A GENERALIZATION OF INFORMATION CONSERVATION LAWS

We propose the following iformation theoretic relations:

Proposition 1 *Let $(X^n, Y^n) \sim P_{X^n,Y^n}$ and let $k < n$. Then :*

1. TE decomposition of DI:
$$I(\mathrm{D} \circ X^n \to Y^n) = \sum_{i=1}^{n-1} T_{i+1}^{X \to Y}(i,i). \tag{6}$$

2. TE conservation:
$$I(X^n; Y^n) = \sum_{i=1}^{n-1} T_{i+1}^{X \to Y}(i,i) + T_{i+1}^{Y \to X}(i,i) + I_{\mathsf{inst}}(X^n, Y^n). \tag{7}$$

3. DI chain rule:
$$I(\mathrm{D}^k \circ X^n \to Y^n) = I(\mathrm{D}^{k+1} \circ X^n \to Y^n) + \sum_{i=1}^{n} I(X_{i-k}; Y_i | Y^{i-1}). \tag{8}$$

The visual proof is given separately for each equation:

Transfer entropy decomposition of directed information (equation (6) Note that an $X \to Y$ ($Y \to X$) TE term corresponds to a sub-column (row), whose length, orientation and starting index are determined by (k, l, n). Specifically, $T_{i+1}^{X \to Y}(i,i)$ corresponds to a sub-column of length i that begins on the $(i+1)$th column. The proof therefore follows simply from decomposing the subtriangular matrix that corresponds to $I(X^n \to Y^n)$ into column elements along the first row.

$$\begin{pmatrix} & \boldsymbol{I}_{1,2}^{X,Y} & \boldsymbol{I}_{1,3}^{X,Y} & \cdots & \boldsymbol{I}_{1,n}^{X,Y} \\ & & \boldsymbol{I}_{2,3}^{X,Y} & \ddots & \vdots \\ & & & \ddots & \vdots \\ & & & & \boldsymbol{I}_{n-1,n}^{X,Y} \end{pmatrix} = \begin{pmatrix} & \boldsymbol{I}_{1,2}^{X,Y} & \boldsymbol{I}_{1,3}^{X,Y} & \cdots & \boldsymbol{I}_{1,n}^{X,Y} \\ & & \boldsymbol{I}_{2,3}^{X,Y} & \ddots & \vdots \\ & & & \ddots & \vdots \\ & & & & \boldsymbol{I}_{n-1,n}^{X,Y} \end{pmatrix} \tag{9}$$

$$I(\mathrm{D} \circ X^n \to Y^n) = T_2^{X \to Y}(1,1) + T_3^{X \to Y}(2,2) + \cdots + T_n^{X \to Y}(n-1, n-1)$$

We have a similar decomposition of $I(\mathrm{D} \circ Y^n \to X^n)$ in terms of $T^{Y \to X}$, which corresponds to decomposition of the corresponding lower subtriangular matrix into rows.

2. Conservation of transfer entropy (equation (7)): The proof utilizes the observation of Proposition 1. We decompose MI, which is given by the entire matrix, into the upper and lower subtriangulars (excluding the main diagonal), and the main diagonal. As noted in the main text, $I_{inst}(X^n, Y^n)$ corresponds to the main diagonal (black), $T_{i+1}^{X \to Y}(i,i)$ corresponds to a sub-column and $T_{i+1}^{Y \to X}(i,i)$ corresponds to a sub-row. We therefore have

$$\begin{pmatrix} \boldsymbol{I}_{1,1}^{X,Y} & \boldsymbol{I}_{1,2}^{X,Y} & \boldsymbol{I}_{1,3}^{X,Y} & \boldsymbol{I}_{1,4}^{X,Y} & \cdots & \boldsymbol{I}_{1,n}^{X,Y} \\ \boldsymbol{I}_{2,1}^{X,Y} & \boldsymbol{I}_{2,2}^{X,Y} & \boldsymbol{I}_{2,3}^{X,Y} & \boldsymbol{I}_{2,4}^{X,Y} & \ddots & \vdots \\ \boldsymbol{I}_{3,1}^{X,Y} & \boldsymbol{I}_{3,2}^{X,Y} & \boldsymbol{I}_{3,3}^{X,Y} & \boldsymbol{I}_{3,4}^{X,Y} & \ddots & \vdots \\ \boldsymbol{I}_{4,1}^{X,Y} & \boldsymbol{I}_{4,2}^{X,Y} & \boldsymbol{I}_{4,3}^{X,Y} & \boldsymbol{I}_{4,4}^{X,Y} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \boldsymbol{I}_{n-1,n}^{X,Y} \\ \boldsymbol{I}_{n,1}^{X,Y} & \cdots & \cdots & \cdots & \boldsymbol{I}_{n,n-1}^{X,Y} & \boldsymbol{I}_{n,n}^{X,Y} \end{pmatrix} \tag{10}$$

$$\begin{aligned} I(X^n; Y^n) = &\left(T_2^{X \to Y}(1,1) + T_3^{X \to Y}(2,2) + \cdots + T_n^{X \to Y}(n-1, n-1)\right) \\ &+ I_{inst}(X^n, Y^n) \\ &+ \left(T_2^{Y \to X}(1,1) + T_3^{Y \to X}(2,2) + \cdots + T_n^{Y \to X}(n-1, n-1)\right) \end{aligned}$$

