Appendix

The appendices are organized as follows. Formal proofs of the results stated in the main text are presented in Section A. In Section B, we describe the algorithm to recover the maximal hedge formed for a certain query (Def. 5), which is used as a subroutine of Algorithm 1. A generalization of Assumption 1 is discussed in Section C. Section D provides further details of the heuristic algorithms discussed in the main text. Further evaluations and experimental conditions for our proposed algorithms are presented in Section E.

Table 2: Table of notations

Symbol	Description
$V^{\mathcal{G}}$	Vertices of $\mathcal{G}$
$E_b^{\mathcal{G}}$	The set of bidirected edges of $\mathcal{G}$
$E_d^{\mathcal{G}}$	The set of directed edges of $\mathcal{G}$
$Anc_{\mathcal{G}}(X)$	Ancestors of X in $\mathcal{G}$
$\mathcal{M}(\mathcal{G})$	The set of the all compatible models with $\mathcal{G}$
p_e	Probability of edge e
w_e	Weight of edge e
$P_X(Y)$	Causal effect of X on Y

A Formal Proofs

We begin with presenting the proofs of Proposition 1 and Lemma 1. Proofs of Theorem 1 and Proposition 2 appear at the end of Sections A.1 and A.2, respectively.

Proposition 1. *For any causal query $P_X(Y)$ and ADMG $\mathcal{G}$, if $\mathcal{F}$ is a valid identification formula for $P_X(Y)$ in $\mathcal{G}$ (Def. 2), then $\mathcal{F}$ is a valid identification formula for $P_X(Y)$ in any $\mathcal{G}' \subseteq \mathcal{G}$.*

Proof. Let $\mathcal{H} \subseteq \mathcal{G}$ be an arbitrary edge-induced subgraph of $\mathcal{G}$. Let $\mathcal{F}$ be an identification formula for $P_X(Y)$ in $\mathcal{G}$, i.e., for any model M that induces $\mathcal{G}$,

$$P_X^M(Y) = \mathcal{F}(P^M(V^{\mathcal{G}})). \tag{5}$$

By definition, $P_X(Y)$ is identifiable in $\mathcal{G}$. As a result, there exists and identification formula such as $\mathcal{F}'$ that can be derived for $P_X(Y)$ in $\mathcal{G}$, using a sequence of do calculus rules and basic probability manipulations. Note that this means for any model M that induces $\mathcal{G}$,

$$P_X^M(Y) = \mathcal{F}'(P^M(V^{\mathcal{G}})). \tag{6}$$

Note that an immediate corollary of Equations 5 and 6 is that for any model M that induces $\mathcal{G}$,

$$\mathcal{F}(P^M(V^{\mathcal{G}})) = \mathcal{F}'(P^M(V^{\mathcal{G}})). \tag{7}$$

Now, we first show that this sequence of actions (combination of do calculus rules and probability manipulations) is valid in $\mathcal{H}$. Note that the basic probability manipulations are graph-independent. It only suffices to show that any applied do calculus rule w.r.t. $\mathcal{G}$ can also be applied w.r.t. $\mathcal{H}$. The validity conditions of all three do calculus rules are based on certain d-separations. As a result, it suffices to show that if a d-separation relation is valid in $\mathcal{G}$, it is also valid in $\mathcal{H}$. To do so, it suffices to show that if all paths between Z_1 and Z_2 are blocked in $\mathcal{G}$ given W, they are blocked in $\mathcal{H}$ too, for arbitrary disjoint sets of vertices $Z_1, Z_2, W \subseteq V^{\mathcal{G}}$. Take an arbitrary path, p, between Z_1 and Z_2 in $\mathcal{H}$. Since $\mathcal{H} \subseteq \mathcal{G}$, this path exists in $\mathcal{G}$. Since Z_1 and Z_2 are d-separated given W in $\mathcal{G}$, the path p is blocked by W. As a result, any path between Z_1 and Z_2 in $\mathcal{H}$ is blocked by W. Therefore, any do-calculus rule applied in $\mathcal{G}$, can also be applied in $\mathcal{H}$. Hence, $\mathcal{F}'$ is a valid identification formula for $P_X(Y)$. That is, for any model M that induces $\mathcal{H}$,

$$P_X^M(Y) = \mathcal{F}'(P^M(V^{\mathcal{H}})). \tag{8}$$

Now note that any model M that induces $\mathcal{H}$, i.e., is compatible with $\mathcal{H}$, is also compatible with $\mathcal{G}$. Also, $V^{\mathcal{G}} = V^{\mathcal{H}}$. As a result, from Equations 7 and 8, we know that for any model M that induces $\mathcal{H}$,

$$P_X^M(Y) = \mathcal{F}(P^M(V^{\mathcal{H}})).$$

By definition, $\mathcal{F}$ is a valid identification formula for $P_X(Y)$ in $\mathcal{H}$. □

Lemma 1. *Under Assumption 1, Problem 1 is equivalent to the edge ID problem with the edge weights chosen to be the log propensity ratios, i.e.,* $w_e = \max\{0, \log(\frac{p_e}{1-p_e})\}$, $\forall e \in \mathcal{G}$*. Moreover, Problem 2 is equivalent to the edge ID problem with the choice of weights* $w_e = -\log(1-p_e)$, $\forall e \in \mathcal{G}$*. That is, an instance of Problems 1 and 2 can be reduced to an instance of the edge ID problem in polynomial time, and vice versa.*

Proof. Problem 1. First consider an arbitrary graph $\mathcal{G}_1 \in [\mathcal{G}]_{Id(Q[Y])}$ such that $\mathcal{G}_1$ has an edge e with $p_e < 1/2$. Let G_2 denote the graph G_1 after removing e. Proposition 1 implies that $\mathcal{G}_2 \in [\mathcal{G}]_{Id(Q[Y])}$. According to Equation 1, we have $P(\mathcal{G}_2) = \frac{1-p_e}{p_e} P(\mathcal{G}_1) > P(\mathcal{G}_1)$ (since $p_e < 1/2$). As a result, the solution $\mathcal{G}^*$ to Problem 1 (Eq. 2) has no edges with probability less than $1/2$. We can therefore rewrite Problem 1 as:

$$\mathcal{G}^* := \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \; P(\mathcal{G}_s) = \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \; P(\mathcal{G}_s) \quad \textbf{s.t.} \quad \forall e \in \mathcal{G}_s : \; p_e \geq \frac{1}{2}.$$

Or equivalently, we can always assume that we start with a graph $\mathcal{G}$ that has no edges with probability less than $1/2$, as otherwise we can remove all of those edges and the problem does not change. This indeed is equivalent to choosing weight (cost) 0 for those edges in the equivalent edge ID problem. Now assuming that the edges have probability at least $1/2$,

$$\begin{aligned}
\mathcal{G}^* &= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \; P(\mathcal{G}_s) \\
&= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \; \log(P(\mathcal{G}_s)) \\
&= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \; \log(\prod_{e \in \mathcal{G}_s} p_e \prod_{e \notin \mathcal{G}_s} (1-p_e)) \\
&= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \; \sum_{e \in \mathcal{G}_s} \log(p_e) + \sum_{e \notin \mathcal{G}_s} \log(1-p_e)) \\
&= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \; \sum_{e \in \mathcal{G}_s} \log(p_e) + \sum_{e \notin \mathcal{G}_s} \log(1-p_e)) + \sum_{e \in \mathcal{G}_s} \log(1-p_e)) - \sum_{e \in \mathcal{G}_s} \log(1-p_e))
\end{aligned}$$

Since $\sum_{e \notin \mathcal{G}_s} \log(1-p_e)) + \sum_{e \in \mathcal{G}_s} \log(1-p_e))$ is a constant value that does not depend on $\mathcal{G}_s$, it can be ignored in the maximization and we have:

$$\begin{aligned}
\mathcal{G}^* &= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \; \sum_{e \in \mathcal{G}_s} \log(p_e) - \sum_{e \in \mathcal{G}_s} \log(1-p_e)) \\
&= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \; \sum_{e \in \mathcal{G}_s} \log(\frac{p_e}{1-p_e}) \\
&= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\min} \; \sum_{e \notin \mathcal{G}_s} \log(\frac{p_e}{1-p_e}).
\end{aligned}$$

From the formulation above, it is clear that if we assign the weight $w_e = \log(\frac{p_e}{1-p_e})$ to each edge $e \in E^{\mathcal{G}}$, we will have an instance of the edge ID problem. Note that for edges with probability higher than $1/2$, $\log(\frac{p_e}{1-p_e}) \geq 0$, and this assignment of edge weights satisfies the positivity requirement. For the opposite direction, note that the procedure explained above is reversible by the choice of probabilities $p_e = \frac{\exp(w_e)}{1+\exp(w_e)}$, which is a value between $1/2$ and 1.

