SAPD+ : An Accelerated Stochastic Method for Nonconvex-Concave Minimax Problems

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Abstract

We propose a new stochastic method `SAPD+` for solving nonconvex-concave minimax problems of the form $\min \max \mathcal{L}(x, y) = f(x) + \Phi(x, y) - g(y)$, where f, g are closed convex and $\Phi(x, y)$ is a smooth function that is weakly convex in x, (strongly) concave in y. For both strongly concave and merely concave settings, `SAPD+` achieves the best known oracle complexities of $\mathcal{O}(L\kappa_y\epsilon^{-4})$ and $\mathcal{O}(L^3\epsilon^{-6})$, respectively, without assuming compactness of the problem domain, where κ_y is the condition number and L is the Lipschitz constant. We also propose `SAPD+` with variance reduction, which enjoys the best known oracle complexity of $\mathcal{O}(L\kappa_y^2\epsilon^{-3})$ for weakly convex-strongly concave setting. We demonstrate the efficiency of `SAPD+` on a distributionally robust learning problem with a nonconvex regularizer and also on a multi-class classification problem in deep learning.

1 Introduction

We consider the following saddle-point (SP) problem:

$$\min_{x\in\mathcal{X}} \max_{y\in\mathcal{Y}} \mathcal{L}(x, y) \triangleq f(x) + \Phi(x, y) - g(y), \tag{1}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are, n and m dimensional Euclidean spaces, the function $\Phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is smooth and possibly nonconvex in $x \in \mathcal{X}$ and μ_y-strongly concave in $y \in \mathcal{Y}$ for some $\mu_y \geq 0$ –with the convention that for $\mu_y = 0$, Φ is merely concave (MC) in y, and the functions f and g are closed, convex and possibly nonsmooth. In this paper, we consider a particular case of nonconvexity, i.e., we assume that $\Phi(\cdot, y)$ is weakly convex (WC) for any fixed $y \in \mathbf{dom}\, g \subset \mathcal{Y}$. Weakly convex functions constitute a rich class of non-convex functions and arise naturally in many practical settings for machine learning (ML) applications [9, 35], precise definitions will be given later in Section 2. In practice, WC assumption is widely satisfied, e.g., under smoothness –see remark 1; most of

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the work in related literature considering nonconvex-(strongly) concave SP problems provide their analyses under the premise of weak convexity. The problem (1) with $\mu_y > 0$ is called a weakly convex-strongly concave (WCSC) saddle-point problem, whereas for $\mu_y = 0$, it is called a weakly convex-merely concave (WCMC) saddle-point problem. Both problems arise frequently in many ML settings including constrained optimization of WC objectives based on Lagrangian duality [22], Generative Adversarial Networks (GAN) (where x denotes the parameters of the *generator* network whereas y represents the parameters of the *discriminator* network [13]), distributional robust learning with weakly convex loss functions such as those arising in deep learning [14, 35] and learning problems with non-decomposable losses [35].

There are two important settings for (1): (i) the *deterministic setting*, where the partial gradients of Φ are exactly available, (ii) the *stochastic setting*, where only stochastic estimates of the gradients are available. Although, recent years have witnessed significant advances in the deterministic setting [6, 17, 19, 23, 24, 25, 33, 36, 38]; our focus in this paper will be mainly on the *stochastic setting*, which is more relevant and more applicable to ML problems. Indeed, due to large-dimensions and the sheer size of the modern datasets, computing gradients exactly is either infeasible or impractical in ML practice, and gradients are often estimated stochastically based on mini-batches (randomly sampled subset of data points) as in the case of stochastic gradient-type algorithms.

There is a growing literature on the WCSC and WCMC problems in the stochastic setting. Several metrics for quantifying the quality of an approximate solution to (1) have been proposed in the literature. A common way to assess the performance is to define the *primal function* $\phi(\cdot) \triangleq \max_{y\in\mathcal{Y}} \mathcal{L}(\cdot, y)$ and measure the violation of first-order necessary conditions for the non-convex problem $\min_{x\in\mathcal{X}} \phi(x)$. Given the primal iterate sequence $\{x_k\}_{k\geq 0}$ of a stochastic SP algorithm and a threshold $\epsilon > 0$, a commonly used metric is the *gradient norm of the Moreau envelope* (GNME); indeed, the objective is to provide a bound K_ϵ such that $\mathbb{E}[\|\nabla\phi_\lambda(x_k)\|] \leq \epsilon$ for all $k \geq K_\epsilon$, where ϕ_λ denotes the Moreau envelope of the primal function ϕ –see Definitions 3, 4 and 5. Another commonly used natural metric is the *gradient norm of the primal function* $\phi(\cdot)$ [4, 17, 16, 26, 37], abbreviated as GNP, where the aim is to derive K_ϵ such that $\mathbb{E}[\|\nabla\phi(x_k)\|] \leq \epsilon$ for all $k \geq K_\epsilon$. Other metrics such as the notion of ϵ-first-order Nash equilibrium (FNE) and its generalized versions also exist in the literature [32, 33].

When using any of the aforementioned metrics, the ultimate goal is to establish a bound on the oracle (sampling) complexity, i.e., $\sum_{k=0}^{K_\epsilon} b_k$, where b_k denotes the batch-size for iteration $k \geq 0$. For the WCSC setting, it crucial to note that GNME, GNP and FNE metrics are all equivalent in the sense that convergence in either of them implies convergence in the other two metric for WCSC problems [23]. In this paper, for the WCSC setting, we adopt both GNME and GNP as the main performance metrics to analyze our algorithms; indeed, in Theorem 2 we show that, when the non-smooth part $f(\cdot) = 0$, we can convert a GNME guarantee to a GNP guarantee by incurring only little additional cost compared to the computational cost required for the GNME guarantee, and the overall worst-case complexity (in terms of worst-case dependency to the target accuracy ϵ) remains the same for both metrics. When the non-smooth part $f(\cdot) \neq 0$, we also obtain similar guarantees and show equivalence between the metrics based on GNME and *the generalized gradient mapping*. On the other hand, for the WCMC setting, we provide our guarantees in GNME metric as ϕ is not necessarily differentiable for this scenario. Moreover, our work accounts for the individual effects of L_{xx}, L_{xy}, L_{yx} and L_{yy}, i.e., the Lipschitz constants of $\nabla_x\Phi(\cdot, y)$, $\nabla_x\Phi(x, \cdot)$, $\nabla_y\Phi(x, \cdot)$ and $\nabla_y\Phi(\cdot, y)$ (see Assumption 2), respectively, instead of using the worst-case parameters $L \triangleq \max\{L_{xx}, L_{xy}, L_{yx}, L_{yy}\}$, while the majority of related work ignore the influence of these block Lipschitz constants in their analyses. We emphasize that using the worst-case parameters will lead to a theoretically conservative step sizes, and this phenomenon has been validated in the work [43].

Contributions. Table 1 summarizes the relevant existing work for WCSC and WCMC problems closest to our setting. More specifically, in Table 1, for the stochastic setting, we report the (oracle) complexity with respect to the GNP and GNME as the performance metrics for WCSC and WCMC problems, respectively, and the batch-size (number of data points in the mini-batches) required at every iteration. We also report whether the method is based on a variance-reduction (VR) technique. VR-based methods mentioned in Table 1 use a small batch-size b' all iterations except for few, where they need a large batch-size $b \geq b'$ once in every q iterations. The period q is equal to the number of times small batches are sampled consecutively plus one, and it is also an algorithm parameter. Therefore, for VR-methods, we report the batch size as a triplet (b', b, q). In the column "Compactness", we list whether achieving the specific complexity requires assuming compactness of the primal and/or dual domains.

Ref.	Complexity	Compactness	VR-based	Batchsize
	Weakly Convex-Strongly Concave (WCSC) problems			
*Rafique *et al.* [35]	$\mathcal{O}(\epsilon^{-4}\log(\epsilon^{-1}))$	(n, n)	✗	$\mathcal{O}(1)$
†Yan *et al.* [39]	$\mathcal{O}(\epsilon^{-4}\log(\epsilon^{-1}))$	(y, y)	✗	$\mathcal{O}(1)$
†Yang *et al.* [41]	$\mathcal{O}(L\kappa_y^2\epsilon^{-4})$	(n, n)	✗	$\mathcal{O}(1)$
Lin *et al.* [23]	$\mathcal{O}(L\kappa_y^3\epsilon^{-4})$	(n, y)	✗	$\mathcal{O}(\kappa_y\epsilon^{-2})$
Bot and B̈ohm [4]	$\mathcal{O}(L\kappa_y^3\epsilon^{-4})$	(n, n)	✗	$\mathcal{O}(\kappa_y\epsilon^{-2})$
‡Huang *et al.* [17]	$\mathcal{O}(\kappa_y^5\mu_y^{-1}\epsilon^{-3})$	(n, n)	✓	$\mathcal{O}(\kappa_y\epsilon^{-1}),\ \mathcal{O}(\kappa_y^2\epsilon^{-2}),\ \mathcal{O}(\kappa_y\epsilon^{-1})$
§Huang *et al.* [16]	$\tilde{\mathcal{O}}(L^{1.5}\kappa_y^{3.5}\epsilon^{-3})$	(y, y)	✓	$\mathcal{O}(\sqrt{\kappa_y})$
Luo *et al.* [26]	$\mathcal{O}(L\kappa_y^3\epsilon^{-3})$	(y, y)	✓	$\mathcal{O}(\kappa_y\epsilon^{-1}),\ \mathcal{O}(\kappa_y^2\epsilon^{-2}),\ \mathcal{O}(\kappa_y\epsilon^{-1})$
Xu *et al.* [37]	$\mathcal{O}(L\kappa_y^3\epsilon^{-3})$	(y, y)	✓	$\mathcal{O}(\kappa_y\epsilon^{-1}),\ \mathcal{O}(\kappa_y^2\epsilon^{-2}),\ \mathcal{O}(\kappa_y\epsilon^{-1})$
SAPD+, Theorem 3	$\mathcal{O}(L\kappa_y\epsilon^{-4})$	(n, n)	✗	$\mathcal{O}(1)$
SAPD+, Theorem 4	$\mathcal{O}(L\kappa_y^2\epsilon^{-3})$	(n, n)	✓	$\mathcal{O}(\kappa_y\epsilon^{-1}),\ \mathcal{O}(\kappa_y\epsilon^{-2}),\ \mathcal{O}(\epsilon^{-1})$
	Weakly Convex-Merely Concave (WCMC) problems			
Rafique *et al.* [35]	$\mathcal{O}(L^3\epsilon^{-6}\log^3(L\epsilon^{-2}))$	(y, y)	-	$\mathcal{O}(1)$
Bot and B̈ohm [4]	$\mathcal{O}(L^5\epsilon^{-8})$	(n, y)	-	$\mathcal{O}(1)$
Lin *et al.* [23]	$\mathcal{O}(L^3\epsilon^{-8})$	(n, y)	-	$\mathcal{O}(1)$
SAPD+, Theorem 5	$\mathcal{O}(L^3\epsilon^{-6})$	(n, y)	-	$\mathcal{O}(1)$

Table 1: Summary of relevant work for WCSC and WCMC problems. For the column "Compactness", we use y and n to indicate when the results require compactness and when do not require it, respectively; the first argument is for primal domain and the second is for dual domain. For batchsize, we use (b', b, q) format for VR-based methods to state *small batch* (b'), *large batch* (b), and *frequency* (q) employed within the algorithm.

Table notes: *For WCSC setting, [35] assumes $\Phi(\cdot, y) \triangleq c^\top(\cdot)y$ is weakly convex and $g(\cdot)$ is strongly convex. † In [39], $\mathcal{L} = \Phi$ and Φ need not be smooth, rather second moment of stochastic subgradients is assumed to be uniformly bounded. When Φ is L-smooth, $\Phi(\cdot, y)$ and $\Phi(x, \cdot)$ are L_Φ-Lipschitz, the results in [39] imply $\mathcal{O}(L_\Phi^2\kappa_y^2\epsilon^{-4}\log^2(\sqrt{\kappa_y}L_\Phi/\epsilon))$ complexity. ‡,§The complexity results reported here are different than those in [17, 16]. The issues in their proofs leading to the wrong complexity results are explained in Appendix I. The notation $\tilde{\mathcal{O}}$ ignores logarithmic factors.

To make the comparison of our results with the existing work easier, we provide the results in the table for the worst-case setting, where $\kappa_y \triangleq \frac{L}{\mu_y}$, and we report the ϵ-, κ_y- and L-dependency of the complexity results for the existing algorithms. That being said, our results have finer granularity in terms of their dependence to the individual effects of L_{xx}, L_{xy}, L_{yx} and L_{yy} as we mentioned earlier.

Our contributions (also summarized in section 1) are as follows:

- We propose a new stochastic method, SAPD+, based on the inexact proximal point method (iPPM). In this framework, one inexactly solves strongly convex-strongly concave (SCSC) saddle point sub-problems using an accelerated primal-dual method, SAPD [43]. In Theorem 3, we establish an oracle complexity of $\mathcal{O}(L\kappa_y\epsilon^{-4})$ for WCSC problems, and unlike the majority of existing work we do not require compactness for neither the primal nor the dual domain. To our knowledge, our bound has the best κ_y dependence in the literature; indeed, prior to this work, without using variance reduction, the best known complexity was $\mathcal{O}(L\kappa_y^2\epsilon^{-4})$ shown in [41]; hence, we establish a $\mathcal{O}(\kappa_y)$ improvement.
- We propose a variance-reduced version of SAPD+ in Theorem 4. For WCSC setting, SAPD+ using variance reduction achieves an oracle complexity of $\mathcal{O}(L\kappa_y^2\epsilon^{-3})$ –this bound has the best ϵ-dependency in the literature to our knowledge, and among all the methods with the $\mathcal{O}(\epsilon^{-3})$ complexity, our approach has the best condition number, κ_y, dependency; indeed, prior to this work, the best known complexity was $\mathcal{O}(L\kappa_y^3\epsilon^{-3})$; hence, we establish $\mathcal{O}(\kappa_y)$ factor improvement.
- For the WCMC case, our proposed algorithm SAPD+ results in $\mathcal{O}(L^3\epsilon^{-6})$ complexity, which is the best to our knowledge, improving the best known complexity by $\log^3(L/\epsilon^2)$ factor.
- Finally, we demonstrate the efficiency of SAPD+ on a distributionally robust learning problem and also on a (worst-case) multi-class classification problem in deep learning.

Notation. Throughout the paper, $\|\cdot\|$ denotes the Euclidean norm. Given $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ a closed convex function, $\mathbf{prox}_{\lambda f}(x) \triangleq \operatorname{argmin}_w f(w) + \frac{1}{2\lambda}\|w - x\|^2$ denotes the proximal map of f. Given random ω, let $\tilde{\nabla}_x\Phi(x, y; \omega)$ and $\tilde{\nabla}_y\Phi(x, y; \omega)$ denote unbiased estimators of $\nabla\Phi_x(x, y)$ and $\nabla\Phi_y(x, y)$. Moreover, given a random mini-batch $\mathcal{B} = \{\omega_i\}_{i=1}^b$, we let $\tilde{\nabla}_x\Phi_{\mathcal{B}}(x, y) \triangleq \frac{1}{b}\sum_{i=1}^b \tilde{\nabla}_x\Phi(x, y; \omega_i)$ to denote the stochastic gradient estimate based on the batch $\mathcal{B}$, and we define $\tilde{\nabla}_y\Phi_{\mathcal{B}}(\cdot, \cdot)$ similarly.

2 Preliminaries

We start with describing the notion of weak convexity.

Definition 1. *$h : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is γ-weakly convex if $x \mapsto h(x) + \frac{\gamma}{2}\|x\|^2$ is convex.*

Definition 2. *A differentiable function $h : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is L-smooth if $\exists L > 0$ such that for $\forall x, x' \in \mathbf{dom}\, h$, $\|\nabla h(x) - \nabla h(x')\| \leq L\|x - x'\|$.*

Remark 1. *If a function is L-smooth, then it is also L-weakly convex.*

Remark 1 shows that weak convexity is a rich class containing the class of smooth functions. In the rest of the paper, we consider the SP problem in (1). Next, we introduce our assumptions.

Assumption 1. *$f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ are proper, closed, convex functions. Let $\Phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be such that (i) for any $y \in \mathbf{dom}\, g \subset \mathcal{Y}$, $\Phi(\cdot, y)$ is γ-weakly convex and bounded from below; (ii) for any $x \in \mathbf{dom}\, f \subset \mathcal{X}$, $\Phi(x, \cdot)$ is μ_y-strongly concave for some $\mu_y \geq 0$; (iii) Φ is differentiable on an open set containing $\mathbf{dom}\, f \times \mathbf{dom}\, g$.*

Assumption 2. *There exist $L_{xx}, L_{yy} \geq 0$, $L_{xy}, L_{yx} > 0$ such that $\|\nabla_x \Phi(x, y) - \nabla_x \Phi(\bar{x}, \bar{y})\| \leq L_{xx}\|x - \bar{x}\| + L_{xy}\|y - \bar{y}\|$, and $\|\nabla_y \Phi(x, y) - \nabla_y \Phi(\bar{x}, \bar{y})\| \leq L_{yx}\|x - \bar{x}\| + L_{yy}\|y - \bar{y}\|$ for all $x, \bar{x} \in \mathbf{dom}\, f \subset \mathcal{X}$, and $y, \bar{y} \in \mathbf{dom}\, g \subset \mathcal{Y}$.*

Assumption 1 allows non-convexity in x while requiring (strong) concavity in the y variable. Assumption 2 is standard in the analysis of first-order methods for solving SP problems. It should be noticed that when $L_{yx} = L_{xy} = 0$, the problem in (1) can be solved separately for the primal and dual variables; hence, it is natural to assume $L_{yx}, L_{xy} > 0$.

Suppose that we implement SAPD, stated in Algorithm 1, on the SCSC problem

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \mathcal{L}(x, y) + \frac{\mu_x + \gamma}{2}\|x - x_0\|^2 \tag{2}$$

for some given $\mu_x > 0$ and $x_0 \in \mathcal{X}$ –strong convexity follows from $\mathcal{L}(\cdot, y)$ being γ-weakly convex.

Algorithm 1 SAPD Algorithm

Input: $\tau, \sigma, \theta, \mu_x, x_0, y_0, N$
$\bar{\Phi}(x, y) \leftarrow \Phi(x, y) + \frac{\mu_x + \gamma}{2}\|x - x_0\|^2$
$\tilde{q}_0 \leftarrow 0$
for $k = 0, 1, 2, ..., N$ **do**
  $\tilde{s}_k \leftarrow \tilde{\nabla}_y \Phi(x_k, y_k; \omega_k^y) + \theta \tilde{q}_k$
  $y_{k+1} \leftarrow \mathbf{prox}_{\sigma g}(y_k + \sigma \tilde{s}_k)$
  $x_{k+1} \leftarrow \mathbf{prox}_{\tau f}(x_k - \tau \tilde{\nabla}_x \bar{\Phi}(x_k, y_{k+1}; \omega_k^x))$
  $\tilde{q}_{k+1} \leftarrow \tilde{\nabla}_y \Phi(x_{k+1}, y_{k+1}; \omega_{k+1}^y) - \tilde{\nabla}_y \Phi(x_k, y_k; \omega_k^y)$
end for
Output:$(\bar{x}_N, \bar{y}_N) = \frac{1}{N}\sum_{k=0}^{N-1}(x_{k+1}, y_{k+1})$

We make the following assumption on the statistical nature of the gradient noise as in, e.g., [5, 11, 43].

Assumption 3. *Given arbitrary $x_0 \in \mathcal{X}$ and $\mu_x > 0$, let $\{x_k, y_k\}$ sequence be generated by SAPD, stated in Algorithm 1, running on (2). There exist $\delta_x, \delta_y \geq 0$ such that for all $k \geq 0$, the stochastic gradients $\tilde{\nabla}_x \Phi(x_k, y_{k+1}; \omega_k^x)$, $\tilde{\nabla}_y \Phi(x_k, y_k; \omega_k^y)$ and random sequences $\{\omega_k^x\}_k$, $\{\omega_k^y\}_k$ satisfy the conditions:*

(i) $\mathbb{E}[\tilde{\nabla}_x \Phi(x_k, y_{k+1}; \omega_k^x) | x_k, y_{k+1}] = \nabla_x \Phi(x_k, y_{k+1})$;
(ii) $\mathbb{E}[\tilde{\nabla}_y \Phi(x_k, y_k; \omega_k^y) | x_k, y_k] = \nabla_y \Phi(x_k, y_k)$;
(iii) $\mathbb{E}[\|\tilde{\nabla}_x \Phi(x_k, y_{k+1}; \omega_k^x) - \nabla_x \Phi(x_k, y_{k+1})\|^2 | x_k, y_{k+1}] \leq \delta_x^2$;
(iv) $\mathbb{E}[\|\tilde{\nabla}_y \Phi(x_k, y_k; \omega_k^y) - \nabla_y \Phi(x_k, y_k)\|^2 | x_k, y_k] \leq \delta_y^2$.

Assumption 3 says that the gradient noise conditioned on the iterates is unbiased with a finite variance[1]. Such assumptions are common in the literature, e.g., [5, 11, 43], and are satisfied when gradients are estimated from randomly sampled data points with replacement.

For WCSC minimax problems, a commonly adopted definition for ϵ-stationary is based on Moreau envelope, e.g., see [23, 39]. It is inspired by Davis and Drusvyatskiy's work [9] for solving weakly convex minimization problems. For the sake of completeness, we briefly review this idea below.

Definition 3. *Let $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be γ-weakly convex. Then, for any $\lambda \in (0, \gamma^{-1})$, Moreau envelope of ϕ is defined as $\phi_\lambda : \mathbb{R}^d \to \mathbb{R}$ such that $\phi_\lambda(x) \triangleq \min_{w \in \mathcal{X}} \phi(w) + \frac{1}{2\lambda}\|w - x\|^2$.*

[1]When we run SAPD, stated in Algorithm 1, on (2), we use the convention that $\tilde{\nabla}_x \bar{\Phi}(x_k, y_{k+1}; \omega_k^x) \triangleq \tilde{\nabla}_x \Phi(x_k, y_{k+1}; \omega_k^x) + (\mu_x + \gamma)(x_k - x_0)$.

Lemma 1. *Let $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a γ-weakly convex function. For any given $\lambda \in (0, \gamma^{-1})$, $\phi_\lambda(\cdot)$ is well-defined on $\mathcal{X}$. Moreover, $\nabla \phi_\lambda(x) = \lambda^{-1}(x - \mathbf{prox}_{\lambda\phi}(x))$ for $x \in \mathcal{X}$; hence, ϕ_λ is λ^{-1}-smooth, where $\mathbf{prox}_{\lambda\phi}(x) \triangleq \mathrm{argmin}_{w\in\mathcal{X}}\{\phi(w) + \frac{1}{2\lambda}\|w - x\|^2\}$.*

Definition 4. *Under Assumption 1, let $\phi, \phi^s : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\phi(x) \triangleq \max_{y\in\mathcal{Y}} \mathcal{L}(x, y)$ and $\phi^s(x) = \phi(x) - f(x)$ for $x \in \mathbf{dom}\, f$, i.e., $\phi^s(x) \triangleq \max_{y\in\mathcal{Y}} \Phi(x, y) - g(y)$ for $x \in \mathbf{dom}\, f$.*

Remark 2. *Under Assumption 1, since $\Phi(\cdot, y)$ is γ-weakly convex for any $y \in \mathbf{dom}\, g$, ϕ^s is γ-weakly convex[2]; hence, ϕ is also γ-weakly convex. Note that*

$$\mathbf{prox}_{\lambda\phi}(x) = \underset{w\in\mathcal{X}}{\mathrm{argmin}}\{\phi(w) + \tfrac{1}{2\lambda}\|w - x\|^2\} = \underset{w\in\mathcal{X}}{\mathrm{argmin}}\max_{y\in\mathcal{Y}} \mathcal{L}(w, y) + \tfrac{1}{2\lambda}\|w - x\|^2. \quad (3)$$

Furthermore, when $\mu_y > 0$, ϕ^s is differentiable on $\mathbf{dom}\, f$.

In the following definition, we introduce the notion of ϵ-stationary with respect to the GNME metric.

Definition 5. *A point x_ϵ is an ϵ-stationary point of a γ-weakly convex function ϕ if $\|\nabla \phi_\lambda(x_\epsilon)\| \leq \epsilon$ for some $\lambda \in (0, \gamma^{-1})$. If $\epsilon = 0$, then x_ϵ is a stationary point of ϕ.*

Thus, from Lemma 1, computing an ϵ-stationary point x_ϵ for ϕ is equivalent to searching for x_ϵ such that $\|x_\epsilon - \mathbf{prox}_{\lambda\phi}(x_\epsilon)\|$ is small. Recall that for any $\lambda \in (0, \gamma^{-1})$, $\mathbf{prox}_{\lambda\phi}(x)$ is well-defined and unique. We also observe from (3) that $\mathbf{prox}_{\lambda\phi}(\cdot)$ computation is indeed an SCSC SP problem. To compute x_ϵ such that $\|x_\epsilon - \mathbf{prox}_{\lambda\phi}(x_\epsilon)\|$ is small, it is natural to consider the iPPM algorithm – e.g., see [18]. A generic iPPM generates $\{x_0^t\}_{t\geq 0}$ such that $x_0^{t+1} \approx \mathbf{prox}_{\lambda\phi}(x_0^t)$, i.e., proximal steps are "inexactly" computed for $t \geq 0$, starting from an arbitrary given point $x_0^0 \in \mathcal{X}$.

In the next section, we describe the proposed `SAPD+` method, an iPPM algorithm employing `SAPD` to *inexactly* solve the SCSC subproblems arising in the iPPM iterations.

3 The proposed algorithm `SAPD+` and its analysis

The convergence and robustness properties of `SAPD` for SCSC SP problems are analyzed in [43]. For the WCSC SP problems, as we explained in the previous section, the main idea is to apply the iPPM framework as stated in `SAPD+` (see Algorithm 2) which requires successively solving SCSC SP problems. In the rest, the counter for iPPM outer iterations is denoted with $t \in \mathbb{Z}_+$. At each outer iteration $t \geq 1$, we inexactly compute the prox map, i.e., $x_0^{t+1} \approx \mathbf{prox}_{\lambda\phi}(x_0^t)$, which is well-defined for $\lambda \in (0, \gamma^{-1})$; hence, to derive our preliminary results, we fix $\lambda = (\mu_x + \gamma)^{-1}$ for some given $\mu_x > 0$ – thus, $\mathcal{L}(x, y) + \frac{\mu_x+\gamma}{2}\|x - x_0^t\|^2$ is SCSC in (x, y) with moduli (μ_x, μ_y) and has a unique saddle point. Consider the following SCSC SP problem:

$$\min_{x\in\mathcal{X}}\max_{y\in\mathcal{Y}} \mathcal{L}^t(x, y) \triangleq f(x) + \Phi^t(x, y) - g(y), \text{ where } \Phi^t(x, y) \triangleq \Phi(x, y) + \frac{\mu_x + \gamma}{2}\|x - x_0^t\|^2. \quad (4)$$

We will construct $\{x_0^t\}_{t=1}^T \subset \mathbf{dom}\, f$ by *inexactly* solving (4) at each outer iteration $t \in \mathbb{Z}_+$ through running `SAPD` for $N_t \in \mathbb{Z}_+$ iterations –we will specify $N_t \in \mathbb{Z}_+$ later. Next, we briefly explain the main step of `SAPD+` with `VR-flag`=**false**. The statement in line 4 of Algorithm 2 means that (x_0^{t+1}, y_0^{t+1}) is generated using `SAPD`, where is dispalyed in Algorithm 1 –indeed, `SAPD` is run on (4) for N_t iterations with `SAPD` parameters (τ, σ, θ) and starting from the initial point (x_0^t, y_0^t). To analyze the convergence of `SAPD+`, we first define the gap function $\mathcal{G}^t$ for t-th `SAPD+` iteration:

Algorithm 2 `SAPD+` Algorithm

```
1: Input: {τ, σ, θ, μ_x}, (x_0^0, y_0^0) ∈ X × Y, {N_t}_{t≥0} ∈ Z^+
2: for t = 0, 1, 2, ..., T do
3:     if VR-flag == false then
4:         (x_0^{t+1}, y_0^{t+1}) ← SAPD(τ, σ, θ, μ_x, x_0^t, y_0^t, N_t)
5:     else
6:         (x_0^{t+1}, y_0^{t+1}) ← VR-SAPD(τ, σ, θ, μ_x, x_0^t, y_0^t, N_t)
7:     end if
8: end for
```

$$\mathcal{G}^t(x, y) \triangleq \max_{y'\in\mathcal{Y}} \mathcal{L}^t(x, y') - \min_{x'\in\mathcal{X}} \mathcal{L}^t(x', y). \quad (5)$$

Recall that $\mathcal{L}^t$ is an SCSC function; therefore, *i)* it has a unique saddle point denoted by (x_*^t, y_*^t), and it is important to note that $x_*^t = \mathbf{prox}_{\lambda\phi}(x_0^t)$ for $\phi(x) = \max_{y\in\mathcal{Y}} \mathcal{L}(x, y)$ and $\lambda = (\gamma + \mu_x)^{-1}$; *ii)*

[2]One can argue that $\phi^s(\cdot) + \frac{\gamma}{2}\|\cdot\|^2$ is a pointwise supremum of convex functions.

for any $(x, y) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$, the following quantities are well-defined:

$$x_*^t(y) \triangleq \operatorname*{argmin}_{x' \in \mathcal{X}} \mathcal{L}^t(x', y), \quad y_*(x) \triangleq \operatorname*{argmax}_{y' \in \mathcal{Y}} \mathcal{L}^t(x, y') = \operatorname*{argmax}_{y' \in \mathcal{Y}} \mathcal{L}(x, y'). \tag{6}$$

Thus, it follows that $\mathcal{G}^t(x, y) = \mathcal{L}^t(x, y_*(x)) - \mathcal{L}^t(x_*^t(y), y)$. Moreover, for $(x, y) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$, we also define $\mathcal{G}(x, y) \triangleq \sup_{y' \in \mathcal{Y}} \mathcal{L}(x, y') - \inf_{x' \in \mathcal{X}} \mathcal{L}(x', y)$. Assumption 1 ensures that $\mathcal{G}$ is well defined.

Next, we first provide our oracle complexity in the GNME metric under the compactness assumption of the primal-dual domains; later, in section 3.1, we show that under a particular subdifferentiability assumption compactness requirement can be avoided.