2. Directed information chain rule (equation (8)): The relation follows from noting that a delayed DI term $I(D^{k+1} \circ X^n \to Y^n)$ corresponds to a subtriangular element, which forms the one-step reduced DI element $I(D^k \circ X^n \to Y^n)$ when combined with the appropriate sub-diagonal, which, in turn, correspond to a 'delayed' instantaneous MI term. For example, when $k = 0$, it is given by

$$\begin{pmatrix} \boldsymbol{I}^{X,Y}_{1,1} & \boldsymbol{I}^{X,Y}_{1,2} & \boldsymbol{I}^{X,Y}_{1,3} & \cdots & \boldsymbol{I}^{X,Y}_{1,n} \\ & \boldsymbol{I}^{X,Y}_{2,2} & \boldsymbol{I}^{X,Y}_{2,3} & \ddots & \vdots \\ & & \boldsymbol{I}^{X,Y}_{3,3} & \ddots & \vdots \\ & & & \ddots & \boldsymbol{I}^{X,Y}_{n-1,n} \\ & & & & \boldsymbol{I}^{X,Y}_{n,n} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}^{X,Y}_{1,1} & \boldsymbol{I}^{X,Y}_{1,2} & \boldsymbol{I}^{X,Y}_{1,3} & \cdots & \boldsymbol{I}^{X,Y}_{1,n} \\ & \boldsymbol{I}^{X,Y}_{2,2} & \boldsymbol{I}^{X,Y}_{2,3} & \ddots & \vdots \\ & & \boldsymbol{I}^{X,Y}_{3,3} & \ddots & \vdots \\ & & & \ddots & \boldsymbol{I}^{X,Y}_{n-1,n} \\ & & & & \boldsymbol{I}^{X,Y}_{n,n} \end{pmatrix}$$

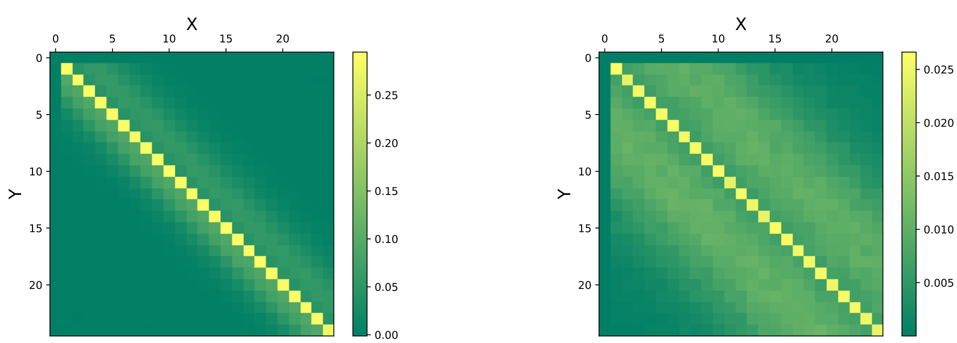
$$I(X^n \to Y^n) = I_{inst}(X^n, Y^n) + I(\mathrm{D} \circ X^n \to Y^n)$$

B Numerical Example - Visual Patterns

We demonstrate the utility of the InfoMat as a visualization tool for information flows in sequential systems through the following example. We let (X^n, Y^n) be jointly Gaussian, whose evolution is determined by the following sequential relation

$$X_i = \sum_{j=0}^{i-1} \alpha_j^X X_{i-j} + \alpha_j^Y Y_{i-j} + N_i^X$$
$$Y_i = \sum_{j=0}^{i-1} \beta_j^X X_{i-j} + \beta_j^Y Y_{i-j} + N_i^Y,$$

where $(N_i^X, N_i^Y)_{i=1}^n$ are samples of independent Gaussian innovation processes. We visualize the InfoMat structure for several settings of $(\alpha_j^X, \alpha_j^Y, \beta_j^X, \beta_j^Y)_{j=1}^n$, large weight values (strong linear connection) in Figure 2a and a small weight value (weak linear connection) in Figure 2a, considering $\alpha_j^X = \beta_j^X = \alpha_j^Y = \beta_j^Y = \gamma$ for $j = 1, \ldots, n$. We note that the symmetric choice of weights translated into a symmetric InfoMat, whose spread depends on the magnitude of γ.

