APPENDIX

A PROPERTIES OF SEPARATORS

PROPOSITION 1. *Let $S_1, S_2 \in \mathcal{S}_{s,t}(G)$. Then $C_s(G–S_1) \subseteq C_s(G–S_2)$ if and only if $C_t(G–S_2) \subseteq C_t(G–S_1)$.*

Proof. If $C_s(G–S_1) \subseteq C_s(G–S_2)$, then $C_s(G–S_1) \cup N_G(C_s(G–S_1)) \subseteq C_s(G–S_2) \cup N_G(C_s(G–S_2))$. By Lemma 1, we have that $S_1 = N_G(C_s(G–S_1))$. Therefore, $C_s(G–S_1) \cup S_1 \subseteq C_s(G–S_2) \cup N_G(C_s(G–S_2))$. In particular, $S_1 \cap C_t(G–S_2) = \emptyset$. This means that $C_t(G–S_2)$ is contained in the connected component of $G–S_1$ that contains t. By definition, $C_t(G–S_2) \subseteq C_t(G–S_1)$. The other direction is symmetrical. □

PROPOSITION 2. *Let S_1, S_2 be two s,t-separators in H. Then $S_1 \trianglelefteq_H S_2$ if and only if $C_s(H–S_1) \subseteq C_s(H–S_2)$ and $C_t(H–S_2) \subseteq C_t(H–S_1)$.*

Proof. If $C_s(H–S_1) \subseteq C_s(H–S_2)$, then $\emptyset = C_s(H–S_2) \cap S_2 \supseteq C_s(H–S_1) \cap S_2$, and hence $C_s(H–S_1) \cap S_2 = \emptyset$. Consequently, $(S_2 \backslash S_1) \cap (C_s(H–S_1) \cup S_1) = \emptyset$. Every vertex connected to s in $H–S_1$ belongs to $C_s(H–S_1)$. Since $(S_2 \backslash S_1) \cap (C_s(H–S_1) \cup S_1) = \emptyset$, then S_1 separates s from $S_2 \backslash S_1$. Symmetrically, if $C_t(H–S_2) \subseteq C_t(H–S_1)$, then $(S_1 \backslash S_2) \cap (C_t(H–S_2) \cup S_2) = \emptyset$, thus S_2 separates t from $S_1 \backslash S_2$.

If S_1 separates s from $S_2 \backslash S_1$, then $(S_2 \backslash S_1) \cap C_s(H–S_1) = \emptyset$. By definition, $S_1 \cap C_s(H–S_1) = \emptyset$, and hence $S_2 \cap C_s(H–S_1) = \emptyset$. This, in turn, means that $C_s(H–S_1)$ is contained in the connected component of $H–S_2$ that contains s. By definition, $C_s(H–S_1) \subseteq C_s(H–S_2)$. Symmetrically, if S_2 separates t from $S_1 \backslash S_2$, then $C_t(H–S_2) \subseteq C_t(H–S_1)$. So, if $S_1 \trianglelefteq_H S_2$ then $C_s(G–S_1) \subseteq C_s(H–S_2)$ and $C_t(H–S_2) \subseteq C_t(H–S_1)$. □

Proposition 3. Let $S \in \mathcal{S}_{s,t}(G)$ where $S \subseteq N_G(s)$. For every $T \in \mathcal{S}_{s,t}(G)$, it holds that $C_s(G–S) \subseteq C_s(G–T)$.

Proof. Since $S \subseteq N_G(s) \subseteq T \cup C_s(G–T)$, then $C_s(G–S) \subseteq C_s(G–T)$. □

A.1 SEPARATORS BETWEEN VERTEX-SETS

Lemma 3. Let A and B be two disjoint, non-adjacent subsets of $\mathtt{V}(G)$. Then $S \in \mathcal{S}_{A,B}(G)$ if and only if S is an A,B-separator, and for every $w \in S$, there exist two connected components $C_A, C_B \in \mathcal{C}(G–S)$ such that $C_A \cap A \neq \emptyset$, $C_B \cap B \neq \emptyset$, and $w \in N_G(C_A) \cap N_G(C_B)$.

Proof. If $S \in \mathcal{S}_{A,B}(G)$, then for every $w \in S$ it holds that $S \backslash \{w\}$ no longer separates A from B. Hence, there is a path from some $a \in A$ to some $b \in B$ in $G–(S \backslash \{w\})$. Let C_a and C_b denote the connected components of $\mathcal{C}(G–S)$ containing $a \in A$ and $b \in B$, respectively. Since C_a and C_b are connected in $G–(S \backslash \{w\})$, then $w \in N_G(C_a) \cap N_G(C_b)$.

Suppose that for every $w \in S$, there exist two connected components $C_A, C_B \in \mathcal{C}_G(S)$ such that $C_A \cap A \neq \emptyset$, $C_B \cap B \neq \emptyset$, and $w \in N_G(C_A) \cap N_G(C_B)$. If $S \notin \mathcal{S}_{A,B}(G)$, then $S \backslash \{w\}$ separates A from B for some $w \in S$. Since w connects C_A to C_B in $G–(S \backslash \{w\})$, no such $w \in S$ exists, and thus $S \in \mathcal{S}_{A,B}(G)$. □

Observe that Lemma 3 implies Lemma 1. By Lemma 3, it holds that $S \in \mathcal{S}_{s,t}(G)$ if and only if S is an s,t-separator and $S \subseteq N_G(C_s(G–S)) \cap N_G(C_t(G–S))$. By definition, $N_G(C_s(G–S)) \subseteq S$ and $N_G(C_t(G–S)) \subseteq S$, and hence $S = N_G(C_s(G–S)) \cap N_G(C_t(G–S))$, and $S = N_G(C_s(G–S)) = N_G(C_t(G–S))$.

Lemma 4. Let G and H be graphs where $\mathtt{V}(G) = \mathtt{V}(H)$ and $\mathtt{E}(G) \subseteq \mathtt{E}(H)$. Let $A, B \subseteq \mathtt{V}(G)$ disjoint and non-adjacent. Let $S \in \mathcal{S}_{A,B}(G)$. If S is an A,B-separator in H, then $S \in \mathcal{S}_{A,B}(H)$.