Problem 2. First note that under Assumption 1, for any graph $\mathcal{G}_s$,

$$\sum_{\hat{\mathcal{G}} \subseteq \mathcal{G}_s} P(\hat{\mathcal{G}}) = \prod_{e \notin \mathcal{G}_s} (1-p_e)[\sum_{\hat{E} \subseteq E^{\mathcal{G}_s}} \prod_{e \in \hat{E}} p_e \prod_{e \notin \hat{E}} (1-p_e)] = \prod_{e \notin \mathcal{G}_s} (1-p_e).$$

This is because the inner summation goes over all the possible subsets of $E^{\mathcal{G}_s}$, and the summation adds up to 1. Therefore, we can rewrite Problem 2 (Eq. 3)as

$$\begin{aligned}
\mathcal{H}^* &= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \sum_{\hat{\mathcal{G}} \subseteq \mathcal{G}_s} P(\hat{\mathcal{G}}) \\
&= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \prod_{e \notin \mathcal{G}_s} (1 - p_e) \\
&= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \log(\prod_{e \notin \mathcal{G}_s} (1 - p_e)) \\
&= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\max} \sum_{e \notin \mathcal{G}_s} \log(1 - p_e) \\
&= \underset{\substack{\mathcal{G}_s \subseteq \mathcal{G}, \\ \mathcal{G}_s \in [\mathcal{G}]_{Id(Q[Y])}}}{\arg\min} \sum_{e \notin \mathcal{G}_s} -\log(1 - p_e).
\end{aligned}$$

With the same reasoning as before, assigning the weights $w_e = -\log(1 - p_e)$ to each edge $e \in E^{\mathcal{G}}$, we end up with an instance of the edge ID problem. Note that again $0 \leq -\log(1 - p_e) \leq \infty$. It is noteworthy that this procedure is also reversible with the choice of edge probabilities $p_e = 1 - \exp(-w_e)$, which reduces the edge ID problem to an instance of Problem 2. Again note that $0 \leq 1 - \exp(-w_e) \leq 1$ for any non-negative w_e. $\square$

A.1 Reduction from MCIP to edge ID

Theorem 1. *The edge ID problem is NP-hard.*

To prove Theorem 1, we first present a polynomial-time reduction from MCIP to the edge ID problem. It has been shown that the minimum vertex cover problem can be reduced to MCIP in polynomial time [1]. Combining the two reductions, we show that there exists a polynomial-time redcution from the minimum vertex cover problem to the edge ID problem. Since the minimum vertex cover problem is known to be NP-hard [11], it follows that the edge ID problem is also NP-hard.

We propose the following reduction from MCIP to the edge ID problem. Assume we want to solve MCIP given ADMG $\mathcal{G} = (V^{\mathcal{G}}, E_d^{\mathcal{G}}, E_b^{\mathcal{G}})$, query $Q[Y]$, and the intervention costs $C(v)$ for $v \in V^{\mathcal{G}}$. We construct a graph, denoted by $\mathcal{H} = \mathcal{T}_1(\mathcal{G}, Y)$, through the following steps.

a. For every vertex $x \in V^{\mathcal{G}} \setminus Y$, add two vertices x^1, x^2 to $V^{\mathcal{H}}$.

b. For any bidirected edge $\{x, z\} \in E_b^{\mathcal{G}}$ where $x \in V^{\mathcal{G}} \setminus Y$ and $z \in V^{\mathcal{G}}$, add the bidirected edge $\{x^2, z^2\}$ to $E_b^{\mathcal{H}}$.

c. For any directed edge $(x, z) \in E_d^{\mathcal{G}}$ where $x \in V^{\mathcal{G}} \setminus Y$ and $z \in V^{\mathcal{G}}$, add the directed edge (x^1, z^1) to $E_d^{\mathcal{H}}$.

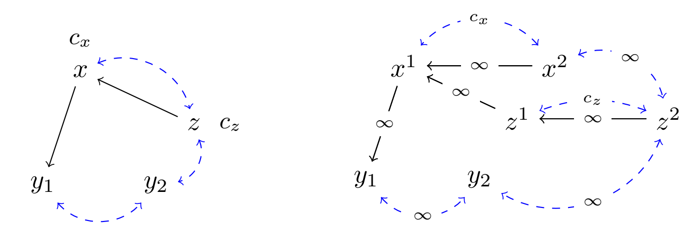
d. For any bidirected edge $\{y_1, y_2\} \in E_b^{\mathcal{G}}$ where $y_1, y_2 \in Y$, add the bidirected edge $\{y_1, y_2\}$ to $E_b^{\mathcal{H}}$.

e. For every $x^1, x^2 \in V^{\mathcal{G}} \setminus Y$, draw the two edges $\{x^1, x^2\} \in E_b^{\mathcal{H}}$ and $(x^2, x^1) \in E_d^{\mathcal{H}}$. Furthermore, the weight of $\{x^1, x^2\}$ is $C(x)$.

f. The costs of the all other edges in $\mathcal{H}$ are assigned to be infinite.

With abuse of notation, for any vertex $x \in V^{\mathcal{G}} \setminus Y$, we define $\mathcal{T}_1(x) = \{x^2, x^1\} \in E_b^{\mathcal{H}}$, where $\{x^2, x^1\}$ is the bidirected edge in $\mathcal{H}$ that corresponds to x in $\mathcal{G}$, and inherits the same weight (cost).

Example 2. *Consider graph $\mathcal{G}$ in Figure 4a. Vertices x and z are mapped to x^1, x^2, and z^1, z^2, respectively. Both a directed and a bidirected edge are drawn between these pairs. The bidirected edge $\{x^1, x^2\}$ is assigned the weight $C(x) = c_x$, and the bidirected edge $\{z^1, z^2\}$ is assigned the weight $C(z) = c_z$. Infinite weights are assigned to the rest of the edges in $\mathcal{H}$ (Figure 4b).*



(a) The model $\mathcal{G}$ with costs on each vertex

(b) The model $\mathcal{G}$ with costs on each edge

Figure 4: Reduction of MCIP to edge ID

Proposition 3. *Suppose $\mathcal{G}'$ is an ADMG, $Y \subseteq V^{\mathcal{G}'}$ is a set of its vertices such that Y is a district in $\mathcal{G}'[Y]$, and $\mathcal{H}' = \mathcal{T}_1(\mathcal{G}', Y)$. Consider $X \subseteq V^{\mathcal{G}'} \setminus Y$ as an arbitrary subset of vertices of $\mathcal{G}'$, and define $\mathcal{G} = \mathcal{G}'[V^{\mathcal{G}'} \setminus X]$. Let $E''_b = \{e \in E^{\mathcal{H}'}_b | \exists v \in X, e = \mathcal{T}_1(v)\}$ and define $E^{\mathcal{H}}_b = E^{\mathcal{H}'}_b \setminus E''_b$. Let $\mathcal{H}$ be the edge-induced subgraph of $\mathcal{H}'$ defined as $\mathcal{H} = (V^{\mathcal{H}'}, E^{\mathcal{H}}_d, E^{\mathcal{H}}_b)$. $Q[Y]$ is identifiable in $\mathcal{G}$ if and only if $Q[Y]$ is identifiable in $\mathcal{H}$.*

Proof. We prove the contrapositive, i.e., $Q[Y]$ is not identifiable in $\mathcal{G}$ iff $Q[Y]$ is not identifiable in $\mathcal{H}$. Note that by construction, Y is a district in both $\mathcal{G}[Y]$ and $\mathcal{H}[Y]$. That is, it suffices to show that there exists a hedge formed for $Q[Y]$ in $\mathcal{G}$ iff there exists a hedge formed for $Q[Y]$ in $\mathcal{H}$.

To this end, we first prove the following claim. Let $W \in V^{\mathcal{H}}$ form a hedge for $Q[Y]$. If $x^1 \in W$ for some $x \in V^{\mathcal{G}'}$, then $x^2 \in W$ and vice versa. That is, the two vertices x^1 and x^2 corresponding to the same vertex x in $V^{\mathcal{G}'}$ appear only simultaneously in any hedge. To see this, note that by construction, x^1 is the only child of x^2. By definition of hedge, if $x^2 \in W$, then it has a directed path to Y within $\mathcal{H}[W]$, and this path can only go through x^1. For the other direction, note that x^1 has only one bidirected edge, which is with x^2. Again, by definition of hedge, if $x^1 \in W$, then it has a bidirected path to Y within $\mathcal{H}[W]$, and this path can only go through x^2. Hence, in the sequel, when there is a hedge W formed for $Q[Y]$ in $\mathcal{H}$, we will without loss of generality assume that there exists a set of variables $Z \subseteq V^{\mathcal{G}'}$ such that $W = Z^1 \cup Z^2 \cup Y$, where $Z^1 = \{z^1 | z \in Z\}$ and $Z^2 = \{z^2 | z \in Z\}$.

If part. Let $W = Z^1 \cup Z^2 \cup Y$ form a hedge for $Q[Y]$ in $\mathcal{H}$. First note that since none of the bidirected edges between Z^1 and Z^2 are removed in $\mathcal{H}$, by construction, all vertices Z are present in $\mathcal{G}$, i.e., $Z \subseteq V^{\mathcal{G}}$. Now we show that $Z \cup Y$ forms a hedge for $Q[Y]$ in $\mathcal{G}$. To this end, we prove $\mathcal{G}[Z \cup Y]$ is a district and $Z \cup Y = Anc_{\mathcal{G}[Z \cup Y]}(Y)$. First note that any vertex in Z^1 has only one bidirected edge to a vertex in Z^2. That is, if we consider the edge-induced subgraph of $\mathcal{H}[W]$ over its bidirected edges, vertices of Z^1 are leaf nodes. As a result, $Z^2 \cup Y$ must be connected in this graph. That is, $Z^2 \cup Y$ is a district in $\mathcal{H}[Z^2 \cup Y]$. This implies by construction of $\mathcal{H}$ that $\mathcal{G}[Z \cup Y]$ is a single district. With a similar reasoning, note that vertices in Z^2 have no parents. As result, $Z^1 \cup Y = Anc_{\mathcal{H}[Z^1 \cup Y]}(Y)$ (since the directed paths cannot go through Z^2). Again, by construction, the edge-induced subgraph of $\mathcal{G}[Z \cup Y]$ over its directed edges is a copy of $\mathcal{H}[Z^1 \cup Y]$. As a result, $Z \cup Y = Anc_{\mathcal{G}[Z \cup Y]}(Y)$.