Assumption 4. $\mathbf{dom}\, f$ *and* $\mathbf{dom}\, g$ *are compact sets.*

Theorem 1. *Suppose Assumptions 1, 2, 3, and 4 hold. Let $\mu_x = \gamma$, $\theta = 1$, τ, σ and N be chosen as*

$$N = 33 \max\{\tfrac{4}{\gamma\tau}, \tfrac{8}{\mu_y \sigma}\}, \quad \tau = \min\{\tfrac{1}{L_{yx} + L_{xx} + 2\gamma}, \tfrac{1}{L_{xy}}, \tfrac{1}{480\gamma} \cdot \tfrac{\epsilon^2}{\delta_x^2}\}, \quad \sigma = \min\{\tfrac{1}{L_{yx} + 2L_{yy}}, \tfrac{1}{4512\gamma} \cdot \tfrac{\epsilon^2}{\delta_y^2}\}. \tag{7}$$

Then, for any $\epsilon > 0$, when `VR-flag=false`, `SAPD+` guarantees ϵ-stationary, $\min_{t=0,\ldots,T} \mathbb{E}\,[\|\nabla \phi_\lambda(x_0^t)\|] \leq \epsilon$, for $T \geq 96\mathcal{G}(x_0^0, y_0^0) \cdot \frac{\gamma}{\epsilon^2} + 1$, which requires C_ϵ stochastic first-order oracle calls in total where

$$C_\epsilon = \mathcal{O}\Big(\Big(\tfrac{\max\{L_{xx}, L_{yx}, L_{xy}\}}{\gamma} + \tfrac{\max\{L_{yy}, L_{yx}\}}{\mu_y}\Big)\gamma \cdot \epsilon^{-2} + \Big(\tfrac{\delta_x^2}{\gamma} + \tfrac{\delta_y^2}{\mu_y}\Big)\gamma^2 \cdot \epsilon^{-4}\Big)\mathcal{G}(x_0^0, y_0^0).$$

Proof. See appendix A for the proof. $\square$

Remark 3. *Since $\mathbb{E}\,[\min_{t=0,\ldots,T} \|\nabla \phi_\lambda(x_0^t)\|] \leq \min_{t=0,\ldots,T} \mathbb{E}\,[\|\nabla \phi_\lambda(x_0^t)\|]$, the guarantees given in Theorem 1 also hold for achieving $\mathbb{E}\big[\min_{t=0,\ldots,T} \|\nabla \phi_\lambda(x_0^t)\|\big] \leq \epsilon$.*

Remark 4. *For any $y \in \mathbf{dom}\, g$, since $\Phi(\cdot, y)$ L_{xx}-smooth, it is necessarily L_{xx}-weakly convex; hence, $\gamma \leq L_{xx}$. To get a worst-case complexity, let*

$$L \triangleq \max\{L_{xy}, L_{yx}, L_{xx}, L_{yy}\},\ \kappa_y \triangleq L/\mu_y,\ \delta \triangleq \max\{\delta_x, \delta_y\},\ \gamma = L. \tag{8}$$

Our oracle complexity C_ϵ in Theorem 1 can be simplified as $C_\epsilon = \mathcal{O}\left(\max\{1, \frac{\delta^2}{\epsilon^2}\} \frac{\kappa_y L \mathcal{G}(x_0^0, y_0^0)}{\epsilon^2}\right)$.

In fact, Li et al. [21] (see also [42]) provide a lower complexity bound for a class of first-order stochastic algorithms that do not use variance reduction. The lower bound for finding ϵ-stationary points of smooth WCSC problems in GNP metric is $\Omega(L\Delta_\phi(\sqrt{\kappa_y}\epsilon^{-2} + \kappa_y^{\frac{1}{3}}\epsilon^{-4}))$, where $\Delta_\phi \triangleq \phi(x_0) - \min_{x \in \mathcal{X}} \phi(x)$ and x_0 is an arbitrary initial point.

Consider $\phi = f + \phi^s$ as given in definition 4. For $\lambda > 0$, the map $G_\lambda : \mathbb{R}^d \to \mathbb{R}^d$ defined as

$$G_\lambda(\tilde{x}) \triangleq \frac{1}{\lambda}[\tilde{x} - \mathbf{prox}_{\lambda f}(\tilde{x} - \lambda \nabla \phi^s(\tilde{x}))] \tag{9}$$

is called the *generalized gradient mapping* and its norm is frequently used in optimization for assessing stationarity (see e.g. [10]). Theorem 1 provides guarantees in the GNME metric. Theorem 2 shows that given x_ϵ, an ϵ-stationary point in GNME metric (see definition 5) in expectation, we can generate $\tilde{x}$ such that $\mathbb{E}[\|G_\lambda(\tilde{x})\|] \leq \epsilon$ for some $\lambda > 0$, i.e., an ϵ-stationary point in *generalized gradient mapping* metric, within $\tilde{\mathcal{O}}(1/\epsilon^2)$ `SAPD` iterations. Indeed, when $f(\cdot) = 0$, this metric and the GNP metric are the same.

Theorem 2. *Suppose Assumptions 1, 2, 3 hold, and x_ϵ, an ϵ-stationary point for the γ-weakly convex function $\phi(\cdot) = \max_{y \in \mathcal{Y}} \mathcal{L}(\cdot, y)$ in expectation, i.e., $\mathbb{E}[\|\nabla \phi_\lambda(x_\epsilon)\|] \leq \frac{\epsilon}{2}$ for some fixed $\lambda \in (0, \gamma^{-1})$ is given. Then, there exists some τ, σ, θ – see eq.* (35) *in appendix B, such that initialized from x_ϵ, `SAPD`, stated in Algorithm 1, can generate $\tilde{x}$ such that $\mathbb{E}\big[\|G_\lambda(\tilde{x})\| \leq \epsilon$ within $\tilde{\mathcal{O}}(\frac{1}{\epsilon^2})$ stochastic first-order oracle calls, where $\phi^s(\cdot) = \max_{y \in \mathcal{Y}} \Phi(\cdot, y) - g(y)$ so that $\phi = f + \phi^s$ as in Definition 4.*

Proof. See appendix B for the proof. $\square$

Remark 5. *Based on Remark 3, the random vector x_ε in Theorem 2 can be chosen as $x_0^{t_*}$ where $t_* \triangleq \operatorname{argmin}_{0 \leq t \leq T} \|\nabla \phi_\lambda(x_0^t)\|$. However, since t_* can not be computed in practice, we provide an alternative method in the appendix to generate a point x_ϵ such that $\mathbb{E}[\|\nabla \phi_\lambda(x_\epsilon)\|] \leq \epsilon$ within $\tilde{\mathcal{O}}\left(\frac{L\kappa_y \mathcal{G}(x_0^0, y_0^0)}{\epsilon^2} + \frac{L\kappa_y \delta^2 \mathcal{G}(x_0^0, y_0^0)}{\epsilon^4}\right)$ stochastic first-order oracle calls – see Theorem 7 in appendix D.*

3.1 Relaxing the compactness assumption

In Theorem 1, we assume that $\mathbf{dom}\, f$ and $\mathbf{dom}\, g$ are compact sets, e.g., $f(\cdot) = \mathbb{1}_X(\cdot)$ and $g(\cdot) = \mathbb{1}_Y(\cdot)$, where $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ are compact convex sets. In this section, we show that `SAPD+` can also handle unbounded domains under the following assumption.

Assumption 5. *For f and g closed convex, suppose $\exists B_f,\ B_g > 0$ such that $\inf\{\|s_f\| :\ s_f \in \partial f(x)\} \leq B_f$ for all $x \in \mathbf{dom}\, f$ and $\inf\{\|s_g\| :\ s_g \in \partial g(y)\} \leq B_g$ for all $y \in \mathbf{dom}\, g$.*

Remark 6. *Assumption 5 holds when f is an indicator function of a closed convex set (not necessarily bounded) or for $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\mathbf{dom}\, f$ is open and f is Lipschitz. Two important examples for this scenario are: (i) $f(\cdot) = 0$, (ii) f is a norm, e.g., ℓ_1-, ℓ_2-, or the Nuclear norms.*

The existing work based on iPPM framework either require compactness, e.g., [39], or some special structure on $\mathcal{L}$, e.g., [35]. This is also true for VR-based methods, e.g.,[16, 26, 37]. To our knowledge, ours is the first one to overcome this difficulty and strictly improve the best known complexity bound for the WCSC setting without compactness assumption; moreover, the same idea also works simultaneously with a variance reduction technique that will be discussed later (see section 4). Finally, the same trick for removing compactness assumption for the WCSC setting also helps removing the compactness assumption for the primal domain in WCMC setting and we still improve the best known complexity for this setting as well (see section 5).

Remark 7. *In [23], when $f = g = 0$, boundedness of dual space is required while Assumption 5 is a weaker requirement. Furthermore, based on the discussion with the authors of [39], compactness of the domain is needed for their proof to hold. In [17], the sub-level set $\{x : \phi(x) + f(x) \leq \alpha\}$ is required to be compact for all $\alpha > 0$. There are simple convex functions that do not satisfy this condition such as $f(x) = \max\{0, x\}$. Bot and Böhm [4] use milder assumptions than [23] without requiring compactness; however, their complexity is the same as the complexity of [23].*

Theorem 3. *The result of Theorem 1 continues to hold, if one replaces the compact domain assumption, i.e., Assumption 4, with Assumption 5.*

Proof. See appendix E for the proof. □

4 Variance reduction

Variance reduction techniques have been found useful for solving SCSC problems in finite sum form, e.g., [34] –see also [5] using Richardson-Romberg extrapolation in solving SCSC problems with noisy gradients to obtain improved practical performance.

In this section, we equip `SAPD+` with SPIDER variance reduction technique [12], a variant of SARAH [31, 31] More precisely, for inexactly solving SCSC subproblems given in (4), we propose using `VR-SAPD` as stated in Algorithm 3. Note `VR-SAPD` employs a large batchsize of b in every q iterations and use small batchsizes of b'_x and b'_y for the rest. We prove that `SAPD+` using variance reduction, i.e., with `VR-flag`=**true**, achieves an oracle complexity of $\mathcal{O}(L\kappa_y^2\epsilon^{-3})$; hence, we show an $\mathcal{O}(\kappa_y)$ factor improvement over the best known complexity in the literature to our knowledge.

Algorithm 3 `VR-SAPD` Algorithm

Input: $\tau, \sigma, \theta, \mu_x, x_0, y_0, N, b, b'_x, b'_y, q$
$\bar{\Phi}(x, y) \leftarrow \Phi(x, y) + \frac{\mu_x+\gamma}{2}\|x - x_0\|^2$
Let $\mathcal{B}_0^x$,$\mathcal{B}_0^y$ be random mini-batch samples with $|\mathcal{B}_0^x| = |\mathcal{B}_0^y| = b$
$w_0 \leftarrow \tilde{\nabla}_y \Phi_{\mathcal{B}_0^y}(x_0, y_0), \quad \tilde{s}_0 \leftarrow w_0$
for $k \geq 0$ **do**
  $y_{k+1} \leftarrow \mathbf{prox}_{\sigma g}(y_k + \sigma\tilde{s}_k)$
  if $\mathrm{mod}(k, q)$ == 0 **then**
    $v_k \leftarrow \tilde{\nabla}_x \bar{\Phi}_{\mathcal{B}_k^x}(x_k, y_{k+1})$
  else
    Let $\mathcal{I}_k^x$ be random mini-batch sample with $|\mathcal{I}_k^x| = b'_x$
    $v_k \leftarrow \tilde{\nabla}_x \bar{\Phi}_{\mathcal{I}_k^x}(x_k, y_{k+1}) - \tilde{\nabla}_x \bar{\Phi}_{\mathcal{I}_k^x}(x_{k-1}, y_k) + v_{k-1}$
  end if
  $x_{k+1} \leftarrow \mathbf{prox}_{\tau f}(x_k - \tau v_k)$
  Let $\mathcal{B}_{k+1}^x$,$\mathcal{B}_{k+1}^y$ be random mini-batch samples with $|\mathcal{B}_{k+1}^x| = |\mathcal{B}_{k+1}^y| = b$
  if $\mathrm{mod}(k+1, q)$ == 0 **then**
    $w_{k+1} \leftarrow \tilde{\nabla}_y \Phi_{\mathcal{B}_{k+1}^y}(x_{k+1}, y_{k+1})$
  else
    Let $\mathcal{I}_{k+1}^y$ be mini-batch sample with $|\mathcal{I}_{k+1}^y| = b'_y$
    $\tilde{q}_{k+1} \leftarrow \tilde{\nabla}_y \Phi_{\mathcal{I}_{k+1}^y}(x_{k+1}, y_{k+1}) - \tilde{\nabla}_y \Phi_{\mathcal{I}_{k+1}^y}(x_k, y_k)$
    $w_{k+1} \leftarrow w_k + \tilde{q}_{k+1}$
  end if
  $\tilde{s}_{k+1} \leftarrow (1 + \theta)w_{k+1} - \theta w_k$
end for
Output: $(\bar{x}_N, \bar{y}_N) = \frac{1}{N}\sum_{k=0}^{N-1}(x_{k+1}, y_{k+1})$

Here, we use $\tilde{\nabla}_y \Phi^t_{\mathcal{B}_k^y}(x_k, y_k)$ to represent $\frac{1}{|\mathcal{B}_k^y|}\sum_{\omega_k^i \in \mathcal{B}_k^y} \tilde{\nabla}_y \Phi(x_k, y_y; \vartheta_k^{y,i})$, where $\mathcal{B}_k^y = \{\vartheta_k^{y,i}\}_{i=1}^b$ is the mini-batch with $|\mathcal{B}_k^y| = b$ and we define $\tilde{\nabla}_x \Phi^t_{\mathcal{B}_k^x}(x_k, y_{k+1})$ similarly. In addition, $\mathcal{I}_k^x = \{\omega_k^{x,i}\}$ and

$\mathcal{I}_k^y = \{\omega_k^{y,i}\}$ with $|\mathcal{I}_k^x| = b'_x$ and $|\mathcal{I}_k^y| = b'_y$ denote the small mini-batches for generating $\tilde{\nabla}_y \Phi^t_{\mathcal{I}_k^y}(x_k, y_k)$ and $\tilde{\nabla}_x \Phi^t_{\mathcal{I}_k^x}(x_k, y_{k+1})$. When we run VR-SAPD on a generic subproblem as in (2), we use the convention that $\tilde{\nabla}_x \bar{\Phi}_{\mathcal{B}_k^x}(x_k, y_{k+1}) \triangleq \tilde{\nabla}_x \Phi_{\mathcal{B}_k^x}(x_k, y_{k+1}) + (\mu_x + \gamma)(x_k - x_0)$.

Throughout this section we make a continuity assumption on the stochastic first-order oracles similar to [17, 16, 26, 37].

Assumption 6. *$\exists L_{xx}, L_{xy}, L_{yx}, L_{yy} \geq 0$ such that $\forall x, \bar{x} \in \mathbf{dom}\, f \subset \mathcal{X}$ and $\forall y, \bar{y} \in \mathbf{dom}\, g \subset \mathcal{Y}$,*

$$\begin{aligned} \|\tilde{\nabla}_y \Phi(x, y; \omega) - \tilde{\nabla}_y \Phi(\bar{x}, \bar{y}; \omega)\| &\leq L_{yx}\|x - \bar{x}\| + L_{yy}\|y - \bar{y}\|, \quad w.p.\ 1, \\ \|\tilde{\nabla}_x \Phi(x, y; \omega) - \tilde{\nabla}_x \Phi(\bar{x}, \bar{y}; \omega)\| &\leq L_{xx}\|x - \bar{x}\| + L_{xy}\|y - \bar{y}\|, \quad w.p.\ 1. \end{aligned} \tag{10}$$

Assumption 7. *Consider SAPD+ with VR-flag = **true**. We assume (i) for any $k \geq 0$, the random mini-batches $\mathcal{B}_k^x$, $\mathcal{B}_k^x$, $\mathcal{I}_k^x$ and $\mathcal{I}_k^y$ consist of independent elements, and $\mathcal{B}_x^k$ is independent from $\mathcal{B}_k^y$; (ii) for $i \in \{k-1, k\}$ $\mathcal{B}_k^x$, $\mathcal{I}_k^x$ are independent of (x_i, y_{i+1}), and $\mathcal{B}_k^y$, $\mathcal{I}_k^y$ are independent of (x_i, y_i).*

Remark 8. *For finite-sum type problems of the form $\min_x \max_y \frac{1}{n}\sum_{i=1}^n \Phi_i(x, y)$, we can set the stochastic gradient according to $\tilde{\nabla}_x \Phi(x, y; \omega) = \nabla_x \Phi_\omega(x, y)$ and $\tilde{\nabla}_y \Phi(x, y; \omega) = \nabla_y \Phi_\omega(x, y)$ where ω is uniformly drawn at random from $\{1, \ldots, n\}$. Therefore, if mini-batch samples are drawn from $\{1, \ldots, n\}$ uniformly at random with replacement; batches will be independent of the past iterates satisfying Assumption 7.*

Theorem 4. *Suppose Assumptions 1,3,6 and 7 hold. Moreover, either Assumption 4 or Assumption 5 holds. Let $\mu_x = \gamma$, $\theta = 1$, and τ, σ, b and N be chosen as follows:*

$$\begin{aligned} \tau &= \Big(L_{yx} + L_{xx} + 2\gamma + 2(q-1)\Big(\frac{(L_{xx} + 2\gamma)^2}{\gamma b'_x} + \frac{10 L_{yx}^2}{\mu_y b'_y}\Big)\Big)^{-1}, \ N = 2(1+\zeta)\max\Big\{\frac{1}{\gamma\tau} - 1,\ \frac{1}{\mu_y \sigma}\Big\}, \\ \sigma &= \left(2L_{yy} + L_{yx} + 2(q-1)\Big(\frac{L_{xy}^2}{\gamma b'_x} + \frac{10 L_{yy}^2}{\mu_y b'_y}\Big)\right)^{-1}, \qquad b \geq \Big\lceil \max\Big\{\frac{144\delta_x^2}{\gamma},\ 360\delta_y^2 \frac{1}{\mu_y}\Big\}\frac{\gamma}{\epsilon^2}\Big\rceil. \end{aligned} \tag{11}$$

*For any $\epsilon > 0$ and parameters $b'_x, b'_y, q \in \mathbb{N}^+$, when VR-flag = **true**, SAPD+ guarantees ϵ-stationary, $\min_{t=0,\ldots,T} \mathbb{E}\left[\|\nabla\phi_\lambda(x_0^t)\|\right] \leq \epsilon$, for $T \geq 288\mathcal{G}(x_0^0, y_0^0) \cdot \frac{\gamma}{\epsilon^2}$, which requires $T(Nb/q + N(b'_x + b'_y))$ stochastic first-order oracle calls in total, where*

$$N = \mathcal{O}\Big(\max\Big\{\frac{L_{yx} + L_{xx}}{\gamma} + \frac{q}{b'_x}\frac{L_{xx}^2}{\gamma^2} + \frac{q}{b'_y}\frac{L_{yx}^2}{\gamma\mu_y},\quad \frac{L_{yy} + L_{yx}}{\mu_y} + \frac{q}{b'_y}\frac{L_{yy}^2}{\mu_y^2} + \frac{q}{b'_x}\frac{L_{xy}^2}{\gamma\mu_y}\Big\}\Big). \tag{12}$$

Proof. See appendix F for the proof. □

Remark 9. *For any $y \in \mathbf{dom}\, g$, since $\Phi(\cdot, y)$ L_{xx}-smooth, it is necessarily L_{xx}-weakly convex; hence, $\gamma \leq L_{xx}$. To get a worst-case complexity, consider the setting in (8), and let $b'_x = b'_y = b'$. Then, Theorem 4 implies that setting $b = \mathcal{O}\Big(\kappa_y \frac{\delta^2}{\epsilon^2}\Big)$, $N = \mathcal{O}\big(\kappa_y + \kappa_y^2 \frac{q}{b'}\big)$, and $T = \mathcal{O}\Big(\frac{L\mathcal{G}(x_0^0, y_0^0)}{\epsilon^2}\Big)$ leads to $Nb/q + Nb' = \mathcal{O}\Big(\kappa_y \frac{b}{q} + \kappa_y^2 \frac{b}{b'} + \kappa_y b' + \kappa_y^2 q\Big)$. Thus, setting $q = \sqrt{\frac{b}{\kappa_y}}$ and $b' = \sqrt{b\kappa_y}$ leads to the oracle complexity of $T(Nb/q + Nb') = \mathcal{O}\Big(\kappa_y^2 \frac{\delta}{\epsilon} \ \cdot \frac{L\mathcal{G}(x_0^0, y_0^0)}{\epsilon^2}\Big)$.*

Remark 10. *The results in Theorem 4 continues to hold under a weaker form of Assumption 6 as in [26, 37], i.e., we replace eq. (10) with*

$$\begin{aligned} \mathbb{E}\Big[\|\tilde{\nabla}_y \Phi(x, y; \omega) - \tilde{\nabla}_y \Phi(\bar{x}, \bar{y}; \omega)\|^2\Big] &\leq 2L_{yx}\|x - \bar{x}\|^2 + 2L_{yy}\|y - \bar{y}\|^2, \\ \mathbb{E}\Big[\|\tilde{\nabla}_x \Phi(x, y; \omega) - \tilde{\nabla}_x \Phi(\bar{x}, \bar{y}; \omega)\|^2\Big] &\leq 2L_{xx}\|x - \bar{x}\|^2 + 2L_{xy}\|y - \bar{y}\|^2. \end{aligned}$$

5 Weakly convex-merely concave (WCMC) problems

In this section, we state the convergence guarantees of SAPD+ for solving WCMC problems. In particular, we will consider (1) such that $f(\cdot) = 0$ and $\mu_y = 0$, i.e., $\Phi(x, \cdot)$ is *merely* concave for all $x \in \mathcal{X}$. Instead of directly solving (1) in WCMC setting, we will solve an approximate model obtained by smoothing the primal problem in a similar spirit to the technique in [30]. More precisely, we approximate (1) with the following WCSC problem: given an arbitrary $\hat{y} \in \mathbf{dom}\, g$, consider

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \hat{\mathcal{L}}(x, y) \triangleq \hat{\Phi}(x, y) - g(y), \quad \text{where} \quad \hat{\Phi}(x, y) \triangleq \Phi(x, y) - \frac{\hat{\mu}_y}{2}\|y - \hat{y}\|^2. \tag{13}$$

Theorem 5. *Under Assumptions 1, 2, 3, consider* (1) *such that* $f(\cdot) \equiv 0$*,* $\mu_y = 0$*, and* $\mathcal{D}_\mathcal{Y} \triangleq \sup_{y_1,y_2 \in \mathbf{dom}\, g} \|y_1 - y_2\| < \infty$*. When either Assumption 4 or Assumption 5 holds, for any given* $\epsilon > 0$*,* `SAPD+` *with* `VR-flag` = ***false****, applied to* (13) *with* $\hat{\mu}_y = \Theta(\epsilon^2/(L\mathcal{D}_y^2))$*, is guaranteed to generate* $x_\epsilon \in \mathcal{X}$ *such that* $\mathbb{E}[\|\nabla\phi_\lambda(x_\epsilon)\|] \leq \epsilon$ *for* $\lambda = 1/(2\gamma)$ *within* $\mathcal{O}(L^3\epsilon^{-6})$ *stochastic first-order oracle calls.*

Proof. See appendix G for the proof. □

6 Numerical experiments

The experiments are conducted on a PC with 3.6 GHz Intel Core i7 CPU and NVIDIA RTX2070 GPU. We consider distributionally robust optimization and fair classification. In the rest, n and d represent the number of samples in the dataset and the dimension of each data point, respectively. In this section, `SAPD+` means calling `SAPD+` with `VR-flag`=**false**, and `SAPD+VR` means calling `SAPD+` with `VR-flag`=**true**.

Distributionally Robust Optimization (DRO). First, we consider nonconvex-regularized variant of DRO problem [1, 28, 20, 26, 43, 40] which arises in distributionally robust learning. Let $\{\mathbf{a}_i, b_i\}_{i=1}^n$ be the dataset where $\mathbf{a}_i \in \mathbb{R}^d$ are the features and $b_i \in \{-1, 1\}$ are labels. The DRO problem is

$$\text{(DRO):} \min_{x \in \mathbb{R}^d} \max_{y \in Y} \frac{1}{n} \sum_{i=1}^{n} y_i \ell_i(x) + f(x) - g(y), \tag{14}$$

where $\ell_i(x) = \log(1 + \exp(-b_i \mathbf{a}_i^\top \mathbf{x}))$ is the logistic loss, $f(x) = \eta_1 \sum_{i=1}^d \frac{\alpha x_i^2}{1+\alpha x_i^2}$ is a nonconvex regularizer [2], $g(y) = \frac{1}{2}\eta_2 \|ny - \mathbf{1}\|^2$, and $Y \triangleq \{y \in \mathbb{R}_+^d : \mathbf{1}^\top y = 1\}$ – here, $\mathbf{1}$ denotes the vector with all entries equal to one. This problem can be viewed as a robust formulation of empirical risk minimization where the weights y_i are allowed to deviate from $1/n$; and the aim is to minimize the worst-case empirical risk. We perform experiments on three data sets: $i)$ `a9a` with $n = 32561$, $d = 123$; $ii)$ `gisette` with $n = 6000$, $d = 5000$; $iii)$ `sido0` with $n = 12678$, $d = 4932$. The dataset `sido0` is obtained from Causality Workbench[3] while the others can be downloaded from LIBSVM repository[4].

Parameter tuning. We set the parameters according to [40, 26, 20], i.e., , $\alpha = 10$, $\eta_1 = 10^{-3}$, $\eta_2 = 1/n^2$. We compare `SAPD+` and `SAPD+VR` against `PASGDA` [4], `SREDA` [26], `SMDA`, `SMDA-VR` [17] algorithms. As suggested in [26], we tune the primal stepsizes of all the algorithms based on a grid-search over the set $\{10^{-3}, 10^{-2}, 10^{-1}\}$ and the ratio of the primal stepsize to dual stepsize, i.e., τ/σ, is varied to take values from the set $\{10, 10^2, 10^3, 10^4\}$. For all variance reduction-based algorithms, i.e., for `SAPD+VR`, `SREDA`, `SMDA-VR`, we tune the large batch size $b \triangleq |\mathcal{B}|$ from the set $\{3000, 6000\}$, and the small batch size $b' \triangleq |I|$ from grid search over the set $\{10, 100, 200\}$. For the frequency parameter q, we let $q = b' = |I|$ for `SAPD+VR` and `SMDA-VR` (as suggested in [17]); for `SREDA`, when we set q and m (`SREDA`'s inner loop iteration number) to $\mathcal{O}(n/|I|)$ as suggested in [26], we noticed that `SREDA` does not perform well against `SAPD+VR` and `SMDA-VR`. Therefore, to optimize the performance of `SREDA` further, we tune q, m from a grid search over $\{10, 100, 200\}$. For methods without variance reduction, i.e., for `SAPD+`, `SMDA` and `PASGDA`, we also use mini-batch to estimate the gradients and tune the batch size from $\{10, 100, 200\}$ as well. For `SAPD+` and `SAPD+VR`, we tune the momentum θ from $\{0.8, 0.85, 0.9\}$ and the inner iteration number from $N = \{10, 50, 100\}$.

Results. To fairly compare the performances of algorithms using different batch sizes, we plot loss against epochs in x-axis[5]. In fig. 1, we plot the average loss against the epoch number based on 30 simulations (runs). The standard deviations of the runs are also illustrated around the average in lighter color as shaded regions. We observe that `SAPD+` and `SAPD+VR` consistently outperforms over other algorithms. For `a9a`, `gisette`, `sido0` datasets, the average training accuracy of `SAPD+` are 84.06%, 95.41%, 96.43%, and of `SAPD+VR` are 84.33%, 97.69%, 97.46%, respectively. The best performance for `a9a`, `gisette`, `sido0` among all the other algorithms are 75.92%, 93.07%, 96.43%, respectively. More importantly, we observe that as an accelerated method, `SAPD+VR` enjoys fast convergence properties while still being robust to gradient noise.

[3]http://www.causality.inf.ethz.ch/challenge.php?page=datasets

[4]https://www.csie.ntu.edu.tw/ cjlin/libsvmtools/datasets/binary.html

[5]an epoch is completed whenever an algorithm does one pass over the whole data set through sampling mini-bathes without replacement.

Figure 1: Comparison of SAPD+ and SAPD+VR against PASGDA [4], SREDA [26], SMDA, SMDA-VR [17] on real-data for solving eq. (14) with 30 times simulation.

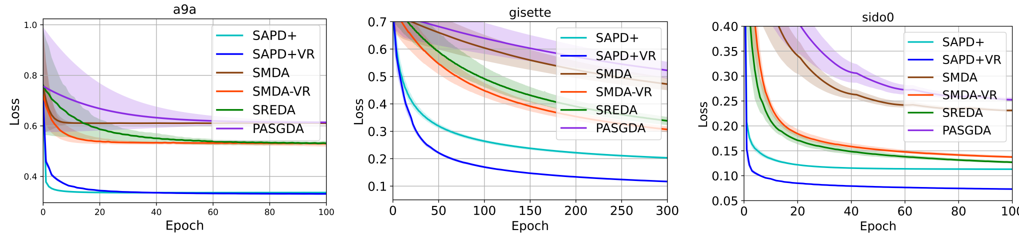
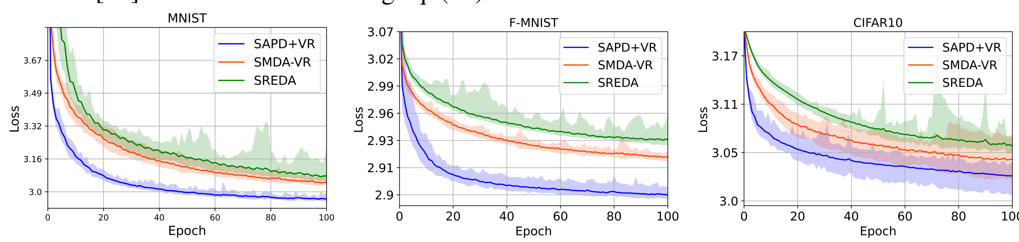


Figure 2: Comparison of SAPD+VR against other Variance Reduction algorithms, SREDA [26], SMDA-VR [17] on real-data for solving eq. (15) with 30 times simulation.



Fair Classification. In the context of multi-class classification, Mohri *et al.* [27] propose training a fair classifier thorough minimizing the worst-case loss over the classification categories. In the spirit of [32, 17], we adopt a nonconvex convolutional neural network (CNN) model as a classifier and set the number of categories to 3, resulting in a minimax problem of the form:

$$\min_{x\in\mathcal{X}} \max_{y\in\mathcal{Y}} \sum_{i=1}^{3} y_i \ell_i(x) - g(y), \quad s.t. \quad \sum_{i=1}^{3} y_i = 1,\ y_i \geq 0,\ \forall\, i \tag{15}$$

where $x \in \mathbb{R}^p$ represents the parameters of the CNN, and ℓ_1, ℓ_2, ℓ_3 correspond to the loss of three categories whose details are given in appendix H, $g(y) = \frac{\eta}{2}\|y\|_2^2$ is a regularizer with $\eta > 0$. We train (15) on the datasets to classify: i) gray-scale hand-written digits $\{0, 2, 3\}$ from MNIST; ii) fashion images with target classes {T-shirt/top, Sandal, Ankle boot} from F-MNIST; iii) RBG colored images with target classes {Plane, Truck, Deer} from CIFAR10. For both MNIST and F-MNIST $p = 43831$, $n = 18000$ and $d = 28 \times 28 \times 1$, and for CIFAR10 $p = 61411$, $n = 15000$, and $d = 32 \times 32 \times 3$.

