^2\sigma^2}{2n_{j,i}^2(t)}\}\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})]\cdot$$
$$\Pi_{j\notin N_m(t)}\exp\{\frac{n_{j,i}(t_{m,j})}{M+2}\frac{\lambda^2 d_{m,t}^2(M+2)^2\sigma^2}{2n_{j,i}^2(t_{m,j})}\}\cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})].$$

Building on that, we establish

$$
\begin{aligned}
(40) \leq & (\exp\{\frac{\lambda^2(P_t'(m,j))^2(M+2)^2}{2}\frac{C\sigma^2}{\min_j n_{j,i}(t_{m,j})}\})^{\frac{M}{M+2}} \cdot \\
& (\exp\{\frac{\lambda^2 d_{m,t}^2(M+2)\sigma^2}{2\min_j n_{j,i}(t)}\})^{|N_m(t)|} \cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \cdot \\
& (\exp\{\frac{\lambda^2 d_{m,t}^2(M+2)\sigma^2}{2\min_j n_{j,i}(t_{m,j})}\})^{|M-N_m(t)|} \cdot E[1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
= & E[(\exp\{\frac{\lambda^2(P_t'(m,j))^2 M(M+2)}{2}\frac{C\sigma^2}{\min_j n_{j,i}(t_{m,j})}\}) \cdot (\exp\{\frac{\lambda^2 d_{m,t}^2(M+2)|N_m(t)|}{2\min_j n_{j,i}(t)}\}) \\
& \cdot (\exp\{\frac{\lambda^2 d_{m,t}^2(M+2)\sigma^2|M-N_m(t)|}{2\min_j n_{j,i}(t_{m,j})}\})1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
\leq & E[(\exp\{\frac{\lambda^2(P_t'(m,j))^2 M(M+2)}{2(1-c_0)}\frac{C\sigma^2}{\min_j n_{j,i}(t+1)}\}) \cdot (\exp\{\frac{\lambda^2 d_{m,t}^2(M+2)|N_m(t)|\sigma^2}{2\frac{L/K}{L/K+1}\min_j n_{j,i}(t+1)}\}) \\
& \cdot (\exp\{\frac{\lambda^2 d_{m,t}^2(M+2)|M-N_m(t)|\sigma^2}{2(1-c_0)\min_j n_{j,i}(t+1)}\})1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})]
\end{aligned}
$$

where the first inequality uses the fact that $n_{j,i}(t) \geq \min_j n_{j,i}(t)$ and $n_{j,i}(t_{m,j}) \geq \min_j n_{j,i}(t_{m,j})$. For the second inequality, the first term is a result of $\frac{\min_j n_{j,i}(t)}{\min_j n_{j,i}(t+1)} \geq \frac{\min_j n_{j,i}(t)}{\min_j n_{j,i}(t)+1} \geq \frac{L/K}{L/K+1}$ since $n_{j,i}(t) > n_{j,i}(L) = L/K$ and the ratio is monotone increasing in n, and the second term is bounded through applying Proposition 6, which holds on event $A_{\epsilon,\delta}$ and leads to

$$
\begin{aligned}
\min_j n_{j,i}(t_{m,j}) &\geq \min_j n_{j,i}(t+1-t_0) \\
&\geq \min_j n_{j,i}(t+1) - t_0 \\
&\geq \min_j n_{j,i}(t+1) - c_0 \min_j n_{j,i}(t+1) \\
&= (1-c_0)\min_j n_{j,i}(t+1).
\end{aligned}
$$

Therefore, we can rewrite the above expression as

$$
\begin{aligned}
(40) = & E[(\exp\{\frac{\lambda^2\sigma^2}{2\min_j n_{j,i}(t+1)} \cdot (\frac{C\lambda^2(P_t'(m,j))^2 M(M+2)}{2(1-c_0)} + \\
& \frac{d_{m,t}^2(M+2)|N_m(t)|}{\frac{L/K}{L/K+1}} + \frac{d_{m,t}^2(M+2)|M-N_m(t)|}{(1-c_0)})\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
\leq & E[\exp\{\frac{C\lambda^2\sigma^2}{2\min_j n_{j,i}(t+1)}\}1_{A_{\epsilon,\delta}}|\sigma(\{n_{m,i}(t)\}_{t,i,m})] \\
\leq & \exp\{\frac{C\lambda^2\sigma^2}{2\min_j n_{j,i}(t+1)}\}
\end{aligned}
$$

where the first inequality holds again by the choice of $P_t'(m,j), d_{m,t}, L$ and c_0 and the second inequality uses the fact that $1_{A_{\epsilon,\delta}} \leq 1$.

This completes the induction step and subsequently concludes the proof.

□

Proposition 12. *Assume the parameter δ satisfies that $0 < \delta < c = f(\epsilon, M, T)$. In setting S_1, S_2, and S_3, for any m, i and $t > L$ where L is the length of the burn-in period, the deviation of $\tilde{\mu}_{m,i}(t)$*

satisfies that if $n_{m,i}(t) \geq 2(K^2 + KM + M)$*, then with* $P(A_{\epsilon,\delta}) = 1 - 7\epsilon$*,*

$$P(\tilde{\mu}_{m,i}(t) - \mu_i \geq \sqrt{\frac{C_1 \log t}{n_{m,i}(t)}} + \frac{C_2 \log t}{n_{m,i}(t)} | A_{\epsilon,\delta}) \leq \frac{1}{P(A_{\epsilon,\delta})} \frac{1}{T^4},$$

$$P(\mu_i - \tilde{\mu}_{m,i}(t) \geq \sqrt{\frac{C_1 \log t}{n_{m,i}(t)}} + \frac{C_2 \log t}{n_{m,i}(t)} | A_{\epsilon,\delta}) \leq \frac{1}{P(A_{\epsilon,\delta})} \frac{1}{T^4}.$$

Proof. By Proposition 7, we have $E[\tilde{\mu}_{m,i}(t) - \mu_i | A_{\epsilon,\delta}] = 0$, which allows us to consider the tail bound of the global estimator $\tilde{\mu}_i^m(t)$ conditional on event $A_{\epsilon,\delta}$. It is worth mentioning that by the choice of C_1 and C_2, we have

$$C_1^2 \cdot \frac{\alpha^2}{\tilde{\sigma}^4} \leq C_2^2.$$

where $\tilde{\sigma}^2$ is $\frac{2C\sigma^2}{n_{m,i}(t)}$.

Note that since we set $Rad = \sqrt{\frac{C_1 \ln T}{n_{m,i}(t)}} + \frac{C_2 \ln T}{n_{m,i}(t)}$, we obtain

$$P(|\tilde{\mu}_i^m(t) - \mu_i| > Rad | A_{\epsilon,\delta}) < P(|\tilde{\mu}_i^m(t) - \mu_i| > \sqrt{\frac{C_1 \ln T}{n_{m,i}(t)}} | A_{\epsilon,\delta}), \tag{41}$$

$$P(|\tilde{\mu}_i^m(t) - \mu_i| > Rad | A_{\epsilon,\delta}) < P(|\tilde{\mu}_i^m(t) - \mu_i| > \frac{C_2 \ln T}{n_{m,i}(t)} | A_{\epsilon,\delta}) \tag{42}$$