Proof. Since $S \in \mathcal{S}_{A,B}(G)$, then by Lemma 3, for every $u \in S$ there exist two distinct connected components $C_A^u, C_B^u \in \mathcal{C}(G–S)$ where $C_A^u \cap A \neq \emptyset$, $C_B^u \cap B \neq \emptyset$, and $u \in N_G(C_A^u) \cap N_G(C_B^u)$. Since $\mathtt{E}(H) \supseteq \mathtt{E}(G)$, and since S is an A,B-separator in H, then $H–S$ contains two distinct connected components D_A^u, D_B^u where $C_A^u \subseteq D_A^u$ and $C_B^u \subseteq D_B^u$. Therefore, $w \in N_H(D_A^u) \cap N_H(D_B^u)$. By Lemma 3, we have that $S \in \mathcal{S}_{A,B}(H)$. □

Lemma 5. Let $u \in \mathtt{V}(G) \backslash sB$ such that $N_G(u) \subseteq N_G(s)$. Then $\mathcal{S}_{s,B}(G) = \mathcal{S}_{s,B}(G–u)$

Proof. Let $S \in \mathcal{S}_{s,B}(G)$. We first show that $u \notin S$. Suppose, by way of contradiction, that $u \in S$. By Lemma 3, there exist two distinct vertices $x, y \in N_G(u)$ such that $x \in C_s(G{-}S)$ and $y \in C_B(G{-}S)$, where $C_B(G{-}S) \cap B \neq \emptyset$. By the assumption of the lemma that $N_G(u) \subseteq N_G(s)$, then $y \in N_G(s)$. But then, S is not an s, B-separator of G; a contradiction. Hence $u \notin S$ for any $S \in \mathcal{S}_{s,B}(G)$.

Let $T \in \mathcal{S}_{s,B}(G{-}u)$. We show that T is an s, B-separator of G. If it is not, then since every s, B-path of $G{-}u$ is also an s, B-path of G, then $T \cup \{u\} \in \mathcal{S}_{s,B}(G)$. But this contradicts the fact that $u \notin S$ for every $S \in \mathcal{S}_{s,B}(G)$. Hence, T is an s, B-separator of G. By Lemma 4, we have that $T \in \mathcal{S}_{s,B}(G)$. Hence, we have that $\mathcal{S}_{s,B}(G{-}u) \subseteq \mathcal{S}_{s,B}(G)$. For the other direction, let $T \in \mathcal{S}_{s,B}(G)$. Clearly T is an s, B-separator of $G{-}u$. If $T \notin \mathcal{S}_{s,B}(G{-}u)$, then there exist a $T' \subset T$ s.t. $T' \in \mathcal{S}_{s,B}(G{-}u)$. By the previous direction, we have that $T' \in \mathcal{S}_{s,B}(G{-}u) \subseteq \mathcal{S}_{s,B}(G)$. But then, $T' \in \mathcal{S}_{s,B}(G)$ contradicting the minimality of T. Hence, $\mathcal{S}_{s,B}(G{-}u) = \mathcal{S}_{s,B}(G)$. ☐

Lemma 6. Let $A \subseteq \mathtt{V}(G) \backslash Bs$. Let H be the graph that results from G by (1) adding all edges between s and $N_G(A)$, and (2) removing the vertices A and their adjacent edges from H. Then $\mathcal{S}_{sA,B}(G) = \mathcal{S}_{s,B}(H)$.

Proof. Let $T \in \mathcal{S}_{sA,B}(G)$, and let $C_1, \dots, C_k$ denote the connected components of $\mathcal{C}(G{-}T)$ containing vertices from sA. By definition, $B \cap C_i = \emptyset$ for all $i \in \{1, 2, \dots, k\}$. Assume wlog that $s \in C_1$. Let H' denote the graph that results from G by adding all edges between s and $N_G(A)$. By definition, the edges added to G to form H' are between C_1 and $C_1 \cdots C_k \cup T$. Therefore, T separates sA from B in H'. Since $\mathtt{E}(H') \supseteq \mathtt{E}(G)$, then by Lemma 4, if $T \in \mathcal{S}_{sA,B}(G)$ and T is an sA, B-separator in H', then $T \in \mathcal{S}_{sA,B}(H')$. Therefore, we have that $\mathcal{S}_{sA,B}(G) \subseteq \mathcal{S}_{sA,B}(H')$.

We now claim that $\mathcal{S}_{sA,B}(H') = \mathcal{S}_{s,B}(H')$. Take $T \in \mathcal{S}_{s,B}(H')$. We claim that T is an sA, B-separator in H'. Suppose it is not, and let $C \in \mathcal{C}(H'{-}T)$ such that $a, b \in C$ where $a \in A$ and $b \in B$. Let $y \in N_{H'}(a) \cap C$. By construction, $y \in N_{H'}(s)$. But then, $s \in N_{H'}(C)$ and hence T is not an s, B-separator in H'; a contradiction. Since $T \in \mathcal{S}_{s,B}(H')$, then by Lemma 3, we have that for every $u \in T$ there exists a connected component $C_B^u \in \mathcal{C}(H'{-}T)$ s.t. $B \cap C_B^u \neq \emptyset$ and $u \in N_{H'}(C_s(H'{-}T)) \cap N_{H'}(C_B^u)$. By Lemma 3, we have that $T \in \mathcal{S}_{sA,B}(H')$. Hence $\mathcal{S}_{s,B}(H') \subseteq \mathcal{S}_{sA,B}(H')$. For the other direction, let $T \in \mathcal{S}_{sA,B}(H')$. Clearly, T is an s, B-separator of H'. If $T \notin \mathcal{S}_{s,B}(H')$, then there exists a $T' \subset T$ such that $T' \in \mathcal{S}_{s,B}(H')$. By the previous direction, we have that $T' \in \mathcal{S}_{sA,B}(H')$, but this contradicts the minimality of T. Hence, $\mathcal{S}_{s,B}(H') = \mathcal{S}_{sA,B}(H')$. Overall, we have shown that $\mathcal{S}_{sA,B}(G) \subseteq \mathcal{S}_{sA,B}(H') = \mathcal{S}_{s,B}(H')$.

Let $T \in \mathcal{S}_{s,B}(H')$. We first show that T separates sA from B in G; if not, there is a path from $x \in sA$ to B in $G{-}T$. Let u be the first vertex on this path such that $u \notin sA$. Note that such a vertex $u \notin sA$ must exist because $B \cap sA = \emptyset$. In particular, $u \in N_G(sA)$, and by construction, $u \in N_{H'}(s)$. This means that there is a path from s to B (via u) in $H'{-}T$, which is a contradiction. Therefore, T is an sA, B-separator in G. If $T \notin \mathcal{S}_{sA,B}(G)$, then there is a $T' \in \mathcal{S}_{sA,B}(G)$ where $T' \subset T$. By the previous direction, $T' \in \mathcal{S}_{sA,B}(G) \subseteq \mathcal{S}_{s,B}(H')$, and hence $T' \in \mathcal{S}_{s,B}(H')$, contradicting the minimality of $T \in \mathcal{S}_{s,B}(H')$. Therefore, $\mathcal{S}_{sA,t}(G) = \mathcal{S}_{s,B}(H')$.