Only if part. Let $Z \cup Y$ form a hedge for $Q[Y]$ in $\mathcal{G}$, where $Z \subseteq V^{\mathcal{G}} \setminus Y$. Define $Z^1 = \{z^1 | z \in Z\}$ and $Z^2 = \{z^2 | z \in Z\}$. We show that $Z^1 \cup Z^2 \cup Y$ forms a hedge for $Q[Y]$ in $\mathcal{H}$. First, by definition of hedge, $Anc_{\mathcal{G}[Z \cup Y]}(Y) = Z \cup Y$. Since the edge-induced subgraph of $\mathcal{H}[Z^1 \cup Y]$ is a copy of $\mathcal{G}[Z \cup Y]$ by construction, we know $Anc_{\mathcal{G}[Z^1 \cup Y]}(Y) = Z^1 \cup Y$. Further, each vertex $z^2 \in Z^2$ is a parent of $z^1 \in Z^1$. As a result, $Anc_{\mathcal{G}[Z^1 \cup Z^2 \cup Y]}(Y) = Z^1 \cup Z^2 \cup Y$. Now it suffices to show that $Z^1 \cup Z^2 \cup Y$ is a district in $\mathcal{H}[Z^1 \cup Z^2 \cup Y]$. By definition of hedge, $Z \cup Y$ is a district in $\mathcal{G}[Z \cup Y]$. By construction of $\mathcal{H}$, exactly the same bidirected edges (and therefore bidirected paths) exist in $\mathcal{H}[Z^2 \cup Y]$. Therefore, $Z^2 \cup Y$ is a district in $\mathcal{H}[Z^2 \cup Y]$. Now note that by construction of $\mathcal{H}'$, each vertex $z^1 \in Z^1$ has a bidirected edge to $z^2 \in Z^2$. And by definition of $\mathcal{G}$ and $\mathcal{H}$, since the vertices Z exist in $\mathcal{G}$, none of these edges are removed in $\mathcal{H}$. As a result, $Z^1 \cup Z^2 \cup Y$ is a district in $\mathcal{H}[Z^1 \cup Z^2 \cup Y]$, which completes the proof.

□

Proof of Theorem 1. A polynomial-time reduction from MCIP to the edge ID problem follows immediately from Proposition 3. MCIP is shown to be NP-hard [1]. As a result, the edge ID problem is Np-hard. $\square$

A.2 Reduction from edge ID to MCIP

Proposition 2. *There exists a polynomial-time reduction from edge ID to MCIP and vice versa.*

To prove Proposition 2, we begin with presenting a transformation $\mathcal{T}_2(\mathcal{G}, Y)$ which is in the core of reduction from edge ID to MCIP.

Suppose we want to solve the edge ID problem given ADMG $\mathcal{G} = (V^{\mathcal{G}}, E_d^{\mathcal{G}}, E_b^{\mathcal{G}})$, query $Q[Y]$, and edge weights $W_{\mathcal{G}} = \{w_e | e \in \mathcal{G}\}$. Let $X = V^{\mathcal{G}} \setminus Y$ denote the set of vertices of $\mathcal{G}$ excluding Y. We define the transformation $(\mathcal{H}, Y^{mcip}) = \mathcal{T}_2(\mathcal{G}, Y)$ where $\mathcal{H} = (V^{\mathcal{H}}, E_d^{\mathcal{H}}, E_b^{\mathcal{H}})$ is an ADMG and $Y^{mcip} \subseteq V^{\mathcal{H}}$ as follows. Note that $V^{\mathcal{H}}$ will consist of two disjoint set of vertices, namely $V_{top}^{\mathcal{H}}$ and $V_{bot}^{\mathcal{H}}$, i.e., $V^{\mathcal{H}} = V_{top}^{\mathcal{H}} \cup V_{bot}^{\mathcal{H}}$.

a. Begin with $V_{top}^{\mathcal{H}} = V_{bot}^{\mathcal{H}} = \emptyset, Y^{mcip} = \emptyset$. For any vertex $v \in V^{\mathcal{G}}$, add a vertex v to $V_{top}^{\mathcal{H}}$ with cost $C(v) = \infty$. If $v \in Y$, add v to Y^{mcip}.

b. For any directed edge $(v_i, v_j) \in E_d^{\mathcal{G}}$ with weight w_{ij}^d in $\mathcal{G}$, add a new vertex v_{ij}^d to $V_{top}^{\mathcal{H}}$, with cost $C(v_{ij}^d) = w_{ij}^d$, where

$$v_{ij}^d = \begin{cases} x_{ij}^d & \text{if } v_i, v_j \in X, \\ z_{ij}^d & \text{if } v_i \in Y \text{ or } v_j \in Y, \\ y_{ij}^d & \text{if both } v_i, v_j \in Y. \end{cases}$$

   Draw directed edges (v_i, v_{ij}^d) and (v_{ij}^d, v_j). Further, draw a bidirected edge between v_i and v_{ij}^d.

c. For any bidirected edge $\{x_i, x_j\} \in E_b^{\mathcal{G}}$ with weight w_{ij}^b, add a new vertex, x_{ij}^b to $V_{top}^{\mathcal{H}}$ with cost $C(x_{ij}^b) = w_{ij}^b$. Add two bidirected edges $\{x_i, x_{ij}^b\}$ and $\{x_j, x_{ij}^b\}$. Further, draw two directed edges (x_{ij}^b, x_i) and (x_{ij}^b, x_j) in $\mathcal{H}$.

d. For any bidirected edge $\{x_i, y_j\}$ with weight w_{ij}^b, add a new vertex z_{ij}^b to $V_{top}^{\mathcal{H}}$ with cost $C(z_{ij}^b) = w_{ij}^b$. Draw bidirected edges $\{z_{ij}^b, x_i\}$ and $\{z_{ij}^b, y_j\}$. Then draw a directed edge from z_{ij}^b to x_i.

e. For any bidirected edge between $\{y_i, y_j\} \in E_b^{\mathcal{G}}$ with weight w_{ij}^b in $\mathcal{G}$, add a new vertex, y_{ij}^b to $V_{top}^{\mathcal{H}}$ with cost $C(y_{ij}^b) = w_{ij}^b$. Draw bidirected edges $\{y_{ij}^b, y_i\}$ and $\{y_{ij}^b, y_j\}$. Further, for any $x \in X$, draw a directed edge from y_{ij}^b to x.

f. Let $y_1 \prec ... \prec y_k$ denote a topological ordering among vertices of Y. For every pair $\{y_i, y_j\}$ of vertices of Y, where $i < j$, add vertices $y_i^{ij}, y_{i+1}^{ij}, \dots, y_j^{ij}$ to $V_{bot}^{\mathcal{H}}$. Add y_j^{ij} to Y^{mcip}. Draw the directed edges (y_k, y_k^{ij}) for every $i \leq k \leq j$. Draw the directed edges (y_k^{ij}, y_i^{ij}) for every $i < k < j$, and the directed edge (y_i^{ij}, y_j^{ij}). Draw a bidirected edge between y_j and y_i^{ij}. Further, for any bidirected edge $\{y_k, y_l\} \in E_b^{\mathcal{G}}$ where $i \leq k, l \leq j$, add a new vertex y_{kl}^{ij} to $V_{bot}^{\mathcal{H}}$, draw two bidirected edges $\{y_{kl}^{ij}, y_k^{ij}\}$ and $\{y_{kl}^{ij}, y_l^{ij}\}$, and a directed edge (y_{kl}^{ij}, y_{ij}^b). The costs of the all of the vertices in $V_{bot}^{\mathcal{H}}$ are infinite.

With abuse of notation, for any bidirected edge $e_{ij}^b = \{v_i, v_j\} \in E_b^{\mathcal{G}}$ and any directed edge $e_{ij}^d = (v_i, v_j) \in E_d^{\mathcal{G}}$, we define $\mathcal{T}_2(e_{ij}^b) = v_{ij}^b$ and $\mathcal{T}_2(e_{ij}^d) = v_{ij}^d$, respectively, where $v_{ij}^b, v_{ij}^d \in V^{\mathcal{H}}$ are the vertices representing their corresponding edges.

We will utilize the following results to prove Proposition 2. More precisely, Lemmas 2 through 9 are used to prove Proposition 4, which in turn is used to prove Proposition 2.