On the one hand, if $\frac{\sqrt{\frac{C_1 \log T}{n_{m,i}(t)}}}{\tilde{\sigma}^2} > \frac{1}{\alpha}$, i.e. $n_{m,i}(t) \leq C_1 \log T \frac{\alpha^2}{(\tilde{\sigma})^4}$, we have

$$\begin{aligned}
&P(|\tilde{\mu}_i^m(t) - \mu_i| > \frac{C_2 \ln T}{n_{m,i}(t)} | A_{\epsilon,\delta}) \\
&= E[1_{|\tilde{\mu}_i^m(t) - \mu_i| > \frac{C_2 \ln T}{n_{m,i}(t)}} | A_{\epsilon,\delta}] \\
&= \frac{1}{P(A_{\epsilon,\delta})} E[1_{|\tilde{\mu}_i^m(t) - \mu_i| > \frac{C_2 \ln T}{n_{m,i}(t)}} 1_{A_{\epsilon,\delta}}] \\
&= \frac{1}{P(A_{\epsilon,\delta})} E[1_{\exp\{\lambda(n)(|\tilde{\mu}_{m,i}(t) - \mu_i|)\} \geq \exp\{\lambda(n) \frac{C_2 \ln T}{n_{m,i}(t)}\}} 1_{A_{\epsilon,\delta}}] \\
&\leq \frac{1}{P(A_{\epsilon,\delta})} E[\frac{\exp\{\lambda(n)(|\tilde{\mu}_{m,i}(t) - \mu_i|)\}}{\exp\{\lambda(n) \frac{C_2 \ln T}{n_{m,i}(t)}\}} 1_{A_{\epsilon,\delta}}]
\end{aligned} \tag{43}$$

where the last inequality is by the fact that $1_{\exp\{\lambda(n)(|\tilde{\mu}_{m,i}(t) - \mu_i|)\} \geq \exp\{\lambda(n) \frac{C_2 \ln T}{n_{m,i}(t)}\}} \leq \frac{\exp\{\lambda(n)(|\tilde{\mu}_{m,i}(t) - \mu_i|)\}}{\exp\{\lambda(n) \frac{C_2 \ln T}{n_{m,i}(t)}\}}$.

By the assumption that $\delta < c$, we have Proposition 11 holds. Subsequently, by Proposition 11 and Lemma 2 which holds since $n_{m,i}(t) \geq 2(K^2 + KM + M)$, we have for any $|\lambda| < \frac{1}{\alpha}$

$$E[\exp\{\lambda(\tilde{\mu}_i^m(t) - \mu_i)\} 1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(t)\}_{t,i,m})] \leq \exp\{\frac{\lambda^2}{2} \tilde{\sigma}^2\}. \tag{44}$$

Likewise, we obtain that by taking $\lambda = -\lambda$,

$$E[\exp\{\lambda(-\tilde{\mu}_i^m(t) + \mu_i)\} 1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(t)\}_{t,i,m})] \leq \exp\{\frac{\lambda^2}{2} \tilde{\sigma}^2\}. \tag{45}$$

With (44) and (45), we arrive at for any $|\lambda| < \frac{1}{\alpha}$ that

$$E[\exp\{\lambda(|\tilde{\mu}_i^m(t) - \mu_i)|\} 1_{A_{\epsilon,\delta}} | \sigma(\{n_{m,i}(t)\}_{t,i,m})] \leq 2 \exp\{\frac{\lambda^2}{2} \tilde{\sigma}^2\}. \tag{46}$$

Again, we utilize the law of total expectation and further obtain that $|\lambda(n)| < \frac{1}{\alpha}$

$$
\begin{aligned}
(43) &= \frac{1}{P(A_{\epsilon,\delta})} E[E[\frac{\exp\{\lambda(n)(|\tilde{\mu}_{m,i}(t) - \mu_i|)\}}{\exp\{\lambda(n)\frac{C_2 \ln T}{n_{m,i}(t)}\}} 1_{A_{\epsilon,\delta}} |\sigma(\{n_{m,i}(t)\}_{m,i,t})]] \\
&= \frac{1}{P(A_{\epsilon,\delta})} E[\frac{1}{\exp\{\lambda(n)\frac{C_2 \ln T}{n_{m,i}(t)}\}} E[\exp\{\lambda(n)(|\tilde{\mu}_{m,i}(t) - \mu_i|)\} 1_{A_{\epsilon,\delta}} |\sigma(\{n_{m,i}(t)\}_{m,i,t})]] \\
&\leq 2\frac{1}{P(A_{\epsilon,\delta})} E[\frac{1}{\exp\{\lambda(n)\frac{C_2 \ln T}{n_{m,i}(t)}\}} \cdot \exp\{\frac{\lambda^2(n)}{2}\tilde{\sigma}^2\}] \qquad (47)
\end{aligned}
$$

where the first inequality holds by (46).

Note that the condition $\frac{\sqrt{\frac{C_1 \log T}{n_{m,i}(t)}}}{\tilde{\sigma}^2} > \frac{1}{\alpha}$ implies that $n_{m,i}(t) < \frac{C_1 \ln T}{\frac{\tilde{\sigma}^2}{\alpha}}$ which is the global optima of the function in (47). This is true since $n_{m,i}(t) \leq C_1 \log T \frac{\alpha^2}{(\tilde{\sigma})^4} \leq \frac{(C_2 \log T)^2}{C_1}$. Henceforth, (47) is monotone decreasing in $\lambda(n) \in (0, \frac{1}{\alpha})$ and we obtain a minima when choosing $\lambda(n) = \frac{1}{\alpha}$ and using the continuity of (47).

Formally, it yields that

$$
\begin{aligned}
(47) &\leq 2\frac{1}{P(A_{\epsilon,\delta})} E[\frac{1}{\exp\{\frac{1}{\alpha}\frac{C_2 \ln T}{n_{m,i}(t)}\}} \cdot \exp\{\frac{\frac{1}{2\alpha^2}}{1}\tilde{\sigma}^2\}] \\
&= 2\frac{1}{P(A_{\epsilon,\delta})} E[\exp\{\frac{1}{2\alpha^2}\tilde{\sigma}^2 - \frac{1}{\alpha}\frac{C_2 \ln T}{n_{m,i}(t)}\}] \\
&\leq 2\frac{1}{P(A_{\epsilon,\delta})} \exp\{-4\log T\} = \frac{2}{P(A_{\epsilon,\delta})T^4} \qquad (48)
\end{aligned}
$$

where the last inequality uses the choice of C_2 and the condition that $\frac{1}{2\alpha^2}\tilde{\sigma}^2 - \frac{1}{\alpha}\frac{C_2 \ln T}{n_{m,i}(t)} \leq -4 \ln T$ which holds by the following derivation. Notably, we have

$$
\begin{aligned}
&\frac{1}{2\alpha^2}\tilde{\sigma}^2 - \frac{1}{\alpha}\frac{C_2 \ln T}{n_{m,i}(t)} \leq \frac{1}{2\alpha^2}\tilde{\sigma}^2 - \frac{C_2 \ln T}{\alpha}\frac{\frac{\tilde{\sigma}^2}{\alpha}}{C_1 \ln T} \\
&= \frac{1}{2\alpha^2}\tilde{\sigma}^2 - \frac{C_2}{C_1}\frac{\tilde{\sigma}^2}{\alpha^2} \\
&= (\frac{1}{2} - \frac{C_2}{C_1})(\tilde{\sigma}^2 \cdot \frac{\frac{C_1 \log T}{n_{m,i}(t)}}{\tilde{\sigma}^4}) = (\frac{1}{2} - \frac{C_2}{C_1})(\frac{\frac{C_1 \log T}{n_{m,i}(t)}}{\tilde{\sigma}^2}) \\
&= (\frac{1}{2} - \frac{C_2}{C_1}) \cdot \frac{C_1 \log T}{n_{m,i}(t)} \cdot \frac{1}{\frac{2C\sigma^2}{n_{m,i}(t)}} = (\frac{1}{2} - \frac{C_2}{C_1})\frac{C_1}{2C\sigma^2}\log T \leq -4\log T
\end{aligned}
$$

where the first inequality uses $n_{m,i}(t) < \frac{C_1 \ln T}{\frac{\tilde{\sigma}^2}{\alpha}}$ and the last inequality is by the choices of parameters $(\frac{1}{2} - \frac{C_2}{C_1})\frac{C_1}{2C\sigma^2} \leq -4$.

