Appendix

Anonymous submission

In this appendix, we provide the full theoretical results, which have been partly omitted from the paper due to the space limit.

Definition (CVN set) *Given a set of joint paths $\mathbb{P}$ and a node n with constraints Ω, let $CVN(n, \mathbb{P})$ be the set of all joint paths that (i) satisfy all constraints in Ω, (ii) are conflict-free, and (iii) whose costs are not weakly dominated by the apex of any joint path in $\mathbb{P}$.*

We say a node n *permits* a joint path P with respect to $\mathbb{P}$ if $P \in CVN(n, \mathbb{P})$.

Lemma 1. *For agent a^i and constraints Ω, let $\Pi^i := ApproxLowLevelSearch(i, \Omega, \varepsilon)$. We have (1) for each path π' of agent a^i that satisfies Ω, there exists a path $\pi \in \Pi^i$ with $\mathbf{A}(\pi) \preceq \mathbf{c}(\pi')$, and (2) all paths in Π^i are ε-bounded (regarding the output apexes).*

Proof. The lemma is shown by Theorem 1 in the paper of A*pex (Zhang et al. 2022). □

Lemma 2. *Let n_{new} denote node $\langle \mathbb{P}_{\text{new}}, \Omega_{\text{new}}, \{\Pi^i_{\text{new}} | i \in I\}\rangle$ that Algorithm 2 inserts to Open on Line 26. We have (1) for any solution P' that satisfies Ω_{new}, there exists a joint path $P \in \mathbb{P}_{\text{new}}$ with $\mathbf{A}(P) \preceq \mathbf{c}(P')$ and (2) all joint paths in $\mathbb{P}_{\text{new}}$ are ε-bounded.*

Proof. BB-MO-CBS-pex uses $MergeJointPaths$ to compute $\mathbb{P}_{\text{new}}$. Therefore, to prove this lemma, we inductively show that, in Algorithm 3, for $i = 1, 2, \ldots, |I|$, $\mathbb{P}_i$ satisfy that (Condition 1) for any conflict-free joint path P' for agents $a^1, a^2, \ldots, a^i$ that satisfies Ω_{new}, there exists a joint path $P \in \mathbb{P}_i$ with $\mathbf{A}(P) \preceq \mathbf{c}(P')$ and, (Condition 2) all joint paths in $\mathbb{P}_i$ are ε-bounded. Note that $\mathbb{P}_{\text{new}}$ is equal to $\mathbb{P}_{|I|}$.

Conditions 1 and 2 hold for $\mathbb{P}_1$ because it is initialized with Π^1_{new} (Line 1), which is computed by $ApproxLowLevelSearch$, and Lemma 1 holds. Assume that conditions 1 and 2 holds for $i = 1, 2, \ldots, j-1$. For any conflict-free joint path $P' = [\pi'^1, \pi'^2, \ldots, \pi'^j]$ that satisfies Ω_{new}, there exists a joint path $P_{j-1} \in \mathbb{P}_{j-1}$ with $\mathbf{A}(P_{j-1}) \preceq \mathbf{c}([\pi'^1, \pi'^2, \ldots, \pi'^{j-1}])$. From Lemma 1, there also exists a path $\pi^j \in \Pi^j_{\text{new}}$ with $\mathbf{A}(\pi^j) \preceq \mathbf{c}(\pi'^j)$. When Algorithm 3 reaches Line 7 with the combination of P_{j-1} and π^j, the resulting joint path P is either added to $\mathbb{P}_j$ or merged with another joint path in $\mathbb{P}_j$. In either case, there exists a joint path whose apex weakly dominates $\mathbf{c}(P')$. Algorithm 3 might merge this joint path several (more) times with other joint paths before returning, but the apex of this joint path will still weakly dominate $\mathbf{c}(P')$. Therefore, condition 1 holds for $i = j$. Since both P_{j-1} and π^j are ε-bounded and $\mathbf{A}(P) = \mathbf{A}(P_{j-1}) + \mathbf{A}(\pi^j)$, P is also ε-bounded. We can see that Algorithm 3 inserts only ε-bounded joint paths to $\mathbb{P}_j$ on Line 9. When Algorithm 3 merge any joint path in $\mathbb{P}_j$ with others on Line 8, the resulting joint path also needs to be ε-bounded. Put together, condition 2 holds for $i = j$, too. □

Lemma 3. *When Algorithm 2 reaches Line 13, for any joint path $P \in CVN(n, \mathcal{S})$, there exists a joint path $P' \in \mathbb{P}'$ with $\mathbf{A}(P') \preceq \mathbf{c}(P)$.*

Proof. Before Algorithm 2 reaches Line 13, n might have been previously extracted from and reinserted to $Open$ with different sets of joint paths. Let $\mathbb{P}_{\text{gen}}$ denote the set of joint paths computed by $MergeJointPaths$ when node n was generated on Line 26. From Lemma 2, for any solution P that satisfies Ω, there exists a joint path $P' \in \mathbb{P}_{\text{gen}}$ with $\mathbf{A}(P') \preceq \mathbf{c}(P)$. Assume that P' is in $\mathbb{P}_{\text{gen}}$ but not in $\mathbb{P}'$, which happens only if P' has been removed on Lines 16 or 33. If P' was removed on Lines 16, Algorithm 2 then added it to $\mathcal{S}$ on Line 15. If P' was removed on Lines 33, the apex of some solution was updated to weakly dominate $\mathbf{A}(P')$ (Line 32). In both cases, there existed a solution in $\mathcal{S}$ whose apex weakly dominates $\mathbf{A}(P')$. Algorithm 2 might later merge this solution several (more) times with other solutions on Line 37 or update its apex on Line 32, but the apex of this solution will still weakly dominate $\mathbf{A}(P')$. We hence find a contradiction because, by the definition of CVN sets, the cost of P is not weakly dominated by the apex of any solution in $\mathcal{S}$.