Lemma 2. *Suppose $\mathcal{G}$ is an ADMG, Y is a set of its vertices, and $(\mathcal{H}, Y^{mcip}) = \mathcal{T}(\mathcal{G}, Y)$. Each vertex $y \in Y^{mcip}$ is a district in $\mathcal{H}$.*

Proof. It suffices to show that for every pair of $v_1, v_2 \in Y^{mcip}$ there is no bidirected edge between them in $\mathcal{H}$. Suppose first that $v_1, v_2 \in Y$. Any bidirected edge between v_1 and v_2 in $\mathcal{G}$ (if it exists) is removed in step (e) of the transformation, and none of the steps (a) through (f) add a bidirected edge between them. Otherwise, at least one of v_1, v_2, w.l.o.g. v_1, is in $Y^{mcip} \setminus Y$. Suppose w.l.o.g. that $v_1 = y_j^{ij}$. From step (f) of the transformation $\mathcal{T}$, we know that v_1 has bidirected edges only to vertices y_{kj}^{ij}, where none of them is a member of Y^{mcip}. $\square$

Lemma 3. *Suppose $\mathcal{G}$ is an ADMG, Y is a set of its vertices, and $(\mathcal{H}, Y^{mcip}) = \mathcal{T}_2(\mathcal{G}, Y)$. Suppose there is a hedge formed for $Q[y]$ in $\mathcal{H}$, where $y \in Y$. Let H denote the set of vertices of this hedge. H does not include any of the vertices added in the step (f) of the transformation. That is, $H \cap V_{bot}^{\mathcal{H}} = \emptyset$.*

Proof. Define $V_1 = \{y_{kl}^{ij} \in V_{bot}^{\mathcal{H}}, \forall i,j,k,l\}$, and $V_2 = V_{bot}^{\mathcal{H}} \setminus V_1$. By construction of $\mathcal{H}$, the vertices of V_2 have directed edges only to vertices in V_2. Therefore, for each vertex $v \in V_2$, we have $v \notin Anc_{\mathcal{H}[H]}(y)$. As a result, $V_2 \cap H = \emptyset$, since by definition of hedge, any vertex of H is an ancestor of y in $\mathcal{H}[H]$. Now, consider an arbitrary vertex $v \in V_1$. By construction of $\mathcal{H}$, if there exists a bidirected edge $\{v, v'\} \in E_b^{\mathcal{H}}$, we must have that $v' \in V_2$. Therefore, if $v \in H$, there must be at least one vertex $v' \in V_2 \cap H$. Since we proved $V_2 \cap H = \emptyset$, v cannot be in H. Consequently, $V_1 \cap H = \emptyset$.

$\square$

Lemma 4. *Suppose $\mathcal{G}$ is an ADMG, Y is a set of its vertices, and $(\mathcal{H}, Y^{mcip}) = \mathcal{T}(\mathcal{G}, Y)$. Suppose there is a hedge formed for $Q[y_j^{ij}]$ in $\mathcal{H}$, where $y_i, y_j \in Y$ and y_j^{ij} is the vertex corresponding to the pair (y_i, y_j) added in step (f) of the transform $\mathcal{T}$. Let H denote the set of vertices of this hedge. If $v \in H \cap V_{bot}^{\mathcal{H}}$, then v has the superscript ij, that is, v is either one of the vertices y_k^{ij}, or one of the vertices y_{kl}^{ij}, where $i \leq k, l \leq j$. In the latter case, $y_{kl}^b \in H$.*

Proof. Define $V_1 = \{y_{kl}^{mn} \in V_{bot}^{\mathcal{H}}, \forall m,n,k,l\}$, and $V_2 = V_{bot}^{\mathcal{H}} \setminus V_1$. Suppose $V_1^* = \{v_{kl}^{ij} \in V_{bot}^{\mathcal{H}}, \forall k,l\}$ and $V_2^* = \{v_k^{ij} \in V_{bot}^{\mathcal{H}}, \forall k\}$. Also define $V_1' = V_1 \setminus V_1^*$, $V_2' = V_2 \setminus V_2^*$. For the first part of the claim, it suffices to show that $V_1' \cap H = \emptyset, V_2' \cap H = \emptyset$. By construction of $\mathcal{H}$, the vertices of $V_2^{'}$ do not have any child out of $V_2^{'}$. Therefore, $V_2' \cap Anc_{\mathcal{H}[H]}(y_j^{ij}) = \emptyset$. This implies that $V_2^{'} \cap H = \emptyset$. Now let $v_1^{i^{'}j^{'}}$ be an arbitrary vertex in $V_1^{'}$. By construction of $\mathcal{H}$, $v_1^{i^{'}j^{'}}$ has bidirected edges only to vertices of V_2'. This implies that if $v_1^{i^{'}j^{'}} \in H$, there must be at least one vertex of V_2' in H which is in contradiction with $V_2' \cap H = \emptyset$. Therefore, $v_1^{i^{'}j^{'}} \notin H$. Since $v_1^{i^{'}j^{'}}$ is an arbitrary vertex in V_1', we conclude $V_1' \cap H = \emptyset$.

Now, we prove that if $v \in H$ is one of the vertices y_{kl}^{ij}, we have $y_{kl}^b \in H$. Since $y_{kl}^{ij} \in H$, there exists a directed path from y_{kl}^{ij} to y_j^{ij} in $\mathcal{H}[H]$. Since y_{kl}^b is the only child of y_{kl}^{ij}, the aforementioned path passes through y_{kl}^b. Therefore, $y_{kl}^b \in H$.

$\square$

Lemma 5. *Suppose $\mathcal{G}' = (V^{\mathcal{G}'}, E_d^{\mathcal{G}'}, E_b^{\mathcal{G}'})$ is an ADMG, $Y \subseteq V^{\mathcal{G}'}$ is a set of its vertices, and $(\mathcal{H}', Y^{mcip}) = \mathcal{T}(\mathcal{G}', Y)$. Let $E_d'' \subseteq E_d^{\mathcal{G}'}$ and $E_b'' \subseteq E_b^{\mathcal{G}'}$ be arbitrary edges of $\mathcal{G}$, and define $E_d^{\mathcal{G}} = E_d^{\mathcal{G}'} \setminus E_d''$, $E_b^{\mathcal{G}} = E_b^{\mathcal{G}'} \setminus E_b''$. Define $\mathcal{G} = (V^{\mathcal{G}}, E_d^{\mathcal{G}}, E_b^{\mathcal{G}})$ and $\mathcal{H} = \mathcal{H}'[V^{\mathcal{H}'} \setminus V']$, where $V^{\mathcal{G}} = V^{\mathcal{G}'}$ and $V' = \{v \in V^{\mathcal{H}'} | \exists e \in E_b'' \cup E_d'', v = \mathcal{T}_2(e)\}$. Suppose there is a hedge formed for $Q[y_j^{ij}]$ in $\mathcal{H}$ for some i,j. Let H denote the set of vertices of this hedge in $\mathcal{H}$. The set of vertices $Y^* = \{y_k | y_k^{ij} \in H\}$ is a district in $\mathcal{G}[Y]$. Moreover, $H_{top} = Anc_{\mathcal{H}[H_{top}]}(Y^*)$, where $H_{top} = H \cap V_{top}^{\mathcal{H}}$.*

Proof. First we prove that Y^* is a district in $\mathcal{G}[Y]$. Consider an arbitrary vertex y_k^{ij} in H. By definition of hedge, there exists a bidirected path, p_1, between y_k^{ij} and y_j^{ij} in $\mathcal{H}[H]$. Let Y^{ij} denotes the set of vertices in H such that their superscript is ij. Lemma 4 implies that $H \subseteq V_{top}^{\mathcal{H}} \cup Y^{ij}$. Furthermore, by construction of $\mathcal{H}$, there is only one bidirected edge between Y^{ij} and $H \setminus Y^{ij}$, which is $\{y_j, y_i^{ij}\}$. Therefore, all of the vertices on the path p_1 are in Y^{ij}. Now, we define $Y_1^{'} = \{y_k | y_k^{ij} \in p_1\}$ and

$Y_2' = \{y_{kl}^b | y_{kl}^{ij} \in p_1\}$, i.e., the $V_{top}^{\mathcal{H}}$ counterparts of the vertices in p_1. Since the vertices on p_1 are in H, $Y_1' \subseteq Y^*$. From Lemma 4, we know that if $y_{kl}^{ij} \in H$, then, $y_{kl}^b \in H$. It implies that $Y_2^{'} \subseteq H$. As a result, Y_1' and Y_2' are both vertices of $\mathcal{H}$. Now if we replace all the vertices in p_1 with their corresponding counterpart in $Y_1' \cup Y_2'$, we arrive at a bidirected path p_2 between y_k and y_j in $\mathcal{H}[Y_1' \cup Y_2']$ (as by construction the same edges exist in $V_{top}^{\mathcal{H}}$). By definition of $\mathcal{G}$ and $\mathcal{H}$, if a vertex y_{kl}^b exists in $\mathcal{H}$, the corresponding edge $\{y_k, y_l\}$ exists in $\mathcal{G}$. As a result, a bidirected path between y_k and y_l exists in $\mathcal{G}[Y_1']$. Noting that y_k is an arbitrary vertex in Y^* and $Y_1' \subseteq Y^*$, this implies that all of the vertices of Y^* are in the same district as y_j in $\mathcal{G}[Y^*]$, which completes the proof.