On the other hand, if $\frac{\sqrt{\frac{C_1 \log T}{n_{m,i}(t)}}}{\tilde{\sigma}^2} < \frac{1}{\alpha}$, i.e. $n_{m,i}(t) \geq C_1 \log T \frac{\alpha^2}{(\tilde{\sigma})^4}$, we observe for $|\lambda(n)| < \frac{1}{\alpha}$

$$
\begin{aligned}
&P(|\tilde{\mu}_i^m(t) - \mu_i| > \sqrt{\frac{C_1 \ln T}{n_{m,i}(t)}}|A_{\epsilon,\delta}) \\
&\leq 2\frac{1}{P(A_{\epsilon,\delta})} E[\frac{1}{\exp\{\lambda(n)\sqrt{\frac{C_1 \ln T}{n_{m,i}(t)}}\}} \cdot \exp\{\frac{\lambda^2(n)}{2}\tilde{\sigma}^2\}] \qquad (49)
\end{aligned}
$$

by a same argument from (43) to (47) replacing $\frac{C_2 \ln T}{n_{m,i}(t)}$ with $\sqrt{\frac{C_1 \ln T}{n_{m,i}(t)}}$. When choosing $\lambda(n) = \frac{\sqrt{\frac{C_1 \log T}{n_{m,i}(t)}}}{\tilde{\sigma}^2}$ that meets the condition $\lambda < \frac{1}{\alpha}$ under the assumption $\frac{\sqrt{\frac{C_1 \log T}{n_{m,i}(t)}}}{\tilde{\sigma}^2} < \frac{1}{\alpha}$ and noting that

$\frac{C_1}{2C\sigma^2} \geq 4$, we obtain

$$(49) \leq 2\frac{1}{P(A_{\epsilon,\delta})}E[\exp\{-\frac{C_1 \log T}{\tilde{\sigma}^2 n_{m,i}(t)}\}]$$
$$= 2\frac{1}{P(A_{\epsilon,\delta})}E[\exp\{-\frac{C_1 \log T}{n_{m,i}(t)}\frac{1}{\frac{2C\sigma^2}{n_{m,i}(t)}}\}]$$
$$\leq 2\frac{1}{P(A_{\epsilon,\delta})}\exp\{-4\log T\} = \frac{2}{P(A_{\epsilon,\delta})T^4}.$$

To conclude, by (41) and (42), we have

$$P(|\tilde{\mu}_i^m(t) - \mu_i| > Rad|A_{\epsilon,\delta}) \leq \frac{2}{P(A_{\epsilon,\delta})T^4}$$

which completes the proof. □

Proposition 13. *Assume the parameter δ satisfies that $0 < \delta < c = f(\epsilon, M, T)$. An arm k is said to be sub-optimal if $k \neq i^*$ where i^* is the unique optimal arm in terms of the global reward, i.e. $i^* = \arg\max \frac{1}{M}\sum_{j=1}^{M}\mu_i^j$. Then in setting S_1, S_2 and S_3, when the game ends, for every client m, $0 < \epsilon < 1$ and $T > L$, the expected numbers of pulling sub-optimal arm k after the burn-in period satisfies with $P(A_{\epsilon,\delta}) = 1 - 7\epsilon$*

$$E[n_{m,k}(T)|A_{\epsilon,\delta}]$$
$$\leq \max([\frac{16C_1 \log T}{\Delta_i^2}], [\frac{4C_2 \log T}{\Delta_i}], 2(K^2 + MK + M)) + \frac{4}{P(A_{\epsilon,\delta})T^3} + K^2 + (2M-1)K$$
$$\leq O(\log T).$$

Proof. Recall that $Rad = \sqrt{\frac{C_1 \ln T}{n_{m,i}(t)} + \frac{C_2 \ln T}{n_{m,i}(t)}}$. We again have $a_t^m = i$ holds when any of the following conditions is met: Case 1: $n_{m,i}(t) \leq \mathcal{N}_{m,i}(t) - K$, Case 2: $\tilde{\mu}_{m,i} - \mu_i > \sqrt{\frac{C_1 \ln T}{n_{m,i}(t)} + \frac{C_2 \ln T}{n_{m,i}(t)}}$, Case 3: $-\tilde{\mu}_{m,i^*} + \mu_{i^*} > \sqrt{\frac{C_1 \ln T}{n_{m,i^*}(t)} + \frac{C_2 \ln T}{n_{m,i^*}(t)}}$, and Case 4: $\mu_{i^*} - \mu_i < 2(\sqrt{\frac{C_1 \ln T}{n_{m,i}(t)} + \frac{C_2 \ln T}{n_{m,i}(t)}})$.

By (30), the expected value of $n_{m,i}(t)$ conditional on $A_{\epsilon,\delta}$ reads as

$$E[n_{m,i}(T)|A_{\epsilon,\delta}]$$
$$= l + \sum_{t=L+1}^{T} P(Case2, n_{m,i}(t-1) \geq l|A_{\epsilon,\delta}) + \sum_{t=L+1}^{T} P(Case3, n_{m,i}(t-1) \geq l|A_{\epsilon,\delta})$$
$$+ \sum_{t=L+1}^{T} P(Case1, a_t^m = i, n_{m,i}(t-1) \geq l|A_{\epsilon,\delta}) + \sum_{t=L+1}^{T} P(Case4, n_{m,i}(t-1) \geq l|A_{\epsilon,\delta}) \tag{50}$$

where l is specified as $l = \max\{[\frac{4C_1 \log T}{\Delta_i^2}], 2(K^2 + MK + M)\}$ with $\Delta_i = \mu_{i^*} - \mu_i$.

For the last term in the above upper bound, we have

$$\sum_{t=L+1}^{T} P(Case4 : \mu_i + 2(\sqrt{\frac{C_1 \ln T}{n_{m,i}(t)}} + \frac{C_2 \ln T}{n_{m,i}(t)}) > \mu_{i^*}, n_{m,i}(t-1) \geq l) = 0 \tag{51}$$

since the choice of l satisfies $l \geq \max([\frac{16C_1 \log T}{\Delta_i^2}], [\frac{4C_2 \log T}{\Delta_i}], 2(K^2 + MK))$.

For the first two terms, we have on event $A_{\epsilon,\delta}$

$$\begin{aligned}
&\sum_{t=L+1}^{T} P(Case2, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta}) + \sum_{t=1}^{T} P(Case3, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta}) \\
&\leq \sum_{t=L+1}^{T} P(\tilde{\mu}_{m,i} - \mu_i > \sqrt{\frac{C_1 \ln T}{n_{m,i}(t)}} + \frac{C_2 \ln T}{n_{m,i}(t)} | A_{\epsilon,\delta}) + \\
&\quad \sum_{t=1}^{T} P(-\tilde{\mu}_{m,i^*} + \mu_{i^*} > \sqrt{\frac{C_1 \ln T}{n_{m,i^*}(t)}} + \frac{C_2 \ln T}{n_{m,i^*}(t)} | A_{\epsilon,\delta}) \\
&\leq \sum_{t=1}^{T} (\frac{1}{P(A_{\epsilon,\delta})T^4}) + \sum_{t=1}^{T} (\frac{1}{P(A_{\epsilon,\delta})T^4}) \leq \frac{2}{P(A_{\epsilon,\delta})T^3}
\end{aligned} \tag{52}$$

where the first inequality holds by the property of the probability measure when removing the event $n_{m,i}(t-1) \geq l$ and the second inequality holds by Proposition 12, which holds by the assumption that $\delta < c$.

For Case 1, we note that Lemma 1 implies that

$$n_{m,i}(t) > N_{m,i}(t) - K(K+2M)$$

with the definition of $N_{m,i}(t+1) = \max\{n_{m,i}(t+1), N_{j,i}(t), j \in \mathcal{N}_m(t)\}$.

Departing from the result that the difference between $N_{m,i}(t)$ and $n_{m,i}(t)$ is at most $K(K+2M)$, we then present the following analysis on how long it takes for the value $-n_{m,i}(t) + N_{m,i}(t)$ to be smaller than K.