We let the regularization parameter $\eta = 0.1$ as suggested in [17]. We compare SAPD+VR against the other VR-based algorithms SREDA and SMDA-VR over 30 runs.We tune the primal stepsizes of SAPD+VR and SREDA by a grid search over the set $\{10^{-2}, 5 \times 10^{-3}, 10^{-3}\}$ and the ratio of primal to dual stepsizes, i.e., τ/σ, is chosen from $\{10, 10^2, 5 \times 10^2, 10^3\}$. For SMDA-VR, the primal and dual stepsizes are 10^{-3} and 10^{-5} as suggested in [17] –we also tried stepsizes bigger than the suggested; but, it caused convergence issues in the experiments. We set the large batchsize $|\mathcal{B}| = 3000$ and the small batchsize $|\mathcal{I}| = 200$ for all algorithms and data sets; the frequency $q = 200$ is used for SAPD+VR and SMDA-VR, and we tune q for SREDA taking values from $\{10, 50, 100, 200\}$. The momentum θ for SAPD+VR is tuned taking values from $\{0.8, 0.85, 0.9\}$ and inner iteration number is tuned from $N = \{10, 50, 100\}$. For SREDA, we tune the inner loop iteration from $\{10, 50, 100\}$. Fig. 2 shows that SAPD+VR outperforms the other VR-based algorithms clearly in terms of both the average loss and the standard deviation of the loss.

7 Conclusion

In this paper, we considered both WCSC and WCMC saddle-point problems assuming we only have an access to an unbiased stochastic first-oracle with a finite variance. This setting arises in many applications ranging from distributionally robust learning to GANs. We proposed a new method SAPD+, which achieves an improved complexity in terms of target accuracy ϵ for both WCSC and WCMC problems; moreover, our bound for SAPD+ has a better dependency to the condition number κ_y for the WCSC scenario. We also showed that our algorithm SAPD+ can support the SPIDER variance-reduction technique. Finally, we provided numerical experiments demonstrating that SAPD+ can achieve a state-of-the-art performance on distributionally robust learning and on multi-class classification problems arising in ML.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] Although we improve the existing best known bounds, there is still room to improve κ_y dependence based on the existing lower complexity bounds [21, 42], unless the lower complexity bounds are loose.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes]
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes]
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes]
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes]
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes]
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   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

A The general construction used in the proof of Theorem 1

In general, the proof of Theorem 1 can be divided into two parts: (1) inner loop and outer loop convergence analysis, (2) combining these results to derive the overall complexity.

- We first study the convergence properties of Algorithm 1 for solving the SCSC subproblems in eq. (4). In Lemma 2, we provide guarantees for the inner loop iterates using the expected gap function as our metric.
- Since the convergence guarantee for the inner loop is provided in terms of $\mathcal{G}^t$, we also consider the relationship between $\mathcal{G}^t(x_0^t, y_0^t)$ and GNME, i.e., $||\nabla_x \phi_\lambda(x_0^t)||$. Indeed, Lemmas 3,4, and 5 allow us to translate the expected gap result of inner loops to the convergence in terms of GNME for the outer loops. In Theorem 6, we provide the convergence result in the GNME metric and state the requirements on the parameters to be able to derive the complexity bound in Theorem 1.
- In Lemma 6, we provide a particular step size rule for solving the SCSC subproblems in eq. (4), and we use this specific choice to compute the overall complexity for solving the WCSC problem eq. (1) by using `SAPD+`.

A.1 The construction for the convergence analysis

Based on Lemma 1, the key step for establishing `SAPD+` convergence is to bound $\|x_0^t - \mathbf{prox}_{\lambda\phi}(x_0^t)\|$, where $\phi(x) \triangleq \max_{y\in\mathcal{Y}} \mathcal{L}(x,y)$ for every $x \in \mathcal{X}$ and $\lambda = (\gamma + \mu_x)^{-1}$. To achieve this, we first give a bound on the gap function $\mathcal{G}^t$ at the t-th outer iteration.

Lemma 2. *Suppose Assumptions 1, 2, 3 hold. Given $\{N_t\}_{t\geq 0} \subset \mathbb{Z}_+$, let $\{x_0^t, y_0^t\}_{t\geq 1}$ be generated by `SAPD+`, stated in Algorithm 2, when `VR-flag`=**false**, initialized from $(x_0^0, y_0^0) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$ and using $\tau, \sigma, \theta, \mu_x > 0$ that satisfy*

$$\begin{pmatrix} \mu_y & (\theta-1)L_{yx} & (\theta-1)L_{yy} & 0 \\ (\theta-1)L_{yx} & \frac{1}{\tau} - L'_{xx} & 0 & -\theta L_{yx} \\ (\theta-1)L_{yy} & 0 & \frac{1}{\sigma} - \alpha & -\theta L_{yy} \\ 0 & -\theta L_{yx} & -\theta L_{yy} & \alpha \end{pmatrix} \succeq 0 \tag{16}$$

for some $\alpha \in [0, \frac{1}{\sigma})$, where $L'_{xx} \triangleq L_{xx} + \mu_x + \gamma$. Then for all $t \geq 0$, it holds that

$$\mathbb{E}\left[\mathcal{G}^t(x_0^{t+1}, y_0^{t+1})\right] \leq \frac{M_{\tau,\sigma,\theta}}{N_t}\left(\frac{\mu_x}{4}\mathbb{E}\left[\|x_*^t(y_0^{t+1}) - x_0^t\|^2\right] + \frac{\mu_y}{4}\mathbb{E}\left[\|y_*(x_0^{t+1}) - y_0^t\|^2\right]\right) + \Xi_{\tau,\sigma,\theta}, \tag{17}$$

where $N_t \in \mathbb{N}^+$ and $M_{\tau,\sigma,\theta} \triangleq \max\{\frac{4}{\mu_x\tau}, \frac{4+4\theta}{\mu_y\sigma}\}$,

$$\Xi_{\tau,\sigma,\theta} \triangleq \tau\left(\Xi^x_{\tau,\sigma,\theta} + \frac{1}{2}\right)\delta_x^2 + \sigma\left(\Xi^y_{\tau,\sigma,\theta} + \frac{1+2\theta}{2}\right)\delta_y^2,$$

$$\Xi^x_{\tau,\sigma,\theta} \triangleq \left(1 + \frac{\sigma\theta(1+\theta)L_{yx}}{2}\right), \tag{18a}$$

$$\Xi^y_{\tau,\sigma,\theta} \triangleq (1 + 3\theta + \sigma\theta(1+\theta)L_{yy} + \tau\sigma\theta(1+\theta)L_{yx}L_{xy})(1+2\theta) + \frac{\tau\theta(1+\theta)L_{yx}}{2}. \tag{18b}$$

Proof. For easier readability, we provide the proof in a separate subsection, see appendix C. □

The following lemma provides a relation between $\mathcal{G}^t(x_0^t, y_0^t)$ and $\mathcal{G}^t(x_0^{t+1}, y_0^{t+1})$.

Lemma 3. *Under the premise of Lemma 2 and Assumption 4, for all $t \geq 0$,*

$$\left(1 - \frac{M_{\tau,\sigma,\theta}}{N_t}\right)\mathbb{E}[\mathcal{G}^t(x_0^{t+1}, y_0^{t+1})] \leq \frac{M_{\tau,\sigma,\theta}}{N_t}\mathbb{E}[\mathcal{G}^t(x_0^t, y_0^t)] + \Xi_{\tau,\sigma,\theta}.$$

Proof. It is shown in [39, Lemma 1] that

$$\frac{\mu_x}{4}\|x_*^t(y) - x'\|^2 + \frac{\mu_y}{4}\|y_*(x) - y'\|^2 \leq \mathcal{G}^t(x,y) + \mathcal{G}^t(x',y')$$

holds for all $(x, y), (x', y') \in \mathbf{dom}\, f \times \mathbf{dom}\, g$. It is important to note that since $\mathbf{dom}\, f$ and $\mathbf{dom}\, g$ are compact sets, (17) implies that $\mathbb{E}[\mathcal{G}^t(x_0^{t+1}, y_0^{t+1})] < \infty$. Furthermore, since $\mathcal{G}^t(\cdot,\cdot) \geq 0$, we also have $\mathbb{E}[\mathcal{G}^t(x_0^{t+1}, y_0^{t+1})] > -\infty$; hence, $-\infty < \mathbb{E}[\mathcal{G}^t(x_0^{t+1}, y_0^{t+1})] < \infty$ for all $t \geq 0$. Then (17) and above inequality with the choice of $x = x_0^{t+1}$, $y = y_0^{t+1}$, $x' = x_0^t$, $y' = y_0^t$ together yield the desired result –one can subtract $\frac{M_{\tau,\sigma,\theta}}{N_t}\mathbb{E}[\mathcal{G}^t(x_0^{t+1}, y_0^{t+1})]$ from both sides $\mathbb{E}[\mathcal{G}^t(x_0^{t+1}, y_0^{t+1})]$ is finite. $\square$

For the sake of completeness, we state [39, Lemma 8] below, which will be used in our analysis.

Lemma 4. ***[39, Lemma 8].*** *Under the premise of Lemma 2, for any $\beta_1, \beta_2 \in (0,1)$ and $t \geq 0$,*

$$\begin{aligned}
&\mathcal{G}^t(x_0^{t+1}, y_0^{t+1}) \geq \left(1 - \frac{\gamma+\mu_x}{\gamma}\Big(\frac{1}{\beta_1} - 1\Big)\mathcal{G}_{t+1}(x_0^{t+1}, y_0^{t+1})\right) - \frac{\gamma+\mu_x}{2}\frac{\beta_1}{1-\beta_1}\|x_0^{t+1} - x_0^t\|^2,\\
&\mathcal{G}^t(x_0^{t+1}, y_0^{t+1}) \geq \phi(x_0^{t+1}) - \phi(x_0^t) + \frac{\gamma+\mu_x}{2}\|x_0^{t+1} - x_0^t\|^2,\\
&\mathcal{G}^t(x_0^{t+1}, y_0^{t+1}) \geq \frac{\gamma\beta_2}{2}\|x_0^t - x_*^t\|^2 - \frac{\gamma\beta_2}{2(1-\beta_2)}\|x_0^{t+1} - x_0^t\|^2,
\end{aligned} \tag{19}$$

hold w.p. 1, where $x_*^t = \mathbf{prox}_{\lambda\phi}(x_0^t)$.

Recall that we aim to control $x_0^t - \mathbf{prox}_{\lambda\phi}(x_0^t)$ as it directly determines $\nabla\phi_\lambda(x_0^t)$, and we also have $\|x_0^t - \mathbf{prox}_{\lambda\phi}(x_0^t)\| = \|x_0^t - x_*^t\|$. Thus, in the following result, we bound $\mathbb{E}[\|x_0^t - x_*^t\|^2]$. Moreover, this result will also help us construct a telescoping sum for analyzing the convergence of $\{x_0^t\}_{t\geq 0}$ to a stationary point.

Lemma 5. *Under the premise of Lemma 2 and Assumption 4, for any $\beta_1, \beta_2 \in (0,1)$, and $p_1, p_2, p_3 > 0$ such that $p_1 + p_2 + p_3 = 1$, it holds for all $t \geq 0$ that*

$$\begin{aligned}
&\left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N_t}\right)\frac{\gamma p_3\beta_2}{2}\mathbb{E}\left[\|x_0^t - x_*^t\|^2\right]\\
\leq &\tfrac{M_{\tau,\sigma,\theta}}{N_t}\mathbb{E}\left[\mathcal{G}^t(x_0^t, y_0^t)\right] - \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N_t}\right)p_1\Big(1 - \frac{\gamma+\mu_x}{\gamma}\Big(\frac{1}{\beta_1} - 1\Big)\Big)\mathbb{E}\left[\mathcal{G}^{t+1}(x_0^{t+1}, y_0^{t+1})\right]\\
&+ \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N_t}\right)p_2\mathbb{E}\left[\phi(x_0^t) - \phi(x_0^{t+1})\right]\\
&+\frac{1}{2}\left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N_t}\right)\left(p_1(\gamma+\mu_x)\frac{\beta_1}{1-\beta_1} - p_2(\gamma+\mu_x) + p_3\gamma\frac{\beta_2}{1-\beta_2}\right)\mathbb{E}\left[\|x_0^{t+1} - x_0^t\|^2\right] + \Xi_{\tau,\sigma,\theta}.
\end{aligned} \tag{20}$$

Proof. Using Lemma 4 and $\mathcal{G}^t(x_0^{t+1}, y_0^{t+1}) = (p_1+p_2+p_3)\mathcal{G}^t(x_0^{t+1}, y_0^{t+1})$ leads to

$$\begin{aligned}
\mathbb{E}\left[\mathcal{G}^t(x_0^{t+1}, y_0^{t+1})\right] \geq &-\left(p_1\frac{\gamma+\mu_x}{2}\frac{\beta_1}{1-\beta_1} - p_2\frac{\gamma+\mu_x}{2} + p_3\frac{\gamma\beta_2}{2(1-\beta_2)}\right)\mathbb{E}\left[\|x_0^{t+1} - x_0^t\|^2\right]\\
&+ p_1\left(1 - \frac{\gamma+\mu_x}{\gamma}\Big(\frac{1}{\beta_1} - 1\Big)\right)\mathbb{E}\left[\mathcal{G}^{t+1}(x_0^{t+1}, y_0^{t+1})\right]\\
&+ p_2\mathbb{E}\left[\phi(x_0^{t+1}) - \phi(x_0^t)\right] + p_3\frac{\gamma\beta_2}{2}\mathbb{E}\left[\|x_0^t - x_*^t\|^2\right].
\end{aligned}$$

Then, combining this inequality with Lemma 3 yields the desired result. $\square$

Finally, in the following result, we establish a preliminary convergence result for `SAPD+` under compactness assumption stated in Assumption 4.

Theorem 6. *Under the premise of Lemma 2, given $T \in \mathbb{Z}_+$, suppose $N_t = N$ for all $t = 0, \ldots T$ for some $N \in \mathbb{Z}_+$ such that $N \geq (1+\zeta)M_{\tau,\sigma,\theta}$ for some $\zeta > 0$, and the inequality system,*

$$\tfrac{M_{\tau,\sigma,\theta}}{N} - \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N}\right)p_1\Big(1 - \frac{\gamma+\mu_x}{\gamma}\Big(\frac{1}{\beta_1} - 1\Big)\Big) \leq 0, \tag{21a}$$

$$(\gamma+\mu_x)\Big(p_1\frac{\beta_1}{1-\beta_1} - p_2\Big) + p_3\gamma\frac{\beta_2}{1-\beta_2} \leq 0, \tag{21b}$$

has a solution for some $\beta_1, \beta_2 \in (0,1)$ and $p_1, p_2, p_3 > 0$ such that $p_1 + p_2 + p_3 = 1$. Then, for $\lambda = (\gamma+\mu_x)^{-1}$, under Assumption 4, the following bound holds for all $T \geq 1$:

$$\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[\|\nabla\phi_\lambda(x_0^t)\|^2\right] \leq \frac{2(1+\zeta)(\gamma+\mu_x)^2}{\zeta\gamma p_3\beta_2}\left(\frac{1}{T+1}\mathcal{G}(x_0^0, y_0^0) + \Xi_{\tau,\sigma,\theta}\right). \tag{22}$$

Proof. Since $\mathbf{dom}\, f$ and $\mathbf{dom}\, g$ are compact sets, $\mathbb{E}[\mathcal{G}^t(x_0^t, y_0^t)] \in \mathbb{R}$ exist for $t = 0, \ldots, T$, i.e., $-\infty < \mathbb{E}[\mathcal{G}^t(x_0^t, y_0^t)] < \infty$ for all t. Therefore, if we sum up equation (20) from 0 to T, we get

$$
\begin{aligned}
&\sum_{t=0}^{T} \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N_t}\right) \frac{\gamma p_3 \beta_2}{2} \mathbb{E}\left[\|x_0^t - x_*^t\|^2\right] \\
&\leq \tfrac{M_{\tau,\sigma,\theta}}{N_0} \mathcal{G}^0(x_0^0, y_0^0) - \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N_t}\right) p_1 \left(1 - \frac{\gamma + \mu_x}{\gamma}\Big(\frac{1}{\beta_1} - 1\Big)\right) \mathbb{E}\left[\mathcal{G}^{T+1}(x_0^{T+1}, y_0^{T+1})\right] \\
&+ \sum_{t=0}^{T-1} \left(\tfrac{M_{\tau,\sigma,\theta}}{N_{t+1}} - \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N_t}\right) p_1 \left(1 - \frac{\gamma + \mu_x}{\gamma}\Big(\frac{1}{\beta_1} - 1\Big)\right)\right) \mathbb{E}\left[\mathcal{G}^{t+1}(x_0^{t+1}, y_0^{t+1})\right] \\
&+ \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N_0}\right) p_2 \phi(x_0^0) - \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N_T}\right) p_2 \mathbb{E}\left[\phi(x_0^{T+1})\right] + p_2 \sum_{t=0}^{T-1} \underbrace{\left(\tfrac{M_{\tau,\sigma,\theta}}{N_t} - \tfrac{M_{\tau,\sigma,\theta}}{N_{t+1}}\right)}_{\textbf{part 1}} \mathbb{E}\left[\phi(x_0^{t+1})\right] \\
&+ \sum_{t=0}^{T} \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N_t}\right) \left(p_1 \frac{\gamma + \mu_x}{2} \frac{\beta_1}{1 - \beta_1} - p_2 \frac{\gamma + \mu_x}{2} + p_3 \gamma \frac{\beta_2}{2(1 - \beta_2)}\right) \mathbb{E}\left[\|x_0^{t+1} - x_0^t\|^2\right] \\
&+ (T+1)\Xi_{\tau,\sigma,\theta}
\end{aligned}
\tag{23}
$$

Thus, using $N_t = N$ for $t = 0, \ldots, N$, it follows from the conditions in (21) that

$$
\begin{aligned}
&\frac{1}{T+1} \sum_{t=0}^{T} \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N}\right) \frac{\gamma p_3 \beta_2}{2} \mathbb{E}\left[\|x_0^t - x_*^t\|^2\right] \\
\leq & \frac{1}{T+1} \tfrac{M_{\tau,\sigma,\theta}}{N} \mathbb{E}\left[\mathcal{G}^0(x_0^0, y_0^0)\right] \\
& - \frac{1}{T+1} \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N}\right) p_1 \left(1 - \frac{\gamma + \mu_x}{\gamma}\Big(\frac{1}{\beta_1} - 1\Big)\right) \mathbb{E}\left[\mathcal{G}^{T+1}(x_0^{T+1}, y_0^{T+1})\right] \\
& + \frac{p_2 \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N}\right)}{T+1} \mathbb{E}\left[\phi(x_0^0) - \phi(x_0^{T+1})\right] + \Xi_{\tau,\sigma,\theta} \\
\leq & \frac{1}{T+1} \tfrac{M_{\tau,\sigma,\theta}}{N} \mathcal{G}^0(x_0^0, y_0^0) + \frac{p_2 \left(1 - \tfrac{M_{\tau,\sigma,\theta}}{N}\right)}{T+1} \mathcal{G}(x_0^0, y_0^0) + \Xi_{\tau,\sigma,\theta},
\end{aligned}
\tag{24}
$$

which follows from (i) $\mathcal{G}_{T+1}(x_0^{T+1}, y_0^{T+1}) \geq 0$, (ii) $\phi(x_0^0) - \phi(x_0^{T+1}) = \mathcal{L}(x_0^0, y_*(x_0^0)) - \mathcal{L}(x_0^{T+1}, y_*(x_0^{T+1})) \leq \mathcal{L}(x_0^0, y_*(x_0^0)) - \mathcal{L}(x_0^{T+1}, y_0^0) \leq \sup_{y' \in \mathcal{Y}} \mathcal{L}(x_0^0, y') - \inf_{x' \in \mathcal{X}} \mathcal{L}(x', y_0^0) = \mathcal{G}(x_0^0, y_0^0)$, and also from the fact that (21a) implies $\left(1 - \frac{M_{\tau,\sigma,\theta}}{N}\right) p_1 \left(1 - \frac{\gamma + \mu_x}{\gamma}\left(\frac{1}{\beta_1} - 1\right)\right) \geq 0$. Then dividing both sides by $\left(1 - \frac{M_{\tau,\sigma,\theta}}{N}\right) \frac{\gamma p_3 \beta_2}{2}$ gives us

$$
\begin{aligned}
&\frac{1}{T+1} \sum_{t=1}^{T} \|x_0^t - x_*^t\|^2 \\
&\leq \frac{2}{(1 - \frac{M_{\tau,\sigma,\theta}}{N}) \gamma p_3 \beta_2} \Big(\frac{1}{T+1} \tfrac{M_{\tau,\sigma,\theta}}{N} \mathcal{G}^0(x_0^0, y_0^0) + \frac{p_2 \left(1 - \frac{M_{\tau,\sigma,\theta}}{N}\right)}{T+1} \mathcal{G}(x_0^0, y_0^0) + \Xi_{\tau,\sigma,\theta}\Big), \\
&\leq \frac{2(1+\zeta)}{\zeta \gamma p_3 \beta_2} \left(\frac{1}{T+1} \mathcal{G}(x_0^0, y_0^0) + \Xi_{\tau,\sigma,\theta}\right),
\end{aligned}
\tag{25}
$$

where the second inequality follows from $\mathcal{G}(x_0^0, y_0^0) \geq \mathcal{G}^0(x_0^0, y_0^0)$, and for $p_2 \in (0, 1)$, we have $N \geq (1+\zeta) M_{\tau,\sigma,\theta}$. Finally, we get the desired result using Lemma 1. □

A.2 A particular parameter choice

We employ the matrix inequality (MI) in eq. (16) to describe the admissible set of algorithm parameters that guarantee convergence of Algorithm 1, i.e., inner loop of `SAPD+` when `VR-flag` is **false**. In this subsection, we compute a particular solution by exploiting the structure of MI in eq. (16). This particular solution is for solving the SCSC subproblems in eq. (4).

Lemma 6. *For any $\mu_x \geq 0$, let $L'_{xx} = L_{xx} + \gamma + \mu_x$. Suppose $\theta = 1$, and $\tau, \sigma > 0$, satisfy*

$$\tau \leq \frac{1}{L'_{xx} + L_{yx}}, \quad \sigma \leq \frac{1}{2L_{yy} + L_{yx}}. \tag{26}$$

Then $\{\tau, \sigma, \theta, \alpha\}$ is a solution to (16) *for $\alpha = L_{yx} + L_{yy}$.*

Proof. It follows from the choice of τ and σ in (26) and $\theta = 1$ that a sufficient condition for (16) is given by the following smaller matrix inequality for $\alpha = L_{yx} + L_{yy}$,

$$\mathbf{0} \preceq \begin{pmatrix} \frac{1}{\tau} - L'_{xx} & 0 & -L_{yx} \\ 0 & \frac{1}{\sigma} - \alpha & -L_{yy} \\ -L_{yx} & -L_{yy} & \alpha \end{pmatrix} = \begin{pmatrix} \frac{1}{\tau} - L'_{xx} & 0 & -L_{yx} \\ 0 & \frac{1}{\sigma} - L_{yx} - L_{yy} & -L_{yy} \\ -L_{yx} & -L_{yy} & L_{yx} + L_{yy} \end{pmatrix} \triangleq \quad M_1 + M_2,$$

where $M_1 \triangleq \begin{pmatrix} \frac{1}{\tau} - L'_{xx} & 0 & -L_{yx} \\ 0 & 0 & 0 \\ -L_{yx} & 0 & L_{yx} \end{pmatrix}$ and $M_2 \triangleq \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sigma} - L_{yx} - L_{yy} & -L_{yy} \\ 0 & -L_{yy} & L_{yy} \end{pmatrix}$. Therefore, the Schur complement conditions together with eq. (26) imply $M_1 \succeq 0$ and $M_2 \succeq 0$, respectively. Thus, $M_1 + M_2 \succeq 0$. □

A.3 Proof of Theorem 1

Proof. Using the results we derived in the previous two subsections, we are now ready to provide the proof of Theorem 1.

For the inner loop iterations, Lemma 6 ensures that eq. (16) holds for our $\{\tau, \sigma, \theta\}$ choice in eq. (7). For the outer loop, if we set N as in eq. (7) and

$$p_1 = \frac{1}{16},\ p_2 = \frac{19}{32},\ p_3 = \frac{11}{32},\ \beta_1 = \frac{4}{5},\ \beta_2 = \frac{1}{2},\ \zeta = 32, \tag{27}$$

all assumptions of Theorem 6 are satisfied, i.e., both the inequality system eq. (21) and $N \geq (1+\zeta)M_{\tau,\sigma,\theta}$ hold.

Specifically, because $\mu_x = \gamma$ and $\theta = 1$, we have $M_{\tau,\sigma,\theta} = \max\{\frac{4}{\gamma\tau}, \frac{8}{\mu_y\sigma}\}$. Therefore, we know that $N \geq (1+\zeta)M_{\tau,\sigma,\theta}$ is trivially true. Moreover, using $M_{\tau,\sigma,\theta}/N \leq (1+\zeta)^{-1}$, it follows that eq. (21a) holds for $\mu_x = \gamma$, $p_1 = \frac{1}{16}$ and $\beta_1 = \frac{4}{5}$, i.e.,

$$\frac{M_{\tau,\sigma,\theta}}{N} - \left(1 - \frac{M_{\tau,\sigma,\theta}}{N}\right) p_1 \Big(1 - \frac{\gamma + \mu_x}{\gamma}\Big(\frac{1}{\beta_1} - 1\Big)\Big) = \frac{33}{32}\frac{M_{\tau,\sigma,\theta}}{N} - \frac{1}{32} \leq \frac{33}{32}\frac{1}{1+\zeta} - \frac{1}{32} = 0.$$

Moreover, it is trivial to check that eq. (21b) holds for the parameter values given in eq. (27).

Since all assumptions of Theorem 6 are satisfied for parameters chosen as in eq. (7) and eq. (27), if we substitute eq. (27) into eq. (22), if follows that

$$\frac{1}{T+1}\sum_{t=0}^{T} \mathbb{E}\left[\|\nabla\phi_\lambda(x_0^t)\|^2\right] \leq 48\gamma \left(\frac{1}{T+1}\mathcal{G}(x_0^0, y_0^0) + \Xi_{\tau,\sigma,\theta}\right).$$

Thus, for any $\epsilon > 0$, the right side of the above inequality can be bounded by ϵ^2 when

$$\frac{48\gamma}{T+1}\mathcal{G}(x_0^0, y_0^0) \leq \frac{\epsilon^2}{2}, \qquad 48\gamma\Xi_{\tau,\sigma,\theta} \leq \frac{\epsilon^2}{2}. \tag{28}$$

Note that because $\Xi_{\tau,\sigma,\theta} = \tau\left(\Xi^x_{\tau,\sigma,\theta} + \frac{1}{2}\right)\delta_x^2 + \sigma\left(\Xi^y_{\tau,\sigma,\theta} + \frac{3}{2}\right)\delta_y^2$, a sufficient condition for the second inequality in eq. (28) is that

$$24\gamma\tau(1 + 2\Xi^x_{\tau,\sigma,\theta})\delta_x^2 \leq \frac{\epsilon^2}{4}, \qquad 24\gamma\sigma(3 + 2\Xi^y_{\tau,\sigma,\theta})\delta_y^2 \leq \frac{\epsilon^2}{4}. \tag{29}$$

Moreover, recall that $\Xi^x_{\tau,\sigma,\theta}$ and $\Xi^y_{\tau,\sigma,\theta}$ are defined in Lemma 2; for $\theta = 1$, they can be simplified as follows:

$$\Xi^x_{\tau,\sigma,\theta} = 1 + \sigma L_{yx}, \quad \Xi^y_{\tau,\sigma,\theta} = 3\left(4 + 2\sigma L_{yy} + 2\tau\sigma L_{yx}L_{xy}\right) + \tau L_{yx}.$$

Because the choice of $\{\tau,\ \sigma\}$ in eq. (7) implies that

$$\tau L_{yx} \le 1, \quad \tau L_{xy} \le 1, \quad \sigma L_{yy} \le \frac{1}{2}, \quad \sigma L_{yx} \le 1,$$

we can upper bound $\Xi^x_{\tau,\sigma,\theta}$ and $\Xi^y_{\tau,\sigma,\theta}$ as follows:

$$\Xi^x_{\tau,\sigma,\theta} \le 2, \quad \Xi^y_{\tau,\sigma,\theta} \le 22.$$

Therefore, with the choice of $\{\tau, \sigma\}$ in eq. (7), we have a sufficient condition for eq. (29) as follows:

$$120\gamma\tau\delta_x^2 \le \frac{\epsilon^2}{4}, \qquad 1128\gamma\sigma\delta_y^2 \le \frac{\epsilon^2}{4}.$$

Indeed, the above condition is trivially satisfied by our choice of $\{\tau, \sigma\}$ given in eq. (7). Therefore, the second condition in (28), i.e., $48\gamma\Xi_{\tau,\sigma,\theta} \le \frac{\epsilon^2}{2}$, holds for the choice of $\{\tau, \sigma\}$ in eq. (7). Thus, from the first inequality in eq. (28), we get $\min_{t=0,\dots,T} \mathbb{E}[\|\nabla\phi_\lambda(x_0^t)\|^2] \le \epsilon^2$ for

$$T \ge 96\mathcal{G}(x_0^0, y_0^0) \cdot \frac{\gamma}{\epsilon^2} + 1. \tag{30}$$

Note that from Jensen's inequality, we have $(\mathbb{E}[\|\nabla\phi_\lambda(x_0^t)\|])^2 \le \mathbb{E}[\|\nabla\phi_\lambda(x_0^t)\|^2]$ for all $t = 0,\dots,T$; hence, it follows that $\min_{t=0,\dots,T} \mathbb{E}[\|\nabla\phi_\lambda(x_0^t)\|] \le \epsilon$ for all $T \in \mathbb{Z}_+$ satisfying (30). Finally, to show the complexity result, recall that $N = 33\max\{\frac{4}{\gamma\tau}, \frac{8}{\mu_y\sigma}\}$. Using the the choice of $\{\tau, \sigma\}$ in eq. (7) we derive that

$$N = \mathcal{O}\Big(\frac{\max\{L_{xx}, L_{yx}, L_{xy}\}}{\gamma} + \frac{\max\{L_{yy}, L_{yx}\}}{\mu_y} + \Big(\frac{\delta_x^2}{\gamma} + \frac{\delta_y^2}{\mu_y}\Big)\frac{\gamma}{\epsilon^2}\Big). \tag{31}$$

Moreover, since `SAPD+` requires NT oracle calls in total, combining (30) with (31) leads to $\mathcal{O}(\epsilon^{-4})$ bound on C_ϵ as stated in Theorem 1, which completes the proof. □

B Proof of Theorem 2 and preliminary technical results

Suppose Assumptions 1, 2, 3 hold. Given x_ϵ, an ϵ-stationary point for the γ-weakly convex function $\phi(\cdot) = \max_{y\in\mathcal{Y}} \mathcal{L}(\cdot, y)$, i.e., $\mathbb{E}\big[\|\nabla\phi_\lambda(x_\epsilon)\|\big] \le \frac{\epsilon}{2}$ for some fixed $\lambda \in (0, \gamma^{-1})$. Let $\phi^s(\cdot) \triangleq \max_{y\in\mathcal{Y}} \Phi(\cdot, y) - g(y)$ so that $\phi = f + \phi^s$. In this section we show that initialized from x_ϵ and using appropriately selected step size parameters, within $\tilde{\mathcal{O}}(\frac{1}{\epsilon^2})$ stochastic first-order oracle calls, `SAPD`, stated in Algorithm 1, can generate $\tilde{x}$ such that $\mathbb{E}\big[\|G_\lambda(\tilde{x})\| \le \epsilon$, where generalized gradient mapping G_λ is defined in (9).