Next, we prove that $H_{top} = Anc_{\mathcal{H}[H_{top}]}(Y^*)$. To this end, it suffices to show that there is a directed path form an arbitrary vertex $v \in H_{top}$ to Y^* in $\mathcal{H}[H_{top}]$. Since H forms a hedge for $Q[y_j^{ij}]$ in $\mathcal{H}$, there exists a directed path from v to y_j^{ij} in $\mathcal{H}[H]$. This path must go through the only parent of y_j^{ij}, which is y_i^{ij}. Then, the last vertex on the path is one of the parents of y_i^{ij}. If this parent is y_i, we are done as we have a directed path from v to y_i, where $y_i \in Y^*$ and it has no ancestors in $H \setminus H_{top}$. Otherwise, let this parent be y_k^{ij} for some $i < k < j$. Now the last vertex on the path before y_k^{ij} must be y_k, which is the only parent of y_k^{ij}. Note that by definition of Y^*, $y_k \in Y^*$. Therefore, v has a directed path to Y^* in $\mathcal{H}[H_{top}]$. □

Lemma 6. *Suppose $\mathcal{G} = (V^{\mathcal{G}}, E_d^{\mathcal{G}}, E_b^{\mathcal{G}})$ is an ADMG, Y is a set of its vertices, and $(\mathcal{H}, Y^{mcip}) = \mathcal{T}_2(\mathcal{G}, Y)$. Suppose there is a hedge formed for $Q[y]$ in $\mathcal{H}$ for some $y \in Y^{mcip}$. Let H denote the set of vertices of this hedge. Then $H \cap X \neq \emptyset$, where $X = V^{\mathcal{G}} \setminus Y$.*

Proof. Since H forms a hedge for $Q[y]$ in $\mathcal{H}$, there exists a vertex $h \in H$ such that $\{y, h\} \in E_b^{\mathcal{H}}$. There are two possibilities for $y \in Y^{mcip}$:

- $y = y_i \in Y$. From Lemma 4 we have $h \notin V_{bot}^{\mathcal{H}}$. Therefore, by construction of $\mathcal{H}$, $h = y_{ij}^b$ for some j.
- $y = y_j^{ij} \in V_{bot}^{\mathcal{H}}$. By construction of $\mathcal{H}$, $h = y_{kj}^{ij}$ for some k. Vertex h must have a directed path to y in H by definition of hedge, which must go through the only child of h, i.e., y_{kl}^b.

In both cases, we showed that there exists a vertex $v = y_{ij}^b \in H$ for some i, j. By definition of hedge, there is a bidirected path, p, from v to y in $\mathcal{H}$ because $v \in Anc_{\mathcal{H}}(y)$. Since all of the children of v are in X, there is at least one vertex in X on path p. Therefore, H includes at least one vertex of X.

□

Lemma 7. *[Inverse transform preserves hedges.] Suppose $\mathcal{G}' = (V^{\mathcal{G}'}, E_d^{\mathcal{G}'}, E_b^{\mathcal{G}'})$ is an ADMG, $Y \subseteq V^{\mathcal{G}'}$ is a set of its vertices, and $(\mathcal{H}', Y^{mcip}) = \mathcal{T}_2(\mathcal{G}', Y)$. Let $E_d'' \subseteq E_d^{\mathcal{G}'}$ and $E_b'' \subseteq E_b^{\mathcal{G}'}$ be arbitrary edges of $\mathcal{G}$, and define $E_d^{\mathcal{G}} = E_d^{\mathcal{G}'} \setminus E_d''$, $E_b^{\mathcal{G}} = E_b^{\mathcal{G}'} \setminus E_b''$. Define $\mathcal{G} = (V^{\mathcal{G}}, E_d^{\mathcal{G}}, E_b^{\mathcal{G}})$ and $\mathcal{H} = \mathcal{H}'[V^{\mathcal{H}'} \setminus V']$, where $V^{\mathcal{G}} = V^{\mathcal{G}'}$ and $V' = \{v \in V^{\mathcal{H}'} | \exists e \in E_b'' \cup E_d'', v = \mathcal{T}_2(e)\}$. Let $W \subseteq V_{top}^{\mathcal{H}}$ be a set of vertices of $\mathcal{H}$. Let $W_s \subseteq W \cap V^{\mathcal{G}}$ be a subset of W such that W_s are vertices of $\mathcal{G}$ as well. Consider the inverse transform of $\mathcal{H}[W]$ in the ADMG $\mathcal{G}$, i.e., for any $v = v_{ij}^b \in W$, delete v and all edges incident to it and draw a bidirected edge between v_i and v_j, and for any $v = v_{ij}^d$, delete v and all edges incident to it and draw a directed edge from v_i to v_j. Let the resulting subgraph (which is a subgraph of $\mathcal{G}$) be denoted by $\mathcal{G}[W^{-1}]$ with the set of vertices $W^{-1} \subseteq V^{\mathcal{G}}$. If $Anc_{\mathcal{H}[W]}(W_s) = W$, then $Anc_{\mathcal{G}[W^{-1}]}(W_s) = W^{-1}$. Moreover, if W is a district in $\mathcal{H}[W]$, then W^{-1} is a district in $\mathcal{G}[W^{-1}]$.*

Proof. First, we show that if $Anc_{\mathcal{H}[W]}(W_s) = W$, then $Anc_{\mathcal{G}[W^{-1}]}(W_s) = W^{-1}$. Let v be an arbitrary vertex in W^{-1}. Vertex v is in W because $W^{-1} \subseteq W$. Since $v \in W$ and $v \in Anc_{\mathcal{H}[W]}(W_s)$, v has a directed path $v \to \dots v_i \to v_{ij}^d \to v_j \dots \to w$, denoted by l, to a vertex $w \in W_s$ in $\mathcal{H}[W]$. For each vertex v_{ij}^d on path l, we have $v_i, v_j \in \mathcal{G}[W^{-1}]$ and since $v_{ij}^d \in V^{\mathcal{H}}$, by definition of $\mathcal{G}$ and $\mathcal{H}$, there exists $(v_i, v_j) \in E_d^{\mathcal{G}}$ s.t. $i \prec j$, and consequently, $(v_i, v_j) \in E_d^{\mathcal{G}[W^{-1}]}$. Therefore, there exists a directed path from v to w in $\mathcal{G}[W^{-1}]$. Noting that v is an arbitrary vertex in W^{-1}, we conclude that $Anc_{\mathcal{G}[W^{-1}]}(W_s) = W^{-1}$.

Now, we prove that if W is a district in $\mathcal{H}[W]$, then W^{-1} is a district in $\mathcal{G}[W^{-1}]$. Consider two vertices $v_1, v_2 \in W^{-1}$. Since $v_1, v_2 \in W$ and W is a district, there exists a bidirected path $v_1 \leftrightarrow \dots v_i \leftrightarrow v^b_{ij} \leftrightarrow v_j \dots \leftrightarrow v_2$, denoted by p, between v_1 and v_2 in $\mathcal{H}[W]$. Each vertex v^b_{ij} on path p is in $\mathcal{H}$ and $v_i, v_j \in \mathcal{G}[W^{-1}]$. By definition of $\mathcal{G}$ and $\mathcal{H}$, we have $\{v_i, v_j\} \in E^{\mathcal{G}}_b$. Therefore, $\{v_i, v_j\} \in E^{\mathcal{G}[W^{-1}]}_b$. Then, there is a bidirected path between v_1 and v_2 in $\mathcal{G}[W^{-1}]$. Since v_1 and v_2 are two arbitrary vertices in W^{-1}, it implies that W^{-1} is a district in $\mathcal{G}[W^{-1}]$. □

Lemma 8. *[Transform preserves hedges.] Suppose $\mathcal{G}' = (V^{\mathcal{G}'}, E^{\mathcal{G}'}_d, E^{\mathcal{G}'}_b)$ is an ADMG, $Y \subseteq V^{\mathcal{G}'}$ is a set of its vertices, and $(\mathcal{H}', Y^{mcip}) = \mathcal{T}_2(\mathcal{G}', Y)$. Let $E''_d \subseteq E^{\mathcal{G}'}_d$ and $E''_b \subseteq E^{\mathcal{G}'}_b$ be arbitrary edges of $\mathcal{G}$, and define $E^{\mathcal{G}}_d = E^{\mathcal{G}'}_d \setminus E''_d$, $E^{\mathcal{G}}_b = E^{\mathcal{G}'}_b \setminus E''_b$. Define $\mathcal{G} = (V^{\mathcal{G}}, E^{\mathcal{G}}_d, E^{\mathcal{G}}_b)$ and $\mathcal{H} = \mathcal{H}'[V^{\mathcal{H}'} \setminus V']$, where $V^{\mathcal{G}} = V^{\mathcal{G}'}$ and $V' = \{v \in V^{\mathcal{H}'} | \exists e \in E''_b \cup E''_d, v = \mathcal{T}_2(e)\}$. Let $W \subseteq V^{\mathcal{G}}$ be a set of vertices of $\mathcal{G}$ such that $W \setminus Y \neq \emptyset$. Let $W_s \subseteq W$ be a subset of W. Let the transformed graph of $\mathcal{G}[W]$ under $\mathcal{T}_2$ be denoted by $\mathcal{H}''$, where $\mathcal{H}'' \subseteq \mathcal{H}$. Define $W^* = V^{\mathcal{H}''}_{top}$. If $Anc_{\mathcal{G}[W]}(W_s) = W$, then $Anc_{\mathcal{H}[W^*]}(W_s) = W^*$. Moreover, if W is a district in $\mathcal{G}[W]$, then W^* is a district in $\mathcal{H}[W^*]$.*

Proof. First, we prove that if $Anc_{\mathcal{G}[W]}(W_s) = W$, then $Anc_{\mathcal{H}[W^*]}(W_s) = W^*$. Take an arbitrary vertex $v \in W^*$. There are two possibilities for v:

- $v \in W$. That is, vertex v is in $\mathcal{G}[W]$.
- $v \notin W$. This implies that v represents an edge e between two vertices v_i and v_j in $\mathcal{G}[W]$. There are three possibilities for e:
  - $e = (v_i, v_j)$. By construction of $\mathcal{H}$, v is parent of v_j in $\mathcal{H}[W^*]$, where v_j is a vertex of $\mathcal{G}[W]$.
  - $e = \{v_i, v_j\}$ and $v_i \in X$ or $v_j \in X$. In this case, v is parent of at least one of v_i and v_j in $\mathcal{H}[W^*]$, w.l.o.g. v_i, where v_i is a vertex of $\mathcal{G}[W]$.
  - $e = \{v_i, v_j\}$ and $v_i, v_j \in Y$. By construction of $\mathcal{H}$, v is parent of all vertices in $V^{\mathcal{G}} \setminus Y$. Since $W \setminus Y \neq \emptyset$, there exists a vertex x in $\mathcal{G}[W]$ such that v is a parent of x.