At time step t, if Case 1 holds for client m, then $n_{m,i}(t+1)$ is increasing by 1 on the basis of $n_{m,i}(t)$. What follows characterizes the change of $N_{m,i}(t+1)$. Client m satisfying $n_{m,i}(t) \leq N_{m,i}(t) - K$ will not change the value of $N_{m,i}(t+1)$ by the definition $N_{m,i}(t+1) = \max\{n_{m,i}(t+1), N_{j,i}(t), j \in \mathcal{N}_m(t)\}$. Moreover, for client $j \in \mathcal{N}_m(t)$ with $n_{j,i}(t) < N_{j,i}(t) - K$, i.e. $N_{j,i}(t+1)$ will not be affected by $n_{j,i}(t+1) \leq n_{j,i}(t) + 1$. Thus, the value of $N_{m,i}(t+1) = \max\{n_{m,i}(t+1), N_{j,i}(t), j \in \mathcal{N}_m(t)\}$ is independent of such clients. We observe that for client $j \in \mathcal{N}_m(t)$ with $n_{j,i}(t) > N_{j,i}(t) - K$, the value $N_{j,i}(t)$ will be the same if the client does not sample arm i, which leads to a decrease of 1 in the difference $-n_{m,i}(t) + N_{m,i}(t)$. Otherwise, if such a client samples arm i which brings an increment of 1 to $N_{m,i}(t)$, the difference between $n_{m,i}(t)$ and $N_{m,i}(t)$ will remain the same. However, the latter has just been discussed and must be the cases as in Case 2 and Case 3, the total length of which has already been upper bounded by $\frac{2}{P(A_{\epsilon,\delta})T^3}$ as shown in (52).

Therefore, the gap is at most $K(K+2M) - K + \frac{2}{P(A_{\epsilon,\delta})T^3}$, i.e.

$$\sum_{t=1}^{T} P(Case1, a_t^m = i, n_{m,i}(t-1) \geq l | A_{\epsilon,\delta}) \leq K(K+2M) - K + \frac{2}{P(A_{\epsilon,\delta})T^3}. \tag{53}$$

Subsequently, we derive that

$$\begin{aligned}
&E[n_{m,i}(T) | A_{\epsilon,\delta}] \\
&\leq l + \frac{2}{P(A_{\epsilon,\delta})T^3} + K(K+2M) - K + \frac{2}{P(A_{\epsilon,\delta})T^3} + 0 \\
&= l + \frac{2\pi^2}{3} + K^2 + (2M-1)K \\
&= \max([\frac{16C_1 \log T}{\Delta_i^2}], [\frac{4C_2 \log T}{\Delta_i}], 2(K^2 + MK + M)) + \frac{4}{P(A_{\epsilon,\delta})T^3} + K^2 + (2M-1)K
\end{aligned}$$

where the inequality results from (50), (51), (52) and (53).

□

F.2 Proof of Theorems

Theorem 1. *For event $A_{\epsilon,\delta}$ and any $1 > \epsilon, \delta > 0$, we have $P(A_{\epsilon,\delta}) \geq 1 - 7\epsilon$.*

Proof. Recall that we define events

$$\begin{aligned}
&A_1 = \{\forall t \geq L, |P_t - cE| \leq \delta\},\\
&A_2 = \{\forall t \geq L, \forall j, m, t + 1 - \min_j t_{m,j} \leq t_0 \leq c_0 \min_l n_{l,i}(t+1)\},\\
&A_3 = \{\forall t \geq L, G_t \text{ is connected}\}
\end{aligned}$$

where A_1, A_2, A_3 belong to the σ-algebra in the probability space since the time horizon is countable, i.e. for the probability space (Ω, Σ, P), $A_1, A_2, A_3 \in \Sigma$.

Meanwhile, we obtain

$$\begin{aligned}
P(A_1) &= P(\{\forall t \geq L, |P_t - cE| \leq \delta\})\\
&\geq P(\cap_i \{\forall t \geq L_{s_i}, |P_t - cE| \leq \delta\})\\
&\geq 1 - \sum_i (1 - P(\cap_i \{\forall t \geq L_{s_i}, |P_t - cE| \leq \delta\}))\\
&\geq 1 - \sum_i (1 - (1 - \epsilon))\\
&= 1 - 3\epsilon
\end{aligned} \tag{54}$$

where the first inequality includes all settings and $L \geq L_{s_i}$, the second inequality results from the Bonferroni's inequality and the third inequality holds by Proposition 1 and Proposition 3.

At the same time, note that

$$\begin{aligned}
P(A_2) &= P(\{\forall t \geq L, \forall j, m, t + 1 - \min_j t_{m,j} \leq t_0 \leq c_0 \min_l n_{l,i}(t+1)\})\\
&\geq P(\cap_i \{\forall t \geq L_{s_i}, \forall j, m, t + 1 - \min_j t_{m,j} \leq t_0 \leq c_0 \min_l n_{l,i}(t+1)\})\\
&\geq 1 - \sum_i (1 - P(\{\forall t \geq L_{s_i}, \forall j, m, t + 1 - \min_j t_{m,j} \leq t_0 \leq c_0 \min_l n_{l,i}(t+1)\}))\\
&\geq 1 - \sum_i (1 - (1 - \epsilon)) = 1 - 3\epsilon
\end{aligned} \tag{55}$$

where the first inequality is by the definition of L, the second inequality again uses the Bonferroni's inequality, and the third inequality results from Proposition 6.

Moreover, we observer that

$$\begin{aligned}
P(A_3) &= P(\{\forall t \geq L, G_t \text{ is connected}\})\\
&\geq P(\cap_i \{\forall t \geq L_{s_i}, G_t \text{ is connected}\})\\
&\geq 1 - \sum_i (1 - P(\{\forall t \geq L_{s_i}, G_t \text{ is connected}\}))\\
&\geq 1 - (1 - (1 - \epsilon)) - 0 = 1 - \epsilon
\end{aligned} \tag{56}$$

where the first inequality uses the definition of L, the second inequality is by the Bonferroni's inequality and the third inequality holds by Proposition 5 and the definition of s_2, s_3 where all graphs are guaranteed to be connected.

Consequently, we arrive at

$$\begin{aligned}
P(A_{\epsilon,\delta}) &= P(A_1 \cap A_2 \cap A_3)\\
&= 1 - P(A_1^c \cup A_2^c \cup A_3^c)\\
&\geq 1 - (P(A_1^c) + P(A_2^c) + P(A_3^c))\\
&\geq 1 - (3\epsilon + 3\epsilon + \epsilon) = 1 - 7\epsilon
\end{aligned}$$

where the first inequality utilizes the Bonferroni's inequality, the second inequality results from (54), (55), and (56).

This concludes the proof and shows the validness of the statement.