By construction, for every $u \in sA$, we have that $N_{H'}(u) \subseteq N_{H'}(s)$. From Lemma 5, we have that $\mathcal{S}_{s,B}(H') = \mathcal{S}_{s,B}(H)$. Therefore, $\mathcal{S}_{sA,t}(G) = \mathcal{S}_{s,B}(H)$. ☐

THEOREM 3. *Let $A, B \subseteq \mathtt{V}(G)$ be disjoint and non-adjacent, where $s \in A$ and $t \in B$. Let H be the graph that results from G by: (1) adding all edges between s and $N_G(A)$, (2) adding all edges between t and $N_G(B)$, and (3) removing vertices $AB \backslash \{s, t\}$ and their adjacent edges. Then $\mathcal{S}_{s,t}(H) = \mathcal{S}_{A,B}(G)$.*

Proof. Let H_1 be the graph that results from G by adding all edges between s and $N_G(A)$, and removing vertices $A \backslash \{s\}$ from the graph. By Lemma 6, we have that $\mathcal{S}_{A,B}(G) = \mathcal{S}_{s,B}(H_1)$. By the assumption that A and B are disjoint and non-adjacent, then $N_G[B] = N_{H_1}[B]$. Now, let H_2 be the graph that results from H_1 by adding all edges between t and $N_{H_1}(B) = N_G(B)$, and removing vertices $B \backslash \{t\}$ from the graph H_2. By Lemma 6, we have that $\mathcal{S}_{s,t}(H_2) = \mathcal{S}_{s,B}(H_1) = \mathcal{S}_{A,B}(G)$. ☐

A.2 MINIMUM SEPARATORS

THEOREM 4. *There exists a unique minimum s, t-separator $S^* \in \mathcal{L}_{s,t}(G)$ such that $S^* \preceq S$ for all $S \in \mathcal{L}_{s,t}(G)$, and S^* can be found in time $O(n \cdot \ell \cdot T(n, m))$, where ℓ is the maximum-cardinality of any minimum-weight s, t-separator; $\ell \stackrel{\text{def}}{=} \max \{|S| : S \in \mathcal{L}_{s,t}(G)\}$.*

Theorem 4 is a straightforward extension of the following Theorem.

Theorem 14. (Cygan et al. [2015]) Let G be a non-weighted graph (i.e., $w(v) = 1$ for every $v \in \mathbb{V}(G)$). There exists a unique minimum-cardinality s,t-separator $S^* \in \mathcal{L}_{s,t}(G)$ such that $S^* \preceq S$ for all $S \in \mathcal{L}_{s,t}(G)$, and S^* can be found in time $O(n \cdot T(n,m))$.

For completeness, we provide the proof of Theorem 4 herein.

Theorem 15. (Theorem 8.3 in Cygan et al. [2015]) For $X, Y \subseteq \mathbb{V}(G)$. It holds that:

$$|N_G(X)| + |N_G(Y)| \geq |N_G(X \cap Y)| + |N_G(X \cup Y)|.$$

Proof Overview. The proof establishes that for every vertex $v \in \mathbb{V}(G)$, the number of times it is accounted for in the left-hand-side (LHS) is at least as large as the number of times it is accounted for in the right-hand-side (RHS), thereby proving the claim. □

Lemma 7. Let G be an undirected, weighted graph, with weight function $w : \mathbb{V}(G) \to \mathbb{N}_{\geq 1}$. For $X, Y \subseteq \mathbb{V}(G)$. It holds that:

$$w(N_G(X)) + w(N_G(Y)) \geq w(N_G(X \cap Y)) + w(N_G(X \cup Y)).$$

Proof Overview. The proof is identical to that of Theorem 15, establishing that for every vertex $v \in \mathbb{V}(G)$, the number of times it is accounted for in the left-hand-side (LHS) is at least as large as the number of times it is accounted for in the right-hand-side (RHS), thereby proving the claim. Since the weights are positive, the claim follows. □

Recall from Definition 1 that for two minimal s,t-separators $S_1, S_2 \in \mathcal{S}_{s,t}(G)$, it holds:

$$S_1 \preceq S_2 \text{ if and only if } C_s(G{-}S_1) \subseteq C_s(G{-}S_2).$$

Theorem 16. (Theorem 8.4 in Cygan et al. [2015]) Let G be an undirected, uweighted graph. There exists a minimum-cardinality s,t-separator $S^* \in \mathcal{L}_{s,t}(G)$, such that $S^* \preceq S$ for every $S \in \mathcal{L}_{s,t}(G)$.

Lemma 8 presents the weighted version of Theorem 16. The proof is similar to that of Theorem 16, and is provided below for completeness.

Lemma 8. Let G be an undirected, weighted graph, with weight function $w : \mathbb{V}(G) \to \mathbb{N}_{\geq 1}$. There exists a minimum-weight s,t-separator $S^* \in \mathcal{L}_{s,t}(G)$, such that $S^* \preceq S$ for every $S \in \mathcal{L}_{s,t}(G)$.

Proof. Let $S_1, S_2 \in \mathcal{L}_{s,t}(G)$. By Lemma 7, and Lemma 1, we have that:

$$\begin{aligned} w(S_1) + w(S_2) &\underbrace{=}_{\text{Lemma 1}} w(N(C_s(G{-}S_1))) + w(N(C_s(G{-}S_2))) \\ &\underbrace{\geq}_{\text{Lemma 7}} w(N(C_s(G{-}S_1) \cap C_s(G{-}S_2))) + w(N(C_s(G{-}S_1) \cup C_s(G{-}S_2))). \end{aligned} \tag{11}$$

Define $S^- \stackrel{\text{def}}{=} N(C_s(G{-}S_1) \cap C_s(G{-}S_2))$ and $S^+ \stackrel{\text{def}}{=} N(C_s(G{-}S_1) \cup C_s(G{-}S_2))$. Since $s \in C_s(G{-}S_1) \cap C_s(G{-}S_2)$, and $t \notin C_s(G{-}S_1) \cup C_s(G{-}S_2)$, then both S^- and S^+ are s,t-seprators of G. Therefore, $w(S^-) \geq \kappa_{s,t}(G) = w(S_1) = w(S_2)$, and $w(S^+) \geq \kappa_{s,t}(G) = w(S_1) = w(S_2)$.