  In all three cases above, we proved that there exists a vertex $x \in W$ such that v is a parent of x.

Therefore, we showed that any vertex $v \in W^*$ either is itself a vertex in W or is a parent of a vertex in W. As a result, it suffices to show that every $w \in W$ has a directed path to W_s in $\mathcal{H}[W^*]$. We know that w has a directed path to W_s in $\mathcal{G}[W]$ such as p. Take an arbitrary pair of consecutive vertices on this path, such as v_1 and v_2. The directed edge (v_1, v_2) exists in $\mathcal{G}[W]$. As a result, the directed path $v_1 \rightarrow v^d_{12} \rightarrow v_2$ exists in $\mathcal{H}[W^*]$. Starting at w and repeating this argument for every pair of consecutive vertices on p, we conclude that there exists a directed path from w to W_s, which completes the proof.

Now, we show that if W is a district in $\mathcal{G}[W]$, then W^* is a district in $\mathcal{H}[W^*]$. Take an arbitrary vertex $v \in W^*$. There are two possibilities for v:

- $v \in W$. That is, v is a vertex in $\mathcal{G}[W]$.
- $v \notin W$. In this case, at least one of the vertices v represents an edge e between two vertices v_i and v_j in $\mathcal{G}[W]$. By construction of $\mathcal{H}$, v is connected to at least one of v_i or v_j, w.l.o.g. v_i, by a bidirected edge, where $v_i \in W$.

We showed that any vertex $v \in W^*$ either is in W, or is connected to a vertex in W through a bidirected edge. Therefore, it suffices to show that for any two vertices $w_1, w_2 \in W$ there exists a bidirected path between w_1 and w_2 in $\mathcal{H}[W^*]$. Since $w_1, w_2 \in W$, there is a bidirected path, p, between w_1 and w_2 in $\mathcal{G}[W]$. Take an arbitrary pair of consecutive vertices on this path, such as v_1 and v_2. The bidirected edge $\{v_1, v_2\}$ exists in $\mathcal{G}[W]$. As a result, the bidirected path $v_1 \leftrightarrow v^b_{12} \leftrightarrow v_2$

exists in $\mathcal{H}[W^*]$. Starting at w and repeating this argument for every pair of consecutive vertices on p, we conclude that there exists a bidirected path from w_1 to w_2, which completes the proof. $\square$

Lemma 9. *Suppose $\mathcal{G}$ is an ADMG, and Y is a subset of its vertices. Also let Y^* be a district in $\mathcal{G}[Y]$. If the set of vertices H form a hedge for $Q[Y^*]$, then $H \setminus Y \neq \emptyset$.*

Proof. Assume by contradiction $H \setminus Y = \emptyset$, i.e., $H \subseteq Y$. By definition of hedge, we know $H \setminus Y^* \neq \emptyset$. Take an arbitrary vertex $v \in H \setminus Y^*$. Furthermore, $v \in Y \setminus Y^*$ because $H \subseteq Y$. Since H forms a hedge for $Q[Y^*]$, H is a district in $\mathcal{G}[H]$. Therefore, there exists a bidirected path between v and a vertex $y^* \in Y^*$ in $Q[Y]$ which is in contradiction with the assumption that Y^* is a district in $\mathcal{G}[Y]$. $\square$

Proposition 4. *Suppose $\mathcal{G}' = (V^{\mathcal{G}'}, E_d^{\mathcal{G}'}, E_b^{\mathcal{G}'})$ is an ADMG, $Y \subseteq V^{\mathcal{G}'}$ is a set of its vertices, and $(\mathcal{H}', Y^{mcip}) = \mathcal{T}_2(\mathcal{G}', Y)$. Let $E_d'' \subseteq E_d^{\mathcal{G}'}$ and $E_b'' \subseteq E_b^{\mathcal{G}'}$ be arbitrary edges of $\mathcal{G}$, and define $E_d^{\mathcal{G}} = E_d^{\mathcal{G}'} \setminus E_d''$, $E_b^{\mathcal{G}} = E_b^{\mathcal{G}'} \setminus E_b''$. $Q[Y]$ is identifiable in $\mathcal{G} = (V^{\mathcal{G}}, E_d^{\mathcal{G}}, E_b^{\mathcal{G}})$ if and only if $Q[Y^{mcip}]$ is identifiable in $\mathcal{H} = \mathcal{H}'[V^{\mathcal{H}'} \setminus V']$, where $V^{\mathcal{G}} = V^{\mathcal{G}'}$ and $V' = \{v \in V^{\mathcal{H}'} | \exists e \in E_b'' \cup E_d'', v = \mathcal{T}_2(e)\}$.*

Proof. We prove the contrapositive, i.e., $Q[Y]$ is not identifiable in $\mathcal{G}$ iff $Q[Y^{mcip}]$ is not identifiable in $\mathcal{H}$.

If part. Suppose $Q[Y^{mcip}]$ is not identifiable in $\mathcal{H}$. That is, there exists a hedge formed for $Q[Y^{mcip}]$ in $\mathcal{H}$. From Lemma 2, this hedge is formed for $Q[y']$ for some $y' \in Y^{mcip}$. Denote the set of vertices of this hedge by H. We consider two possibilities separately:

- $y' = y_i$, where $y_i \in Y$. From Lemma 3, $H \subseteq V_{top}^{\mathcal{H}}$. Taking $W = H$ in Lemma 7, W^{-1} is a set of vertices in $\mathcal{G}$ such that $Anc_{\mathcal{G}[W^{-1}]}(y) = W^{-1}$, and W^{-1} is a district in $\mathcal{G}$. Now take Y^* to be the district of $\mathcal{G}[Y]$ that includes y_i. By definition of hedge, $\mathcal{G}[W^{-1} \cup Y^*]$ forms a hedge for $Q[Y^*]$ in $\mathcal{G}$. Note that from Lemma 6, $W^{-1} \setminus Y \neq \emptyset$. As a result, $Q[Y]$ is not identifiable in $\mathcal{G}$.
- $y' = y_j^{ij}$, where $y_i, y_j \in Y$ and y' is one of the vertices added to $\mathcal{H}$ in the last step of the transformation $\mathcal{T}$ (step (f)). Define the set $Y^* = \{y_k | y_k^{ij} \in H\}$. From Lemma 5, Y^* is a district in $\mathcal{G}$, and therefore a district in $\mathcal{G}[Y]$. As a result, it suffices to show that there exists a hedge formed for $Q[Y^*]$ in $\mathcal{G}$. Now define $H_{top} = H \cap V_{top}^{\mathcal{H}}$. By definition of hedge, H is a district in $\mathcal{H}[H]$, i.e., it is connected over its bidirected edges. By construction of $\mathcal{H}$, there is only one bidirected edge between the vertices in H_{top} and $H \setminus H_{top}$, which is the bidirected edge between y_j and y_i^{ij}. Therefore, this edge is a cut set that partitions the graph $\mathcal{H}[H]$ into two connected components $\mathcal{H}[H_{top}]$ and $\mathcal{H}[H \setminus H_{top}]$. That is, $\mathcal{H}[H_{top}]$ is connected over its bidirected edges and therefore H_{top} is a district in $\mathcal{H}[H_{top}]$. Further, from Lemma 5, $H_{top} = Anc_{\mathcal{H}[H_{top}]}(Y^*)$. Noting that $H_{top} \subseteq V_{top}^{\mathcal{H}}$, taking $W = H_{top}$ in Lemma 7, W^{-1} is a district in $\mathcal{G}$ and $Anc_{\mathcal{G}[W^{-1}]}(Y^*) = W^{-1}$. Note that from Lemma 6, $W^{-1} \setminus Y \neq \emptyset$. Therefore, the set of vertices W^{-1} form a hedge for $Q[Y^*]$ in $\mathcal{G}$. Hence, $Q[Y]$ is not identifiable in $\mathcal{G}$.