□

Theorem 2. *Let f be a function specific to a setting and detailed later. For every $0 < \epsilon < 1$ and $0 < \delta < f(\epsilon, M, T)$, in setting s_1 with $c \geq \frac{1}{2} + \frac{1}{2}\sqrt{1 - (\frac{\epsilon}{MT})^{\frac{2}{M-1}}}$, s_2 and s_3, with the time horizon T satisfying $T \geq L$, the regret of Algorithm 2 with $F(m, i, t) = \sqrt{\frac{C_1 \ln t}{n_{m,i}(t)}}$ satisfies that*

$$E[R_T|A_{\epsilon,\delta}] \leq L + \sum_{i \neq i^*}(\max\{[\frac{4C_1 \log T}{\Delta_i^2}], 2(K^2 + MK + M)\} + \frac{2\pi^2}{3P(A_{\epsilon,\delta})} + K^2 + (2M-1)K)$$

where the length of the burn-in period is explicitly

$$L = \max\Bigg\{ \underbrace{\frac{\ln \frac{T}{2\epsilon}}{2\delta^2}, \frac{4K \log_2 T}{c_0}}_{L_{s_1}}, \underbrace{\frac{\ln \frac{\delta}{10}}{\ln p^*} + 25\frac{1+\lambda}{1-\lambda}\frac{\ln \frac{T}{2\epsilon}}{2\delta^2}, \frac{4K \log_2 T}{c_0}}_{L_{s_2}}, \underbrace{\frac{\ln \frac{\delta}{10}}{\ln p^*} + 25\frac{1+\lambda}{1-\lambda}\frac{\ln \frac{T}{2\epsilon}}{2\delta^2}, \frac{\frac{K \ln(\frac{MT}{\epsilon})}{\ln(\frac{1}{1-\frac{2\log M}{M-1}})}}{c_0}}_{L_{s_3}} \Bigg\}$$

with λ being the spectral gap of the Markov chain in s_2, s_3 that satisfies $1 - \lambda \geq \frac{1}{2\frac{\ln 2}{\ln 2p^*}\ln 4 + 1}$, $p^* = p^*(M) < 1$ and $c_0 = c_0(K, \min_{i \neq i^*} \Delta_i, M, \epsilon, \delta)$, and the instance-dependent constant $C_1 = 8\sigma^2 \max\{12\frac{M(M+2)}{M^4}\}$.

Proof. The optimal arm is denoted as i^* satisfying

$$i^* = \arg\max_i \sum_{m=1}^{M} \mu_i^m.$$

For the proposed regret, we have that for any constant L,

$$\begin{aligned}
R_T &= \frac{1}{M}(\max_i \sum_{t=1}^{T}\sum_{m=1}^{M} \mu_i^m - \sum_{t=1}^{T}\sum_{m=1}^{M} \mu_{a_t^m}^m) \\
&= \sum_{t=1}^{T}\frac{1}{M}\sum_{m=1}^{M} \mu_{i^*}^m - \sum_{t=1}^{T}\frac{1}{M}\sum_{m=1}^{M} \mu_{a_t^m}^m \\
&\leq \sum_{t=1}^{L}|\frac{1}{M}\sum_{m=1}^{M} \mu_{i^*}^m - \frac{1}{M}\sum_{m=1}^{M} \mu_{a_t^m}^m| + \sum_{t=L+1}^{T}(\frac{1}{M}\sum_{m=1}^{M} \mu_{i^*}^m - \frac{1}{M}\sum_{m=1}^{M} \mu_{a_t^m}^m) \\
&\leq L + \sum_{t=L+1}^{T}(\frac{1}{M}\sum_{m=1}^{M} \mu_{i^*}^m - \frac{1}{M}\sum_{m=1}^{M} \mu_{a_t^m}^m) \\
&= L + \sum_{t=L+1}^{T}(\mu_{i^*} - \frac{1}{M}\sum_{m=1}^{M} \mu_{a_t^m}^m) \\
&= L + ((T-L)\cdot \mu_{i^*} - \frac{1}{M}\sum_{m=1}^{M}\sum_{i=1}^{K} n_{m,i}(T)\mu_i^m)
\end{aligned}$$

where the first inequality is by taking the absolute value and the second inequality results from the assumption that $0 < \mu_i^j < 1$ for any arm i and client j.

Note that $\sum_{i=1}^{K}\sum_{m=1}^{M} n_{m,i}(T) = M(T-L)$ where by definition $n_{m,i}(T)$ is the number of pulls of arm i at client m from time step $L+1$ to time step T, which yields that

$$\begin{aligned}
R_T &\leq L + \sum_{i=1}^{K}\frac{1}{M}\sum_{m=1}^{M} n_{m,i}(T)\mu_{i^*}^m - \sum_{i=1}^{K}\frac{1}{M}\sum_{m=1}^{M} n_{m,i}(T)\mu_i^m \\
&= L + \sum_{i=1}^{K}\frac{1}{M}\sum_{m=1}^{M} n_{m,i}(T)(\mu_{i^*}^m - \mu_i^m) \\
&\leq L + \frac{1}{M}\sum_{i=1}^{K}\sum_{m:\mu_{i^*}^m - \mu_i^m > 0} n_{m,i}(T)(\mu_{i^*}^m - \mu_i^m) \\
&= L + \frac{1}{M}\sum_{i \neq i*}\sum_{m:\mu_{i^*}^m - \mu_i^m > 0} n_{m,i}(T)(\mu_{i^*}^m - \mu_i^m).
\end{aligned}$$

where the second inequality uses the fact that $\sum_{m:\mu_{i^*}^m - \mu_i^m \leq 0} n_{m,i}(T)(\mu_{i^*}^m - \mu_i^m) \leq 0$ holds for any arm i and the last equality is true since $n_{m,i}(T)(\mu_{i^*}^m - \mu_i^m) = 0$ for $i = i^*$ and any m.

Meanwhile, by the choices of δ such that $\delta < c = f(\epsilon, M, T)$, we apply Proposition 10 which leads to for any client m and arm $i \neq i^*$,

$$E[n_{m,i}(T)|A_{\epsilon,\delta}] \leq \max\{[\frac{4C_1\log T}{\Delta_i^2}], 2(K^2+MK+M)\} + \frac{2\pi^2}{3} + K^2 + (2M-1)K. \quad (57)$$

As a result, the upper bound on R_T can be derived as by taking the conditional expectation over R_T on $A_{\epsilon,\delta}$

$$\begin{aligned}
&E[R_T|A_{\epsilon,\delta}] \\
&\leq L + \frac{1}{M}\sum_{i\neq i^*}\sum_{m:\mu_{i^*}^m - \mu_i^m > 0} E[n_{m,i}(T)|A_{\epsilon,\delta}](\mu_{i^*}^m - \mu_i^m) \quad (58)\\
&\leq L + \\
&\quad \frac{1}{M}\sum_{i\neq i^*}\sum_{m:\mu_{i^*}^m - \mu_i^m > 0}(\max\{[\frac{4C_1\log T}{\Delta_i^2}], 2(K^2+MK)\} + \frac{2\pi^2}{3} + K^2 + (2M-1)K)(\mu_{i^*}^m - \mu_i^m) \\
&= L + \\
&\quad \frac{1}{M}\sum_{i\neq i^*}(\max\{[\frac{4C_1\log T}{\Delta_i^2}], 2(K^2+MK)\} + \frac{2\pi^2}{3} + K^2 + (2M-1)K)\sum_{m:\mu_{i^*}^m - \mu_i^m > 0}(\mu_{i^*}^m - \mu_i^m) \\
&\quad (59)
\end{aligned}$$

where the second inequality holds by plugging in (57).

Meanwhile, we note that for any $i \neq i^*$,

$$\begin{aligned}
&\sum_{m:\mu_{i^*}^m - \mu_i^m > 0}(\mu_{i^*}^m - \mu_i^m) + \sum_{m:\mu_{i^*}^m - \mu_i^m \leq 0}(\mu_{i^*}^m - \mu_i^m) \\
&= \sum_{m=1}^{M}(\mu_{i^*}^m - \mu_i^m) \\
&= M\Delta_i > 0
\end{aligned}$$

and

$$|\sum_{m:\mu_{i^*}^m - \mu_i^m \leq 0}(\mu_{i^*}^m - \mu_i^m)| \leq M$$

which gives us that

$$
\begin{aligned}
&\sum_{m:\mu_{i^*}^m-\mu_i^m>0}(\mu_{i^*}^m-\mu_i^m) \\
&= M\Delta_i - \sum_{m:\mu_{i^*}^m-\mu_i^m\le 0}(\mu_{i^*}^m-\mu_i^m) \\
&= M\Delta_i + |\sum_{m:\mu_{i^*}^m-\mu_i^m\le 0}(\mu_{i^*}^m-\mu_i^m)| \\
&\le M\Delta_i + M = M(\Delta_i+1).
\end{aligned} \tag{60}
$$

Hence, the regret can be upper bounded by

$$
\begin{aligned}
&(59) \\
&\le L + \sum_{i\ne i^*}(\Delta_i+1)(\max\{[\frac{4C_1\log T}{\Delta_i^2}], 2(K^2+MK+M)\} + \frac{2\pi^2}{3} + K^2 + (2M-1)K) \\
&= O(\max\{L, \log T\})
\end{aligned}
$$

where the inequality is derived from (60) and L is the same constant as in the definition of $A_{\epsilon,\delta}$. This completes the proof.