From (11), we have that

$$2\kappa_{s,t}(G) = w(S_1) + w(S_2) \geq w(S^-) + w(S^+) \geq 2\kappa_{s,t}(G),$$

and hence, $w(S^-) = w(S^+) = \kappa_{s,t}(G)$. Since $S^- = N(C_s(G{-}S_1) \cap C_s(G{-}S_2))$, then by definition, $S^- \preceq S_1$ and $S^- \preceq S_2$. Since $\mathcal{L}_{s,t}(G)$, the set of minimum-weight s,t-separators of G, is finite, this proves the claim. □

We are now eready to prove Theorem 4.

THEOREM 4. *There exists a unique minimum s,t-separator $S^* \in \mathcal{L}_{s,t}(G)$ such that $S^* \preceq S$ for all $S \in \mathcal{L}_{s,t}(G)$, and S^* can be found in time $O(n \cdot \ell \cdot T(n,m))$, where ℓ is the maximum-cardinality of any minimum-weight s,t-separator;* $\ell \stackrel{\text{def}}{=} \max\{|S| : S \in \mathcal{L}_{s,t}(G)\}$.

Proof. From Lemma 8, we have that $S^* \in \mathcal{L}_{s,t}(G)$ exists and is unique. We show that it can be found in time $O(n \cdot T(n,m))$. Finding a minimum-weight s,t-separator can be reduced, by standard techniques to the maximum-flow problem. Let $S_1 \in \mathcal{L}_{s,t}(G)$ be a minimum-weight s,t-separator found in this way. Now, we need to check whether there is another $S_2 \in \mathcal{L}_{s,t}(G)$ such that $S_2 \prec S_1$. If $C_s(G{-}S_2) \subset C_s(G{-}S_1)$, then by Proposition 1, it holds that $C_t(G{-}S_1) \subset C_t(G{-}S_2)$. In particular, $S_1 = N(C_t(G{-}S_1)) \subseteq C_t(G{-}S_2) \cup N(C_t(G{-}S_2)) = C_t(G{-}S_2) \cup S_2$. Since $S_1, S_2 \in \mathcal{L}_{s,t}(G)$, then $S_1 \not\subseteq S_2$, and hence $S_1 \cap C_t(G{-}S_2) \neq \emptyset$. In other words, if $S_2 \prec S_1$, then there must be a vertex $v \in S_1$ that belongs to $C_t(G{-}S_2)$. We check if this is the case by iterating over all vertices $v \in S_1$, and contracting $C_t(G{-}S_1) \cup \{v\}$ to the vertex t, and finding a minimum-weight s,t-separator in the resulting graph. If, for all $v \in S_1$, this results in a separator whose weight is strictly larger than $\kappa_{s,t}(G)$, then we have identified the minimum-weight s,t-separator that is closest to s. Otherwise, we repeat this procedure until no such vertex $v \in S_1$ is found – indicating that the computed s,t-seprator is both minimum-weight, and closest to s. $\square$

B PROOFS FROM SECTION 5: CORRECTNESS OF ALGORITHM SmallMinimalSeps

Lemma 9. Let $v \in N_G(s)$, and let G' denote the graph that results from G by contracting the edge (s,v) to s. Then $\mathcal{S}_{s,t}(G') = \{S \in \mathcal{S}_{s,t}(G) : v \notin S\}$.

Proof. Let G'' be the graph that results from G by adding all edges between s and $N_G(v)$. By definition, this means that $N_{G''}(v) \subseteq N_{G''}(s)$. We first show that $\mathcal{S}_{s,t}(G'') = \{S \in \mathcal{S}_{s,t}(G) : v \notin S\}$.

Let $S \in \mathcal{S}_{s,t}(G)$ such that $v \notin S$. Since $v \in N_G(s)$, then $v \in C_s(G{-}S)$, and hence $N_G(v) \subseteq S \cup C_s(G{-}S)$. Therefore, S is an s,t-separator in G'' as well. Since $\mathtt{E}(G'') \supseteq \mathtt{E}(G)$, then by Lemma 4, $S \in \mathcal{S}_{s,t}(G'')$.

Now, let $T \in \mathcal{S}_{s,t}(G'')$. Since $\mathtt{E}(G) \subseteq \mathtt{E}(G'')$ then clearly T is an s,t-separator of G. Since $N_{G''}(v) \subseteq N_{G''}(s)$, then by Lemma 5, it holds that $\mathcal{S}_{s,t}(G'') = \mathcal{S}_{s,t}(G''{-}v)$. Therefore, we have that $v \notin T$. If $T \notin \mathcal{S}_{s,t}(G)$, then there exists a $T' \subset T$ such that $T' \in \mathcal{S}_{s,t}(G)$. Since $v \notin T$, then $v \notin T'$. We have previously established that $\mathcal{S}_{s,t}(G'') \supseteq \{S \in \mathcal{S}_{s,t}(G) : v \notin S\}$, and hence $T' \in \mathcal{S}_{s,t}(G'')$. But this contradicts the minimality of T. Therefore, $T \in \{S \in \mathcal{S}_{s,t}(G) : v \notin S\}$, and we have that $\mathcal{S}_{s,t}(G'') = \{S \in \mathcal{S}_{s,t}(G) : v \notin S\}$.

By construction, we have that $N_{G''}(v) \subseteq N_{G''}(s)$. By Lemma 5, we have that $\mathcal{S}_{s,t}(G'') = \mathcal{S}_{s,t}(G''{-}v) = \mathcal{S}_{s,t}(G')$. Therefore, we get that $\mathcal{S}_{s,t}(G') = \{S \in \mathcal{S}_{s,t}(G) : v \notin S\}$. $\square$

Lemma 10. Let $S, T \in \mathcal{S}_{s,t}(G)$. Then:

$$C_s(G{-}S) \subseteq C_s(G{-}T) \text{ if and only if } T \subseteq S \cup C_t(G{-}S).$$

Proof. If $T \subseteq S \cup C_t(G{-}S)$, then by definition $T \cap C_s(G{-}S) = \emptyset$. Therefore, $C_s(G{-}S)$ remains connected in $G{-}T$. This means that $C_s(G{-}S) \subseteq C_s(G{-}T)$.