□

Lemma 4. *At the beginning of each iteration of BB-MO-CBS-pex (i.e., before executing Line 7), for any solution P, if there does not exist a solution $P_{sol} \in \mathcal{S}$ with $\mathbf{A}(P_{sol}) \preceq \mathbf{c}(P)$, there exists a node $n \in Open$, which permits P with respect to $\mathcal{S}$.*

Proof. We prove this lemma by induction. After the initialization, $Open$ contains only the root node n_o, which has an empty constraint set and thus permits any solution with respect to $\mathcal{S}$ because $\mathcal{S}$ is empty. Therefore, the lemma holds for the first iteration. Assuming lemma holds at the beginning of an iteration, if there exists a solution $P_{sol} \in \mathcal{S}$ with $\mathbf{A}(P_{sol}) \preceq \mathbf{c}(P)$, as we have already proved, there will always exist a solution whose apex weakly dominates $\mathbf{c}(P)$ afterwards, and hence the lemma holds for the next iteration. Otherwise, there must exist a node $n = \langle \mathbb{P}, \Omega, \{\Pi^i | i \in I\} \rangle$ in $Open$ that permits P with respect to $\mathcal{S}$. If n is not extracted from $Open$, it still permits P till the next iteration. Therefore, the lemma holds for the next iteration. There are three cases if n is extracted from $Open$: First, some joint paths in $\mathbb{P}$ are removed on Lines 8. Because of Lemma 3 and because $P \in CVN(n, \mathcal{S})$, $\mathbb{P}'$ is not empty, and hence node n is reinserted into $Open$. The lemma holds for the next iteration. Second, $\mathbb{P}'.lexFirst$ is conflict-free. If $\mathbf{A}(\mathbb{P}'.lexFirst) \preceq \mathbf{c}(P)$, then $\mathbb{P}'.lexFirst$ is added to $\mathcal{S}$ and the lemma holds for the next iteration. Otherwise, from Lemma 3, $\mathbb{P}'$ is not empty, node n is reinserted into $Open$ (Line 18), and the lemma also holds for the next iteration. Third, Algorithm 2 generates two child nodes to resolve conflict cft (Lines 20-26). P cannot violate both ω^i and ω^j. Thus, π does not violate at least one of the two constraints, and the corresponding child node of this constraint permits π. Both nodes are then added to $Open$, and the lemma holds for the next iteration.

□

Theorem 1. *Given an MO-MAPF instance that has at least one solution, when BB-MO-CBS-pex terminates, $\mathcal{S}$ is an ε-approximate Pareto frontier.*

Proof. From Lemma 2, all joint paths in the joint path set of a generated node are ε-bounded. Additionally, whenever BB-MO-CBS-pex merges joint paths (Line 37) or updates the apex of an joint path (Line 32), the resulting joint path or the updated joint path is always ε-bounded. Therefore, for any $P_{sol} \in \mathcal{S}$, we have $\mathbf{c}(P_{sol}) \preceq_{\boldsymbol{\varepsilon}} \mathbf{A}(P_{sol})$. From Lemma 4, we know that, for any solution P, there exists a solution $P_{sol} \in \mathcal{S}$ with $\mathbf{A}(P_{sol}) \preceq \mathbf{c}(P)$, and hence $\mathbf{c}(P_{sol}) \preceq_{\boldsymbol{\varepsilon}} \mathbf{c}(P)$, when OPEN is empty, i.e., when BB-MO-CBS-pex terminates. Therefore, $\mathcal{S}$ is an ε-approximate Pareto frontier. □

Lemma 5. *BB-MO-CBS-pex never reaches Line 20 with a node n if there exists a solution P with $\mathbf{c}(P) \prec \mathbf{g}(n)$.*

Proof. If BB-MO-CBS-pex reaches Line 20 with node n and there exists a solution P with $\mathbf{c}(P) \preceq \mathbf{g}(n)$ and $\mathbf{c}(P) \neq \mathbf{g}(n)$. There does not exist a solution $P_{sol} \in \mathcal{S}$ with $\mathbf{A}(P_{sol}) \preceq \mathbf{A}(P)$ (and hence $\preceq \mathbf{c}(P)$) because, otherwise, as guaranteed by Line 8. From Lemma 4, there exists a node $n' \in Open$ that permits P with respect to $\mathcal{S}$. From Lemma 3, we know that there exists P' in the joint path set of n' with $\mathbf{A}(P') \preceq \mathbf{c}(P)$ and hence $\mathbf{g}(n')$ is lexicographically smaller than $\mathbf{c}(P)$ and hence lexicographically smaller than $\mathbf{g}(n)$, which contradicts that n has the lexicographically smallest g-value when it is extracted from $Open$ (Line 7). □

Theorem 2. *Given an MO-MAPF instance that has at least one solution, BB-MO-CBS-pex terminates in finite time.*

Proof. Because the given graph G is finite (i.e., has finite vertices and edges) and the cost of each edge in G is a positive M-dimensional vector, there are only a finite number of ε-bounded joint paths whose apexes are not dominated by the cost of any solution. Because of Lemma 5, when BB-MO-CBS-pex reaches Line 20 with node n, the current joint path of n must be a joint path whose apex is not dominated by the cost of any solution. When generating a child node for a node n, BB-MO-CBS-pex adds a new constraint, which prevents at least one joint path (i.e., the current joint path of n), whose apex is not dominated by the cost of any solution. Therefore, the CT of BB-MO-CBS-pex must contain a finite number of node. Because each node in BB-MO-CBS-pex can only be reinserted to $Open$ for finite times, BB-MO-CBS-pex terminates in finite time. □