Only if part. Suppose $Q[Y]$ is not identifiable in $\mathcal{G}$. It implies that there exists a district of $\mathcal{G}[Y]$ such as Y^* such that there is a hedge formed for $Q[Y^*]$ in $\mathcal{G}$. Let H denote the set of vertices of this hedge. From Lemma 9, $H \setminus Y \neq \emptyset$. Define W^* as in Lemma 8, that is the transform $\mathcal{T}(\mathcal{G}[H], Y^*)$ without step (f) (only on the vertices of $V_{top}^{\mathcal{H}}$). Note that $Y^* \subseteq W^*$. We consider the following two cases separately:

- $Y^* = \{y\}$, that is, Y^* is a single vertex. From Lemma 8, W^* is a district in $\mathcal{H}[W^*]$, and $Anc_{\mathcal{H}[W^*]}(y) = W^*$. By definition of hedge, the vertices W^* form a hedge for $Q[y]$ in $\mathcal{H}$. Note that $y \in Y^{mcip}$, and from Lemma 2 it is a district of $\mathcal{H}[Y^{mcip}]$. As a result, $Q[Y^{mcip}]$ is not identifiable in $\mathcal{H}$.

- $|Y^*| \geq 2$. Let y_i and y_j be the first and the last vertices of Y^* in the topological order. Define $Y^{ij*} = \{y_k^{ij} | y_k \in Y^*\} \cup \{y_{kl}^{ij} | y_k, y_l \in Y^*\}$. Y^{ij*} are the vertices in $V_{bot}^{\mathcal{H}}$ with superscript ij corresponding to the vertices in Y^*. Note that $y_i^{ij}, y_j^{ij} \in Y^{ij*}$, since $y_i, y_j \in Y^*$. Since $y_j^{ij} \in Y^{mcip}$ and from Lemma 2 y_j^{ij} is a district in $\mathcal{H}[Y^{mcip}]$, it suffices to show that there is a hedge formed for y_j^{ij} in $\mathcal{H}$. We show that the vertices $W = W^* \cup Y^{ij*}$ form a hedge for y_j^{ij} in $\mathcal{H}$. From Lemma 8, $Anc_{\mathcal{H}[W^*]}(Y^*) = W^*$, that is, all of the vertices in W^* are ancestors of Y^* in $\mathcal{H}[W^*]$, and therefore in $\mathcal{H}[W]$. Also, the vertices y_{kl}^{ij} in Y^{ij*} have a direct edge to their corresponding vertex in W^*, i.e., y_{kl}^b, and therefore are ancestors of Y^* in $\mathcal{H}[W]$ as well. Further, each vertex in Y^* such as y_k is a parent of y_k^{ij}, which is in turn a parent of y_i^{ij} (or is y_i^{ij} itself if $k = i$.) Finally, y_i^{ij} has a directed edge to y_j^{ij} by construction. As a result, all of the vertices W have a direct path to y_j^{ij} in $\mathcal{H}[W]$. That is, $Anc_{\mathcal{H}[W]}(y_j^{ij}) = W$. It now remains to show that W is a district in $\mathcal{H}[W]$. From Lemma 8, W^* is a district in $\mathcal{H}[W^*]$. As a result, the vertices W^* are connected through bidirected edges in $\mathcal{H}[W]$. There is a bidirected edge between y_j and y_i^{ij} by construction of $\mathcal{H}$. It suffices to show that for any $v \in Y^{ij*}$, there exists a bidirected path between v and y_i^{ij} in $\mathcal{H}[W]$. A vertex $y_{kl}^{ij} \in Y^{ij*}$ (with double subscript, which are due to the bidirected edges among Y^*) has bidirected edges to y_k^{ij} and y_l^{ij}, which are both in Y^{ij*} by definition. Now take an arbitrary vertex $y_k^{ij} \in Y^{ij*}$ (with single subscript, due to vertices in Y^*). We know $y_k \in Y^*$, as $y_k^{ij} \in Y^{ij*}$, by definition of Y^{ij*}. Y^* is a district in $\mathcal{G}[Y^*]$. That is, there exists a bidirected path from y_k to y_i in $\mathcal{G}[Y^*]$. From Lemma 8 by taking $W = Y^*$, there is a bidirected path p from y_k to y_i in $\mathcal{H}[Y^* \cup \{y_{lm} | y_l, y_m \in Y^*\}]$. By construction of $\mathcal{H}$, if we replace each vertex v on p by v^{ij}, we achieve a bidirected path p' with vertices in Y^{ij*} from y_k^{ij} to y_i^{ij}, which completes the proof.

□

Proof of Proposition 2. The reduction from the edge ID problem to MCIP was shown through the proof of Proposition 4. The opposite direction is an immediate corollary of Proposition 3. □

Corollary 2. *The edge ID problem and MCIP are equivalent.*

B Maximal Hedge

Algorithm 3 Maximal Hedge.

```
function MH(𝒢, Y)
    Initialize M ← ∅
    for Y_i in districts of 𝒢[Y] do
        M ← M ∪ HHull(𝒢, Y_i)
    return 𝒢[M]

function HHULL(𝒢, Y_i)
    Initialize H ← V^𝒢
    while True do
        C ← connected component (district) of Y_i via bidirected edges in 𝒢[H]
        A ← ancestors of Y_i in 𝒢[C]
        if C ≠ A then
            H ← A
        else
            break
    return H
```

Herein, we present the algorithm for recovering the maximal hedge formed for $Q[Y]$ in a given ADMG $\mathcal{G}$ (see Definition 5). Maximal hedge was initially defined in [1] under the name *hedge hull*.

$$z \xrightarrow{p} x \xrightarrow{q} y$$

Figure 5: An example where the expert is aware that there is no causal path from z to y, e.g., because $z \perp\!\!\!\perp y$ with high confidence.

We adopt the same definition, and when $\mathcal{G}[Y]$ comprises several districts, we define the maximal hedge as the union of the hedge hulls formed for each district of $\mathcal{G}[Y]$. As a result, the complete procedure of recovering the maximal hedge for a query $Q[Y]$, summarized in Algorithm 3, finds the maximal hedge formed for each district Y_i of $\mathcal{G}[Y]$ and returns the union of them. This procedure is used as a subroutine **MH** in Algorithm 1. The function **HHull** is in fact Algorithm 1 borrowed from [1]. This function is proven to recover the union of all hedges formed for Y_i, where Y_i is one of the districts of $\mathcal{G}[Y]$ (see Lemma 6 of [1]).

C Generalizing Assumption 1

Lemma 1 states the equivalence of Problems 1 and 2 with the edge ID problem under Assumption 1. However, as mentioned in the main text, this equivalence holds in the more general setting where we allow for perfect negative correlations among edges. As an example, consider the graph of Figure 5. Suppose that the performed statistical independence tests show that the two variables z and y are independent of each other with high confidence. As a result, the expert believes that the edges (z, x) and (x, y) must not exist simultaneously, as otherwise the causal path from z to y would make them dependent. In such cases, the belief of the expert can be modeled as probabilities p and q assigned to the existence of the edges (z, x) and (x, y), respectively, as well as a perfect negative correlation between them.

Note that the aforementioned constraint, i.e., that the edges do not exist simultaneously, can be specified for any number of edges, not limited to two edges only. For instance, the expert might believe at least one of the edges along a causal path of length n must not exist in the true ADMG describing the system. This belief can be modeled as an extra constraint in the optimization of Equations 2 and 3. We show that with the specification of such negative correlations, Problems 1 and 2 can still be cast as an instance of the edge ID problem. Therefore, the results presented in this work are valid in this setting as well.

Assumption 2. *The edges in $\mathcal{G}$ are assigned probabilities $p_e, \forall e \in \mathcal{G}$, and perfect negative correlations are defined among subsets of edges. More precisely, for any subset $E \subseteq E_d^{\mathcal{G}} \cup E_b^{\mathcal{G}}$, there is either 1) no constraint (mutually independent), or 2) the constraint that at least one of the edges in E must not exist in the true ADMG (perfect negative correlation).*

Proposition 5. *Under Assumption 2, there exists a reduction from Problems 1 and 2 to the edge ID problem and vice versa with the time complexity in the order of $\mathcal{O}(|C| \cdot |V^{\mathcal{G}}| + |E_d^{\mathcal{G}} \cup E_b^{\mathcal{G}}|)$, where C is the set of perfect correlation constraints.*

Proof. First note that we proved the equivalence of Problems 1 and 2 with the edge ID problem without the perfect correlation constraints in Lemma 1. As a result, under assumption 2, i.e., by adding the perfect correlation constraints, Problems 1 and 2 are equivalent to a modified edge ID problem with those constraints. But we claim that there exists and instance of the original unconstrained edge ID problem which is equivalent to these problems. To see this, first note that we know from Corollary 2 that the edge ID problem is equivalent to MCIP. Therefore, it suffices to show that there exists an instance of MCIP which is equivalent to the constrained edge ID mentioned above. To this end, consider the transform $\mathcal{T}_2(\mathcal{G}, Y)$ introduced in Section A.2. This transformation maps an instance of the edge ID problem to an instance of MCIP. Applying this transformation to the constrained edge ID problem, we can map the constrained edge ID to an instance of MCIP with extra constraints, with transforming the constraints as well. That is, if for instance, there is a perfect negative correlation among the edges e_1, e_2 in $\mathcal{G}$, this constraint is mapped to a negative perfect correlation on the corresponding vertices in $\mathcal{H}$, namely $\mathcal{T}_2(e_1), \mathcal{T}_2(e_2)$. In words, this constraint would be that at least one of $\mathcal{T}_2(e_1)$ and $\mathcal{T}_2(e_2)$ must be intervened upon. We show that such constraints can be integrated into the original definition of MCIP.