Now, suppose that $C_s(G{-}S) \subseteq C_s(G{-}T)$. By Lemma 1, it holds that $S = N_G(C_s(G{-}S))$. Since $C_s(G{-}S) \subseteq C_s(G{-}T)$, then $S = N_G(C_s(G{-}S)) \subseteq T \cup C_s(G{-}T)$. Since $S \subseteq T \cup C_s(G{-}T)$ then by definition it holds that $S \cap C_t(G{-}T) = \emptyset$. This, in turn, implies that $C_t(G{-}T)$ remains connected in $G{-}S$. In particular, we have that $C_t(G{-}T) \subseteq C_t(G{-}S)$. By Lemma 1, it holds that $T = N_G(C_t(G{-}T))$. Since $C_t(G{-}T) \subseteq C_t(G{-}S)$, then $T = N_G(C_t(G{-}T)) \subseteq S \cup C_t(G{-}S)$. $\square$

Lemma 11. Let $S \in \mathcal{S}_{s,t}(G)$, and let H_S be the graph that results from G by adding all edges from s to S. That is, $\mathtt{E}(H_S) = \mathtt{E}(G) \cup \{(s,v) : v \in S\}$. Then:

$$\mathcal{S}_{s,t}(H_S) = \{Q \in \mathcal{S}_{s,t}(G) : Q \subseteq S \cup C_t(G{-}S)\}$$

Proof. Let $Q \in \mathcal{S}_{s,t}(G)$ where $Q \subseteq S \cup C_t(G{-}S)$. Since $Q \cap C_s(G{-}S) = \emptyset$, then $C_s(G{-}S)$ remains connected in $G{-}Q$. Therefore, $C_s(G{-}S) \subseteq C_s(G{-}Q)$. By Lemma 1, $S = N_G(C_s(G{-}S))$. Since $C_s(G{-}S) \subseteq C_s(G{-}Q)$, then $S = N_G(C_s(G{-}S)) \subseteq C_s(G{-}Q) \cup Q$. In particular, $S \cap C_t(G{-}Q) = \emptyset$. Consequently, Q separates $C_t(G{-}Q)$ from s in H_S as well. That is, Q is an s,t-separator in H_S. Since $\mathtt{E}(H_S) \supseteq \mathtt{E}(G)$, then $Q \in \mathcal{S}_{s,t}(H_S)$.

Let $T \in \mathcal{S}_{s,t}(H_S)$. By construction, $S \in \mathcal{S}_{s,t}(H_S)$ where $S \subseteq N_H(s)$. By Proposition 3, $C_s(H_S{-}S) \subseteq C_s(H_S{-}T)$. By Lemma 10, it holds that $T \subseteq S \cup C_t(H_S{-}S)$. Since, by construction, $C_t(H_S{-}S) = C_t(G{-}S)$, we get that $T \subseteq S \cup C_t(G{-}S)$. $\square$

Lemma 12. Let $T \in \mathcal{S}_{s,t,k}(G)$. Exactly one of the following holds: (1) $T \in \mathcal{S}^*_{s,t,k}(G)$ or (2) There exists a minimal s,t-separator $S \in \mathcal{S}^*_{s,t,k}(G)$ such that $S \prec T$.

Proof. By induction on $|C_s(G{-}T)|$. If $|C_s(G{-}T)| = 1$, then clearly $T \subseteq N_G(s)$. By Lemma 2, T is the unique minimal s,t-separator that is closest to s, and hence $T \in \mathcal{S}^*_{s,t,k}(G)$. So, we assume that the claim holds for all $T \in \mathcal{S}_{s,t,k}(G)$, where $1 \leq |C_s(G{-}S)| \leq \ell$. Let $T \in \mathcal{S}_{s,t,k}(G)$, where $|C_s(G{-}S)| = \ell + 1$. If $T \in \mathcal{S}^*_{s,t,k}(G)$, then we are done. Otherwise, if $T \notin \mathcal{S}^*_{s,t,k}(G)$, then since $|T| \leq k$, it must hold that $T \notin \mathcal{S}^*_{s,t}(G)$. By definition 2, there exists a $T' \in \mathcal{S}_{s,t}(G)$ such that $T' \prec T$ (i.e., $C_s(G{-}T') \subset C_s(G{-}T)$), and $|T'| \leq |T| \leq k$. Consequently, $|C_s(G{-}T')| < |C_s(G{-}T)| = \ell + 1$, and $|C_s(G{-}T')| \leq \ell$. Since $T' \in \mathcal{S}_{s,t,k}(G)$ and $|C_s(G{-}T')| \leq \ell$, then by the induction hypothesis, either $T' \in \mathcal{S}^*_{s,t,k}(G)$, in which case $T' \prec T$, thus proving the claim. Otherwise, there exists an $S \in \mathcal{S}^*_{s,t,k}(G)$ such that $S \prec T'$. Hence, $S \prec T' \prec T$, and $S \prec T$, thus proving the claim. □

Lemma 13. Let $T \in \mathcal{S}_{s,t,k}(G)$. There exists a $S \in \mathcal{S}^*_{s,t,k}(G)$ such that $S \preceq T$, and $T \subseteq S \cup C_t(G{-}S)$.

Proof. If $T \in \mathcal{S}^*_{s,t,k}(G)$, then the claim is immediate. If $T \notin \mathcal{S}^*_{s,t,k}(G)$ then, by Lemma 12, there exists an $S \in \mathcal{S}^*_{s,t,k}(G)$, such that $S \prec T$. By Lemma 10, $T{\in}S{\cup}C_t(G{-}S)$. □

THEOREM 8. *If $S \subseteq \mathbb{V}(G)$ is printed, then $S \in \mathcal{S}_{s,t,k}(G)$, and S is printed exactly once.*

Proof. Every subset of vertices inserted into the queue (in lines 7 and 18) is pushed exactly once and has cardinality at most k. Therefore, we only need to show that every subset of vertices pushed into the queue Q, and printed by the algorithm, belongs to $\mathcal{S}_{s,t}(G)$. Suppose, by way of contradiction, that this is not the case, and let $T \subseteq \mathbb{V}(G)$ be the first subset of vertices printed where $T \notin \mathcal{S}_{s,t}(G)$. Then T must be inserted into the queue in line 18. Consider the set S that was printed before T is inserted into the queue. By our assumption $S \in \mathcal{S}_{s,t}(G)$. Therefore, $T \in \mathcal{S}^*_{s,t,k}(H^v_S)$, where $v \in S$. By Lemma 11, $\mathcal{S}_{s,t}(H_S) \subseteq \mathcal{S}_{s,t}(G)$. Since $v \in N_{H_S}(s)$, and H^v_S is the graph that results from H_S by contracting the edge (s,v) to vertex v, by Lemma 9, it holds that $\mathcal{S}_{s,t}(H^v_S) \subseteq \mathcal{S}_{s,t}(H_S) \subseteq \mathcal{S}_{s,t}(G)$. Since $T \in \mathcal{S}^*_{s,t,k}(H^v_S) \subseteq \mathcal{S}_{s,t}(H^v_S)$, we get that $T \in \mathcal{S}_{s,t}(G)$, which brings us to a contradiction. □