□

Theorem 3. *Let f be a function specific to a setting and defined in the above remark. For every $0<\epsilon<1$ and $0<\delta<f(\epsilon,M,T)$, in settings S_1 with $c\ge\frac{1}{2}+\frac{1}{2}\sqrt{1-(\frac{\epsilon}{MT})^{\frac{2}{M-1}}}$, S_2, S_3 with the time horizon T satisfying $T\ge L$, the regret of Algorithm 2 with $F(m,i,t)=\sqrt{\frac{C_1\ln T}{n_{m,i}(t)}+\frac{C_2\ln T}{n_{m,i}(t)}}$ satisfies*

$$
\begin{aligned}
E[R_T|A_{\epsilon,\delta}] \le L + \sum_{i\ne i^*}(\Delta_i+1)\cdot(\max([\frac{16C_1\log T}{\Delta_i^2}], [\frac{4C_2\log T}{\Delta_i}], 2(K^2+MK+M)) \\
+ \frac{4}{P(A_{\epsilon,\delta})T^3} + K^2 + (2M-1)K)
\end{aligned}
$$

where L, C_1 are specified as in Theorem 2 and $\frac{C_2}{C_1}\ge\frac{3}{2}$.

Proof. By the regret decomposition as in (58), we obtain that

$$
E[R_T|A_{\epsilon,\delta}] \le L + \frac{1}{M}\sum_{i\ne i^*}\sum_{m:\mu_{i^*}^m-\mu_i^m>0}E[n_{m,i}(T)|A_{\epsilon,\delta}](\mu_{i^*}^m-\mu_i^m). \tag{61}
$$

By Proposition 13, we have that with probability at least $1-7\epsilon$

$$
\begin{aligned}
&E[n_{m,i}(T)|A_{\epsilon,\delta}] \\
&\le \max([\frac{16C_1\log T}{\Delta_i^2}], [\frac{4C_2\log T}{\Delta_i}], 2(K^2+MK+M)) + \frac{4}{P(A_{\epsilon,\delta})T^3} + K^2 + (2M-1)K.
\end{aligned} \tag{62}
$$

Following (60) gives us that

$$
\sum_{m:\mu_{i^*}^m-\mu_i^m>0}(\mu_{i^*}^m-\mu_i^m) \le M\Delta_i + M = M(\Delta_i+1). \tag{63}
$$

Therefore, we derive that with probability at least $P(A_{\epsilon,\delta})=1-7\epsilon$

$$
\begin{aligned}
E[R_T|A_{\epsilon,\delta}] \le L + \sum_{i\ne i^*}(\Delta_i+1)\cdot(\max([\frac{16C_1\log T}{\Delta_i^2}], [\frac{4C_2\log T}{\Delta_i}], 2(K^2+MK+M)) \\
+ \frac{4}{P(A_{\epsilon,\delta})T^3} + K^2 + (2M-1)K)
\end{aligned}
$$

which completes the proof.

□

Theorem 4. *Assume the same conditions as in Theorems 2 and 3. The regret of Algorithm 2 satisfies that*

$$E[R_T|A_{\epsilon,\delta}] \leq L_1 + \frac{4}{P(A_{\epsilon,\delta})T^3} + (1+\max\{\sqrt{C_1 \ln T}, C_2 \ln T\})(K(K+2M) - K + \frac{2}{P(A_{\epsilon,\delta})T^3}) + K(C_2(\ln T)^2 + C_2 \ln T + \sqrt{C_1 \ln T}\sqrt{T(\ln T+1)}) = O(\sqrt{T}\ln T).$$

where $L_1 = \max(L, K(2(K^2+MK+M)))$, L, C_1 is specified as in Theorem 2 and $\frac{C_2}{C_1} \geq \frac{3}{2}$. The involved constants depend on σ^2 but not on Δ_i.

Proof. Define $U_m^t(i)$ and $L_m^t(i)$ as $\tilde{\mu}_i^m(t) + Rad(i,m,t)$ and $\tilde{\mu}_i^m(t) - Rad(i,m,t)$, respectively, where Rad is previously defined as $Rad(i,m,t) = \sqrt{\frac{C_1 \ln T}{n_{m,i}(t)}} + \frac{C_2 \ln T}{n_{m,i}(t)}$. We observe that by definition, the regret R_T can be written as

$$\begin{aligned} R_T &= \frac{1}{M}\sum_{t=1}^{T}\sum_{m=1}^{M}(\mu_{i^*} - \mu_{a_m^t}^t) \\ &= \frac{1}{M}\sum_{t=1}^{T}\sum_{m=1}^{M}(\mu_{i^*} - U_m^t(a_m^t) + U_m^t(a_m^t) - L_m^t(a_m^t) + L_m^t(a_m^t) - \mu_{a_m^t}^t). \end{aligned}$$

Subsequently, the conditional expectation of R_T has the following decomposition

$$\begin{aligned} &E[R_T|A_{\epsilon,\delta}] \\ &= \frac{1}{M}\sum_{t=1}^{T}\sum_{m=1}^{M}(E[\mu_{i^*} - U_m^t(a_m^t)|A_{\epsilon,\delta}] + E[U_m^t(a_m^t) - L_m^t(a_m^t)|A_{\epsilon,\delta}] + E[L_m^t(a_m^t) - \mu_{a_m^t}^t|A_{\epsilon,\delta}]) \\ &= L_1 + \frac{1}{M}\sum_{t=L_1+1}^{T}\sum_{m=1}^{M}(E[\mu_{i^*} - U_m^t(i^*)|A_{\epsilon,\delta}] + E[U_m^t(i^*) - U_m^t(a_m^t)|A_{\epsilon,\delta}] + \\ &\qquad E[U_m^t(a_m^t) - L_m^t(a_m^t)|A_{\epsilon,\delta}] + E[L_m^t(a_m^t) - \mu_{a_m^t}^t|A_{\epsilon,\delta}]) \end{aligned} \tag{64}$$

where $L_1 = \max(L, 2(K^2+MK+M))$.

For the first term, we derive its upper bound as follows.

Note that

$$\begin{aligned} &E[\mu_{i^*} - U_m^t(i^*)|A_{\epsilon,\delta}] \\ &\leq E[(\mu_{i^*} - U_m^t(i^*))1_{\mu_{i^*} - U_m^t(i^*)>0}|A_{\epsilon,\delta}] \\ &= E[\mu_{i^*}1_{\mu_{i^*} - U_m^t(i^*)>0}|A_{\epsilon,\delta}] - E[U_m^t 1_{\mu_{i^*} - U_m^t(i^*)>0}|A_{\epsilon,\delta}] \\ &\leq E[\mu_{i^*}1_{\mu_{i^*} - U_m^t(i^*)>0}|A_{\epsilon,\delta}] \\ &\leq E[1_{\mu_{i^*} - U_m^t(i^*)>0}|A_{\epsilon,\delta}] \\ &= P(\mu_{i^*} - U_m^t(i^*) > 0|A_{\epsilon,\delta}) \\ &= P(\mu_{i^*} - \tilde{\mu}_{i^*}^m(t) > Rad|A_{\epsilon,\delta}) \\ &\leq P(|\mu_{i^*} - \tilde{\mu}_{i^*}^m(t)| > Rad|A_{\epsilon,\delta}) \leq \frac{2}{P(A_{\epsilon,\delta})T^4} \end{aligned} \tag{65}$$

where the first inequality uses the monotone property of $E[\cdot]$, the second inequality omits the latter negative quantity, the third inequality holds by the fact that $0 \leq \mu_i^* \leq 1$, and the last inequality is by Proposition 12.