Lemma 7. *Suppose Assumptions 1, 2, 3 hold. Given some $(x_0, y_0) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$, consider the SCSC problem in (2) for some $\mu_x > 0$. Let $\{x_k, y_k\}_{k\ge 0}$ be generated by `SAPD`, stated in Algorithm 1, initialized from (x_0, y_0) and using $\tau, \sigma, \theta > 0$ that satisfy*

$$G \triangleq \begin{pmatrix} \frac{1}{\tau}(1-\frac{1}{\rho}) + \frac{\mu_x}{\rho} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sigma}(1-\frac{1}{\rho}) + \mu_y & (\frac{\theta}{\rho}-1)L_{yx} & (\frac{\theta}{\rho}-1)L_{yy} & 0 \\ 0 & (\frac{\theta}{\rho}-1)L_{yx} & \frac{1}{\tau} - L'_{xx} & 0 & -\frac{\theta}{\rho}L_{yx} \\ 0 & (\frac{\theta}{\rho}-1)L_{yy} & 0 & \frac{1}{\sigma} - \alpha & -\frac{\theta}{\rho}L_{yy} \\ 0 & 0 & -\frac{\theta}{\rho}L_{yx} & -\frac{\theta}{\rho}L_{yy} & \frac{\alpha}{\rho} \end{pmatrix} \succeq 0 \tag{32}$$

for some $\alpha \in [0, \frac{1}{\sigma})$ and $\rho \in (0,1)$, where $L'_{xx} \triangleq L_{xx} + \mu_x + \gamma$. Define $\phi(x) = \max_{y\in\mathcal{Y}} \mathcal{L}(x,y)$; and let $\hat{x} = \mathbf{prox}_{\lambda\phi}(x_0)$ for $\lambda = (\mu_x+\gamma)^{-1}$ and $y_*(\hat{x}) = \operatorname{argmax}_{y\in\mathcal{Y}} \mathcal{L}(\hat{x}, y)$. Then for all $N \in \mathbb{Z}_+$, it holds that

$$\begin{aligned} &\mathbb{E}\Big[\Big(\frac{1}{\tau} - \mu_x\Big)\|x_N - \hat{x}\|^2 + \Big(\frac{1}{\sigma} - \alpha\Big)\|y_N - y_*(\hat{x})\|^2\Big] \\ &\quad \le \rho^N\left(\frac{1}{\tau}\|x_0 - \hat{x}\|^2 + \frac{1}{\sigma}\|y_0 - y_*(\hat{x})\|^2\right) + \frac{\rho}{1-\rho}\Big(\tau\Xi^x_{\tau,\sigma,\theta}\delta_x^2 + \sigma\Xi^y_{\tau,\sigma,\theta}\delta_y^2\Big), \end{aligned} \tag{33}$$

where $\Xi^x_{\tau,\sigma,\theta}$ and $\Xi^y_{\tau,\sigma,\theta}$ are defined in (18a) *and* (18b)*, respectively.*

Proof. For easier readability, we provide the proof in a separate subsection, see appendix C. □

In the following part, we will compute a particular solution by exploiting the structure of MI in eq. (32) and use this particular solution for the rest of the proof. First, in Lemma 8, we give an intermediate condition to help us construct the particular solution subsequently provided in Lemma 9 for solving the generic SCSC subproblems in eq. (2).

Lemma 8. *For any $\mu_x > 0$, let $L'_{xx} = L_{xx} + \gamma + \mu_x$. Suppose $\rho = \theta$, and $\tau, \sigma > 0$, $\theta \in (0,1)$ satisfy*

$$\tau \geq \frac{1-\theta}{\mu_x}, \quad \sigma \geq \frac{1-\theta}{\mu_y \theta}, \quad \frac{1}{\tau} \geq L'_{xx} + \pi_1 L_{yx}, \quad \frac{1}{\sigma} \geq \frac{\theta L_{yx}}{\pi_1} + \Big(\frac{\theta}{\pi_2} + \pi_2\Big) L_{yy}, \tag{34}$$

for some $\pi_1, \pi_2 > 0$. Then $\{\tau, \sigma, \theta, \alpha\}$ is a solution to (32) *for $\alpha = \frac{\theta L_{yx}}{\pi_1} + \frac{\theta L_{yy}}{\pi_2}$.*

Proof. It follows from the choice of τ and σ in (34) and $\rho = \theta$ that a sufficient condition for eq. (32), i.e., for $G \succeq 0$, is given by the following smaller matrix inequality for $\alpha = \frac{\theta L_{yx}}{\pi_1} + \frac{\theta L_{yy}}{\pi_2}$,

$$\mathbf{0} \preceq \begin{pmatrix} \frac{1}{\tau} - L'_{xx} & 0 & -L_{yx} \\ 0 & \frac{1}{\sigma} - \alpha & -L_{yy} \\ -L_{yx} & -L_{yy} & \frac{\alpha}{\theta} \end{pmatrix} = \begin{pmatrix} \frac{1}{\tau} - L'_{xx} & 0 & -L_{yx} \\ 0 & \frac{1}{\sigma} - \frac{\theta L_{yx}}{\pi_1} - \frac{\theta L_{yy}}{\pi_2} & -L_{yy} \\ -L_{yx} & -L_{yy} & \frac{L_{yx}}{\pi_1} + \frac{L_{yy}}{\pi_2} \end{pmatrix} \triangleq M_1 + M_2,$$

where $M_1 \triangleq \begin{pmatrix} \frac{1}{\tau} - L'_{xx} & 0 & -L_{yx} \\ 0 & 0 & 0 \\ -L_{yx} & 0 & \frac{L_{yx}}{\pi_1} \end{pmatrix}$ and $M_2 \triangleq \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sigma} - \frac{\theta L_{yx}}{\pi_1} - \frac{\theta L_{yy}}{\pi_2} & -L_{yy} \\ 0 & -L_{yy} & \frac{L_{yy}}{\pi_2} \end{pmatrix}$. Therefore, since $\pi_1, \pi_2 > 0$, the Schur complement conditions in (34), i.e., the third and the fourth inequalities, imply $M_1 \succeq 0$ and $M_2 \succeq 0$, respectively. Thus, $M_1 + M_2 \succeq 0$. □

Lemma 8 shows that every solution to (34) can be converted to a solution to (32). Next, based on Lemma 8, we will give another explicit parameter choice for Algorithm 1 in addition to the solution we provided earlier in Lemma 6.

Lemma 9. *For any $\mu_x > 0$, let $L'_{xx} = L_{xx} + \gamma + \mu_x$. For any given $\beta \in (0,1]$, let $\tau, \sigma > 0$ and $\theta \in (0,1)$ be chosen satisfying*

$$\tau = \frac{1-\theta}{\mu_x}, \quad \sigma = \frac{1-\theta}{\mu_y \theta}, \quad \theta \geq \bar{\theta}(\beta), \tag{35}$$

where $\bar{\theta}(\beta) \triangleq \max\{\bar{\theta}_1(\beta),\ \bar{\theta}_2(\beta)\} \in (0,1)$ such that

$$\bar{\theta}_1(\beta) \triangleq 1 - \tfrac{\beta \mu_y L'_{xx}}{2L_{yx}^2}\Big(\sqrt{1 + \tfrac{4L_{yx}^2 \mu_x}{\beta {L'_{xx}}^2 \mu_y}} - 1\Big),$$

$$\bar{\theta}_2(\beta) \triangleq \begin{cases} 1 - \frac{(1-\beta)^2}{8} \frac{\mu_y^2}{L_{yy}^2}\left(\sqrt{1 + \frac{16 L_{yy}^2}{(1-\beta)^2 \mu_y^2}} - 1\right) & L_{yy} > 0 \\ 0 & L_{yy} = 0. \end{cases}$$

Then, $\{\tau, \sigma, \theta, \alpha, \rho\}$ with $\alpha = \frac{1}{\sigma} - \sqrt{\theta} L_{yy} > 0$ and $\rho = \theta$ is a solution to (32).

Proof. Consider arbitrary $\tau, \sigma, \pi_1, \pi_2 > 0$ and $\theta \in (0,1)$. By a straightforward calculation, $\{\tau, \sigma, \theta, \pi_1, \pi_2\}$ is a solution to (34) if and only if

$$\tau \geq \frac{1-\theta}{\mu_x}, \quad \sigma \geq \frac{1-\theta}{\theta \mu_y}, \quad \pi_1 \geq \frac{\sigma \theta L_{yx}}{1 - \sigma(\pi_2 + \frac{\theta}{\pi_2}) L_{yy}}, \tag{36a}$$

$$\sigma(\pi_2 + \frac{\theta}{\pi_2}) L_{yy} < 1, \quad \frac{1}{\tau} - L'_{xx} \geq \pi_1 L_{yx}. \tag{36b}$$

In the remainder of the proof, we fix (π_1, π_2) as follows:

$$\pi_1 = \frac{\sigma \theta L_{yx}}{1 - \sigma\left(\pi_2 + \frac{\theta}{\pi_2}\right) L_{yy}}, \quad \pi_2 = \sqrt{\theta}. \tag{37}$$

Note the definition of $\bar{\theta}(\beta)$ implies that $\bar{\theta}(\beta) \in (0,1)$. Next, we show that $\theta \in [\bar{\theta}(\beta), 1)$ implies $\pi_1, \pi_2 > 0$; furthermore, we also show that $\tau, \sigma > 0$ defined as in (35) for $\theta \in [\bar{\theta}(\beta), 1)$ together with (π_1, π_2) as in (37) is a solution to (36).

First, setting τ, σ as in (35) and π_1, π_2 as in (37) imply that (36a) is trivially satisfied. Next, by substituting $\{\tau, \sigma, \pi_1, \pi_2\}$, chosen as in (35) and (37), into (36b), we conclude that $\{\tau, \sigma, \theta, \pi_1, \pi_2\}$ satisfies (36) for any $\theta \in (0,1)$ such that

$$\frac{2L_{yy}}{\mu_y} \cdot \frac{1-\theta}{\sqrt{\theta}} \leq 1 - \beta, \tag{38}$$

$$\frac{\mu_x}{1-\theta} - L'_{xx} \geq (1-\theta)\frac{L_{yx}^2}{\mu_y} \cdot \Big(1 - \frac{2L_{yy}}{\mu_y} \cdot \frac{1-\theta}{\sqrt{\theta}}\Big)^{-1}, \tag{39}$$

for some $\beta \in (0,1]$. Clearly, a sufficient condition for (39) is

$$\frac{\mu_x}{1-\theta} - L'_{xx} \geq (1-\theta)\frac{L_{yx}^2}{\mu_y} \cdot \frac{1}{\beta}. \tag{40}$$

Note that (38) implies that $\pi_1 > 0$. We also have $\pi_2 = \sqrt{\theta} > 0$ trivially.

When $L_{yy} > 0$, given any $\beta \in (0,1)$, solving eqs. (38) and (40) for $\theta \in (0,1)$, we get the third condition in (35). Indeed, it can be checked that $\theta \in [\bar{\theta}_2(\beta), 1)$ satisfies (38) and $\theta \in [\bar{\theta}_1(\beta), 1)$ satisfies (40); thus, $\theta \in [\bar{\theta}(\beta), 1)$ satisfies (38) and (40) simultaneously. Moreover, when $L_{yy} = 0$, one does not need to solve eq. (38) as the first inequality in (36b) holds trivially; thus, the only condition on θ comes from (39) which is equivalent to (40) with $\beta = 1$. The rest follows from Lemma 8 by setting $\alpha = \frac{\theta L_{yx}}{\pi_1} + \frac{\theta L_{yy}}{\pi_2}$. Indeed, the particular choice of (π_1, π_2) in (37) gives us $\alpha = \frac{1}{\sigma} - \sqrt{\theta} L_{yy}$. $\square$

Now that we have provided a particular solution to eq. (32), we will next use this particular solution within Lemma 7 to derive an error bound customized for this choice of parameters. The following two technical results, i.e., Lemmas 10 and 11, will be used later within the proof of Theorem 2.

Lemma 10. *Consider $\mathcal{L}$ defined in* (1)*. Suppose Assumptions 1, 2, 3 hold. Given arbitrary x_0, let $\hat{x} = \mathbf{prox}_{\lambda\phi}(x_0)$, where $\phi(\cdot) = \max_{y\in\mathcal{Y}} \mathcal{L}(\cdot, y)$ and $\lambda = (2\gamma)^{-1}$. For any given $\hat{\epsilon} > 0$,* `SAPD`*, displayed in Algorithm 1, can generate $\tilde{x}_* \in \mathcal{X}$ such that $\mathbb{E}[\|\tilde{x}_* - \hat{x}\|] \leq \hat{\epsilon}$ within $\tilde{\mathcal{O}}(\frac{1}{\hat{\epsilon}^2})$ stochastic first-order oracle calls.*

Proof. Recall that $y_*(x) = \operatorname{argmax}_{y\in\mathcal{Y}} \mathcal{L}(x, y)$ for $x \in \mathbf{dom}\, f$. Hence, $(\hat{x}, y_*(\hat{x}))$ is the unique saddle point to the SCSC problem:

$$\min_{x\in\mathcal{X}} \max_{y\in\mathcal{Y}} \bar{\mathcal{L}}(x,y) \triangleq f(x) + \Phi(x,y) + \gamma\|x - x_0\|^2 - g(y), \tag{41}$$

which is equivalent to the SCSC problem in eq. (2) with $\mu_x = \gamma$. Let $\{x_k, y_k\}$ be the iterate sequence generated by `SAPD` running on (41), initialized from an arbitrary point (x_0, y_0), with parameters $\{\tau, \sigma, \theta\}$ chosen as follows:

$$\tau = \frac{1-\theta}{\gamma}, \quad \sigma = \frac{1-\theta}{\mu_y\theta}, \quad \theta = \max\{\bar{\theta}(\beta),\ \hat{\theta}_1,\ \hat{\theta}_2\}, \tag{42}$$

for $\beta = \min\{\frac{1}{2},\ \frac{\mu_y}{\gamma},\ \frac{\gamma}{\mu_y},\ \frac{L_{yx}}{L_{xy}}\}$, where $\bar{\theta}(\beta) \triangleq \max\{\bar{\theta}_1(\beta),\ \bar{\theta}_2(\beta)\} \in (0,1)$ such that

$$\bar{\theta}_1(\beta) \triangleq 1 - \tfrac{\beta\mu_y L'_{xx}}{2L_{yx}^2}\Big(\sqrt{1 + \tfrac{4L_{yx}^2\gamma}{\beta {L'_{xx}}^2 \mu_y}} - 1\Big),$$

$$\bar{\theta}_2(\beta) \triangleq \begin{cases} 1 - \frac{(1-\beta)^2}{8}\frac{\mu_y^2}{L_{yy}^2}\Big(\sqrt{1 + \frac{16L_{yy}^2}{(1-\beta)^2\mu_y^2}} - 1\Big) & L_{yy} > 0 \\ 0 & L_{yy} = 0, \end{cases}$$

with $L'_{xx} = L_{xx} + 2\gamma$ and

$$\hat{\theta}_1 \triangleq \max\Big\{0, 1 - \frac{1}{8} \cdot \gamma^2 \cdot \frac{\hat{\epsilon}^2}{\delta_x^2}\Big\}, \quad \hat{\theta}_2 \triangleq \Big(1 + \frac{1}{8} \cdot \frac{\mu_y\gamma}{11} \cdot \frac{\hat{\epsilon}^2}{\delta_y^2}\Big)^{-1}. \tag{43}$$

In fact, in Lemma 7 we provide a convergence guarantee for solving the above problem in (41) using Algorithm 1. Since the parameter choice above satisfies the condition (32) in Lemma 7, we can invoke eq. (33) to complete the rest of the analysis. To be more precise, the problem in eq. (41) is a generic form of the SCSC subproblems given in eq. (4) with $\mu_x = \gamma$; furthermore, by Lemma 9, (τ, σ, θ) chosen as in (42) satisfies (32) with $\rho = \theta$, $\mu_x = \gamma$, $\alpha = \frac{1}{\sigma} - \sqrt{\theta} L_{yy} > 0$, and $L'_{xx} = L_{xx} + 2\gamma$.

Since $(\hat{x}, y_*(\hat{x}))$ is the saddle point of $\bar{\mathcal{L}}$, then by Lemma 7, we get

$$\mathbb{E}\Big[\Big(\frac{1}{\tau} - \gamma\Big)\|x_N - \hat{x}\|^2\Big] \le \theta^N \left(\frac{1}{\tau}\|x_0 - \hat{x}\|^2 + \frac{1}{\sigma}\|y_0 - y_*(\hat{x})\|^2\right) + \frac{\theta}{1-\theta}\Big(\tau \Xi^x_{\tau,\sigma,\theta}\delta_x^2 + \sigma \Xi^y_{\tau,\sigma,\theta}\delta_y^2\Big).$$

If we substitute the choice of $\{\tau, \sigma\}$ in eq. (42) into the above inequality, it follows that

$$\mathbb{E}\Big[\|x_N - \hat{x}\|^2\Big] \le \theta^{N-1} \max\Big\{1, \frac{\mu_y}{\gamma}\Big\}\left(\|x_0 - \hat{x}\|^2 + \|y_0 - y_*(\hat{x})\|^2\right) + \frac{1}{\gamma}\Big(\tau \Xi^x_{\tau,\sigma,\theta}\delta_x^2 + \sigma \Xi^y_{\tau,\sigma,\theta}\delta_y^2\Big).$$

Then, by Jensen's inequality, it follows that

$$\Big(\mathbb{E}\left[\|x_N - \hat{x}\|\right]\Big)^2 \le \mathbb{E}\left[\|x_N - \hat{x}\|^2\right] \le \theta^{N-1}\max\Big\{1, \frac{\mu_y}{\gamma}\Big\}\mathcal{D}_0^2 + \frac{1}{\gamma}\Big(\tau \Xi^x_{\tau,\sigma,\theta}\delta_x^2 + \sigma \Xi^y_{\tau,\sigma,\theta}\delta_y^2\Big),$$

where $\mathcal{D}_0 \triangleq \left(\|\hat{x} - x_0\|^2 + \|y_*(\hat{x}) - y_0\|^2\right)^{1/2}$. Thus, for any given $\hat{\epsilon} > 0$, $\mathbb{E}\left[\|x_N - \hat{x}\|\right]$ can be bounded by $\hat{\epsilon}$ when

$$\frac{1}{\gamma}\Big(\tau \Xi^x_{\tau,\sigma,\theta}\delta_x^2 + \sigma \Xi^y_{\tau,\sigma,\theta}\delta_y^2\Big) \le \frac{\hat{\epsilon}^2}{2}, \tag{44a}$$

$$\theta^{N-1}\max\Big\{1, \frac{\mu_y}{\gamma}\Big\}\mathcal{D}_0^2 \le \frac{\hat{\epsilon}^2}{2}. \tag{44b}$$

Recall that $\Xi^x_{\tau,\sigma,\theta}$, $\Xi^y_{\tau,\sigma,\theta}$ are defined in Lemma 7. Thus, the choice of τ and σ in (42) further implies that

$$\Xi^x_{\tau,\sigma,\theta} = 1 + (1-\theta^2)\frac{L_{yx}}{2\mu_y},$$
$$\Xi^y_{\tau,\sigma,\theta} = \Big(1 + 3\theta + (1-\theta^2)\frac{L_{yy}}{\mu_y} + (1+\theta)(1-\theta)^2\frac{L_{yx}L_{xy}}{\gamma\mu_y}\Big)(1+2\theta) + \theta(1-\theta^2)\frac{L_{yx}}{2\gamma}.$$

Since $0 < \theta < 1$ and $1 - \theta^2 \le 2(1-\theta)$, we have

$$\Xi^x_{\tau,\sigma,\theta} \le 1 + (1-\theta)\frac{L_{yx}}{\mu_y}, \tag{45a}$$

$$\Xi^y_{\tau,\sigma,\theta} \le 3\Big(4 + 2(1-\theta)\frac{L_{yy}}{\mu_y} + 2(1-\theta)^2\frac{L_{yx}L_{xy}}{\gamma\mu_y}\Big) + (1-\theta)\frac{L_{yx}}{\gamma}. \tag{45b}$$

On the other hand, since $\theta \ge \bar{\theta}(\beta) = \max\{\bar{\theta}_1(\beta), \bar{\theta}_2(\beta)\}$, the inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for all $a, b \ge 0$, and the definition of $\bar{\theta}(\beta)$ together imply that

$$1 - \theta \le \min\Big\{\frac{\sqrt{\beta\gamma\mu_y}}{L_{yx}}, \frac{1-\beta}{2}\frac{\mu_y}{L_{yy}}\Big\}. \tag{46}$$

Therefore, by eq. (46), we can derive that

$$(1-\theta)\frac{L_{yx}}{\mu_y} \le \sqrt{\frac{\beta\gamma}{\mu_y}},\ (1-\theta)\frac{L_{yy}}{\mu_y} \le \frac{1-\beta}{2},\ (1-\theta)^2\frac{L_{yx}L_{xy}}{\gamma\mu_y} \le \frac{\beta L_{xy}}{L_{yx}},\ (1-\theta)\frac{L_{yx}}{\gamma} \le \sqrt{\frac{\beta\mu_y}{\gamma}};$$

thus, using those inequalities within eq. (45a) and eq. (45b), we get

$$\Xi^x_{\tau,\sigma,\theta} \le 1 + \sqrt{\frac{\beta\gamma}{\mu_y}}, \tag{47a}$$

$$\Xi^y_{\tau,\sigma,\theta} \le 15 - 3\beta + 6\beta\frac{L_{xy}}{L_{yx}} + \sqrt{\frac{\beta\mu_y}{\gamma}}. \tag{47b}$$

Note that $\beta = \min\{\frac{1}{2},\ \frac{\mu_y}{\gamma},\ \frac{\gamma}{\mu_y},\ \frac{L_{yx}}{L_{xy}}\} \in (0,1)$ implies that

$$\Xi^x_{\tau,\sigma,\theta} \leq 2, \qquad \Xi^y_{\tau,\sigma,\theta} \leq 22.$$

Therefore, using the choice of $\{\tau,\sigma\}$ in eq. (42), we obtain a sufficient condition for eq. (44a) as given below:

$$\frac{1-\theta}{\gamma}\frac{2}{\gamma}\delta_x^2 + \frac{1-\theta}{\mu_y\theta}\frac{22}{\gamma}\delta_y^2 \leq \frac{\hat{\epsilon}^2}{2}. \tag{48}$$

Our choice of $\theta \in (0,1)$ in (42) implies that $\theta \geq \max\{\hat{\theta}_1, \hat{\theta}_2\}$, where $\hat{\theta}_1$ and $\hat{\theta}_2$ are defined in eq. (43). Note $\theta \geq \max\{\hat{\theta}_1, \hat{\theta}_2\}$ immediately implies that the above sufficient condition in (48) holds. Therefore, with the choice of $\{\tau,\sigma,\theta\}$ in eq. (42) we obtain that eq. (44a) holds, i.e.,

$$\frac{1}{\gamma}\Big(\tau\Xi^x_{\tau,\sigma,\theta}\delta_x^2 + \sigma\Xi^y_{\tau,\sigma,\theta}\delta_y^2\Big) \leq \frac{\hat{\epsilon}^2}{2}.$$

Furthermore, (44b) holds when $N \geq \ln\Big(\frac{2\max\{1,\ \mu_y/\gamma\}\mathcal{D}_0^2}{\hat{\epsilon}^2}\Big)/\ln\Big(\frac{1}{\theta}\Big) + 1$. Thus, we conclude that for any $\hat{\epsilon} > 0$, `SAPD`, stated in Algorithm 1, can generate x_N such that $\mathbb{E}\left[\|x_N - \hat{x}\|\right] \leq \hat{\epsilon}$ within $N_{\hat{\epsilon}}$ iterations for $\theta = \max\{\overline{\theta}(\beta),\ \hat{\theta}_1,\ \hat{\theta}_2\}$, where

$$N_{\hat{\epsilon}} = \mathcal{O}\Big(\ln\Big(\frac{\max\{1, \mu_y/\gamma\}}{\hat{\epsilon}}\Big)/\ln\Big(\frac{1}{\theta}\Big) + 1\Big). \tag{49}$$

Note $\frac{1}{\ln(\frac{1}{\theta})} \leq (1-\theta)^{-1}$ for $\theta \in (0,1)$ implies that

$$\frac{1}{\ln(\frac{1}{\theta})} \leq \mathcal{O}\Big(\max\{(1-\overline{\theta}_1(\beta))^{-1}, (1-\overline{\theta}_2(\beta))^{-1}, (1-\hat{\theta}_1)^{-1}, (1-\hat{\theta}_2)^{-1}\}\Big).$$

First, we equivalently rewrite $(1-\overline{\theta}_1(\beta))^{-1}$ and $(1-\overline{\theta}_2(\beta))^{-1}$ as follows:

$$(1-\overline{\theta}_1(\beta))^{-1} = \frac{1}{2}\frac{L'_{xx}}{\gamma} + \sqrt{\frac{1}{4}\frac{{L'_{xx}}^2}{\gamma^2} + \frac{L_{yx}^2}{\beta\gamma\mu_y}}, \quad (1-\overline{\theta}_2(\beta))^{-1} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{4L_{yy}^2}{(1-\beta)^2\mu_y^2}};$$

thus,

$$(1-\overline{\theta}_1)^{-1} \leq \frac{L'_{xx}}{\gamma} + \frac{L_{yx}}{\sqrt{\beta\gamma\mu_y}}, \qquad (1-\overline{\theta}_2)^{-1} \leq 1 + \frac{2}{1-\beta}\cdot\frac{L_{yy}}{\mu_y}.$$

Finally,

$$(1-\hat{\theta}_1)^{-1} = \mathcal{O}\Big(\frac{1}{\gamma^2}\cdot\frac{\delta_x^2}{\hat{\epsilon}^2}\Big), \qquad (1-\hat{\theta}_2)^{-1} = \mathcal{O}\Big(\frac{1}{\gamma\mu_y}\cdot\frac{\delta_y^2}{\hat{\epsilon}^2} + 1\Big).$$

Recall that $L'_{xx} = 2\gamma + L_{xx}$, using the above four identities that and our choice of $\beta = \min\{\frac{1}{2},\ \frac{\mu_y}{\gamma},\ \frac{\gamma}{\mu_y},\ \frac{L_{yx}}{L_{xy}}\}$ we derive that

$$\frac{1}{\ln(\frac{1}{\theta})} = \mathcal{O}\Big(\frac{\max\{L_{xx}, L_{yx}\}}{\gamma} + \frac{\max\{L_{yx}, L_{xy}\}}{\sqrt{\gamma\mu_y}} + \frac{\max\{L_{yy}, L_{yx}\}}{\mu_y} + \Big(\frac{\delta_x^2}{\gamma} + \frac{\delta_y^2}{\mu_y}\Big)\frac{1}{\gamma\hat{\epsilon}^2}\Big),$$

From (49), we conclude that

$$N_{\hat{\epsilon}} = \mathcal{O}\left(\frac{\max\{L_{xx}, L_{yx}\}}{\gamma} + \frac{\max\{L_{yx}, L_{xy}\}}{\sqrt{\gamma\mu_y}} + \frac{\max\{L_{yy}, L_{yx}\}}{\mu_y} + \Big(\frac{\delta_x^2}{\gamma} + \frac{\delta_y^2}{\mu_y}\Big)\frac{1}{\gamma\hat{\epsilon}^2}\right)\cdot\ln\Big(\frac{\max\{1, \mu_y/\gamma\}}{\hat{\epsilon}}\Big),$$

which completes the proof. □

Lemma 11. *Suppose $f : \mathcal{X} \to \mathbb{R}\cup\{+\infty\}$ is closed convex, and V is a strictly convex function on* $\mathbf{dom}\, f$ *and differentiable on an open set containing* $\mathbf{dom}\, f$. *Let $x_* = \operatorname{argmin}_{x\in\mathcal{X}} f(x) + V(x)$. Then, for any $\alpha > 0$, it holds that $x_* = \mathbf{prox}_{\alpha f}(x_* - \alpha\nabla V(x_*))$.*

Proof. From the first-order optimality condition, we have

$$0 \in \partial f(x_*) + \nabla V(x_*). \tag{50}$$

Moreover, from the definition of $\mathbf{prox}_{\alpha f}(\cdot)$ operator, it follows that

$$\mathbf{prox}_{\alpha f}(x_* - \alpha\nabla V(x_*)) = \operatorname*{argmin}_{x\in\mathcal{X}} f(x) + \nabla V(x_*)^\top(x - x_*) + \frac{1}{2\alpha}\|x - x_*\|^2. \tag{51}$$

Finally, (50) implies that x_* is the unique minimizer of the problem on the rhs of (51). Therefore, we get that $x_* = \mathbf{prox}_{\alpha f}(x_* - \alpha\nabla V(x_*))$, which completes the proof. □

B.1 Proof of Theorem 2

We are now ready to prove Theorem 2.