THEOREM 9. *Let $T \in \mathcal{S}_{s,t,k}(G)$. Then T is printed by* SmallMinimalSeps *in Figure 2.*

Proof. If $T \in \mathcal{S}^*_{s,t,k}(G)$, then T is inserted into the queue in line 7, and will be printed. Therefore, assume that $T \notin \mathcal{S}^*_{s,t,k}(G)$. Suppose that T is not printed. Let $T' \in \mathcal{S}_{s,t}(G)$ be the largest minimal s,t-separator, with respect to $\prec$, that is printed by the algorithm, such that $T' \preceq T$. In other words, there does not exist a $T'' \in \mathcal{S}_{s,t}(G)$, that is printed by the algorithm where $T' \prec T'' \preceq T$. By Lemma 13, and the fact that $T \notin \mathcal{S}^*_{s,t,k}(G)$ such a separator T' exists.

Since $C_s(G{-}T') \subset C_s(G{-}T)$, then by Lemma 10, it holds that $T \in T' \cup C_t(G{-}T')$. By Lemma 11, it holds that $T \in \mathcal{S}_{s,t}(H_{T'})$. Consider what happens when T' is popped from the queue in line 9, and the graph $H_{T'}$ is generated in line 12. Since $T \neq T'$ (we assume that T is not printed), $T' \subseteq N_{H_{T'}}(s)$, and $T \in \mathcal{S}_{s,t}(H_{T'})$, then there exists a vertex $v \in T'$, such that $T \in \mathcal{S}_{s,t}(H^v_{T'})$ (see line 14). If $T \in \mathcal{S}^*_{s,t,k}(H^v_{T'})$, then T is pushed into the queue in line 18, and will therefore be printed. Otherwise, by Lemma 13, there exists an $S \in \mathcal{S}^*_{s,t,k}(H^v_{T'})$, such that $C_s(H^v_{T'}{-}S) \subseteq C_s(H^v_{T'}{-}T)$. By construction, we have that $C_s(H_{T'}{-}T') \subset C_s(H^v_{T'}{-}S) \subseteq C_s(H_{T'}{-}T)$. Since S is pushed into the queue in line 18, then it will be printed by the algorithm in line 10. By Theorem 8, we have that $S \in \mathcal{S}_{s,t,k}(G)$ is printed by the algorithm, where $T' \prec S \preceq T$, contradicting our assumption that T' is maximal with respect to the partial order $\prec$. □

THEOREM 10. *Let $S_1, S_2 \in \mathcal{S}_{s,t,k}(G)$. If $S_1 \prec S_2$, then S_1 is printed before S_2 by Algorithm* SmallMinimalSeps.

Proof. By Theorem 9, both S_1 and S_2 are printed by the algorithm. Consider the point in time where S_2 is pushed into the queue Q.

1. **Case 1**: $S_1 \in \mathcal{M}$. In that case, when S_2 is pushed into the queue, S_1 has already been printed, and hence S_1 is printed before S_2.
2. **Case 2**: $S_1 \in Q$. Since Q is a priority queue sorted according to $\prec$, then S_1 will be popped from the queue Q (in line 9), and printed (in line 10) before S_2 is popped (and printed).

3. **Case 3**: S_1 is generated and inserted into the queue *after* S_2 is printed. In that case, by the workings of the algorithm, $S_1 \in \mathcal{S}_{s,t,k}(H^v_{S_2})$ for some $v \in S_2$ (see lines 13–18). By Lemma 9, $S_1 \in \mathcal{S}_{s,t,k}(H^v_{S_2}) \subseteq \mathcal{S}_{s,t,k}(H_{S_2})$. By Lemma 11, if $S_1 \in \mathcal{S}_{s,t,k}(H_{S_2})$, then $S_1 \in \mathcal{S}_{s,t,k}(G)$ where $S_1 \subseteq S_2 \cup C_t(G{-}S_2)$. By Lemma 10, we have that $C_s(G{-}S_2) \subseteq C_s(G{-}S_1)$; a contradiction. Therefore, only cases 1 and 2 are possible, which means that S_1 is printed before S_2.

□

THEOREM 11. *The delay between the printing of minimal s,t-separators whose size is at most k is $O(k^2 4^k(n+m))$.*

Proof. The size of the queue Q and the data structure $\mathcal{M}$, can be at most n^k. We make the standard assumption that these data structures allow logarithmic insertion and extraction, which take time $O(k \log n)$. Applying Theorem 5, which states that there are at most 4^k important separators that can be found in time $O(k4^k(n+m))$, we get that the loop in lines (13)-(18) runs in time: $O(k \cdot (n + 4^k \cdot k \cdot (n+m) + k \cdot 4^k \cdot \log n)$. Overall, the delay is $O(4^k k^2(n+m))$. □

C PROOFS FROM SECTION 6

We prove that $\mathcal{S}_{s,t}(G, \overline{U}) = \mathcal{S}_{s,t}(\text{Sat}(G,U))$. We proceed by a series of lemmas.

Lemma 14. Let $u \in \texttt{V}(G)$ such that $N_G[u]$ forms a clique. Then $u \notin S$ for every $S \in \mathcal{S}_{s,t}(G)$.

Proof. Let $S \in \mathcal{S}_{s,t}(G)$. By Lemma 1, $G{-}S$ contains two full connected components $C_s(G{-}S)$ and $C_t(G{-}S)$ containing s and t respectively, such that $S = N_G(C_s(G{-}S)) = N_G(C_t(G{-}S))$. Therefore, if $u \in S$, then it has two neighbors $v_1 \in C_s(G{-}S)$ and $v_2 \in C_t(G{-}S)$ that are connected by an edge (because $N_G[u]$ is a clique). But then, there is an s,t-path in $G{-}S$ that avoids S, which contradicts the fact that S is an s,t-separator. □

Lemma 15. If $S \in \mathcal{S}_{s,t}(G, \overline{u})$, there exists a connected component $C_u \in \mathcal{C}(G{-}S)$ such that $N_G[u] \subseteq C_u \cup S$.