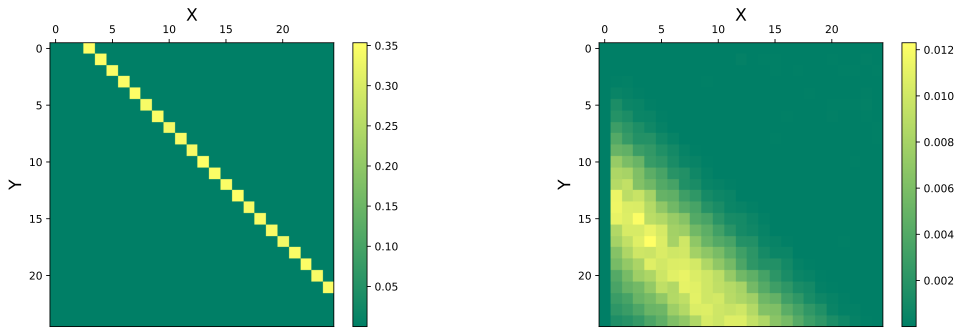


(a) Linear strong, $\gamma = 0.5$

(b) Linear weak, $\gamma = 0.12$

Figure 2: Visualization of the InfoMat under the sequential linear relation with $n = 25$.

Next, we consider a case in which the influence in one direction is bigger than the other, by changing the magnitude of the weights. By choosing bigger values for (α^X, β^X), we increase the effect in the direction $X \to Y$, which is then translated into bigger values in the upper triangular of $\boldsymbol{I}^{X,Y}$. Similarly, while increasing (α^Y, β^Y) increases the effect in $Y \to X$, with a similar effect on $\boldsymbol{I}^{X,Y}$ for the lower triangular. In Figure 3a we consider a simple delayed relation $Y_i = \gamma X_{i-3} + N_i$, resulting in a single shifted diagonal, whose shift is determined by the delay value. In Figure 3b we consider increasing weights that form a trade-off with the inherent decay in dependence, i.e., $\alpha_i^X = \beta_i^X = 3i/10$ and $\alpha_i^Y = \beta_i^Y = i/100$, resulting in bigger transfer in the direction $Y \to X$. This confirms that the underlying information transfer can be visualized through the InfoMat.



(a) X^n are i.i.d. and $Y_i = \gamma X_{i-3} + N_i$

(b) Linear dependence, increasing weights.

Figure 3: InfoMat under asymmetric linear relations.

C VISUALIZATION ON PHYSIOLOGICAL DATA

We demonstrate the utility of the visualization tool on physiological data that consists of $34,000$ heart and breath rate samples Rigney et al. (1993). We divide the long sequence into overlapping subsequences of length $n = 20$ and estimate the InfoMat elements via Gaussian MI approximation. We consider X^n and Y^n to be the heart and breath rates, respectively. As seen in Figure 4, most information transfer occurs in the lower triangular, which corresponds to a bigger effect in the direction 'breath'→'heart'. This can be also verified by straightforward calculation

$$\hat{I}(\mathrm{D} \circ X^n \to Y^n) = 0.14 < 0.41 = \hat{I}(\mathrm{D} \circ Y^n \to X^n), \quad \hat{I}_{inst}(X^n, Y^n) = 0.034$$

This further demonstrates the power of the InfoMat as a visualization tool, while providing a finer granularity to information transfer than previous tools.

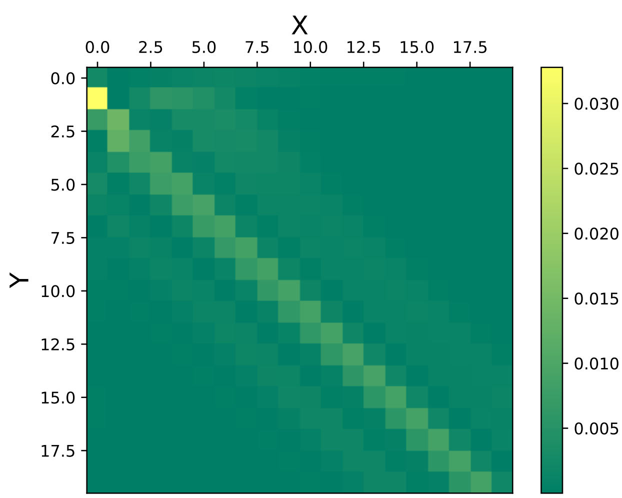


Figure 4: Physiological data. Larger effect in observed in 'breath'→'heart' direction.