Proof. Let $\hat{x} = \mathbf{prox}_{\lambda\phi}(x_\epsilon)$, and ϕ^s be the smooth part of ϕ, i.e., $\phi = f + \phi^s$. Moreover, since $\Phi(x,\cdot) - g(\cdot)$ is strongly concave and $\Phi(\cdot, y)$ is differentiable, we have that ϕ^s is differentiable; hence, for any $x \in \mathbf{dom}\, f$,

$$\nabla \phi^s(x) = \nabla_x \Phi(x, y_*(x)), \quad \text{where} \quad y_*(x) = \operatorname*{argmax}_{y \in \mathcal{Y}} \Phi(x,y) - g(y).$$

Then we can explicitly write $\hat{x}$ as

$$\hat{x} = \operatorname*{argmin}_{x \in \mathcal{X}} f(x) + \phi^s(x) + \frac{1}{2\lambda}\|x - x_\epsilon\|^2.$$

Since $\phi^s(\cdot) + \frac{1}{2\lambda}\|\cdot - x_\epsilon\|^2$ is smooth and strongly convex, for any $\alpha > 0$, Lemma 11 implies that

$$\hat{x} = \mathbf{prox}_{\alpha f}\left(\hat{x} - \alpha\big(\nabla\phi^s(\hat{x}) + \frac{1}{\lambda}(\hat{x} - x_\epsilon)\big)\right).$$

If we let $\alpha = \lambda$, it follows that

$$\hat{x} = \mathbf{prox}_{\lambda f}\big(x_\epsilon - \lambda \nabla_x \phi^s(\hat{x})\big).$$

Moreover, since f is closed convex, $\mathbf{prox}_f(\cdot)$ is nonexpansive; hence,

$$\mathbb{E}\left[\|\hat{x} - \mathbf{prox}_{\lambda f}\big(\hat{x} - \lambda\nabla_x\phi^s(\hat{x})\big)\|\right] \le \mathbb{E}\left[\|x_\epsilon - \hat{x}\|\right] \le \frac{\lambda\epsilon}{2}, \tag{52}$$

where we used Lemma 1 for the last inequality, i.e., $\|x_\epsilon - \hat{x}\| = \lambda\|\nabla\phi_\lambda(x_\epsilon)\|$. On the other hand, for any $\tilde{x} \in \mathbf{dom}\, f$,

$$\begin{aligned}
&\mathbb{E}\left[\|\tilde{x} - \mathbf{prox}_{\lambda f}\big(\tilde{x} - \lambda\nabla_x\phi^s(\tilde{x})\big)\|\right] \\
\le&\mathbb{E}\left[\|\tilde{x} - \mathbf{prox}_{\lambda f}\big(\tilde{x} - \lambda\nabla_x\phi^s(\tilde{x})\big) - \hat{x} + \mathbf{prox}_{\lambda f}\big(\hat{x} - \lambda\nabla_x\phi^s(\hat{x})\big)\|\right] + \frac{\lambda\epsilon}{2} \\
\le&2\mathbb{E}\left[\|\tilde{x} - \hat{x}\|\right] + \lambda\mathbb{E}\left[\|\nabla_x\Phi(\tilde{x}, y_*(\tilde{x})) - \nabla_x\Phi(\hat{x}, y_*(\hat{x}))\|\right] + \frac{\lambda\epsilon}{2}.
\end{aligned} \tag{53}$$

According to [7, Proposition 1], $y_*(\cdot)$ is Lipschitz with constant $\kappa_{yx} = \frac{L_{yx}}{\mu_y}$. Therefore, we get

$$\|\nabla_x\Phi(\tilde{x}, y_*(\tilde{x})) - \nabla_x\Phi(\hat{x}, y_*(\hat{x}))\| \le L_{xx}\|\tilde{x} - \hat{x}\| + L_{xy}\|y_*(\tilde{x}) - y_*(\hat{x})\| \le \big(L_{xx} + L_{xy}\kappa_{yx}\big)\|\tilde{x} - \hat{x}\|,$$

which together with eq. (53) implies that

$$\frac{1}{\lambda}\mathbb{E}\left[\|\tilde{x} - \mathbf{prox}_{\lambda f}\big(\tilde{x} - \lambda\nabla_x\phi^s(\tilde{x})\big)\|\right] \le \big(\frac{2}{\lambda} + L_{xx} + L_{xy}\kappa_{yx}\big)\mathbb{E}\left[\|\tilde{x} - \hat{x}\|\right] + \frac{\epsilon}{2}. \tag{54}$$

Let $\lambda^{-1} = 2\gamma$, and $C \triangleq (4\gamma + L_{xx} + L_{xy}\kappa_{yx})^{-1}/2$. Thus, for any $\tilde{x} \in \mathbf{dom}\, f$ such that $\mathbb{E}\left[\|\tilde{x} - \hat{x}\|\right] \le C\epsilon$, we have

$$\mathbb{E}\left[\frac{1}{\lambda}\|\tilde{x} - \mathbf{prox}_{\lambda f}\big(\tilde{x} - \lambda\nabla_x\phi^s(\tilde{x})\big)\|\right] \le \epsilon.$$

Indeed, when $f(x) = 0$ for all $x \in \mathcal{X}$, we get $\phi(x) = \phi^s(x)$ and the above inequality implies that

$$\mathbb{E}\left[\|\nabla\phi(\tilde{x})\|\right] \le \epsilon.$$

The rest directly follows from invoking Lemma 10 with $\hat{\epsilon} = C\epsilon$, and $x_0 = x_\epsilon$. $\square$

C Proofs of Lemma 2 and Lemma 7

We first discuss the proof of Lemma 7 and later establish Lemma 2 through specializing some parts of this proof. Indeed recall that Lemma 7 is stated for a generic SAPD+ subproblem of the form (2). In Lemma 12 below, we restate Lemma 7 and rather than using a generic subproblem, we state it for the specific subproblems as in (4), which arise while implementing SAPD+. It is crucial to remind that the matrix inequality (MI) we establish in Lemma 7, i.e., eq. (32), helps us describe the admissible set of algorithm parameters that guarantee the *linear* convergence of inner loop iterates generated by SAPD, i.e., $\left\{\mathbb{E}\left[\|x_k^t - x_*^t\|^2 + \|y_k^t - y_*^t\|^2\right]\right\}_{k\geq 0}$, for any $t \geq 0$.

Lemma 12. *Suppose Assumptions 1, 2, 3 hold. For any given $\mu_x > 0$ and $t \in \mathbb{Z}_+$, consider solving the SCSC subproblem in* (4) *using SAPD, displayed in Algorithm 1. Let (x_*^t, y_*^t) denote the unique saddle point of* (4)*, and let $\{x_k^t, y_k^t\}_{k\geq 0}$ be the iterate sequence when initialized from $(x_0^t, y_0^t) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$ and using τ, σ, θ that satisfy* (32) *for some $\alpha \in [0, \frac{1}{\sigma})$ and $\rho \in (0, 1)$, where $L'_{xx} \triangleq L_{xx} + \mu_x + \gamma$. Then for all $N \geq \mathbb{Z}_+$, it holds that*

$$\begin{aligned} &\mathbb{E}\Big[\Big(\frac{1}{\tau} - \mu_x\Big)\|x_N^t - x_*^t\|^2 + \Big(\frac{1}{\sigma} - \alpha\Big)\|y_N^t - y_*^t\|^2\Big] \\ &\quad\leq \rho^N \mathbb{E}\left[\frac{1}{\tau}\|x_0^t - x_*^t\|^2 + \frac{1}{\sigma}\|y_0^t - y_*^t\|^2\right] + \frac{\rho}{1-\rho}\Big(\tau \Xi^x_{\tau,\sigma,\theta}\delta_x^2 + \sigma \Xi^y_{\tau,\sigma,\theta}\delta_y^2\Big), \end{aligned} \tag{55}$$

where $\Xi^x_{\tau,\sigma,\theta}$ and $\Xi^y_{\tau,\sigma,\theta}$ are defined in (18a) *and* (18b)*, respectively.*

Proof. The proof is provided in appendix C.2. □

Clearly, it is sufficient to prove Lemma 12 in order to establish Lemma 7. Moreover, as stated earlier, we will exploit the techniques used in the proof of Lemma 12 when showing Lemma 2 –this is why we restated Lemma 7.

C.1 Preliminary technical results

Recall that given some $x_0^t \in \mathbf{dom}\, f$ and $\mu_x > 0$, we define

$$\mathcal{L}^t(x, y) \triangleq f(x) + \Phi^t(x, y) - g(y), \tag{56a}$$

$$\Phi^t(x, y) \triangleq \Phi(x, y) + \frac{\mu_x + \gamma}{2}\|x - x_0^t\|^2, \tag{56b}$$

where $\gamma > 0$ is the weak-convexity constant of $\Phi(\cdot, y)$ for any $y \in \mathbf{dom}\, g$. It follows from Assumption 2 that $\nabla_y \Phi^t$ and $\nabla_x \Phi^t$ are Lipschitz such that

$$\|\nabla_y \Phi^t(x, y) - \nabla_y \Phi^t(x', y')\| \leq L_{yx}\|x - x'\| + L_{yy}\|y - y'\|, \tag{57}$$

$$\|\nabla_x \Phi^t(x, y) - \nabla_x \Phi^t(x', y')\| \leq L'_{xx}\|x - x'\| + L_{xy}\|y - y'\| \tag{58}$$

such that $L'_{xx} \triangleq L_{xx} + \mu_x + \gamma$. Furthermore, (56b) implies that for any $y \in \mathbf{dom}\, g$, $\Phi^t(\cdot, y)$ is strongly convex with modulus $\mu_x > 0$.

We will derive some key inequalities below for SAPD iterates $\{x_k^t, y_k^t\}_{k\geq 0}$ generated by Algorithm 1 to solve $\min_x \max_y \mathcal{L}^t(x, y)$. Let $x_{-1}^t = x_0^t$, $y_{-1}^t = y_0^t$, and for $k \geq 0$, define

$$q_k^t \triangleq \nabla_y \Phi^t(x_k^t, y_k^t) - \nabla_y \Phi^t(x_{k-1}^t, y_{k-1}^t), \qquad s_k^t \triangleq \nabla_y \Phi^t(x_k^t, y_k^t) + \theta q_k^t. \tag{59}$$

Thus $q_0^t = \mathbf{0}$; and for $k \geq 0$, Assumption 2 implies that

$$\|q_{k+1}^t\| \leq L_{yx}\|x_{k+1}^t - x_k^t\| + L_{yy}\|y_{k+1}^t - y_k^t\|. \tag{60}$$

Lemma 13. *Suppose Assumptions 1, 2, 3 hold. Let $\{x_k^t, y_k^t\}_{k\geq 0}$ be SAPD iterates generated according to Algorithm 1 for solving $\min_x \max_y \mathcal{L}^t(x, y)$. Then for all $x \in \mathbf{dom}\, f \subset \mathcal{X}$, $y \in \mathbf{dom}\, g \subset \mathcal{Y}$, and $k \geq 0$,*

$$\begin{aligned} &\mathcal{L}^t(x_{k+1}^t, y) - \mathcal{L}^t(x, y_{k+1}^t) \\ &\quad\leq -\langle q_{k+1}^t, y_{k+1}^t - y\rangle + \theta\langle q_k^t, y_k^t - y\rangle + \Lambda_k^t(x, y) - \Sigma_{k+1}^t(x, y) + \Gamma_{k+1}^t + \varepsilon_k^{t,x}(x) + \varepsilon_k^{t,y}(y), \end{aligned} \tag{61}$$

where

$$\varepsilon_k^{t,x}(x) \triangleq \langle \tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t; \omega_k^x) - \nabla_x \Phi^t(x_k^t, y_{k+1}^t),\ x - x_{k+1}^t \rangle, \quad \varepsilon_k^{t,y}(y) \triangleq \langle \tilde{s}_k^t - s_k^t, y_{k+1}^t - y \rangle,$$

q_k^t and s_k^t are defined as in (59)*, and*

$$\begin{aligned}
\Lambda_k^t(x,y) &\triangleq (\frac{1}{2\tau} - \frac{\mu_x}{2})\|x - x_k^t\|^2 + \frac{1}{2\sigma}\|y - y_k^t\|^2, \\
\Sigma_{k+1}^t(x,y) &\triangleq \frac{1}{2\tau}\|x - x_{k+1}^t\|^2 + (\frac{1}{2\sigma} + \frac{\mu_y}{2})\|y - y_{k+1}^t\|^2, \\
\Gamma_{k+1}^t &\triangleq (\frac{L'_{xx}}{2} - \frac{1}{2\tau})\|x_{k+1}^t - x_k^t\|^2 - \frac{1}{2\sigma}\|y_{k+1}^t - y_k^t\|^2 \\
&\quad + \theta L_{yx}\|x_k^t - x_{k-1}^t\|\|y_{k+1}^t - y_k^t\| + \theta L_{yy}\|y_k^t - y_{k-1}^t\|\|y_{k+1}^t - y_k^t\|.
\end{aligned}$$

Proof. Fix $x \in \mathbf{dom}\, f$ and $y \in \mathbf{dom}\, g$. Using Lemma 7.1 from [15] for the $y-$ and $x-$subproblems in Algorithm 1, we get

$$f(x_{k+1}^t) + \langle \tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t; \omega_k^x), x_{k+1}^t - x \rangle \le f(x) + \frac{1}{2\tau}\left[\|x - x_k^t\|^2 - \|x - x_{k+1}^t\|^2 - \|x_{k+1}^t - x_k^t\|^2\right],$$
$$g(y_{k+1}^t) - \langle \tilde{s}_k^t, y_{k+1}^t - y \rangle \le g(y) + \frac{1}{2\sigma}\left[\|y - y_k^t\|^2 - \|y - y_{k+1}^t\|^2 - \|y_{k+1}^t - y_k^t\|^2\right].$$

Thus, by adding and subtracting we further get

$$\begin{aligned}
&f(x_{k+1}^t) + \langle \nabla_x \Phi^t(x_k^t, y_{k+1}^t),\ x_{k+1}^t - x \rangle \\
&\quad \le f(x) + \tfrac{1}{2\tau}(\|x - x_k^t\|^2 - \|x - x_{k+1}^t\|^2 - \|x_{k+1}^t - x_k^t\|^2) + \varepsilon_k^{t,x}(x),
\end{aligned} \tag{62a}$$

$$\begin{aligned}
&g(y_{k+1}^t) - \langle s_k^t, y_{k+1}^t - y \rangle \\
&\quad \le g(y) + \tfrac{1}{2\sigma}(\|y - y_k^t\|^2 - \|y - y_{k+1}^t\|^2 - \|y_{k+1}^t - y_k^t\|^2) + \varepsilon_k^{t,y}(y).
\end{aligned} \tag{62b}$$

Rearranging the terms in (62b), we get

$$\begin{aligned}
&-g(y) + g(y_{k+1}^t) \\
&\quad \le \langle s_k^t, y_{k+1}^t - y \rangle + \tfrac{1}{2\sigma}\left[\|y - y_k^t\|^2 - \|y - y_{k+1}^t\|^2 - \|y_{k+1}^t - y_k^t\|^2\right] + \varepsilon_k^{t,y}(y).
\end{aligned} \tag{63}$$

Since $y_{k+1}^t \in \mathbf{dom}\, g$, the inner product in (62a) can be lower bounded using convexity of $\Phi^t(\cdot, y_{k+1}^t)$ as follows (see Assumption 2):

$$\begin{aligned}
\langle \nabla_x \Phi^t(x_k^t, y_{k+1}^t), x_{k+1}^t - x \rangle &= \langle \nabla_x \Phi^t(x_k^t, y_{k+1}^t), x_k^t - x \rangle + \langle \nabla_x \Phi^t(x_k^t, y_{k+1}^t), x_{k+1}^t - x_k^t \rangle \\
&\ge \Phi^t(x_k^t, y_{k+1}^t) - \Phi^t(x, y_{k+1}^t) + \frac{\mu_x}{2}\|x - x_k^t\|^2 + \langle \nabla_x \Phi^t(x_k^t, y_{k+1}^t),\ x_{k+1}^t - x_k^t \rangle.
\end{aligned}$$

Using this inequality after adding $\Phi^t(x_{k+1}^t, y_{k+1}^t)$ to both sides of (62a), we get

$$\begin{aligned}
&\Phi^t(x_{k+1}^t, y_{k+1}^t) + f(x_{k+1}^t) \\
&\le \Phi^t(x, y_{k+1}^t) + f(x) + \Phi^t(x_{k+1}^t, y_{k+1}^t) - \Phi^t(x_k^t, y_{k+1}^t) - \langle \nabla_x \Phi^t(x_k^t, y_{k+1}^t), x_{k+1}^t - x_k^t \rangle \\
&\quad + \tfrac{1}{2\tau}\left[\|x - x_k^t\|^2 - \|x - x_{k+1}^t\|^2 - \|x_{k+1}^t - x_k^t\|^2\right] - \tfrac{\mu_x}{2}\|x - x_k^t\|^2 + \varepsilon_k^{t,x}(x) \\
&\le \Phi^t(x, y_{k+1}^t) + f(x) + \frac{L'_{xx}}{2}\|x_{k+1}^t - x_k^t\|^2 \\
&\quad + \tfrac{1}{2\tau}\left[\|x - x_k^t\|^2 - \|x - x_{k+1}^t\|^2 - \|x_{k+1}^t - x_k^t\|^2\right] - \tfrac{\mu_x}{2}\|x - x_k^t\|^2 + \varepsilon_k^{t,x}(x),
\end{aligned} \tag{64}$$

where the last step uses Assumption 2. Rearranging the terms gives us

$$\begin{aligned}
&f(x_{k+1}^t) - f(x) - \Phi^t(x, y_{k+1}^t) \le -\Phi^t(x_{k+1}^t, y_{k+1}^t) + \frac{L'_{xx}}{2}\|x_{k+1}^t - x_k^t\|^2 \\
&\quad + \frac{1}{2\tau}\left[\|x - x_k^t\|^2 - \|x - x_{k+1}^t\|^2 - \|x_{k+1}^t - x_k^t\|^2\right] - \frac{\mu_x}{2}\|x - x_k^t\|^2 + \varepsilon_k^{t,x}(x).
\end{aligned} \tag{65}$$

Then, for $k \geq 0$, by summing (63) and (65), we obtain

$$
\begin{aligned}
&\mathcal{L}(x_{k+1}^t, y) - \mathcal{L}(x, y_{k+1}^t) = f(x_{k+1}^t) + \Phi^t(x_{k+1}^t, y) - g(y) - f(x) - \Phi^t(x, y_{k+1}^t) + g(y_{k+1}^t) \\
&\leq \Phi^t(x_{k+1}^t, y) - \Phi^t(x_{k+1}^t, y_{k+1}) + \langle s_k^t, y_{k+1}^t - y\rangle + \frac{L'_{xx}}{2}\|x_{k+1}^t - x_k^t\|^2 \\
&\quad + \frac{1}{2\sigma}\left[\|y - y_k^t\|^2 - \|y - y_{k+1}^t\|^2 - \|y_{k+1}^t - y_k^t\|^2\right] + \varepsilon_k^{t,y}(y) \\
&\quad + \frac{1}{2\tau}\left[\|x - x_k^t\|^2 - \|x - x_{k+1}^t\|^2 - \|x_{k+1}^t - x_k^t\|^2\right] - \frac{\mu_x}{2}\|x - x_k^t\|^2 + \varepsilon_k^{t,x}(x).
\end{aligned} \tag{66}
$$

From Assumption 2, the μ_y-strongly concavity of $\Phi^t(x, \cdot)$ for fixed $x \in \mathbf{dom}\, f \subset \mathcal{X}$ implies

$$
\begin{aligned}
&\Phi^t(x_{k+1}^t, y) - \Phi^t(x_{k+1}^t, y_{k+1}^t) + \langle s_k^t, y_{k+1}^t - y\rangle \\
&\leq \langle \nabla_y \Phi^t(x_{k+1}^t, y_{k+1}^t), y - y_{k+1}^t\rangle - \frac{\mu_y}{2}\|y - y_{k+1}^t\|^2 + \langle \nabla_y \Phi^t(x_k^t, y_k^t) + \theta q_k^t, y_{k+1}^t - y\rangle \\
&= -\langle q_{k+1}^t, y_{k+1}^t - y\rangle - \frac{\mu_y}{2}\|y - y_{k+1}^t\|^2 + \theta\langle q_k^t, y_k^t - y\rangle + \theta\langle q_k^t, y_{k+1}^t - y_k^t\rangle.
\end{aligned}
$$

Thus, using the above inequality within (66), we get

$$
\begin{aligned}
&\mathcal{L}^t(x_{k+1}^t, y) - \mathcal{L}^t(x, y_{k+1}^t) \leq -\langle q_{k+1}^t, y_{k+1}^t - y\rangle + \theta\langle q_k^t, y_k^t - y\rangle + \theta\langle q_k^t, y_{k+1}^t - y_k^t\rangle \\
&\quad + \frac{L'_{xx}}{2}\|x_{k+1}^t - x_k^t\|^2 + \frac{1}{2\sigma}\left[\|y - y_k^t\|^2 - \|y - y_{k+1}^t\|^2 - \|y_{k+1}^t - y_k^t\|^2\right] - \frac{\mu_y}{2}\|y - y_{k+1}^t\|^2 \\
&\quad + \frac{1}{2\tau}\left[\|x - x_k^t\|^2 - \|x - x_{k+1}^t\|^2 - \|x_{k+1}^t - x_k^t\|^2\right] - \frac{\mu_x}{2}\|x - x_k^t\|^2 + \varepsilon_k^{t,x}(x) + \varepsilon_k^{t,y}(y).
\end{aligned}
$$

Finally, (61) follows from using Cauchy-Schwarz for $\left\langle q_k^t, y_{k+1}^t - y_k^t\right\rangle$ and (60). $\square$

Lemma 14. *[3, Theorem 6.42] Let f be proper, closed and convex function. Then for any $x, x' \in \mathcal{X}$, we get $\|\mathbf{prox}_f(x) - \mathbf{prox}_f(x')\| \leq \|x - x'\|$.*

Next, based on the above inequality, we prove an intermediate result, which we use later to bound the variance of the SAPD iterate sequence.

Lemma 15. *Suppose Assumptions 1, 2, 3 hold. Let $\{x_k^t, y_k^t\}_{k\geq 0}$ be SAPD iterates generated as in Algorithm 1 for solving $\min_x \max_y \mathcal{L}^t(x, y)$. For $k \geq 0$, let q_k^t and s_k^t be defined as in (59), and let*

$$
\begin{aligned}
&\hat{x}_{k+1}^t \triangleq \mathbf{prox}_{\tau f}\left(x_k^t - \tau\nabla_x \Phi^t(x_k^t, y_{k+1}^t)\right), \quad \hat{\hat{x}}_{k+1}^t \triangleq \mathbf{prox}_{\tau f}\left(x_k^t - \tau\nabla_x \Phi^t(x_k^t, \hat{y}_{k+1}^t)\right), \\
&\hat{y}_{k+1}^t \triangleq \mathbf{prox}_{\sigma g}\left(y_k^t + \sigma s_k^t\right), \quad \hat{\hat{y}}_{k+1}^t \triangleq \mathbf{prox}_{\sigma g}\Big(\hat{y}_k^t + \sigma(1+\theta)\nabla_y \Phi^t(\hat{\hat{x}}_k^t, \hat{y}_k^t) - \sigma\theta\nabla_y \Phi^t(x_{k-1}^t, y_{k-1}^t)\Big),
\end{aligned}
$$

then the following inequalities hold for $k \geq 0$:

$$
\|x_{k+1}^t - \hat{x}_{k+1}^t\| \leq \tau\|\Delta_k^{t,x}\|, \qquad \|y_{k+1}^t - \hat{y}_{k+1}^t\| \leq \sigma\left((1+\theta)\|\Delta_k^{t,y}\| + \theta\|\Delta_{k-1}^{t,y}\|\right), \tag{67a}
$$

$$
\begin{aligned}
\|y_{k+1}^t - \hat{\hat{y}}_{k+1}^t\| &\leq \sigma\left((1+\theta)\|\Delta_k^{t,y}\| + \theta\|\Delta_{k-1}^{t,y}\| + \tau(1+\theta)L_{yx}\|\Delta_{k-1}^{t,x}\|\right) \\
&\quad + \sigma\left(1 + \sigma(1+\theta)L_{yy} + \tau\sigma(1+\theta)L_{yx}L_{xy}\right)\left((1+\theta)\|\Delta_{k-1}^{t,y}\| + \theta\|\Delta_{k-2}^{t,y}\|\right),
\end{aligned} \tag{67b}
$$

where $\Delta_k^{t,x} \triangleq \tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t; \omega_k^x) - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)$, and $\Delta_k^{t,y} \triangleq \tilde{\nabla}_y \Phi^t(x_k^t, y_k^t; \omega_k^y) - \nabla_y \Phi^t(x_k^t, y_k^t)$.

Proof. The first inequality in eq. (67a) is from Lemma 14; for the second, we have

$$
\|y_{k+1}^t - \hat{y}_{k+1}^t\| \leq \sigma\|\tilde{s}_k^t - s_k^t\| \leq \sigma\left((1+\theta)\|\Delta_k^{t,y}\| + \theta\|\Delta_{k-1}^{t,y}\|\right),
$$

which follows from Lemma 14 and the triangle inequality. To show eq. (67b), we bound $\|y_{k+1}^t - \hat{y}_{k+1}^t\|$ and $\|\hat{y}_{k+1}^t - \hat{\hat{y}}_{k+1}^t\|$ separately. It follows from Lemma 14 that $\|x_{k+1}^t - \hat{\hat{x}}_{k+1}^t\| \leq \tau\|\tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t; \omega_k^x) - \nabla_x \Phi^t(x_k^t, \hat{y}_{k+1}^t)\|$. After adding and subtracting $\nabla_x \Phi^t(x_k^t, y_{k+1}^t)$, Assumption 2 implies that

$$
\|x_{k+1}^t - \hat{\hat{x}}_{k+1}^t\| \leq \tau\left(\|\Delta_k^{t,x}\| + L_{xy}\|y_{k+1}^t - \hat{y}_{k+1}^t\|\right). \tag{68}
$$

We will use this relation to bound $\|\hat{y}_{k+1}^t - \hat{\hat{y}}_{k+1}^t\|$. Indeed, using Lemma 14, we have

$$
\begin{aligned}
\|\hat{y}_{k+1}^t - \hat{\hat{y}}_{k+1}^t\| \leq & \|y_k^t - \hat{y}_k^t + \sigma(1+\theta)\left(\nabla_y \Phi^t(x_k^t, y_k^t) - \nabla_y \Phi^t(\hat{\hat{x}}_k^t, \hat{y}_k^t)\right)\| \\
\leq & (1+\sigma(1+\theta)L_{yy})\|y_k^t - \hat{y}_k^t\| + \sigma(1+\theta)L_{yx}\|x_k^t - \hat{\hat{x}}_k^t\| \\
\leq & \Big(1+\sigma(1+\theta)L_{yy} + \tau\sigma(1+\theta)L_{yx}L_{xy}\Big)\|y_k^t - \hat{y}_k^t\| + \tau\sigma(1+\theta)L_{yx}\|\Delta_{k-1}^{t,x}\| \\
\leq & \sigma\left(1+\sigma(1+\theta)L_{yy} + \tau\sigma(1+\theta)L_{yx}L_{xy}\right)\cdot\left((1+\theta)\|\Delta_{k-1}^{t,y}\| + \theta\|\Delta_{k-2}^{t,y}\|\right) \\
& + \sigma\tau(1+\theta)L_{yx}\|\Delta_{k-1}^{t,x}\|,
\end{aligned}
$$

where the second, third and fourth inequalities follow from Assumption 2, eq. (68) and the second inequality in eq. (67a), respectively. Combining this with $\|y_{k+1}^t - \hat{\hat{y}}_{k+1}^t\| \leq \|y_{k+1}^t - \hat{y}_{k+1}^t\| + \|\hat{y}_{k+1}^t - \hat{\hat{y}}_{k+1}^t\|$, and the second inequality in eq. (67a) give us the desired bound. $\square$

Next, we provide some inequalities to bound the `SAPD` variance term later in our analysis.

Lemma 16. *Suppose Assumptions 1, 2, 3 hold. Let $\{x_k^t, y_k^t\}_{k\geq 0}$ be `SAPD` iterates generated according to Algorithm 1 for solving $\min_x \max_y \mathcal{L}^t(x,y)$. The following inequality holds for all $k \geq 0$:*

$$
\mathbb{E}\left[\langle \Delta_k^{t,x}, \hat{x}_{k+1}^t - x_{k+1}^t\rangle\right] \leq \tau\delta_x^2, \qquad \mathbb{E}\left[\langle \Delta_k^{t,y}, y_{k+1}^t - \hat{y}_{k+1}^t\rangle\right] \leq \sigma(1+2\theta)\delta_y^2,
$$

$$
\begin{aligned}
&\mathbb{E}\left[\langle \Delta_{k-1}^{t,y}, \hat{\hat{y}}_{k+1}^t - y_{k+1}^t\rangle\right] \\
&\quad \leq \sigma\left[\left((2+\sigma(1+\theta)L_{yy} + \tau\sigma(1+\theta)L_{yx}L_{xy})\cdot(1+2\theta) + \frac{\tau(1+\theta)L_{yx}}{2}\right)\delta_y^2 + \frac{\tau(1+\theta)L_{yx}}{2}\delta_x^2\right],
\end{aligned}
$$

where $\Delta_k^{t,x}$ and $\Delta_k^{t,y}$ are defined in Lemma 15.

Proof. With the convention that $y_{-2}^t = y_{-1}^t = y_0^t$, and $x_{-2}^t = x_{-1}^t = x_0^t$, Lemma 15 and Cauchy-Schwarz inequality imply for all $k \geq 0$ that

$$
\begin{aligned}
&\langle \Delta_k^{t,x}, x_{k+1}^t - \hat{x}_{k+1}^t\rangle \leq \tau\|\Delta_k^{t,x}\|^2, \\
&\langle \Delta_k^{t,y}, y_{k+1}^t - \hat{y}_{k+1}^t\rangle \leq \sigma\left((1+\theta)\|\Delta_k^{t,y}\|^2 + \theta\|\Delta_{k-1}^{t,y}\|\|\Delta_k^{t,y}\|\right), \\
&\langle \Delta_{k-1}^{t,y}, y_{k+1}^t - \hat{\hat{y}}_{k+1}^t\rangle \leq \sigma\Bigg((1+\theta)\|\Delta_k^{t,y}\|\|\Delta_{k-1}^{t,y}\| + \theta\|\Delta_{k-1}^{t,y}\|^2 + \tau(1+\theta)L_{yx}\|\Delta_{k-1}^{t,x}\|\|\Delta_{k-1}^{t,y}\| \\
&+ \Big(1+\sigma(1+\theta)L_{yy} + \tau\sigma(1+\theta)L_{yx}L_{xy}\Big)\cdot\Big((1+\theta)\|\Delta_{k-1}^{t,y}\|^2 + \theta\|\Delta_{k-2}^{t,y}\|\|\Delta_{k-1}^{t,y}\|\Big)\Bigg).
\end{aligned}
$$

Next, using Assumption 3 and $\|a\|\|b\| \leq \frac{1}{2}\|a\|^2 + \frac{1}{2}\|b\|^2$, which holds for $a, b \in \mathbb{R}^n$, and taking the expectation leads to the desired result. $\square$

Before we move on to prove our intermediate result in Lemma 12, we give two technical lemmas that help us simplify the `SAPD` parameter selection rule and lead to the matrix inequality in eq. (32).