In like manner, we have that the last term satisfies that

$$E[L_m^t(a_m^t) - \mu_{a_m^t}|A_{\epsilon,\delta}] \leq \frac{2}{P(A_{\epsilon,\delta})T^4} \tag{66}$$

by the same logic as the above and substituting i^* with a_m^t, and thus we omit the details here.

We then proceed to bound the second term. Based on the decision rule in Algorithm 2, we have either $E[U_m^t(i^*) - U_m^t(a_m^t)|A_{\epsilon,\delta}] < 0$ or $n_{m,i}(t) < N_{m,i}(t) - K$. This is equivalent to

$$\begin{aligned}
&E[U_m^t(i^*) - U_m^t(a_m^t)|A_{\epsilon,\delta}] \\
&= E[U_m^t(i^*) - U_m^t(a_m^t)1_{n_{m,i}(t)\geq N_{m,i}(t)-K}|A_{\epsilon,\delta}]+ \\
&\qquad E[U_m^t(i^*) - U_m^t(a_m^t)1_{n_{m,i}(t)<N_{m,i}(t)-K}|A_{\epsilon,\delta}] \\
&\leq E[U_m^t(i^*) - U_m^t(a_m^t)1_{n_{m,i}(t)<N_{m,i}(t)-K}|A_{\epsilon,\delta}] \\
&\leq E[U_m^t(i^*)1_{n_{m,i}(t)<N_{m,i}(t)-K}|A_{\epsilon,\delta}].
\end{aligned} \tag{67}$$

By definition, $U_m^t(i^*) = \tilde{\mu}_{i^*} + Rad(i^*, m, t)$ implies that

$$U_m^t(i^*) \leq 1 + Rad(i^*, m, t)$$

which leads to

$$\begin{aligned}
(67) &\leq E[(1 + Rad(i^*, m, t))1_{n_{m,i}(t)<N_{m,i}(t)-K}|A_{\epsilon,\delta}] \\
&\leq (1 + \max Rad(i^*, m, t))E[1_{n_{m,i}(t)<N_{m,i}(t)-K}|A_{\epsilon,\delta}] \\
&\leq (1 + \max\{\sqrt{C_1 \ln T}, C_2 \ln T\})E[1_{n_{m,i}(t)<N_{m,i}(t)-K}|A_{\epsilon,\delta}]
\end{aligned}$$

and subsequently

$$\begin{aligned}
&\frac{1}{M}\sum_{t=L_1+1}^{T}\sum_{m=1}^{M} E[U_m^t(i^*) - U_m^t(a_m^t)|A_{\epsilon,\delta}] \\
&\leq \frac{1}{M}\sum_{t=L_1+1}^{T}\sum_{m=1}^{M}(1 + \max\{\sqrt{C_1 \ln T}, C_2 \ln T\})E[1_{n_{m,i}(t)<N_{m,i}(t)-K}|A_{\epsilon,\delta}].
\end{aligned} \tag{68}$$

Following (53) that only depends on whether clients stay on the same page that relies on the transmission, we obtain

$$\sum_t E[1_{n_{m,i}(t)<N_{m,i}(t)-K}|A_{\epsilon,\delta}] \leq K(K + 2M) - K + \frac{2}{P(A_{\epsilon,\delta})T^3}$$

which immediately leads to

$$(68) \leq (1 + \max\{\sqrt{C_1 \ln T}, C_2 \ln T\}) \cdot (K(K + 2M) - K + \frac{2}{P(A_{\epsilon,\delta})T^3})$$

Afterwards, we consider the third term and have

$$\begin{aligned}
&E[U_m^t(a_m^t) - L_m^t(a_m^t)|A_{\epsilon,\delta}] \\
&= E[2Rad(a_m^t, m, t)|A_{\epsilon,\delta}]
\end{aligned} \tag{69}$$

Putting (65, 66, 68, 69) all together, we deduce that

$$\begin{aligned}
(64) \leq\, & L_1 + \frac{1}{M}\sum_{t=L_1+1}^{T}\sum_{m=1}^{M}(\frac{2}{P(A_{\epsilon,\delta})T^4} + E[2Rad(a_m^t, m, t)|A_{\epsilon,\delta}] + \frac{2}{P(A_{\epsilon,\delta})T^4}) \\
&+ (1 + \max\{\sqrt{C_1 \ln T}, C_2 \ln T\}) \cdot (K(K + 2M) - K + \frac{2}{P(A_{\epsilon,\delta})T^3}) \\
\leq\, & L_1 + \frac{4}{P(A_{\epsilon,\delta})T^3} + \frac{1}{M}\sum_{t>L_1}\sum_{m}(E[2Rad(a_m^t, m, t)|A_{\epsilon,\delta}])+ \\
&(1 + \max\{\sqrt{C_1 \ln T}, C_2 \ln T\}) \cdot (K(K + 2M) - K + \frac{2}{P(A_{\epsilon,\delta})T^3}).
\end{aligned}$$

Meanwhile, we observe that by definition

$$\frac{1}{M}\sum_{t>L_1}\sum_{m}(E[2Rad(a_m^t, m, t)|A_{\epsilon,\delta}])$$
$$= \frac{1}{M}\sum_{i}\sum_{m}\sum_{\substack{a_m^t=i\\ t>L_1}}(E[2Rad(i, m, t)|A_{\epsilon,\delta}])$$
$$= \frac{1}{M}\sum_{i}\sum_{m}\sum_{\substack{a_m^t=i\\ t>L_1}}E[2\sqrt{\frac{C_1\ln T}{n_{m,i}(t)}} + \frac{C_2\ln T}{n_{m,i}(t)}|A_{\epsilon,\delta}]. \tag{70}$$

By the sum of the Harmonic series, we have

$$\sum_{\substack{a_m^t=i\\ t>L_1}}\frac{C_2\ln T}{n_{m,i}(t)} \leq C_2\ln T\ln n_{m,i}(T) + C_2\ln T \leq C_2(\ln T)^2 + C_2\ln T. \tag{71}$$

Meanwhile, by the Cauchy-Schwartz inequality, we obtain

$$\sum_{\substack{a_m^t=i\\ t>L_1}}\sqrt{\frac{C_1\ln T}{n_{m,i}(t)}}$$
$$\leq \sqrt{C_1\ln T}\sqrt{(\sum_t 1)(\sum_t(\sqrt{\frac{1}{n_{m,i}(t)}})^2)}$$
$$\leq \sqrt{C_1\ln T}\sqrt{T(\ln T + 1)}$$

where the last inequality again uses the result on the Harmonic series as in (71).

Therefore, the cumulative value can be bounded as

$$(70) \leq \frac{1}{M}\sum_{i}\sum_{m}(C_2(\ln T)^2 + C_2\ln T + \sqrt{C_1\ln T}\sqrt{T(\ln T+1)})$$
$$= K(C_2(\ln T)^2 + C_2\ln T + \sqrt{C_1\ln T}\sqrt{T(\ln T+1)})$$

Using the result of (70), we have

$$(64) \leq L_1 + \frac{4}{P(A_{\epsilon,\delta})T^3} + K(C_2(\ln T)^2 + C_2\ln T + \sqrt{C_1\ln T}\sqrt{T(\ln T+1)}) +$$
$$(1 + \max\{\sqrt{C_1\ln T}, C_2\ln T\})\cdot(K(K+2M) - K + \frac{2}{P(A_{\epsilon,\delta})T^3})$$
$$= O(\max\{\sqrt{T}\ln T, (\ln T)^2\})$$

which completes the proof.

□

G Choices of parameter c_0 in Theorem 2

Parameter c_0 We note that c_0 is a pre-specified parameter which are different in different settings. The choices of c_0 are as follows. Meanwhile, we need to study whether the possible choices of c_0 explode in terms of the order of T.