Suppose we have an MCIP problem in ADMG $\mathcal{G}$ with query $Q[Y]$, with the extra constraint that at least one of the vertices $X \subseteq V^{\mathcal{G}}$ must be intervened upon. Consider the example of $X =$

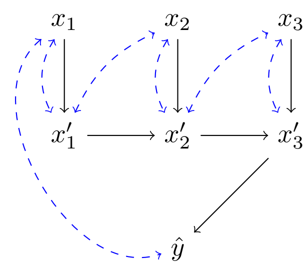


Figure 6: Integrating the perfect negative correlation constraint into MCIP.

$\{x_1, x_2, x_3\}$ in Figure 6. We build a new ADMG $\mathcal{G}'$ by adding vertices $\{x'|x \in X\}$, i.e., a new vertex corresponding to each vertex in X, along with an auxiliary vertex $\hat{y}$ to $\mathcal{G}$. We fix a random ordering over the vertices of X, and denote the set of vertices of X as $x_1, ..., x_m$. We add the directed edges (x_i, x'_i) to $\mathcal{G}'$, as well as the bidirected edges $\{x_i, x'_i\}$. Further, we draw directed edges (x'_i, x'_{i+1}) for every $1 \leq i < m$. Finally, we draw the directed edge $(x'_m, \hat{y})$ and the bidirected edge $\{x_1, \hat{y}\}$. Refer to the graph in Figure 6 for an example with $X = \{x_1, x_2, x_3\}$. Note that the set $X \cup X' \cup \{\hat{y}\}$ forms a hedge for $Q[\hat{y}]$, where $X' = \{x'|x \in X\}$ Now it suffices to set the cost of intervention on vertices of X' to infinity, and consider MCIP for the query $Q[Y \cup \{\hat{y}\}]$ in $\mathcal{G}'$. It is straightforward to see that the objective of this problem would be to find the minimum cost intervention for identification of $Q[Y]$, with the constraint that at least one of the vertices in X must be intervened on. Note that as soon as one vertex in X gets intervened upon, there is no hedge left for $Q[\hat{y}]$. Also it is noteworthy that adding this structure does not add any new hedges formed for $Q[Y]$, since the structure only includes new descendants for X which have no directed paths to Y. Also note that the vertices X' and $\hat{y}$ are specific to the very constraint corresponding to the set of vertices X. For any constraint, we add such a structure to $\mathcal{G}$. The number of vertices (and therefore the time complexity) is at most in the order $\mathcal{O}(|C| \cdot |V^{\mathcal{G}}|)$, where C is the set of constraints.

□

D Heuristic Algorithms

Algorithm 2 was devised considering the fact that every hedge formed for $Q[Y]$ must include a vertex that has a bidirected edge to Y. As mentioned in Section 4.2, an analogous approach, summarized in Algorithm 4, uses the fact that any hedge formed for $Q[Y]$ must include a parent of Y.

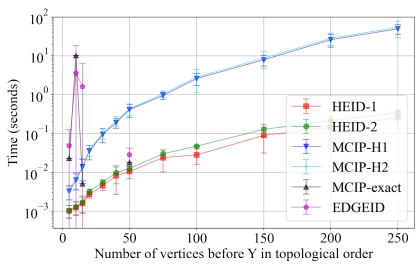
Let $Y \subseteq V^{\mathcal{G}}$ be a set of vertices of $\mathcal{G}$ such that $\mathcal{G}[Y]$ comprises of only one district. Let $Z := \{z \in V^{\mathcal{G}} | \exists\, y \in Y : (z, y) \in E_d^{\mathcal{G}}\} \setminus Y$ denote the set of vertices that have at least one directed edge to a vertex in Y, i.e., the parents of Y excluding Y. Any hedge formed for $Q[Y]$ contains at least one vertex of Z. As a result, in order to eliminate all the hedges formed for $Q[Y]$, it suffices to ensure that none of the vertices in Z appear in the final hedge. To this end, for any $z \in Z$, it suffices to either remove all the directed edges between z and Y, or eliminate all the bidirected paths from z to Y. The problem of eliminating all bidirected paths from Z to Y can be cast as a minimum cut problem between Z and Y in the edge-induced subgraph of $\mathcal{G}$ over its bidirected edges. To add the possibility of removing the directed edges between Z and Y, we add an auxiliary vertex z^* to the graph and draw a bidirected edge between z^* and every $z \in Z$ with weight $w = \sum_{y \in Y} w_{(z,y)}$, i.e., the sum of the weights of all directed edges between z and Y. Note that z can have directed edges to multiple vertices in Y. We then solve the minimum cut problem for z^* and Y. If an edge between z^* and $z \in Z$ is in the solution to this min-cut problem, it translates to removing all the directed edges from z to Y in the original problem. Note that we can run the algorithm on the maximal hedge formed for $Q[Y]$ in $\mathcal{G}$ rather than $\mathcal{G}$ itself.

E Experiments

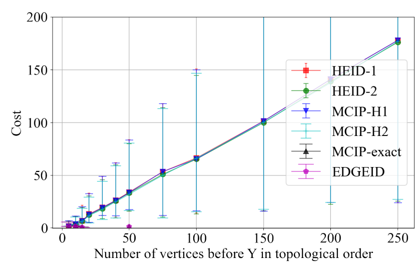
Noting that the synthetic/simulation results in the main paper were for graphs with a $\log(n)/n$ sparsity constraint, we begin this section by providing a set a results on the simulated graphs without the

Algorithm 4 Heuristic algorithm 2.

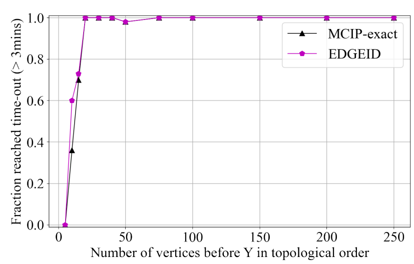
function HEID2$(\mathcal{G}, Y, W_{\mathcal{G}})$
    $\mathcal{G}' \leftarrow$ **MH**$(\mathcal{G}, Y)$
    $Z \leftarrow \{z \in V^{\mathcal{G}'} | \exists y \in Y : (z, y) \in E_d^{\mathcal{G}'}\} \setminus Y$
    $\mathcal{H} \leftarrow$ The induced subgraph of $\mathcal{G}'$ on its bidirected edges.
    $W_{\mathcal{H}} \leftarrow \{w_e \in W_{\mathcal{G}} | e \in \mathcal{H}\}$
    $V^{\mathcal{H}} \leftarrow V^{\mathcal{H}} \cup \{y^*, z^*\}$
    for $z \in Z$ **do**
        $E^{\mathcal{H}} \leftarrow E^{\mathcal{H}} \cup \{z^*, z\}$
        $W_{\mathcal{H}} \leftarrow W_{\mathcal{H}} \cup \{w_{\{z^*,z\}} = \sum_y w_{(z,y)}\}$
    for $y \in Y$ **do**
        $E^{\mathcal{H}} \leftarrow E^{\mathcal{H}} \cup \{y, y^*\}$
        $W_{\mathcal{H}} \leftarrow W_{\mathcal{H}} \cup \{w_{\{y,y^*\}} = \infty\}$
    $E \leftarrow MinCut(\mathcal{H}, W_{\mathcal{H}}, z^*, y^*)$
    return $(E, \sum_{e \in E} w_e)$



(a) Runtimes.



(b) Solution costs.



(c) Fraction for which runtime of 3 minutes exceeded.

Figure 7: Experimental results (for graphs generated without the sparsity constraint) for runtime, solution costs, fraction of graphs for which no solution was found, and fraction of graphs for which runtime limit of 3 minutes was exceeded. Error bars for runtime and cost graphs indicate 5th and 95th percentiles. Best viewed in color.

sparsity penalty for comparison. Then, we provide information about the causal discovery algorithm used to derive the psychology ‘Psych’ real-world graph.

E.1 Additional Simulation Results without Sparsity Constraint

The simulation results for graphs generated without the sparsity constraint are shown in Figure 7. These results illustrate monotonic increases in runtime and cost as the number of nodes increases. Our proposed heuristic algorithms (HEID-1 and HEID-2) maintain runtimes less than 0.5 seconds even for 250 nodes. In contrast, the two exact algorithms (MCIP-exact and EDGEID) exceed the three minute runtime limit at only 20 nodes, and the MCIP heuristic variants (MCIP-H1 and MCIP-H2) have runtimes which increase exponentially with the number of nodes. These results highlight the efficiency of our proposed heuristic algorithms to find solutions with equivalent cost with significantly faster runtimes.

E.2 Psychology Graph Discovery

The settings for deriving the putative structure used on the psychology real-world graph are provided in Table 3.

Table 3: Hyperparameter settings for the Structural Agnostic Model used to generate the putative (directed) structure for the 'Psych' real-world dataset.

Parameter	**Value**
Learning Rate	0.01
DAG Penalty	True
DAG Penalty Weight	0.05
Number of Runs	50
Train Epochs	3000
Test Epochs	800
Mixed Data	True
hlayers	2
dhlayers	2
lambda1	10
lambda2	0.001
dlr	0.001
linear	False
nh	20
dnh	200