Lemma 17. *Given $\tau, \sigma > 0$, $\theta, \alpha \geq 0$, and $\rho \in (0,1)$, let*

$$
G' \triangleq \begin{pmatrix}
\frac{1}{\tau}(1-\frac{1}{\rho}) + \frac{\mu_x}{\rho} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sigma}(1-\frac{1}{\rho}) + \mu_y & -|1-\frac{\theta}{\rho}|\,L_{yx} & -|1-\frac{\theta}{\rho}|\,L_{yy} & 0 \\
0 & -|1-\frac{\theta}{\rho}|\,L_{yx} & \frac{1}{\tau} - L'_{xx} & 0 & -\frac{\theta}{\rho}L_{yx} \\
0 & -|1-\frac{\theta}{\rho}|\,L_{yy} & 0 & \frac{1}{\sigma} - \alpha & -\frac{\theta}{\rho}L_{yy} \\
0 & 0 & -\frac{\theta}{\rho}L_{yx} & -\frac{\theta}{\rho}L_{yy} & \frac{\alpha}{\rho}
\end{pmatrix}, \tag{69}
$$

then $G \succeq 0$ if and only if $G' \succeq 0$, where G is defined in eq. (32).

Proof. $\forall\, \mathbf{y} = (y_1, y_2, y_3, y_4, y_5)^\top \in \mathbb{R}^5$, letting $\tilde{\mathbf{y}} = (y_1, -y_2, y_3, y_4, y_5)^\top$, we have

$$
\mathbf{y}^\top G'\mathbf{y} = \begin{cases} \mathbf{y}^\top G\mathbf{y} & \text{if } \theta \leq \rho, \\ \tilde{\mathbf{y}}^\top G\tilde{\mathbf{y}} & \text{else;} \end{cases} \qquad \mathbf{y}^\top G\mathbf{y} = \begin{cases} \mathbf{y}^\top G'\mathbf{y} & \text{if } \theta \leq \rho, \\ \tilde{\mathbf{y}}^\top G'\tilde{\mathbf{y}} & \text{else.} \end{cases}
$$

Thus, $G \succeq 0$ is equivalent to $G' \succeq 0$. $\square$

Lemma 18. *Given $\tau, \sigma > 0$, $\theta, \alpha \geq 0$, and $\rho \in (0,1)$, consider G defined in eq. (32). If $G \succeq 0$, then $G'' \succeq 0$, where*

$$G'' \triangleq \begin{pmatrix} \frac{1}{\sigma}(1-\frac{1}{\rho}) + \mu_y + \frac{\alpha}{\rho} & (-|1-\frac{\theta}{\rho}| - \frac{\theta}{\rho})L_{yx} & (-|1-\frac{\theta}{\rho}| - \frac{\theta}{\rho})L_{yy} \\ (-|1-\frac{\theta}{\rho}| - \frac{\theta}{\rho})L_{yx} & \frac{1}{\tau} - L'_{xx} & 0 \\ (-|1-\frac{\theta}{\rho}| - \frac{\theta}{\rho})L_{yy} & 0 & \frac{1}{\sigma} - \alpha \end{pmatrix} \succeq 0. \tag{70}$$

Proof. Note that $\mathbf{x}^\top G''\mathbf{x} = \mathbf{x}'^\top G'\mathbf{x}' \geq 0$ for all $\mathbf{x} = [x_1\ x_2\ x_3]^\top \in \mathbb{R}^3$, where $\mathbf{x}' = [0\ x_1\ x_2\ x_3\ x_1]^\top$ and G' is defined in (69). Then the desired result follows from Lemma 17. $\square$

Finally, with the following observation, we will be ready to proceed to the proof of Lemma 12. Let $\{\mathcal{F}_k^{t,x}\}$ and $\{\mathcal{F}_k^{t,y}\}$ be the filtrations such that $\mathcal{F}_k^{t,x} \triangleq \mathcal{F}(\{x_i^t\}_{i=0}^k, \{y_i^t\}_{i=0}^{k+1})$ and $\mathcal{F}_k^{t,y} \triangleq \mathcal{F}(\{x_i^t\}_{i=0}^k, \{y_i^t\}_{i=0}^k)$ denote the σ-algebras generated by the random variables in their arguments. A consequence of Assumption 3 is that for $\mathcal{F}_k^{t,x}$-measurable random variable v, i.e., $v \in \mathcal{F}_k^{t,x}$, we have that $\mathbb{E}\left[\langle \tilde{\nabla}\Phi_x(x_k^t, y_{k+1}^t; \omega_k^x) - \nabla\Phi_x(x_k^t, y_{k+1}^t), v\rangle\right] = 0$; similarly, for $v \in \mathcal{F}_k^{t,y}$, it holds that $\mathbb{E}\left[\langle \tilde{\nabla}\Phi_y(x_k^t, y_k^t; \omega_k^y) - \nabla\Phi_y(x_k^t, y_k^t), v\rangle\right] = 0$.

C.2 Proof of Lemma 12

Proof. Fix arbitrary $(x,y) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$. Since $(x_{k+1}^t, y_{k+1}^t) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$, using the concavity of $\mathcal{L}^t(x_{k+1}^t, \cdot)$ and the convexity of $\mathcal{L}^t(\cdot, y_{k+1}^t)$, Jensen's lemma immediately implies that

$$K_N(\rho)\left(\mathcal{L}^t(\bar{x}_N^t, y) - \mathcal{L}^t(x, \bar{y}_N^t)\right) \leq \sum_{k=0}^{N-1} \rho^{-k}\left(\mathcal{L}^t(x_{k+1}^t, y) - \mathcal{L}^t(x, y_{k+1}^t)\right),\ \forall \rho \in (0,1], \tag{71}$$

where $\bar{x}_N^t = \frac{1}{K_N(\rho)}\sum_{k=0}^{N-1}\rho^{-k}x_{k+1}^t$, $\bar{y}_N^t = \frac{1}{K_N(\rho)}\sum_{k=0}^{N-1}\rho^{-k}y_{k+1}^t$, $K_N(\rho) = \sum_{k=0}^{N-1}\rho^{-k+1}$. Thus, if we multiply both sides of (61) by ρ^{-k} and sum the resulting inequality from $k=0$ to $N-1$, then using (71) we get

$$\begin{aligned} &K_N(\rho)\left(\mathcal{L}^t(\bar{x}_N^t, y) - \mathcal{L}(x, \bar{y}_N^t)\right) \\ &\leq \sum_{k=0}^{N-1}\rho^{-k}\Big(\underbrace{-\langle q_{k+1}^t, y_{k+1}^t - y\rangle + \theta\langle q_k^t, y_k^t - y\rangle}_{\textbf{part 1}} + \Lambda_k^t(x,y) - \Sigma_{k+1}^t(x,y) + \Gamma_{k+1}^t \\ &\qquad \underbrace{-\langle \tilde{\nabla}_x\Phi^t(x_k^t, y_{k+1}^t; \omega_k^x) - \nabla_x\Phi^t(x_k^t, y_{k+1}^t), x_{k+1}^t - x\rangle}_{\textbf{part 2}} + \underbrace{\langle \tilde{s}_k^t - s_k^t, y_{k+1}^t - y\rangle}_{\textbf{part 3}}\Big). \end{aligned} \tag{72}$$

Using Cauchy–Schwarz inequality and (60) leads to

$$|\langle q_{k+1}^t, y_{k+1}^t - y\rangle| \leq S_{k+1}^t(x,y) \triangleq L_{yx}\|x_{k+1}^t - x_k^t\|\|y_{k+1}^t - y\| + L_{yy}\|y_{k+1}^t - y_k^t\|\|y_{k+1}^t - y\| \tag{73}$$

for $k \geq -1$. Recall $x_{-1}^t = x_0^t$, $y_{-1}^t = y_0^t$, thus $q_0 = \mathbf{0}$; therefore, for **part 1**,

$$\begin{aligned} \sum_{k=0}^{N-1}\rho^{-k}(\theta\langle q_k^t, y_k^t - y\rangle - \langle q_{k+1}^t, y_{k+1}^t - y\rangle) &= \sum_{k=0}^{N-2}\rho^{-k}\Big(\frac{\theta}{\rho} - 1\Big)\langle q_{k+1}^t, y_{k+1}^t - y\rangle - \rho^{-N+1}\langle q_N^t, y_N^t - y\rangle \\ &\leq \sum_{k=0}^{N-2}\rho^{-k}|1 - \frac{\theta}{\rho}|\, S_{k+1}^t(x,y) + \rho^{-N+1}S_N^t(x,y) = \sum_{k=0}^{N-1}\rho^{-k}|1 - \frac{\theta}{\rho}|\, S_{k+1}^t(x,y) + \rho^{-N+1}\frac{\theta}{\rho}S_N^t(x,y), \end{aligned} \tag{74}$$

where the first inequality follows from eq. (73).

Next, letting $\Delta_k^{t,x}$ and $\hat{x}_{k+1}$ be defined as in Lemma 15, we equivalently write **part 2** as

$$\sum_{k=0}^{N-1} -\rho^{-k}\langle \Delta_k^{t,x}, x_{k+1}^t - x\rangle = \sum_{k=0}^{N-1}\rho^{-k}\Big(\langle \Delta_k^{t,x}, \hat{x}_{k+1}^t - x_{k+1}^t\rangle - \langle \Delta_k^{t,x}, \hat{x}_{k+1}^t - x\rangle\Big). \tag{75}$$

Moreover, for $\Delta_k^{t,y}$, $\hat{y}_{k+1}^t$ and $\hat{\hat{y}}_{k+1}^t$ defined as in Lemma 15, we also equivalently write **part 3** as

$$\begin{aligned}&\sum_{k=0}^{N-1}\rho^{-k}\langle \tilde{s}_k^t - s_k, y_{k+1}^t - y\rangle\\ &\quad= \sum_{k=0}^{N-1}\rho^{-k}\Big[(1+\theta)\langle \Delta_k^{t,y}, y_{k+1}^t - \hat{y}_{k+1}^t + \hat{y}_{k+1}^t - y\rangle - \theta\langle \Delta_{k-1}^{t,y}, y_{k+1}^t - \hat{\hat{y}}_{k+1}^t + \hat{\hat{y}}_{k+1}^t - y\rangle\Big].\end{aligned} \tag{76}$$

Adding $\rho^{-N+1}D_N^t(x,y)$ to both sides of (72), then using (74), (75) and (76), for any fixed $(x,y) \in \mathcal{X}\times\mathcal{Y}$, we get

$$K_N(\rho)\left(\mathcal{L}^t(\bar{x}_N^t, y) - \mathcal{L}^t(x, \bar{y}_N^t)\right) + \rho^{-N+1}D_N^t(x,y) \leq U_N^t(x,y) + \sum_{k=0}^{N-1}\rho^{-k}(P_k^t(x,y) + Q_k^t), \tag{77}$$

where $U_N^t(x,y)$, $D_N^t(x,y)$ are defined as

$$\begin{aligned}U_N^t(x,y) \triangleq &\sum_{k=0}^{N-1}\rho^{-k}\left(\Gamma_{k+1}^t + \Lambda_k^t(x,y) - \Sigma_{k+1}(x,y) + |1-\frac{\theta}{\rho}|\; S_{k+1}^t(x,y)\right)\\ &- \rho^{-N+1}\Big(- D_N^t(x,y) - \frac{\theta}{\rho}S_N^t(x,y)\Big),\end{aligned} \tag{78a}$$

$$D_N^t(x,y) \triangleq \frac{1}{2\rho}\Big(\frac{1}{\tau} - \mu_x\Big)\|x_N^t - x\|^2 + \frac{1}{2\rho}\left(\frac{1}{\sigma} - \alpha\right)\|y_N^t - y\|^2, \tag{78b}$$

and $P_k^t(x,y)$, Q_k^t for $k = 0, \cdots, N-1$ are defined as

$$P_k^t(x,y) \triangleq -\langle \Delta_k^{t,x}, \hat{x}_{k+1}^t - x\rangle + (1+\theta)\langle \Delta_k^{t,y}, \hat{y}_{k+1}^t - y\rangle - \theta\langle \Delta_{k-1}^{t,y}, \hat{\hat{y}}_{k+1}^t - y\rangle, \tag{79a}$$

$$Q_k^t \triangleq \langle \Delta_k^{t,x}, \hat{x}_{k+1}^t - x_{k+1}^t\rangle + (1+\theta)\langle \Delta_k^{t,y}, y_{k+1}^t - \hat{y}_{k+1}^t\rangle - \theta\langle \Delta_{k-1}^{t,y}, y_{k+1}^t - \hat{\hat{y}}_{k+1}^t\rangle. \tag{79b}$$

For any fixed $(x,y) \in \mathcal{X}\times\mathcal{Y}$, we first analyze $U_N^t(x,y)$. After adding and subtracting $\frac{\alpha}{2}\|y_{k+1}^t - y_k^t\|^2$, and rearranging the terms, we get

$$\begin{aligned}U_N^t(x,y) =&\frac{1}{2}\sum_{k=0}^{N-1}\rho^{-k}\Big(\xi_k^\top A\xi_k - \xi_{k+1}^\top B\xi_{k+1}\Big) - \rho^{-N+1}(-D_N^t(x,y) - \frac{\theta}{\rho}S_N^t(x,y))\\ =&\frac{1}{2}\xi_0^\top A\xi_0 - \frac{1}{2}\sum_{k=1}^{N-1}\rho^{-k+1}[\xi_k^\top(B - \tfrac{1}{\rho}A)\xi_k] - \rho^{-N+1}\Big(\frac{1}{2}\xi_N^\top B\xi_N - D_N^t(x,y) - \frac{\theta}{\rho}S_N^t(x,y)\Big),\end{aligned} \tag{80}$$

where $A, B \in \mathbb{R}^{5\times 5}$ and $\xi_k \in \mathbb{R}^5$ are defined for $k \geq 0$ as follows: $\xi_k \triangleq \begin{pmatrix}\|x_k^t - x\|\\ \|y_k^t - y\|\\ \|x_k^t - x_{k-1}^t\|\\ \|y_k^t - y_{k-1}^t\|\\ \|y_{k+1}^t - y_k^t\|\end{pmatrix}$ such that $x_{-1}^t = x_0^t$, $y_{-1}^t = y_0^t$, and

$$A \triangleq \begin{pmatrix}\frac{1}{\tau} - \mu_x & 0 & 0 & 0 & 0\\ 0 & \frac{1}{\sigma} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \theta L_{yx}\\ 0 & 0 & 0 & 0 & \theta L_{yy}\\ 0 & 0 & \theta L_{yx} & \theta L_{yy} & -\alpha\end{pmatrix}, \quad B \triangleq \begin{pmatrix}\frac{1}{\tau} & 0 & 0 & 0 & 0\\ 0 & \frac{1}{\sigma} + \mu_y & -|1-\frac{\theta}{\rho}|\; L_{yx} & -|1-\frac{\theta}{\rho}|\; L_{yy} & 0\\ 0 & -|1-\frac{\theta}{\rho}|\; L_{yx} & \frac{1}{\tau} - L'_{xx} & 0 & 0\\ 0 & -|1-\frac{\theta}{\rho}|\; L_{yy} & 0 & \frac{1}{\sigma} - \alpha & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

In Lemma 17 we show that eq. (32) is equivalent to $B - \frac{1}{\rho}A \succeq 0$; therefore, it follows from (80) that for any given $(x,y) \in \mathcal{X}\times\mathcal{Y}$,

$$U_N^t(x,y) \leq \frac{1}{2}\xi_0^\top A\xi_0 - \rho^{-N+1}(\tfrac{1}{2}\xi_N^\top B\xi_N - D_N^t(x,y) - \frac{\theta}{\rho}S_N^t(x,y)), \text{ holds w.p. 1.}$$

Furthermore, we have

$$\frac{1}{2}\xi_N^\top B\xi_N - D_N^t(x,y) - \frac{\theta}{\rho}S_N^t(x,y) = \frac{1}{2}\xi_N^\top \begin{pmatrix} \frac{1}{\tau}(1-\frac{1}{\rho}) + \frac{\mu_x}{\rho} & \mathbf{0}_{1\times 3} & 0 \\ \mathbf{0}_{3\times 1} & G'' & \mathbf{0}_{3\times 1} \\ 0 & \mathbf{0}_{1\times 3} & 0 \end{pmatrix} \xi_N \geq 0,$$

which follows from eq. (32) and Lemma 18, where G'' is defined in eq. (70). Finally,

$$\frac{1}{2}\xi_0^\top A\xi_0 \leq \frac{1}{2\tau}\|x - x_0^t\|^2 + \frac{1}{2\sigma}\|y - y_0^t\|^2.$$

Thus, for any $(x,y) \in \mathcal{X} \times \mathcal{Y}$,

$$U_N^t(x,y) \leq \frac{1}{2\tau}\|x - x_0^t\|^2 + \frac{1}{2\sigma}\|y - y_0^t\|^2, \quad \text{w.p. } 1. \tag{81}$$

Now, we are ready to show eq. (55). It follows from eq. (77) and eq. (81) that, for any $(x,y) \in \mathcal{X} \times \mathcal{Y}$,

$$\begin{aligned} &K_N(\rho)\Big(\mathcal{L}^t(\bar{x}_N^t, y) - \mathcal{L}^t(x, \bar{y}_N^t)\Big) + \rho^{-N+1}D_N^t(x,y) \\ &\leq \frac{1}{2\tau}\|x - x_0^t\|^2 + \frac{1}{2\sigma}\|y - y_0^t\|^2 + \sum_{k=0}^{N-1}\rho^{-k}(P_k^t(x,y) + Q_k^t). \end{aligned} \tag{82}$$

Let (x_*^t, y_*^t) be the unique saddle point of $\mathcal{L}^t$. If we substitute $(x,y) = (x_*^t, y_*^t)$ into eq. (82) and use the fact $\mathcal{L}^t(\bar{x}_N^t, y_*^t) - \mathcal{L}^t(x_*^t, \bar{y}_N^t) \geq 0$, we obtain that

$$\rho^{-N+1}D_N^t(x_*^t, y_*^t) \leq \frac{1}{2\tau}\|x_*^t - x_0^t\|^2 + \frac{1}{2\sigma}\|y_*^t - y_0^t\|^2 + \sum_{k=0}^{N-1}\rho^{-k}(P_k^t(x_*^t, y_*^t) + Q_k^t). \tag{83}$$

From Assumption 3, for $k \geq -1$, we have

$$\mathbb{E}\left[\langle \Delta_k^{t,x}, \hat{x}_{k+1}^t - x_*^t\rangle\right] = \mathbb{E}\left[\langle \Delta_k^{t,y}, \hat{y}_{k+1}^t - y_*^t\rangle\right] = \mathbb{E}\Big[\langle \Delta_{k-1}^{t,y}, \hat{\hat{y}}_{k+1}^t - y_*^t\rangle\Big] = 0.$$

Thus,

$$\mathbb{E}[P_k^t(x_*^t, y_*^t)] = 0.$$

Moreover, from Assumption 3, for $k \geq -1$, we have

$$\mathbb{E}\left[\|\Delta_k^{t,x}\|^2\right] \leq \delta_x^2, \quad \mathbb{E}\left[\|\Delta_k^{t,y}\|^2\right] \leq \delta_y^2.$$

Therefore, we uniformly upper bound $\mathbb{E}\left[Q_k^t\right]$ for $k \geq 0$ using Lemma 16, i.e.,

$$\mathbb{E}[\sum_{k=0}^{N-1}\rho^{-k}Q_k^t] \leq \Big(\tau\Xi_{\tau,\sigma,\theta}^x\delta_x^2 + \sigma\Xi_{\tau,\sigma,\theta}^y\delta_y^2\Big)\sum_{k=0}^{N-1}\rho^{-k},$$

where $\Xi_{\tau,\sigma,\theta}^x$ and $\Xi_{\tau,\sigma,\theta}^y$ are defined in (18a) and (18b). Therefore, combining this result with $\mathbb{E}[P_k^t(x_*^t, y_*^t)] = 0$ for any $k \in \{0,\ldots,N-1\}$, we get

$$\mathbb{E}[\sum_{k=0}^{N-1}\rho^{-k}(P_k^t(x_*^t, y_*^t) + Q_k^t)] \leq \sum_{k=0}^{N-1}\rho^{-k}\,\Big(\tau\Xi_{\tau,\sigma,\theta}^x\delta_x^2 + \sigma\Xi_{\tau,\sigma,\theta}^y\delta_y^2\Big). \tag{84}$$

Then, using the definition of $D_N^t(x_*^t, y_*^t)$ in eq. (78b) and the fact

$$\sum_{k=0}^{N-1}\rho^{-k} = \rho^{-N+1}\frac{1-\rho^N}{1-\rho} \leq \rho^{-N+1}\frac{1}{1-\rho},$$

for any $\rho \in (0,1)$, the desired inequality in (55) follows from (83) and (84). □

C.3 Proof of Lemma 2

Throughout this proof, our analysis is based on the proof of Lemma 12. To analyze the expected gap in Lemma 2, we consider the setting with $\rho = 1$, which implies that $K_N(\rho) = N$ and $\bar{x}_N^t = \frac{1}{N}\sum_{k=0}^{N-1} x_{k+1}^t$, $\bar{y}_N^t = \frac{1}{N}\sum_{k=0}^{N-1} y_{k+1}^t$. The proof of Lemma 2 is different than that of Lemma 12 in the way we analyze the variance terms. To be precise, we construct the auxiliary sequences –see $\tilde{x}_k$, $\tilde{y}_k^+$, $\tilde{y}_k^-$ defined in eq. (88) and eq. (91) –for the analysis of **part 2** and **part 3** in eq. (72) to provide guarantees on the expected gap function.

Proof. Fix arbitrary $(x, y) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$. Since $(x_{k+1}^t, y_{k+1}^t) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$, using the concavity of $\mathcal{L}^t(x_{k+1}^t, \cdot)$ and the convexity of $\mathcal{L}^t(\cdot, y_{k+1}^t)$, Jensen's lemma immediately implies that

$$N\left(\mathcal{L}^t(\bar{x}_N^t, y) - \mathcal{L}^t(x, \bar{y}_N^t)\right) \le \sum_{k=0}^{N-1}\left(\mathcal{L}^t(x_{k+1}^t, y) - \mathcal{L}^t(x, y_{k+1}^t)\right), \tag{85}$$

where $\bar{x}_N^t = \frac{1}{N}\sum_{k=0}^{N-1} x_{k+1}^t$, $\bar{y}_N^t = \frac{1}{N}\sum_{k=0}^{N-1} y_{k+1}^t$. Summing eq. (61) from $k = 0$ to $N - 1$ and using (85), we get

$$\begin{aligned} &N\left(\mathcal{L}^t(\bar{x}_N^t, y) - \mathcal{L}(x, \bar{y}_N^t)\right) \\ &\le \sum_{k=0}^{N-1} \underbrace{-\langle q_{k+1}^t, y_{k+1}^t - y\rangle + \theta\langle q_k^t, y_k^t - y\rangle}_{\textbf{part 1}} + \Lambda_k^t(x, y) - \Sigma_{k+1}^t(x, y) + \Gamma_{k+1}^t \\ &\quad \underbrace{-\langle \tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t; \omega_k^x) - \nabla_x \Phi^t(x_k^t, y_{k+1}^t), x_{k+1}^t - x\rangle}_{\textbf{part 2}} + \underbrace{\langle \tilde{s}_k^t - s_k^t, y_{k+1}^t - y\rangle}_{\textbf{part 3}}. \end{aligned} \tag{86}$$

The bound on **Part 1** immediately follows from eq. (74) with $\rho = 1$, i.e.,

$$\sum_{k=0}^{N-1} \theta\langle q_k^t, y_k^t - y\rangle - \langle q_{k+1}^t, y_{k+1}^t - y\rangle \le \sum_{k=0}^{N-1} |1 - \theta|\, S_{k+1}^t(x, y) + \theta S_N^t(x, y). \tag{87}$$

Recall that $x_{-1}^t = x_0^t$ and $y_{-1}^t = y_0^t$; thus, $q_0^t = \mathbf{0}$.

Next we consider **part 2**, let $\Delta_k^{t,x}$ be defined as in Lemma 15. For some arbitrary $\eta_x > 0$, define $\{\tilde{x}_k\}$ sequence as follows:

$$\tilde{x}_0 \triangleq x_0^t, \quad \tilde{x}_{k+1} \triangleq \operatorname*{argmin}_{x' \in \mathcal{X}} -\langle \Delta_k^{t,x}, x'\rangle + \frac{\eta_x}{2}\|x' - \tilde{x}_k\|^2, \quad \forall\, k \ge 0. \tag{88}$$

Then by [29, Lemma 2.1], for all $k \ge 0$ and $x \in \mathcal{X}$, we have that

$$\langle \Delta_k^{t,x}, x - \tilde{x}_k\rangle \le \frac{\eta_x}{2}\|x - \tilde{x}_k\|^2 - \frac{\eta_x}{2}\|x - \tilde{x}_{k+1}\|^2 + \frac{1}{2\eta_x}\|\Delta_k^{t,x}\|^2.$$

Thus, using $\tilde{x}_0 = x_0^t$ we get

$$\begin{aligned} &\sum_{k=0}^{N-1}\langle \Delta_k^{t,x}, x - \tilde{x}_k\rangle \le \sum_{k=0}^{N-1}\left(\frac{\eta_x}{2}\|x - \tilde{x}_k\|^2 - \frac{\eta_x}{2}\|x - \tilde{x}_{k+1}\|^2 + \frac{1}{2\eta_x}\|\Delta_k^{t,x}\|^2\right) \\ &= \frac{\eta_x}{2}(\|x - x_0^t\|^2 - \|x - \tilde{x}_N\|^2) + \sum_{k=0}^{N-1}\frac{1}{2\eta_x}\|\Delta_k^{t,x}\|^2 \le \frac{\eta_x}{2}\|x - x_0^t\|^2 + \frac{1}{2\eta_x}\sum_{k=0}^{N-1}\|\Delta_k^{t,x}\|^2; \end{aligned} \tag{89}$$

hence, **part 2** becomes

$$\begin{aligned} &\sum_{k=0}^{N-1}\langle \Delta_k^{t,x}, x - x_{k+1}^t\rangle \\ &= \sum_{k=0}^{N-1}\langle \Delta_k^{t,x}, \hat{x}_{k+1}{}^t - x_{k+1}^t\rangle - \langle \Delta_k^{t,x}, \hat{x}_{k+1}^t - \tilde{x}_k\rangle + \langle \Delta_k^{t,x}, x - \tilde{x}_k\rangle \\ &\le \frac{\eta_x}{2}\|x - x_0^t\|^2 + \sum_{k=0}^{N-1}\langle \Delta_k^{t,x}, \hat{x}_{k+1}^t - x_{k+1}^t\rangle - \langle \Delta_k^{t,x}, \hat{x}_{k+1}^t - \tilde{x}_k\rangle + \frac{1}{2\eta_x}\|\Delta_k^{t,x}\|^2, \end{aligned} \tag{90}$$

which follows from eq. (89), and $\hat{x}_{k+1}$ is defined in Lemma 15.[6]

[6]When $\delta_x = 0$, clearly $\Delta_k^{t,x} = \mathbf{0}$; thus, **part 2** is equal to 0 and we can set $\eta_x = 0$ for which (90) becomes $0 \le 0$.

Next, we consider **part 3**, let $\Delta_k^{t,y}$ be defined as in Lemma 15. For some arbitrary $\eta_y > 0$, we construct two auxiliary sequences: let $\tilde{y}_0^+ = \tilde{y}_0^- = y_0^t$, and for $k \geq 0$, we define

$$\tilde{y}_{k+1}^+ \triangleq \operatorname*{argmin}_{y' \in \mathcal{Y}} \langle \Delta_k^{t,y}, y' \rangle + \frac{\eta_y}{2} \|y' - \tilde{y}_k^+\|^2, \quad \tilde{y}_{k+1}^- \triangleq \operatorname*{argmin}_{y' \in \mathcal{Y}} -\langle \Delta_k^{t,y}, y' \rangle + \frac{\eta_y}{2} \|y' - \tilde{y}_k^-\|^2. \tag{91}$$

Thus, it follows from [29, Lemma 2.1] that for $y \in \mathcal{Y}$,

$$\begin{aligned}
\langle \Delta_k^{t,y}, \tilde{y}_k^+ - y \rangle &\leq \frac{\eta_y}{2} \|y - \tilde{y}_k^+\|^2 - \frac{\eta_y}{2} \|y - \tilde{y}_{k+1}^+\|^2 + \frac{1}{2\eta_y} \|\Delta_k^{t,y}\|^2, \\
\langle \Delta_k^{t,y}, y - \tilde{y}_k^- \rangle &\leq \frac{\eta_y}{2} \|y - \tilde{y}_k^-\|^2 - \frac{\eta_y}{2} \|y - \tilde{y}_{k+1}^-\|^2 + \frac{1}{2\eta_y} \|\Delta_k^{t,y}\|^2.
\end{aligned}$$

Therefore, as in (89), we get[7]

$$\begin{aligned}
&\sum_{k=0}^{N-1} (1+\theta) \langle \Delta_k^{t,y}, \tilde{y}_k^+ - y \rangle + \theta \langle \Delta_{k-1}^{t,y}, y - \tilde{y}_{k-1}^- \rangle \\
&\leq \frac{\eta_y}{2} (1+2\theta) \|y - y_0^t\|^2 + \frac{1}{2\eta_y} \sum_{k=0}^{N-1} \Big((1+\theta) \|\Delta_k^{t,y}\|^2 + \theta \|\Delta_{k-1}^{t,y}\|^2 \Big).
\end{aligned} \tag{92}$$

Next, using eq. (92), we can bound **part 3** as follows:

$$\begin{aligned}
&\sum_{k=0}^{N-1} \langle \tilde{s}_k^t - s_k^t, y_{k+1}^t - y \rangle \\
&= \sum_{k=0}^{N-1} (1+\theta) \langle \Delta_k^{t,y}, y_{k+1}^t - \hat{y}_{k+1}^t + \hat{y}_{k+1}^t - \tilde{y}_k^+ + \tilde{y}_k^+ - y \rangle - \theta \langle \Delta_{k-1}^{t,y}, y_{k+1}^t - \hat{\hat{y}}_{k+1}^t + \hat{\hat{y}}_{k+1}^t - \tilde{y}_{k-1}^- + \tilde{y}_{k-1}^- - y \rangle \\
&\leq \sum_{k=0}^{N-1} (1+\theta) \langle \Delta_k^{t,y}, y_{k+1}^t - \hat{y}_{k+1}^t + \hat{y}_{k+1}^t - \tilde{y}_k^+ \rangle - \theta \langle \Delta_{k-1}^{t,y}, y_{k+1}^t - \hat{\hat{y}}_{k+1}^t + \hat{\hat{y}}_{k+1}^t - \tilde{y}_{k-1}^- \rangle \\
&\quad + \frac{1}{2\eta_y} \sum_{k=0}^{N-1} \Big((1+\theta) \|\Delta_k^{t,y}\|^2 + \theta \|\Delta_{k-1}^{t,y}\|^2 \Big) + \frac{\eta_y}{2} (1+2\theta) \|y - y_0^t\|^2,
\end{aligned} \tag{93}$$

where $\hat{y}_{k+1}^t$ and $\hat{\hat{y}}_{k+1}^t$ are defined in Lemma 15.