Proof. Let $C_u \in \mathcal{C}(G{-}S)$ be the connected component that contains u. Such a component must exist because $u \notin S$. If $N_G(u) \not\subseteq C_u \cup S$, then there exists a vertex $v \in N_G(u)$ that resides in a connected component $C_v \in \mathcal{C}(G{-}S)$ distinct from C_u. But this is a contradiction because, by definition, $(u,v) \in \texttt{E}(G)$. Hence, $C_v = C_u$, and this proves the claim. □

Lemma 16. Let $u \in \texttt{V}(G)$. Then $\mathcal{S}_{s,t}(G, \overline{u}) = \mathcal{S}_{s,t}(\text{Sat}(G, \{u\}))$.

Proof. Let $S \in \mathcal{S}_{s,t}(G, \overline{u})$. By Lemma 15, there exists a connected component $C_u \in \mathcal{C}(G{-}S)$ that contains u, where $N_G[u] \subseteq C_u \cup S$. Therefore, no added edge in $\texttt{E}(\text{Sat}(G,\{u\})) \setminus \texttt{E}(G)$ connects vertices in distinct connected components in $\mathcal{C}(G{-}S)$. Hence, S separates s and t also in $\text{Sat}(G,\{u\})$. Since the addition of edges cannot eliminate any path between s and t, we get that S is a minimal s,t-separator also in $\text{Sat}(G,\{u\})$.

Now, let $S \in \mathcal{S}_{s,t}(\text{Sat}(G,\{u\}))$. Hence, $N_G[u]$ is a clique in $\text{Sat}(G,\{u\})$. By Lemma 14, $u \notin S$. Since G is a subgraph of $\text{Sat}(G,\{u\})$, then if S separates s from t in $\text{Sat}(G,\{u\})$, it must separate s from t in G. Hence, S is an s,t-separator in G where $u \notin S$. It is left to show that S is a *minimal* s,t-separator in G. To that end, we show that the connected components $C_s \stackrel{\text{def}}{=} C_s(\text{Sat}(G,\{u\}){-}S), C_t \stackrel{\text{def}}{=} C_t(\text{Sat}(G,\{u\}){-}S)$, containing s and t respectively, are full connected components of S also in G. That is, we show that $S = N_G(C_s) = N_G(C_t)$. By Lemma 1, this proves that $S \in \mathcal{S}_{s,t}(G, \overline{u}) \subseteq \mathcal{S}_{s,t}(G)$.

Denote by $D_s, D_t \in \mathcal{C}(G{-}S)$ the connected components containing s and t respectively in $G{-}S$. Since $G[D_s]$ ($G[D_t]$) is connected, $D_s \cap S = \emptyset$ ($D_t \cap S = \emptyset$), and $s \in D_s$ ($t \in D_t$), then $D_s \subseteq C_s$ ($D_t \subseteq C_t$). We now prove that $C_s \subseteq D_s$. We first consider the case where $u \notin C_s$. Hence, by definition of connected component of $G{-}S$, we have that $N_G[u] \cap C_s = \emptyset$. Since the only added edges are between vertices in $N_G(u)$, then $\texttt{E}(\text{Sat}(G,u)[C_s]) = \texttt{E}(G[C_s])$. Therefore C_s is a connected component containing s also in $G{-}S$, thus $C_s \subseteq D_s$, and $C_s = D_s$. Since $N_G[u] \cap C_s = \emptyset$, then $N_G(C_s) = N_{\text{Sat}(G,\{u\})}(C_s) = S$ as required.

We now consider the case where $u \in C_s$, and suppose, by way of contradiction, that $C_s \not\subseteq D_s$. Let $v \in C_s \setminus D_s$. This means that there is a path from s to v in $\text{Sat}(G,\{u\})$ that avoids S. Let P denote the shortest such path. Then P passes through a single edge $(y,w) \in \texttt{E}(\text{Sat}(G,u)) \setminus \texttt{E}(G)$. In other words, there is a path P_{vy} from v to y in G that avoids S, and a path P_{sw} from s to w in G that avoids S. In particular, $\texttt{V}(P_{sw}) \subseteq D_s$. By construction, $\{y,w\} \subseteq N_G(u)$. Since $\texttt{V}(P_{ws}) \cap S = \emptyset$, $w \in N_G(u)$, and $\{u,w,y\} \cap S = \emptyset$, this means that $\{u,w,y\} \subseteq D_s$. But this means that the path $P_{vy}uP_{ws}$ is contained in G, and avoids S. Consequently, $v \in D_s$, and we arrive at a contradiction. Hence, $D_s = C_s$. Since $u \in C_s$, we get that $N_G(C_s) = N_{\mathrm{Sat}(G,\{u\})}(C_s)$, making C_s a full connected component of S also in G. $\square$

THEOREM 12. $\mathcal{S}_{s,t}(G, \overline{U}) = \mathcal{S}_{s,t}(\mathrm{Sat}(G, U))$.

Proof. The fact that $\mathcal{S}_{s,t}(G, \overline{U}) = \mathcal{S}_{s,t}(\mathrm{Sat}(G, U))$ follows from Lemma 16 by induction on $|U|$.

Let $0 \leq k \leq n$ be an integer, and $\mathcal{S}_{s,t}(G, \overline{U})^k$ and $\mathcal{S}_{s,t}(\mathrm{Sat}(G, U))^k$ denote the sets of minimal s,t-separators in $\mathcal{S}_{s,t}(G, \overline{U})$ and $\mathcal{S}_{s,t}(\mathrm{Sat}(G, U))$ whose size is exactly k, respectively. Since $\mathcal{S}_{s,t}(G, \overline{U}) = \mathcal{S}_{s,t}(\mathrm{Sat}(G, U))$, then $\mathcal{S}_{s,t}(G, \overline{U})^k = \mathcal{S}_{s,t}(\mathrm{Sat}(G, U))^k$ for every integer $0 \leq k \leq n$. In particular, this is the case for $k = \kappa_{s,t}(G, \overline{U}) = \kappa_{s,t}(\mathrm{Sat}(G, U))$. Hence, $\mathcal{L}_{s,t}(G, \overline{U}) = \mathcal{L}_{s,t}(\mathrm{Sat}(G, U))$. $\square$

THEOREM 13. *Let S be an s,t-separator of G. There exists an s,t-separator S' printed by the algorithm where $S' \subseteq S$.*

Proof. Let T be an s,t-separator of G, and suppose, by way of contradiction that neither T, nor any of its subsets are printed. Every triple $\langle H, S, I\rangle$ pushed into the queue Q in lines 3 and 11 corresponds to a pair of inclusion/exclusion constraints that restrict the set of s,t-separators to those that include vertices I, and exclude vertices $U \subseteq \mathbb{V}(G)$ that have been saturated in G (i.e., to form H). Let $\langle H, S, I\rangle$ be the triple, inserted into Q, where: (1) $I \subseteq T$, and (2) $U \subseteq \mathbb{V}(G)\backslash T$, which maximizes $|I| + |U|$. Note that such a triple $\langle H, S, I\rangle$ must exist because the first triple pushed into the queue Q in line 3 is $\langle G, S, \emptyset\rangle$ where $S \in \mathcal{L}_{s,t}(G)$, $I = \emptyset \subseteq T$, and no vertex of G has yet been saturated and hence $U = \emptyset \subseteq \mathbb{V}(G)\backslash T$.