Remark (2)**.** *The regret reads*

$$E[R_T|A_{\epsilon,\delta}] \leq L + C_1\sum_{i\neq i^*}([\frac{4\log T}{\Delta_i^2}]) + (K-1)(2(K^2+MK) + \frac{2\pi^2}{3} + K^2 + (2M-1)K)$$

with L denoted as $L = \max\{L_1, L_2, L_3\} = \max\{a_1, a_2, a_3, \frac{b_1}{c_0}, \frac{b_2}{c_0}, \frac{b_3}{c_0}\}$ and C_1 denoted as $\max\{\frac{e}{1-c_0}, f\}$, where parameters $a_1, a_2, a_3, b_1, b_2, b_3, e, f$ are specified as

$$
\begin{aligned}
a_1 &= \frac{\ln \frac{2T}{\epsilon}}{2\delta^2} \\
b_1 &= 4K \log_2 T \\
a_2 &= \frac{\ln \frac{\delta}{10}}{\ln p^*} + 25\frac{1+\lambda}{1-\lambda}\frac{\ln \frac{2T}{\epsilon}}{2\delta^2} \\
b_2 &= 4K \log_2 T \\
a_3 &= \frac{\ln \frac{\delta}{10}}{\ln p^*} + 25\frac{1+\lambda}{1-\lambda}\frac{\ln \frac{2T}{\epsilon}}{2\delta^2} \\
b_3 &= \frac{K \ln(\frac{MT}{\epsilon})}{\ln(\frac{1}{1-c})} \\
e &= 16\frac{4(M+2)}{3M} \\
f &= 16(M+2)(1+4Md_{m,t}^2).
\end{aligned}
$$

This function of c_0 is non-differentiable which brings additional challenges and requires a case-by-case analysis.

Let $a = \{a_1, a_2, a_3\}$ and $b = \{b_1, b_2, b_3\}$. Then continue with the decision rule as in the previous discussion.

- *Case 1: there exists c_0 such that $a \geq \frac{b}{c_0}$,i.e. $c \geq \frac{b}{a}$ and $\frac{b}{a} \leq 1$ Then R_T is monotone increasing in c due to C_1 and $c_0 = \frac{b}{a}$ gives us the optimal regret R_T^1.*
- *Case 2: if $a \leq \frac{b}{c_0}$,i.e. $c_0 \leq \frac{b}{a}$*
  - *if $\frac{e}{1-c_0} < f$,i.e. $c_0 \leq 1 - \frac{e}{f}$, then $c_0 = \min\{\frac{b}{a}, 1 - \frac{e}{f}\}$ is the minima.*
  - *else we have $c_0 \geq 1 - \frac{e}{f}$*
    - *if $1 - \frac{e}{f} > \frac{b}{a}$, it leads to contradiction and this can not be the case.*
    - *else $1 - \frac{e}{f} \leq \frac{b}{a}$, we obtain*

      $$R_T \leq \frac{b}{c_0} + \frac{e}{1-c_0}\sum_{i \neq i^*}([\frac{4 \log T}{\Delta_i^2}]) + (K-1)(2(K^2 + MK) + \frac{2\pi^2}{3} + K^2 + (2M-1)K)$$

      which implies that the optimal choice of c_0 is $\frac{\sqrt{b}}{\sqrt{b} + \sqrt{e \sum_{i \neq i^*}([\frac{4 \log T}{\Delta_i^2}])}}$

      · *if $1 - \frac{e}{f} \leq \frac{\sqrt{b}}{\sqrt{b} + \sqrt{e \sum_{i \neq i^*}([\frac{4 \log T}{\Delta_i^2}])}} \leq \frac{b}{a}$, this gives us the final choice of c_0 and the subsequent local optimal regret R_T^2.*

      · *elif $\frac{\sqrt{b}}{\sqrt{b} + \sqrt{e \sum_{i \neq i^*}([\frac{4 \log T}{\Delta_i^2}])}} < 1 - \frac{e}{f}$, the optimal choice of c_0 is $1 - \frac{e}{f}$ and the subsequent local optimal regret is R_T^2*

      · *else the optimal choice of c_0 is $\frac{b}{a}$ and the subsequent local optimal regret is R_T^2*
- *Compare R_T^1 and R_T^2 and choose the c_0 associated with the smaller value.*

The possible choices of c_0 *are* $\{\frac{b}{a}, \min\{\frac{b}{a}, 1-\frac{e}{f}\}, \frac{\sqrt{b}}{\sqrt{b}+\sqrt{e\sum_{i\neq i^*}([\frac{4\log T}{\Delta_i^2}])}}\}$, *i.e.*

$$
\begin{aligned}
c_0 &= \{\frac{b_1+b_2+b_3}{a_1+a_2+a_3}, \min\{\frac{b_1+b_2+b_3}{a_1+a_2+a_3}, 1-\frac{16^{\frac{4(M+2)}{3M}}}{16(M+2)(1+4Md_{m,t}^2)}\}, \frac{\sqrt{b}}{\sqrt{b}+\sqrt{e\sum_{i\neq i^*}([\frac{4\log T}{\Delta_i^2}])}}\} \\
&= \{\frac{b_1+b_2+b_3}{a_1+a_2+a_3}, \min\{\frac{b_1+b_2+b_3}{a_1+a_2+a_3}, 1-\frac{16^{\frac{4(M+2)}{3M}}}{16(M+2)(1+4Md_{m,t}^2)}\}, \frac{\sqrt{b}}{\sqrt{b}+\sqrt{e\sum_{i\neq i^*}([\frac{4\log T}{\Delta_i^2}])}}\} \\
&= \frac{8K\log T+\frac{K\ln(\frac{MT}{\epsilon})}{\ln(\frac{1}{1-c})}}{\frac{\ln\frac{2T}{\epsilon}}{2\delta^2}+2\frac{\ln\frac{\delta}{10}}{\ln p^*}+50\frac{1+\lambda}{1-\lambda}\frac{\ln\frac{2T}{\epsilon}}{2\delta^2}}, \\
&\qquad \min\{\frac{8K\log T+\frac{K\ln(\frac{MT}{\epsilon})}{\ln(\frac{1}{1-c})}}{\frac{\ln\frac{2T}{\epsilon}}{2\delta^2}+2\frac{\ln\frac{\delta}{10}}{\ln p^*}+50\frac{1+\lambda}{1-\lambda}\frac{\ln\frac{2T}{\epsilon}}{2\delta^2}}, 1-\frac{16^{\frac{4(M+2)}{3M}}}{16(M+2)(1+4Md_{m,t}^2)}\}, \\
&\qquad \frac{\sqrt{8K\log T+\frac{K\ln(\frac{MT}{\epsilon})}{\ln(\frac{1}{1-c})}}}{\sqrt{8K\log T+\frac{K\ln(\frac{MT}{\epsilon})}{\ln(\frac{1}{1-c})}}+\sqrt{e\sum_{i\neq i^*}([\frac{4\log T}{\Delta_i^2}])}}
\end{aligned}
$$

which implies the choice of c_0 *is between* $\frac{\sqrt{8K\log T+\frac{K\ln(\frac{MT}{\epsilon})}{\ln(\frac{1}{1-c})}}}{\sqrt{8K\log T+\frac{K\ln(\frac{MT}{\epsilon})}{\ln(\frac{1}{1-c})}}+\sqrt{e\sum_{i\neq i^*}([\frac{4\log T}{\Delta_i^2}])}}$ *and* $1-\frac{16^{\frac{4(M+2)}{3M}}}{16(M+2)(1+4Md_{m,t}^2)}$. *Meanwhile, we observe that the choice of* c_0 *satisfies*

$$E[R_T|A_{\epsilon,\delta}] \leq R_T(1-\frac{16^{\frac{4(M+2)}{3M}}}{16(M+2)(1+4Md_{m,t}^2)}) = O(\log T).$$