For any fixed $(x, y) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$, we use (87), (90) and (93) to get

$$\begin{aligned}
&N \left(\mathcal{L}^t(\bar{x}_N^t, y) - \mathcal{L}^t(x, \bar{y}_N^t) \right) \\
&\leq \tilde{U}_N^t(x, y) + \frac{\eta_x}{2} \|x - x_0^t\|^2 + \frac{\eta_y}{2} (1+2\theta) \|y - y_0^t\|^2 + \sum_{k=0}^{N-1} (\tilde{P}_k^t + \tilde{Q}_k^t),
\end{aligned} \tag{94}$$

where $\tilde{U}_N^t(x, y)$ and $\tilde{P}_k^t$, $\tilde{Q}_k^t$ for $k = 0, \ldots, N-1$ are defined as follows:

$$\tilde{U}_N^t(x, y) \triangleq \sum_{k=0}^{N-1} \left(\Gamma_{k+1}^t + \Lambda_k^t(x, y) - \Sigma_{k+1}^t(x, y) + |1-\theta|\, S_{k+1}^t(x, y) \right) + \theta S_N^t(x, y), \tag{95a}$$

$$\tilde{P}_k^t \triangleq -\langle \Delta_k^{t,x}, \hat{x}_{k+1}^t - \tilde{x}_k \rangle + (1+\theta) \langle \Delta_k^{t,y}, \hat{y}_{k+1}^t - \tilde{y}_k^+ \rangle - \theta \langle \Delta_{k-1}^{t,y}, \hat{\hat{y}}_{k+1}^t - \tilde{y}_{k-1}^- \rangle, \tag{95b}$$

$$\begin{aligned}
\tilde{Q}_k^t \triangleq\ &\langle \Delta_k^{t,x}, \hat{x}_{k+1}^t - x_{k+1}^t \rangle + (1+\theta) \langle \Delta_k^{t,y}, y_{k+1}^t - \hat{y}_{k+1}^t \rangle - \theta \langle \Delta_{k-1}^{t,y}, y_{k+1}^t - \hat{\hat{y}}_{k+1}^t \rangle \\
&+ \frac{1}{2\eta_x} \|\Delta_k^{t,x}\|^2 + \frac{1+\theta}{2\eta_y} \|\Delta_k^{t,y}\|^2 + \frac{\theta}{2\eta_y} \|\Delta_{k-1}^{t,y}\|^2.
\end{aligned} \tag{95c}$$

The remaining part of the analysis directly follows from the arguments we used in the proof of Lemma 12. For any fixed $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we first analyze $\tilde{U}_N^t(x, y)$. For some given $\alpha > 0$, after

[7] As in **part 2**, when $\delta_y = 0$, we can set $\eta_y = 0$.

adding and subtracting $\frac{\alpha}{2}\|y_{k+1}^t - y_k^t\|^2$, and rearranging the terms, we get

$$\begin{aligned}\tilde{U}_N^t(x,y) =& \frac{1}{2}\sum_{k=0}^{N-1}\Big(\xi_k^\top \tilde{A}\xi_k - \xi_{k+1}^\top \tilde{B}\xi_{k+1}\Big) + \theta S_N^t(x,y) \\ =& \frac{1}{2}\xi_0^\top \tilde{A}\xi_0 - \frac{1}{2}\sum_{k=1}^{N-1}[\xi_k^\top(\tilde{B}-\tilde{A})\xi_k] - \Big(\frac{1}{2}\xi_N^\top \tilde{B}\xi_N - \theta S_N^t(x,y)\Big),\end{aligned} \tag{96}$$

where $A, B \in \mathbb{R}^{5\times 5}$ and $\xi_k \in \mathbb{R}^5$ are defined for $k \geq 0$ as follows:

$$\xi_k \triangleq \begin{pmatrix} \|x_k^t - x\| \\ \|y_k^t - y\| \\ \|x_k^t - x_{k-1}^t\| \\ \|y_k^t - y_{k-1}^t\| \\ \|y_{k+1}^t - y_k^t\| \end{pmatrix}, \qquad \tilde{A} \triangleq \begin{pmatrix} \frac{1}{\tau}-\mu_x & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sigma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta L_{yx} \\ 0 & 0 & 0 & 0 & \theta L_{yy} \\ 0 & 0 & \theta L_{yx} & \theta L_{yy} & -\alpha \end{pmatrix},$$

and

$$\tilde{B} \triangleq \begin{pmatrix} \frac{1}{\tau} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sigma}+\mu_y & -|1-\theta|\,L_{yx} & -|1-\theta|\,L_{yy} & 0 \\ 0 & -|1-\theta|\,L_{yx} & \frac{1}{\tau}-L'_{xx} & 0 & 0 \\ 0 & -|1-\theta|\,L_{yy} & 0 & \frac{1}{\sigma}-\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

such that $x_{-1}^t = x_0^t$, $y_{-1}^t = y_0^t$. Lemma 17 together with $\rho = 1$ implies that eq. (16) is equivalent to $\tilde{B} - \tilde{A} \succeq 0$; therefore, it follows from (96) that, for any given $(x,y) \in \mathcal{X}\times\mathcal{Y}$, we have

$$U_N^t(x,y) \leq \frac{1}{2}\xi_0^\top \tilde{A}\xi_0 - (\tfrac{1}{2}\xi_N^\top \tilde{B}\xi_N - \theta S_N^t(x,y)), \quad \text{w.p. } 1.$$

Furthermore, we also have

$$\frac{1}{2}\xi_N^\top \tilde{B}\xi_N - \theta S_N^t(x,y) \geq \frac{1}{2}\xi_N^\top \begin{pmatrix} \mu_x & \mathbf{0}_{1\times 3} & 0 \\ \mathbf{0}_{3\times 1} & G'' & \mathbf{0}_{3\times 1} \\ 0 & \mathbf{0}_{1\times 3} & 0 \end{pmatrix} \xi_N \geq 0,$$

which follows from eq. (16) and Lemma 18 with $\rho = 1$, where G'' is defined in eq. (70). Finally,

$$\frac{1}{2}\xi_0^\top \tilde{A}\xi_0 \leq \frac{1}{2\tau}\|x - x_0^t\|^2 + \frac{1}{2\sigma}\|y - y_0^t\|^2.$$

Thus, the above three inequalities imply that, for any $(x,y) \in \mathcal{X}\times\mathcal{Y}$,

$$\tilde{U}_N^t(x,y) \leq \frac{1}{2\tau}\|x - x_0^t\|^2 + \frac{1}{2\sigma}\|y - y_0^t\|^2, \quad \text{w.p. } 1. \tag{97}$$

Now, we are ready to show eq. (17). It follows from eq. (94) and eq. (97) that

$$\begin{aligned} & N \sup_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \left\{\mathcal{L}^t(\bar{x}_N^t, y) - \mathcal{L}^t(x, \bar{y}_N^t)\right\} \\ & \leq \left(\frac{1}{2\tau} + \frac{\eta_x}{2}\right)\|x_*^t(\bar{y}_N^t) - x_0^t\|^2 + \left(\frac{1}{2\sigma} + \frac{\eta_y(1+2\theta)}{2}\right)\|y_*(\bar{x}_N^t) - y_0^t\|^2 + \sum_{k=0}^{N-1}(\tilde{P}_k^t + \tilde{Q}_k^t), \end{aligned} \tag{98}$$

where $(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t))$ is the point achieving the supremum on the left hand side. Indeed, to derive the above inequality, we substitute $(x,y) = (x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t))$ into the eq. (94) and use the fact that

$$\sup_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \left\{\mathcal{L}^t(\bar{x}_N^t, y) - \mathcal{L}^t(x, \bar{y}_N^t)\right\} = \mathcal{L}^t(\bar{x}_N^t, y_*(\bar{x}_N^t)) - \mathcal{L}^t(x_*^t(\bar{y}_N^t), \bar{y}_N^t).$$

From Assumption 3, for $k \geq -1$, we have

$$\mathbb{E}\left[\langle \Delta_k^{t,x}, \hat{x}_{k+1}^t - \tilde{x}_k\rangle\right] = \mathbb{E}\left[\langle \Delta_k^{t,y},\ \hat{y}_{k+1}^t - \tilde{y}_k^{\pm}\rangle\right] = \mathbb{E}\left[\langle \Delta_{k-1}^{t,y}, \hat{\hat{y}}_{k+1}^t - \tilde{y}_{k-1}^{-}\rangle\right] = 0.$$

Thus, $\mathbb{E}[\tilde{P}_k^t] = 0$. Moreover, for $k \geq -1$, from Assumption 3 we also have

$$\mathbb{E}\left[\|\Delta_k^{t,x}\|^2\right] \leq \delta_x^2, \qquad \mathbb{E}\left[\|\Delta_k^{t,y}\|^2\right] \leq \delta_y^2.$$

Next, we uniformly upper bound $\mathbb{E}\left[\tilde{Q}_k^t\right]$ for $k \geq 0$ using Lemma 16, i.e.,

$$\mathbb{E}[\sum_{k=0}^{N-1} \tilde{Q}_k^t] \leq N\Big[\Big(\tau \Xi_{\tau,\sigma,\theta}^x + \frac{1}{2\eta_x}\Big)\delta_x^2 + \Big(\sigma \Xi_{\tau,\sigma,\theta}^y + \frac{1+2\theta}{2\eta_y}\Big)\delta_y^2\Big].$$

Therefore, combining this result with $\mathbb{E}[\tilde{P}_k^t] = 0$ for any $k \in \{0, \ldots, N-1\}$, we get

$$\mathbb{E}[\sum_{k=0}^{N-1} (\tilde{P}_k^t + \tilde{Q}_k^t)] \leq N\, \Xi_{\tau,\sigma,\theta}. \tag{99}$$

Finally, setting $\eta_x = \frac{1}{\tau}$, $\eta_y = \frac{1}{\sigma}$, $x_0^{t+1} = \bar{x}_N^t$, and $y_0^{t+1} = \bar{y}_N^t$, the desired result in (17) follows from (98) and (99). □

D Computation of ϵ-stationary point in practice

In this section, we discuss how to compute a point x_ϵ such that $\mathbb{E}[\|\nabla \phi_\lambda(x_\epsilon)\|] \leq \epsilon$ –as the t_* in remark 5 can not be computed in practice. This result is shown in Theorem 7, which directly follows from Theorem 1 and Lemma 10. Below, for the sake of completeness, we state a known technical result that we need for the proof of Theorem 7.

Lemma 19. *Suppose Assumptions 1 and 2 hold. Then $\phi_\lambda(\cdot)$ is $\frac{1}{\lambda}$-smooth for $\lambda \in (0, \gamma^{-1})$, where $\phi_\lambda(\cdot)$ is defined in definition 3 for $\phi(\cdot) = \max_{y \in \mathcal{Y}} \mathcal{L}(\cdot, y)$.*

Proof. Let $R(x) \triangleq x - \mathbf{prox}_{\lambda\phi}(x)$ for $\lambda \in (0, \gamma^{-1})$ and $x \in \mathbf{dom}\, f$. Indeed, by definition 3, we know $R(x) = \lambda \nabla \phi_\lambda(x)$. Then by the optimality condition of $\mathbf{prox}_{\lambda\phi}(x)$, we obtain that

$$R(x) \in \partial f(\mathbf{prox}_{\lambda\phi}(x))$$

holds for $x \in \mathbf{dom}\, f$. Hence, for $x_1, x_2 \in \mathbf{dom}\, f$, we have that

$$\langle R(x_1) - R(x_2), \mathbf{prox}_{\lambda\phi}(x_1) - \mathbf{prox}_{\lambda\phi}(x_2)\rangle \geq 0,$$

which further implies that

$$\begin{aligned}
\|x_1 - x_2\|^2 &= \|R(x_1) - \mathbf{prox}_{\lambda\phi}(x_1) - R(x_2) + \mathbf{prox}_{\lambda\phi}(x_2)\|^2 \\
&\geq \|R(x_1) - R(x_2)\|^2 + \|\mathbf{prox}_{\lambda\phi}(x_1) - \mathbf{prox}_{\lambda\phi}(x_2)\|^2 \\
&\geq \|R(x_1) - R(x_2)\|^2.
\end{aligned}$$

Then using the fact $R(x) = \lambda \nabla \phi_\lambda(x)$ completes the proof. □

Theorem 7. *Consider $\mathcal{L}$ defined in* (1)*. Suppose Assumptions 1, 2, 3 hold. Under the premise of Theorem 1, for any $\epsilon > 0$,* `SAPD+` *can generate an point x_ϵ such that $\mathbb{E}[\|\nabla \phi_\lambda(x_\epsilon)\|] \leq \epsilon$ within $\mathcal{O}\left(\frac{L\kappa_y \mathcal{G}(x_0^0, y_0^0)}{\epsilon^2} \ln(1/\epsilon) + \frac{L\kappa_y \delta^2 \mathcal{G}(x_0^0, y_0^0)}{\epsilon^4} \ln(1/\epsilon)\right)$ stochastic first-order oracle calls.*

Proof. Under the premise of Theorem 1, given $\epsilon > 0$, `SAPD+` generates $\{x_0^t\}_{t=0}^T$ such that $\min_{t=0,\ldots,T} \mathbb{E}[\|\nabla \phi_\lambda(x_0^t)\|] \leq \epsilon/4$, for $T \geq 96\mathcal{G}(x_0^0, y_0^0) \cdot \frac{16\gamma}{\epsilon^2} + 1$. Therefore, for each x_0^t, if we let $\hat{x}_0^t = \mathbf{prox}_{\lambda\phi}(x_0^t)$, then Lemma 10 ensures that we can generate a point $\tilde{x}_*^t$ such that

$$\mathbb{E}[\|\tilde{x}_*^t - \hat{x}_0^t\|] \leq \hat{\epsilon}$$

within $N_{\hat{\epsilon}}$ many iterations, where

$$N_{\hat{\epsilon}} = \mathcal{O}\left(\frac{\max\{L_{xx}, L_{yx}\}}{\gamma} + \frac{\max\{L_{yx}, L_{xy}\}}{\sqrt{\gamma\mu_y}} + \frac{\max\{L_{yy}, L_{yx}\}}{\mu_y} + \Big(\frac{\delta_x^2}{\gamma} + \frac{\delta_y^2}{\mu_y}\Big)\frac{1}{\gamma\hat{\epsilon}^2}\right) \cdot \ln\Big(\frac{\max\{1, \mu_y/\gamma\}}{\hat{\epsilon}}\Big)$$

Moreover, if we compute the GNME of $\phi(\cdot)$ at $\tilde{x}_*^t$, it follows that

$$\begin{aligned}\|\nabla\phi_\lambda(\tilde{x}_*^t)\| \leq & \|\nabla\phi_\lambda(\tilde{x}_*^t)-\nabla\phi_\lambda(\hat{x}_0^t)\|+\|\nabla\phi_\lambda(\hat{x}_*^t)-\nabla\phi_\lambda(x_0^t)\|+\|\nabla\phi_\lambda(x_0^t)\| \\ \leq & \frac{2}{\lambda}\|\tilde{x}_*^t-\hat{x}_0^t\|+\frac{1}{\lambda}\|\hat{x}_*^t-x_0^t\|+\|\nabla\phi_\lambda(x_0^t)\| \\ = & \frac{2}{\lambda}\|\tilde{x}_*^t-\hat{x}_0^t\|+2\|\nabla\phi_\lambda(x_0^t)\| \\ \leq & \frac{2}{\lambda}\hat{\epsilon}+2\|\nabla\phi_\lambda(x_0^t)\|.\end{aligned}$$

where the second inequality is by Lemma 19; the first equality is by definition 3 and the fact $\hat{x}_0^t = \mathbf{prox}_{\lambda\phi}(x_0^t)$. Furthermore, because $\min_{t=0,\ldots,T}\mathbb{E}\left[\|\nabla\phi_\lambda(x_0^t)\|\right] \leq \epsilon/4$, then we have

$$\min_{t=0,\ldots,T}\mathbb{E}[\|\nabla\phi_\lambda(\tilde{x}_*^t)\|] \leq \frac{2}{\lambda}\hat{\epsilon}+\frac{\epsilon}{2}$$

and we let $x_\epsilon = \tilde{x}_*^{\tilde{t}}$, where $\tilde{t}_* \triangleq \operatorname{argmin}_{\{t=0,..,T\}}\mathbb{E}[\|\nabla\phi_\lambda(\tilde{x}_*^t)\|]$. Therefore, setting $\lambda = \frac{1}{2\gamma}$ and $\hat{\epsilon} = \frac{1}{8\gamma}$, Lemma 10 implies that calling `SAPD` T times, each with $\tilde{N}$ iterations, one can generate x_ϵ such that

$$\mathbb{E}\big[\|\nabla\phi_\lambda(x_\epsilon)\|\big] \leq \epsilon,$$

where T is given in Theorem 1 and

$$\tilde{N} = \mathcal{O}\left(\frac{\max\{L_{xx},L_{yx}\}}{\gamma}+\frac{\max\{L_{yx},L_{xy}\}}{\sqrt{\gamma\mu_y}}+\frac{\max\{L_{yy},L_{yx}\}}{\mu_y}+\Big(\frac{\delta_x^2}{\gamma}+\frac{\delta_y^2}{\mu_y}\Big)\frac{\gamma}{\epsilon^2}\right)\cdot\ln\Big(\frac{\max\{\gamma,\mu_y\}}{\epsilon}\Big)$$

Thus, considering the setting in (8), one can compute x_ϵ in practice requiring $T\tilde{N} = \mathcal{O}\left(\frac{L\kappa_y\mathcal{G}(x_0^0,y_0^0)}{\epsilon^2}\ln(1/\epsilon)+\frac{L\kappa_y\delta^2\mathcal{G}(x_0^0,y_0^0)}{\epsilon^4}\ln(1/\epsilon)\right)$ oracle calls; furthermore, $\ln(1/\epsilon)$ can be removed by employing a restarting strategy as in [43]. □

E Proof of Theorem 3

For completeness, we provide a technical lemma below establishing Lipschitz continuity of the best response functions (see also [43, Lemma 2.5] and [24, Lemma B.2(a)]).

Lemma 20. *[7, Proposition 1] Suppose Assumptions 1 and 2 hold. For any given $y \in \mathbf{dom}\, g$, let $x_*^t(y) \triangleq \operatorname{argmin}_{x\in\mathcal{X}}\mathcal{L}^t(x,y)$; and for any given $x \in \mathbf{dom}\, f$, let $y_*(x) \triangleq \operatorname{argmax}_{y\in\mathcal{Y}}\mathcal{L}^t(x,y) = \operatorname{argmax}_{y\in\mathcal{Y}}\mathcal{L}(x,y)$. Then $x_*^t(\cdot)$ and $y_*(\cdot)$ are Lipschitz maps on $\mathbf{dom}\, g$ and $\mathbf{dom}\, f$, with constants κ_{xy} and κ_{yx}, respectively, where $\kappa_{xy} \triangleq L_{xy}/\mu_x$ and $\kappa_{yx} \triangleq L_{yx}/\mu_y$.*

Lemma 21. *For any $t \geq 0$, let $z_*^t \triangleq (x_*^t, y_*^t)$ be the unique saddle point of $\mathcal{L}^t$ defined in eq. (4), and let $\{z_k^t\}_{k=0}^{N_t}$ be generated by running `SAPD` on $\min_{x\in\mathcal{X}}\max_{y\in\mathcal{Y}}\mathcal{L}^t(x,y)$ for $N_t \in \mathbb{Z}_+$ iterations, where $z_k^t \triangleq (x_k^t, y_k^t)$; and define $z_0^{t+1} \triangleq \frac{1}{N_t}\sum_{i=0}^{N_t-1} z_{i+1}^t$. Under the setting of Lemma 2,*

$$\max\Big\{\mathbb{E}\left[\mathcal{G}^t(z_0^{t+1})\right],\ \mathbb{E}\left[\|z_0^{t+1}-z_*^t\|^2\right]\Big\} \leq \frac{1}{N_t}C_{\tau,\sigma,\theta}\mathbb{E}[\|z_0^t-z_*^t\|^2]+C'_{\tau,\sigma,\theta} \tag{100}$$

holds for all $t \geq 0$ and $N_t \geq 1$, for some positive constants $C_{\tau,\sigma,\theta}$ and $C'_{\tau,\sigma,\theta}$.

Proof. For simplicity we assume $N_t = N$ for all $t \geq 0$ –the proof still holds for arbitrary $\{N_t\}_{t\geq 0} \subset \mathbb{Z}_+$. The proof mainly follows the proof of Lemma 2. We first show a bound for $\mathbb{E}\left[\|z_0^{t+1}-z_*^t\|^2\right]$ that is in the form of the rhs of eq. (100); then, we show it for $\mathbb{E}\left[\mathcal{G}^t(z_0^{t+1})\right]$. In addition, given $\{z_k^t\}_{k=0}^{N_t}$, we let $\bar{z}_{N_t}^t = (\bar{x}_{N_t}^t, \bar{y}_{N_t}^t)$, and $\bar{z}_{N_t}^t = z_0^{t+1} = \frac{1}{N_t}\sum_{i=0}^{N_t-1} z_{i+1}^t$ for all $t \geq 0$ and $N_t \geq 1$.

Now, we start with analyzing $\mathbb{E}\left[\|z_0^{t+1}-z_*^t\|^2\right]$. The analysis below mainly relies on the proof of Lemma 2. Indeed, given $z_0^t = (x_0^t, y_0^t)$ for $t \geq 0$, substituting $x = x_*^t$ and $y = y_*^t$ within (94) and then using eq. (97), we obtain that

$$\begin{aligned}& N\mathbb{E}\big[\mathcal{L}^t(\bar{x}_N^t, y_*^t)-\mathcal{L}^t(x_*^t,\bar{y}_N^t)\big] \\ & \leq \mathbb{E}\left[\left(\frac{1}{2\tau}+\frac{\eta_x}{2}\right)\|x_*^t-x_0^t\|^2+\left(\frac{1}{2\sigma}+\frac{\eta_y(1+2\theta)}{2}\right)\|y_*^t-y_0^t\|^2+\sum_{k=0}^{N-1}\rho^{-k}(\tilde{P}_k^t+\tilde{Q}_k^t)\right].\end{aligned} \tag{101}$$

Moreover, since $\mathcal{L}^t(\cdot, y_*^t)$ is μ_x-strongly convex and $\mathcal{L}^t(x_*^t, \cdot)$ is μ_y-strongly concave, and (x_*^t, y_*^t) is the unique saddle point of $\mathcal{L}^t$, we have that

$$\frac{\mu_x}{2}\|\bar{x}_N^t - x_*^t\|^2 + \frac{\mu_y}{2}\|\bar{y}_N^t - y_*^t\|^2 \leq \mathcal{L}^t(\bar{x}_N^t, y_*^t) - \mathcal{L}^t(x_*^t, \bar{y}_N^t). \tag{102}$$

If we let $\eta_x = \frac{1}{\tau}$ and $\eta_y = \frac{1}{\sigma}$, then it follows from eqs. (101, 102,99) and the fact that $\bar{z}_N^t = z_0^{t+1}$ that

$$N\mathbb{E}\left[\frac{\mu_x}{2}\|x_0^{t+1} - x_*^t\|^2 + \frac{\mu_y}{2}\|y_0^{t+1} - y_*^t\|^2\right] \leq \mathbb{E}\left[\overline{U}^t(x_*^t, y_*^t)\right] + N\Xi_{\tau,\sigma,\theta}, \tag{103}$$

where $\Xi_{\tau,\sigma,\theta}$ is defined in Lemma 2 and for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we define

$$\overline{U}^t(x, y) \triangleq \frac{1}{\tau}\|x - x_0^t\|^2 + \frac{1+\theta}{\sigma}\|y - y_0^t\|^2. \tag{104}$$

Therefore, we conclude that

$$\mathbb{E}\left[\|z_0^{t+1} - z_*^t\|^2\right] \leq \frac{1}{N}\overline{C}_{\tau,\sigma,\theta}\mathbb{E}\left[\|z_0^t - z_*^t\|^2\right] + \overline{C}'_{\tau,\sigma,\theta}, \tag{105}$$

where

$$\overline{C}_{\tau,\sigma,\theta} \triangleq \frac{2\max\{\frac{1}{\tau}, \frac{1+\theta}{\sigma}\}}{\min\{\mu_x, \mu_y\}}, \qquad \overline{C}'_{\tau,\sigma,\theta} \triangleq \frac{2}{\min\{\mu_x, \mu_y\}}\Xi_{\tau,\sigma,\theta}.$$

This completes the first part of the proof. Next, we will bound $\mathbb{E}[\mathcal{G}^t(z_0^{t+1})]$ using the bound on $\mathbb{E}[\|z_0^{t+1} - z_*^t\|^2]$ we derived in the first part.

Given z_0^t, using eq. (98) and eq. (99) in the proof Lemma 2 for $\eta_x = \frac{1}{\tau}$ and $\eta_y = \frac{1}{\sigma}$ as above, we obtain that

$$\mathbb{E}\left[\mathcal{G}^t(z_0^{t+1})\right] \leq \frac{1}{N}\mathbb{E}\left[\overline{U}^t\left(x_*^t(y_0^{t+1}), y_*(x_0^{t+1})\right)\right] + \Xi_{\tau,\sigma,\theta}, \tag{106}$$

where $\overline{U}^t(x, y)$ is defined in (104) and $\Xi_{\tau,\sigma,\theta}$ is defined in Lemma 2; furhermore, $x_*^t(\cdot)$ and $y_*(\cdot)$ are defined in eq. (6). Next, we will use eq. (105) to derive an upper bound for the right hand side of eq. (106).

Since z_*^t is the unique saddle point for $\mathcal{L}^t$, we have $x_*^t(y_*^t) = x_*^t$ and $y_*(x_*^t) = y_*^t$. Moreover, according to Lemma 20, $x_*^t(\cdot)$, $y_*(\cdot)$ is Lipschitz with constants $\kappa_{xy} = \frac{L_{xy}}{\mu_x}$ and $\kappa_{yx} = \frac{L_{yx}}{\mu_y}$, respectively. Therefore, Lemma 20 and

$$\overline{U}^t(x_*^t(y_0^{t+1}), y_*(x_0^{t+1})) \leq \frac{2}{\tau}\|x_*^t - x_*^t(y_0^{t+1})\|^2 + \frac{2+2\theta}{\sigma}\|y_*^t - y_*(x_0^{t+1})\|^2 + 2\overline{U}^t(x_*^t, y_*^t), \quad \text{w.p. } 1,$$

together imply that

$$\begin{aligned}
&\mathbb{E}\left[\overline{U}^t(x_*^t(y_0^{t+1}), y_*(x_0^{t+1}))\right] \\
\leq &\mathbb{E}\left[\frac{2\kappa_{xy}^2}{\tau}\|y_*^t - y_0^{t+1}\|^2 + \frac{(2+2\theta)\kappa_{yx}^2}{\sigma}\|x_*^t - x_0^{t+1}\|^2 + 2\overline{U}^t(x_*^t, y_*^t)\right] \\
\leq &\mathbb{E}\left[\max\left\{\frac{2}{\tau}, \frac{(2+2\theta)}{\sigma}\right\}\Big(\max\{\kappa_{xy}^2,\ \kappa_{yx}^2\}\|z_0^{t+1} - z_*^t\|^2 + \|z_0^t - z_*^t\|^2\Big)\right] \\
\leq &\mathbb{E}\left[\max\left\{\frac{2}{\tau}, \frac{(2+2\theta)}{\sigma}\right\}\max\{1, \kappa_{xy}^2,\ \kappa_{yx}^2\}\left(\Big(\frac{1}{N}+1\Big)\overline{C}_{\tau,\sigma,\theta}\|z_0^t - z_*^t\|^2 + \overline{C}'_{\tau,\sigma,\theta}\right)\right],
\end{aligned}$$

where we use eq. (105) for the last inequality. Then, if we use the above inequality within eq. (106), it follows that

$$\mathbb{E}\left[\mathcal{G}^t(z_0^{t+1})\right] \leq \frac{1}{N}\overline{\overline{C}}_{\tau,\sigma,\theta}\mathbb{E}\left[\|z_0^t - z_*^t\|^2\right] + \frac{1}{N}\overline{\overline{C}}'_{\tau,\sigma,\theta} + \Xi_{\tau,\sigma,\theta},$$

where

$$\begin{aligned}
\overline{\overline{C}}_{\tau,\sigma,\theta} &\triangleq 4\ \max\left\{\frac{1}{\tau}, \frac{1+\theta}{\sigma}\right\}\max\{1, \kappa_{xy}^2,\ \kappa_{yx}^2\}\overline{C}_{\tau,\sigma,\theta}, \\
\overline{\overline{C}}'_{\tau,\sigma,\theta} &\triangleq 2\ \max\left\{\frac{1}{\tau}, \frac{1+\theta}{\sigma}\right\}\max\{1, \kappa_{xy}^2,\ \kappa_{yx}^2\}\overline{C}'_{\tau,\sigma,\theta}.
\end{aligned}$$

Thus, for $C_{\tau,\sigma,\theta} \triangleq \max\{\overline{C}_{\tau,\sigma,\theta},\ \overline{\overline{C}}_{\tau,\sigma,\theta}\}$ and $C'_{\tau,\sigma,\theta} \triangleq \max\{\overline{C}'_{\tau,\sigma,\theta},\ \frac{1}{N}\overline{\overline{C}}'_{\tau,\sigma,\theta} + \Xi_{\tau,\sigma,\theta}\}$, we get the desired result in (100). $\square$

Lemma 22. *Under the premise of Lemma 2,* $\mathbb{E}[\|z_0^t - z_*^t\|^2]$, $\mathbb{E}\left[\mathcal{G}^t(z_0^t)\right]$ *and* $\mathbb{E}\left[\mathcal{G}^t(z_0^{t+1})\right]$ *are finite for any* $t \geq 0$ *when either Assumption 4 or Assumption 5 holds.*

Proof. In Lemma 21, we show that

$$\max\left\{\mathbb{E}\left[\mathcal{G}^t(z_0^{t+1})\right],\ \mathbb{E}\left[\|z_0^{t+1} - z_*^t\|^2\right]\right\} \leq \frac{1}{N_t} C_{\tau,\sigma,\theta}\mathbb{E}[\|z_0^t - z_*^t\|^2] + C'_{\tau,\sigma,\theta}, \tag{107}$$

for some $C_{\tau,\sigma,\theta}, C'_{\tau,\sigma,\theta} \in \mathbb{R}_+$ constants, dependent on the SAPD parameters. Next, we show that $\{\mathbb{E}\left[\|z_0^t - z_*^t\|^2\right]\} < \infty$ for all $t \geq 0$ by induction. This is trivially true for $t = 0$, i.e., $\mathbb{E}\left[\|z_0^0 - z_*^0\|^2\right] = \|z_0^0 - z_*^0\|^2 < \infty$. Next, for some $t \geq 0$, suppose $\mathbb{E}\left[\|z_0^t - z_*^t\|^2\right] < \infty$, (107) implies that

$$\mathbb{E}\left[\|z_0^{t+1} - z_*^t\|^2\right] < \infty. \tag{108}$$

The inductive assumption $\mathbb{E}\left[\|z_0^t - z_*^t\|^2\right] < \infty$ and (108) imply that

$$\mathbb{E}\left[\|z_0^{t+1} - z_0^t\|^2\right] \leq 2\mathbb{E}\left[\|z_0^{t+1} - z_*^t\|^2\right] + 2\mathbb{E}\left[\|z_0^t - z_*^t\|^2\right] < \infty. \tag{109}$$

For any $\mu_x > 0$, fix $\lambda = (\mu_x + \gamma)^{-1}$; since we have $x_*^\ell = \mathbf{prox}_{\lambda\phi}(x_0^\ell)$ for $\ell = t, t+1$ and $\mathbf{prox}_{\lambda\phi}(\cdot)$ is non-expansive, we have $\mathbb{E}[\|x_*^{t+1} - x_*^t\|^2] \leq \mathbb{E}[\|x_0^{t+1} - x_0^t\|^2]$. Moreover, Lemma 20 implies that $\mathbb{E}[\|y_*^{t+1} - y_*^t\|^2] \leq \kappa_{yx}^2\mathbb{E}[\|x_*^{t+1} - x_*^t\|^2]$ for $\kappa_{yx} = \frac{L_{yx}}{\mu_y}$; thus, using (109), we get

$$\mathbb{E}[\|z_*^{t+1} - z_*^t\|^2] \leq (\kappa_{yx}^2 + 1)\mathbb{E}[\|x_0^{t+1} - x_0^t\|^2] \leq (\kappa_{yx}^2 + 1)\mathbb{E}[\|z_0^{t+1} - z_0^t\|^2] < \infty. \tag{110}$$

Therefore, we can conclude that $\mathbb{E}[\|z_0^{t+1} - z_*^{t+1}\|^2] \leq 2\mathbb{E}[\|z_0^{t+1} - z_*^t\|^2] + 2\mathbb{E}[\|z_*^{t+1} - z_*^t\|^2] < \infty$, which follows from (108) and (110). This completes induction, providing us with $\mathbb{E}[\|z_0^t - z_*^t\|^2] < \infty$ for all $t \geq 0$. Note that using this result together with the definition of $\mathcal{G}^t$ and (107) implies that $0 \leq \mathbb{E}[\mathcal{G}^t(z_0^{t+1})] < \infty$ for $t \geq 0$.