Let $S\backslash I = \{v_1, \dots, v_q\}$. By our assumption, $S \not\subseteq T$. Let $\ell \leq q$ be the smallest index such that $v_\ell \notin T$. In other words, $\{v_1, \dots, v_{\ell-1}\} \subseteq T$, and $v_\ell \notin T$. In the ℓth iteration of the loop in lines 7-11, the algorithm generates a triple $\langle H_\ell, S_\ell, I_\ell\rangle$, where $I_\ell \stackrel{\text{def}}{=} I \cup \{v_1, \dots, v_{\ell-1}\} \subseteq T$, and H_ℓ is the graph that metrializes the condition of excluding $U - \ell \stackrel{\text{def}}{=} U \cup \{v_\ell\}$. In other wirds, the algoirithm generates a triple with inclusion constraints $I \subseteq I_\ell \subseteq T$, and exclusion constraint $U_\ell \stackrel{\text{def}}{=} U \cup \{v_\ell\} \subset U$, where $U_\ell \subseteq \mathbb{V}(G)\backslash T$, and $|U_\ell| > |U|$. But then, $\langle H, S, I\rangle$ does not maximize $|I| + |U|$; a contradiction. $\square$

D MINIMAL SEPARATORS AND CHORDLESS s, t-PATHS

In this section we show that given a set $I \subseteq \mathbb{V}(G)$, it is NP-hard to decide whether there exists a minimal s,t-separator $S \in \mathcal{S}_{s,t}(G)$ such that $I \subset S$. We prove this by showing a reduction from the problem 3-IN-A-PATH that asks whether there is an induced (or chordless) path containing three given terminals. Bienstock Bienstock [1991] has shown that deciding whether two terminals belong to an induced cycle is NP-hard. From this, it is easy to show that the 3-IN-A-PATH problem is NP-hard even for graphs whose degree is at most three Derhy and Picouleau [2009]. In fact, even deciding whether there is such a path of length at most k was shown to be $W[1]$-complete with respect to the length parameter k Haas and Hoffmann [2006]. The related problem, called THREE-IN-A-TREE, for deciding whether there is an induced tree containing three terminals, is in PTIME Lai et al. [2020].

Theorem 17. Let $v \in \mathbb{V}(G)$. There exists a minimal s,t-separator that includes v if and only if there exists a chordless s,t-path through v.

Proof. Let $S \in \mathcal{S}_{s,t}(G)$ where $v \in S$, and let $C_s(G{-}S)$, $C_t(G{-}S)$ denote the connected components of $G{-}S$ that contain s and t respectively. By Lemma 1, there exists a path from s to v where all the internal vertices belong to $C_s(G{-}S)$. Let P_{sv} denote the shortest such path. Likewise, let P_{vt} denote the shortest path from v to t where all internal vertices belong to $C_t(G{-}S)$. Clearly, P_{sv} and P_{vt} are both chordless paths. Since $C_s(G{-}S) \cap C_t(G{-}S) = \emptyset$, then $\mathbb{V}(P_{sv}) \cap \mathbb{V}(P_{vt}) = \{v\}$. Since $S \in \mathcal{S}_{s,t}(G)$, then there are no edges between vertices in $C_s(G{-}S)$ and vertices in $C_t(G{-}S)$. Consequently, there are no edges between vertices in $\mathbb{V}(P_{sv})$ and $\mathbb{V}(P_{vt})$. Therefore, the path $P_{sv}P_{vt}$ is a chordless s,t-path that passes through v. In other words, if $v \in S$, then there is an induced s,t-path through v.

Let $P = s, a_1, \dots, a_k, v, b_1, \dots, b_\ell, t$ denote a simple, chordless s,t-path through v. If $v \in N_G(s)$ ($v \in N_G(t)$), then $k = 0$ ($\ell = 0$). Contract all edges on the sub-path $P_a \stackrel{\text{def}}{=} (s, a_1, \dots, a_k)$ such that P_a is reduced to an edge (s, v). Likewise, contract all edges on the sub-path $P_b \stackrel{\text{def}}{=} (b_1, \dots, b_\ell, t)$ such that P_b is reduced to an edge (v, t). Denote the resulting graph by G'. Since P is chordless, then there are no edges between (a_i, b_j) for all $i \in [1, k]$ and all $j \in [1, \ell]$. Therefore, following the contraction, s and t are not adjacent in the new graph G', and hence separable.

Let $S' \in \mathcal{S}_{s,t}(G')$ be a minimal s,t-separator in G'. By construction, $v \in N_{G'}(s) \cap N_{G'}(t)$, and hence $v \in S'$. It is left to show that $S' \in \mathcal{S}_{s,t}(G)$. Let $C_s(G'{-}S')$ and $C_t(G'{-}S')$ denote the full connected components of $G'{-}S'$ containing s and t respectively. Define $D_s(G{-}S') \stackrel{\text{def}}{=} C_s(G'{-}S') \cup \{a_1, \dots, a_k\}$ and $D_t(G{-}S') \stackrel{\text{def}}{=} C_t(G'{-}S') \cup \{b_1, \dots, b_\ell\}$. By construction, $D_s(G{-}S')$ and $D_t(G{-}S')$ are disjoint, non-adjacent, and $G[D_s(G{-}S')]$ ($G[D_t(G{-}S')]$) are both connected components in G. Since $C_s(G'{-}S')$ and $C_t(G'{-}S')$ are full components of S' in G', and $D_s(G{-}S') \supseteq C_s(G'{-}S')$ and $D_t(G{-}S') \supseteq C_t(G'{-}S')$, then $D_s(G{-}S')$ and $D_t(G{-}S')$ are full components of S' in G. By Lemma 1, $S' \in \mathcal{S}_{s,t}(G)$. $\square$

Theorem 17 provides a characterization of when a vertex v is included in a minimal s,t-separator. By reduction from the 3-IN-A-PATH problem we conclude that deciding whether there is a minimal s,t-separator containing a subset $I \subseteq \mathbb{V}(G)$ is an NP-complete problem.