Next, we will argue that $\mathbb{E}[\mathcal{G}^t(z_0^t)] < \infty$ for all $t \geq 0$ as well. Recall that $\mathcal{G}^t(z_0^t) = \sup_{y\in\mathcal{Y}} \mathcal{L}^t(x_0^t, y) - \inf_{x\in\mathcal{X}} \mathcal{L}^t(x, y_0^t)$; furthermore, note that $\mathcal{L}^t(x_0^t, y) = \mathcal{L}(x_0^t, y)$ for all $y \in \mathcal{Y}$, and given z_0^t, we have $\mathcal{L}^t(\cdot, y_0^t)$ strongly convex with modulus μ_x and $\mathcal{L}(x_0^t, \cdot)$ strongly concave with modulus μ_y. Therefore, we have

$$\begin{aligned}
\mathcal{L}(x_0^t, y) &\leq \mathcal{L}(x_0^t, y_0^t) + \left\langle \nabla_y\Phi(x_0^t, y_0^t) - s_g(y_0^t),\ y - y_0^t\right\rangle - \frac{\mu_y}{2}\|y - y_0^t\|^2 \\
&\leq \mathcal{L}(x_0^t, y_0^t) + \frac{1}{2\mu_y}\|\nabla_y\Phi(x_0^t, y_0^t) - s_g(y_0^t)\|^2, \qquad (111) \\
\mathcal{L}^t(x, y_0^t) &\geq \mathcal{L}(x_0^t, y_0^t) + \left\langle \nabla_x\Phi(x_0^t, y_0^t) + s_f(x_0^t),\ x - x_0^t\right\rangle + \frac{\mu_x}{2}\|x - x_0^t\|^2 \\
&\geq \mathcal{L}(x_0^t, y_0^t) - \frac{1}{2\mu_x}\|\nabla_x\Phi(x_0^t, y_0^t) + s_f(x_0^t)\|^2, \qquad (112)
\end{aligned}$$

where $s_f(x_0^t) \in \partial f(x_0^t)$ and $s_g(y_0^t) \in \partial g(y_0^t)$ such that $\|s_f(x_0^t)\| \leq B_f$ and $\|s_g(y_0^t)\| \leq B_g$ –see Assumption 5; moreover, we have used the fact that $\mathcal{L}^t(x_0^t, y_0^t) = \mathcal{L}(x_0^t, y_0^t)$ and $\partial_x\mathcal{L}^t(x_0^t, y_0^t) = \nabla_x\Phi(x_0^t, y_0^t) + \partial f(x_0^t)$. Thus, (111) and (112) imply that

$$\begin{aligned}
\mathcal{G}^t(z_0^t) &= \sup_{x\in\mathcal{X}, y\in\mathcal{Y}} \{\mathcal{L}(x_0^t, y) - \mathcal{L}^t(x, y_0^t)\} \leq \left(\|\nabla\Phi(z_0^t)\|^2 + \|s_f(x_0^t)\|^2 + \|s_g(y_0^t)\|^2\right)/\min\{\mu_x, \mu_y\} \\
&\leq \frac{1}{\mu}\left(\|\nabla\Phi(z_0^t) - \nabla\Phi(z_0^0)\|^2 + \|\nabla\Phi(z_0^0)\|^2 + B_f^2 + B_g^2\right) \\
&\leq \frac{L}{\mu}\|z_0^t - z_0^0\|^2 + \frac{1}{\mu}\left(\|\nabla\Phi(z_0^0)\|^2 + B_f^2 + B_g^2\right),
\end{aligned}$$

where $L = \max\{L_{xx}, L_{yy}, L_{yx}, L_{xy}\}$ and $\mu = \min\{\mu_x, \mu_y\}$. Finally, (109) implies that $\mathbb{E}[\|z_0^t - z_0^0\|^2] < \infty$; therefore, we can conclude that $\mathbb{E}[\mathcal{G}^t(z_0^t)] < \infty$ for all $t \geq 0$. □

Thus, Lemma 22 implies that the analysis given in appendix A.1 directly goes through if we replace Assumption 4 with Assumption 5, which does not require compactness of the problem domain.

F Proof of Theorem 4 and preliminary technical results

The general proof structure of Theorem 4 is the same with Theorem 1's. The main difference is the way we bound the variance, which is given in Lemma 24.

F.1 Construction for the iteration complexity result

Lemma 23. *Suppose Assumptions 1, 3, 6 and 7 hold. Given $\{N_t\}_{t\geq 0} \subset \mathbb{Z}_+$, let $\{x_0^t, y_0^t\}_{t\geq 0}$ be generated by* `SAPD+`*, stated in Algorithm 2, when* `VR-flag=`**`true`**, *initialized from $(x_0^0, y_0^0) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$ and using $\tau, \sigma, \theta, \mu_x > 0$ that satisfy*

$$G - \mathrm{diag}(g) \succeq 0, \tag{113}$$

for some $\alpha \in [0, \frac{1}{\sigma})$, $\rho \in (0, 1]$ and $\pi_x, \pi_y > 0$, where G is defined in (32), *$g \triangleq [\pi_x, \pi_y, L'_x, L'_y, 0]^\top$ and*

$$L'_x \triangleq c(\rho)\,\Big(\frac{{L'_{xx}}^2}{\pi_x b'_x} + \frac{2(1+2\theta+2\theta^2)\rho^{-1}L_{yx}^2}{\pi_y b'_y}\Big),\ L'_y \triangleq c(\rho)\,\Big(\frac{\rho L_{xy}^2}{\pi_x b'_x} + \frac{2(1+2\theta+2\theta^2)\rho^{-1}L_{yy}^2}{\pi_y b'_y}\Big),$$

such that $c(\rho) = \frac{2}{1-\rho}(\rho^{-q+1} - 1)$ for $\rho \in (0, 1)$ and $c(\rho) = 2(q-1)$ for $\rho = 1$, where $L'_{xx} \triangleq L_{xx} + \mu_x + \gamma$. Then for all $t \geq 0$, it holds that

$$\mathbb{E}\left[\mathcal{G}^t(x_0^{t+1}, y_0^{t+1})\right] \leq \frac{M^{\texttt{VR}}}{K_{N_t}(\rho)}\Big(\frac{\mu_x}{4}\mathbb{E}\left[\|x_*^t(y_0^{t+1}) - x_0^t\|^2\right] + \frac{\mu_y}{4}\mathbb{E}\left[\|y_*(x_0^{t+1}) - y_0^t\|^2\right]\Big) + \Xi^{\texttt{VR}}, \tag{114}$$

where $K_{N_t}(\rho) = \sum_{k=0}^{N_t-1}\rho^{-k}$, $\Xi^{\texttt{VR}} \triangleq \frac{\delta_x^2}{2\pi_x b} + (1+2\theta+2\theta^2)\frac{\delta_y^2}{\pi_y b}$ and $M^{\texttt{VR}} \triangleq \max\{\frac{2}{\mu_x}(\frac{1}{\tau} - \mu_x), \frac{2}{\mu_y \sigma}\}$.

Proof. For easier readability, we provide the proof in a separate subsection, see appendix F.2. □

Theorem 8. *Under the premise of Lemma 23, given an arbitrary $\zeta > 0$ and $T \in \mathbb{Z}_+$, suppose $N_t = N$ for all $t = 0, \ldots T$ for some $N \in \mathbb{Z}_+$ such that $N \geq (1+\zeta)M^{\texttt{VR}}$, and* (21) *has a solution for some $\beta_1, \beta_2 \in (0, 1)$ and $p_1, p_2, p_3 > 0$ such that $p_1 + p_2 + p_3 = 1$. If either Assumption 4 or Assumption 5 holds, then* (22) *holds with $\Xi_{\tau,\sigma,\theta} = \Xi^{\texttt{VR}}$ for $\lambda = (\gamma + \mu_x)^{-1}$ and for all $T \geq 1$.*

Proof. The proof is omitted as it is essentially the same with the proof of Theorem 6. □

F.2 Proof of Lemma 23 and preliminary technical results

In this section we prove Lemma 23. We first state a technical lemma that will be used in our analysis.

Lemma 24. *Suppose Assumptions 1, 3, 6 and 7 hold. Let $\{x_k^t, y_k^t\}_{k\geq 0}$ be* `VR-SAPD` *iterates generated according to Algorithm 3 for solving $\min_x \max_y \mathcal{L}^t(x, y)$. Then,*

$$\mathbb{E}\left[\|v_k^t - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\|^2\right] \leq \frac{\delta_x^2}{b} + \sum_{i=(n_k-1)q+1}^{k} \frac{2{L'_{xx}}^2}{b'_x}\mathbb{E}\left[\|x_i^t - x_{i-1}^t\|^2\right] + \frac{2L_{xy}^2}{b'_x}\mathbb{E}\left[\|y_{i+1}^t - y_i^t\|^2\right], \tag{115a}$$

$$\mathbb{E}\left[\|w_k^t - \nabla_y \Phi^t(x_k^t, y_k^t)\|^2\right] \leq \frac{\delta_y^2}{b} + \sum_{i=(n_k-1)q+1}^{k} \frac{2L_{yx}^2}{b'_y}\mathbb{E}\left[\|x_i^t - x_{i-1}^t\|^2\right] + \frac{2L_{yy}^2}{b'_y}\mathbb{E}\left[\|y_i^t - y_{i-1}^t\|^2\right], \tag{115b}$$

for all $k \geq 0$ such that $\mathrm{mod}(k, q) \neq 0$, where $n_k \triangleq \lceil k/q \rceil$. Moreover, if $\mathrm{mod}(k, q) = 0$, then

$$\mathbb{E}\left[\|v_k^t - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\|^2\right] \leq \frac{\delta_x^2}{b}, \quad \mathbb{E}\left[\|w_k^t - \nabla_y \Phi^t(x_k^t, y_k^t)\|^2\right] \leq \frac{\delta_y^2}{b}. \tag{116}$$

Proof. Recall that $\tilde{\nabla}_x \Phi_{I_k^x}^t(x_k^t, y_{k+1}^t) \triangleq \frac{1}{|I_k^x|}\sum_{\omega_k^{x,i} \in I_k^x} \tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t; \omega_k^{x,i})$, where $I_k^x = \{\omega_k^{x,i}\}_{i=1}^{b'_x}$ is a randomly generated batch with $|I_k^x| = b'_x$ independent elements which are also independent of (x_{k-1}^t, y_k^t) and (x_k^t, y_{k+1}^t). According to the definition of v_k in Algorithm 3, for $\mathrm{mod}(k, q) > 0$,

$$v_k^t = v_{k-1}^t + \tilde{\nabla}_x \Phi_{I_k^x}^t(x_k^t, y_{k+1}^t) - \tilde{\nabla}_x \Phi_{I_k^x}^t(x_{k-1}^t, y_k^t). \tag{117}$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\left[\|v_k^t - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\|^2\right] \\
&= \mathbb{E}\left[\|v_{k-1}^t + \tilde{\nabla}_x \Phi^t_{I_k^x}(x_k^t, y_{k+1}^t) - \tilde{\nabla}_x \Phi^t_{I_k^x}(x_{k-1}^t, y_k^t) - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\|^2\right] \\
&= \mathbb{E}\left[\|v_{k-1}^t - \nabla_x \Phi^t(x_{k-1}^t, y_k^t) + \nabla_x \Phi^t(x_{k-1}^t, y_k^t) - \tilde{\nabla}_x \Phi^t_{I_k^x}(x_{k-1}^t, y_k^t) + \tilde{\nabla}_x \Phi^t_{I_k^x}(x_k^t, y_{k+1}^t) - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\|^2\right] \\
&= \mathbb{E}\left[\|v_{k-1}^t - \nabla_x \Phi^t(x_{k-1}^t, y_k^t)\|^2\right] \\
&\quad + \mathbb{E}\left[\|\nabla_x \Phi^t(x_{k-1}^t, y_k^t) - \tilde{\nabla}_x \Phi^t_{I_k^x}(x_{k-1}^t, y_k^t) + \tilde{\nabla}_x \Phi^t_{I_k^x}(x_k^t, y_{k+1}^t) - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\|^2\right],
\end{aligned} \tag{118}
$$

where for the last equality we used

$$
\mathbb{E}\left[\nabla_x \Phi^t(x_{k-1}^t, y_k^t) - \tilde{\nabla}_x \Phi^t_{I_k^x}(x_{k-1}^t, y_k^t) + \tilde{\nabla}_x \Phi^t_{I_k^x}(x_k^t, y_{k+1}^t) - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\right] = 0.
$$

Next, we bound the second expectation on the rhs of (118). It follows that

$$
\begin{aligned}
&\mathbb{E}\left[\|\nabla_x \Phi^t(x_{k-1}^t, y_k^t) - \tilde{\nabla}_x \Phi^t_{I_k^x}(x_{k-1}^t, y_k^t) + \tilde{\nabla}_x \Phi^t_{I_k^x}(x_k^t, y_{k+1}^t) - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\|^2\right] \\
=&\frac{1}{{b'_x}^2}\mathbb{E}\left[\|\sum_{i=1}^{b'_x}\left(\tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t; \omega_k^{x,i}) - \tilde{\nabla}_x \Phi^t(x_{k-1}^t, y_k^t; \omega_k^{x,i}) - \nabla_x \Phi^t(x_k^t, y_{k+1}^t) + \nabla_x \Phi^t(x_{k-1}^t, y_k^t)\right)\|^2\right] \\
=&\frac{1}{{b'_x}^2}\sum_{i=1}^{b'_x}\mathbb{E}\left[\|\tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t; \omega_k^{x,i}) - \tilde{\nabla}_x \Phi^t(x_{k-1}^t, y_k^t; \omega_k^{x,i}) - \nabla_x \Phi^t(x_k^t, y_{k+1}^t) + \nabla_x \Phi^t(x_{k-1}^t, y_k^t)\|^2\right] \\
\le&\frac{1}{{b'_x}^2}\sum_{i=1}^{b'_x}\left(\mathbb{E}\left[\|\tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t; \omega_k^{x,i}) - \tilde{\nabla}_x \Phi^t(x_{k-1}^t, y_k^t; \omega_k^{x,i})\|^2\right]\right) \\
\le&\frac{2{L'_{xx}}^2}{b'_x}\mathbb{E}\left[\|x_k^t - x_{k-1}^t\|^2\right] + \frac{2L_{xy}^2}{b'_x}\mathbb{E}\left[\|y_{k+1}^t - y_k^t\|^2\right],
\end{aligned} \tag{119}
$$

where the second equality follows from the stochastic oracle being unbiased –see Assumption 3, which implies

$$
\mathbb{E}\left[\nabla_x \Phi^t(x_{k-1}^t, y_k^t) - \tilde{\nabla}_x \Phi^t(x_{k-1}^t, y_k^t; \omega_k^{x,i}) + \tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t; \omega_k^{x,i}) - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\right] = 0,
$$

for all $i = 1, \ldots, b'_x$ and $\{\omega_i^k\}_{i=1}^{b'_x}$ being independent; the first inequality is because $\mathbb{E}\left[\|\zeta - \mathbb{E}[\zeta]\|^2\right] \le \mathbb{E}\left[\|\zeta\|^2\right]$ for any given random variable ζ with finite second order moment –we invoke this inequality for $\zeta = \tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t; \omega_k^{x,i}) - \tilde{\nabla}_x \Phi^t(x_{k-1}^t, y_k^t; \omega_k^{x,i})$; and finally, the last inequality follows from Assumption 6 and the inequality $(a+b)^2 \le 2a^2 + 2b^2$ for any $a, b \in \mathbb{R}$. Next, if we combine eq. (118) and eq. (119), we get

$$
\begin{aligned}
&\mathbb{E}\left[\|v_k^t - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\|^2\right] \\
\le&\mathbb{E}\left[\|v_{k-1}^t - \nabla_x \Phi^t(x_{k-1}^t, y_k^t)\|^2\right] + \frac{2{L'_{xx}}^2}{b'_x}\mathbb{E}\left[\|x_k^t - x_{k-1}^t\|^2\right] + \frac{2L_{xy}^2}{b'_x}\mathbb{E}\left[\|y_{k+1}^t - y_k^t\|^2\right].
\end{aligned}
$$

Hence, if we sum the above inequality from $(n_k - 1)q + 1$ to k, we get a telescoping sum:

$$
\begin{aligned}
&\mathbb{E}\left[\|v_k^t - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\|^2\right] \\
&\le \sum_{i=(n_k-1)q+1}^{k}\frac{2{L'_{xx}}^2}{b'_x}\mathbb{E}\left[\|x_i^t - x_{i-1}^t\|^2\right] + \sum_{i=(n_k-1)q+1}^{k}\frac{2L_{xy}^2}{b'_x}\mathbb{E}\left[\|y_{i+1}^t - y_i^t\|^2\right] \\
&\quad + \mathbb{E}\left[\|v_{(n_k-1)q} - \nabla_x \Phi^t(x_{(n_k-1)q}^t, y_{(n_k-1)q+1}^t)\|^2\right] \\
&\le \sum_{i=(n_k-1)q+1}^{k}\frac{2{L'_{xx}}^2}{b'_x}\mathbb{E}\left[\|x_i^t - x_{i-1}^t\|^2\right] + \sum_{i=(n_k-1)q+1}^{k}\frac{2L_{xy}^2}{b'_x}\mathbb{E}\left[\|y_{i+1}^t - y_i^t\|^2\right] + \frac{\delta_x^2}{b},
\end{aligned} \tag{120}
$$

where the last inequality follows from Assumption 3 since $\mathrm{mod}((n_k-1)q, q) = 0$ and for $\ell \in \mathbb{Z}_+$ such that $\mathrm{mod}(\ell, q) = 0$, we have $v_\ell = \tilde{\nabla}_x \Phi^t_{\mathcal{B}_\ell^x}(x_\ell^t, y_{\ell+1}^t) = \frac{1}{|\mathcal{B}_\ell^x|}\sum_{\omega_\ell^{x,i} \in \mathcal{B}_\ell^x} \tilde{\nabla}_x \Phi^t(x_\ell^t, y_{\ell+1}^t; \omega_\ell^{x,i})$, where $\mathcal{B}_\ell^x = \{\omega_\ell^{x,i}\}$ is a randomly generated batch with $|\mathcal{B}_\ell^x| = b$ independent elements which are also independent of $(x_\ell^t, y_{\ell+1}^t)$. This completes the proof of the case for k such that $\mathrm{mod}(k,q) > 0$.

When $\mathrm{mod}(k,q) = 0$, it follows from Algorithm 3 that $v_k = \tilde{\nabla}_x \Phi_{\mathcal{B}_k}^t(x_k^t, y_{k+1}^t)$. Hence, above discussion yields

$$\mathbb{E}\left[\|v_k^t - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\|^2\right] \le \frac{\delta_x^2}{b}. \tag{121}$$

Finally, the second inequality in (115b) can be shown similarly. □

Next, we will modify Lemma 13 for VR-SAPD, stated in Algorithm 3. Specifically, instead of using the stochastic oracles $\tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t; \omega_k^x)$ and $\tilde{\nabla}_y \Phi^t(x_k^t, y_k^t; \omega_k^y)$ as in Lemma 13, we adopt v_k^t and w_k^t to estimate $\nabla_x \Phi^t(x_k^t, y_{k+1}^t)$ and $\nabla_y \Phi^t(x_k^t, y_k^t)$, respectively.

Lemma 25. *Suppose Assumptions 1, 3, and 6 hold. Let $\{x_k^t, y_k^t\}_{k\ge 0}$ be VR-SAPD iterates generated according to Algorithm 3 for solving $\min_x \max_y \mathcal{L}^t(x,y)$. Then for all $x \in \mathbf{dom}\, f \subset \mathcal{X}$, $y \in \mathbf{dom}\, g \subset \mathcal{Y}$, and $k \ge 0$,*

$$\begin{aligned}&\mathcal{L}^t(x_{k+1}^t, y) - \mathcal{L}^t(x, y_{k+1}^t)\\ &\quad\le -\langle q_{k+1}^t, y_{k+1}^t - y\rangle + \theta\langle q_k^t, y_k^t - y\rangle + \Lambda_k^t(x,y) - \Sigma_{k+1}^t(x,y) + \Gamma_{k+1}^t + \varepsilon_k^{t,x}(x) + \varepsilon_k^{t,y}(y),\end{aligned} \tag{122}$$

where $\varepsilon_k^{t,x}(x) \triangleq \langle v_k^t - \nabla_x \Phi^t(x_k^t, y_{k+1}^t),\ x - x_{k+1}^t\rangle$ and $\varepsilon_k^{t,y}(y) \triangleq \langle \tilde{s}_k^t - s_k^t, y_{k+1}^t - y\rangle$ for $\tilde{s}_k^t = (1+\theta)w_k^t - \theta w_{k-1}^t$ as defined in Algorithm 3, q_k^t and s_k^t are defined as in (59), and $\Lambda_k^t(x,y)$, $\Sigma_{k+1}^t(x,y)$, Γ_{k+1}^t are the same with those in Lemma 13.

Proof. The proof uses the same arguments as the proof of Lemma 13. One only needs to replace $\tilde{\nabla}_x \Phi^t(x_k^t, y_{k+1}^t)$ and $\tilde{\nabla}_y \Phi^t(x_k^t, y_k^t)$ in the proof of Lemma 13 with v_k^t, w_k^t, respectively. □

F.3 Proof of Lemma 23

Proof. For simplifying the notation, let $N_t = N$ for some $N \in \mathbb{Z}_+$. For arbitrary $(x,y) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$, since $(x_{k+1}^t, y_{k+1}^t) \in \mathbf{dom}\, f \times \mathbf{dom}\, g$, using the concavity of $\mathcal{L}^t(x_{k+1}^t, \cdot)$ and the convexity of $\mathcal{L}^t(\cdot, y_{k+1}^t)$, Lemma 25 and Jensen's lemma immediately implies that

$$K_N(\rho)\left(\mathcal{L}^t(\bar{x}_N^t, y) - \mathcal{L}^t(x, \bar{y}_N^t)\right) \le \sum_{k=0}^{N-1} \rho^{-k}\left(\mathcal{L}^t(x_{k+1}^t, y) - \mathcal{L}^t(x, y_{k+1}^t)\right),\ \forall \rho \in (0,1], \tag{123}$$

where $\bar{x}_N^t = \frac{1}{K_N(\rho)}\sum_{k=0}^{N-1} \rho^{-k} x_{k+1}^t$, $\bar{y}_N^t = \frac{1}{K_N(\rho)}\sum_{k=0}^{N-1} \rho^{-k} y_{k+1}^t$, and $K_N(\rho) = \sum_{i=0}^{N-1} \rho^{-k}$. Thus, if we multiply both sides of (122) by ρ^{-k} and sum the resulting inequality from $k = 0$ to $N-1$, then using (123) we get

$$\begin{aligned}&K_N(\rho)\left(\mathcal{L}^t(\bar{x}_N^t, x) - \mathcal{L}(x, \bar{y}_N^t)\right)\\ &\quad\le \sum_{k=0}^{N-1} \rho^{-k}\Big(-\langle q_{k+1}^t, y_{k+1}^t - x\rangle + \theta\langle q_k^t, y_k^t - x\rangle + \Lambda_k^t(x,y) - \Sigma_{k+1}^t(x,y) + \Gamma_{k+1}^t\\ &\qquad\quad - \langle v_k^t - \nabla_x \Phi^t(x_k^t, y_{k+1}^t), x_{k+1}^t - x\rangle + \langle \tilde{s}_k^t - s_k^t, y_{k+1}^t - x\rangle\Big)\\ &\quad\le \sum_{k=0}^{N-1} \rho^{-k}\Big(\underbrace{-\langle q_{k+1}^t, y_{k+1}^t - x\rangle + \theta\langle q_k^t, y_k^t - x\rangle}_{\textbf{part 1}} + \Lambda_k^t(x,y) - \Sigma_{k+1}^t(x,y) + \Gamma_{k+1}^t\\ &\qquad\quad + \frac{\pi_x}{2}\|x_{k+1}^t - x\|^2 + \frac{\pi_y}{2}\|y_{k+1}^t - x\|^2 + \underbrace{\frac{1}{2\pi_x}\|v_k^t - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\|^2 + \frac{1}{2\pi_y}\|\tilde{s}_k^t - s_k^t\|^2}_{\textbf{part 2}}\Big).\end{aligned} \tag{124}$$

The second inequality follows from Young's inequality for some constants $\pi_x, \pi_y > 0$.

The following bound for **part1** can be obtained from (74). Indeed, for any $k \ge -1$, we get

$$\sum_{k=0}^{N-1} \rho^{-k}(\theta\langle q_k^t, y_k^t - y_*^t\rangle - \langle q_{k+1}^t, y_{k+1}^t - y_*^t\rangle) \le \sum_{k=0}^{N-1} \rho^{-k}|1 - \frac{\theta}{\rho}|\ S_{k+1}^t(x,y) + \rho^{-N+1}\frac{\theta}{\rho} S_N^t(x,y). \tag{125}$$

where $S_{k+1}^t(x,y)$ is defined in (73).

Next we consider **part 2**, recall that $n_k = \lceil k/q \rceil$ such that $(n_k-1)q+1 \le k \le n_k q$, it follows from Lemma 24 that

$$
\begin{aligned}
&\sum_{k=0}^{N-1} \rho^{-k} \mathbb{E}\left[\|v_k^t - \nabla_x \Phi^t(x_k^t, y_{k+1}^t)\|^2\right] \\
&\le \sum_{\substack{k\in\{1,\dots,N-1\}\\ \text{s.t. } \bmod(k,q)\neq 0}} \rho^{-k} \sum_{i=(n_k-1)q+1}^{k} \Big(\frac{2{L'_{xx}}^2}{b'_x}\mathbb{E}\left[\|x_i^t - x_{i-1}^t\|^2\right] + \frac{2L_{xy}^2}{b'_x}\mathbb{E}\left[\|y_{i+1}^t - y_i^t\|^2\right]\Big) + \frac{\delta_x^2}{b}\sum_{k=0}^{N-1}\rho^{-k} \\
&= \sum_{k=1}^{N-1} \rho^{-k} \Big(\frac{2{L'_{xx}}^2}{b'_x}\mathbb{E}\left[\|x_k^t - x_{k-1}^t\|^2\right] + \frac{2L_{xy}^2}{b'_x}\mathbb{E}\left[\|y_{k+1}^t - y_k^t\|^2\right]\Big) \sum_{i=0}^{n_k q-k-1} \rho^{-i} + \frac{\delta_x^2}{b}\sum_{k=0}^{N-1}\rho^{-k} \\
&= \sum_{k=0}^{N-1} \rho^{-k}\cdot\frac{\rho}{1-\rho}(\rho^{-n_k q+k}-1) \Big(\frac{2{L'_{xx}}^2}{b'_x}\mathbb{E}\left[\|x_k^t - x_{k-1}^t\|^2\right] + \frac{2L_{xy}^2}{b'_x}\mathbb{E}\left[\|y_{k+1}^t - y_k^t\|^2\right]\Big) + \frac{\delta_x^2}{b}\sum_{k=0}^{N-1}\rho^{-k} \\
&\le \sum_{k=0}^{N-1} \rho^{-k}\cdot\frac{\rho}{1-\rho}(\rho^{-q+1}-1) \Big(\frac{2{L'_{xx}}^2}{b'_x}\mathbb{E}\left[\|x_k^t - x_{k-1}^t\|^2\right] + \frac{2L_{xy}^2}{b'_x}\mathbb{E}\left[\|y_{k+1}^t - y_k^t\|^2\right]\Big) + \frac{\delta_x^2}{b}\sum_{k=0}^{N-1}\rho^{-k}
\end{aligned}
\tag{126}
$$

where the first inequality follows from Lemma 24, the following equality is by rearranging terms, and for the last inequality we used the following bound: $n_k = \lceil k/q \rceil \le k/q + (q-1)/q$; hence, $-n_k q + k \ge -q+1$. To bound **part 2** in eq. (124), we next consider $\|\tilde{s}_k^t - s_k^t\|^2$. For $k>0$,

$$
\begin{aligned}
&\|\tilde{s}_k^t - s_k^t\|^2 = \|(1+\theta)w_k^t - (1+\theta)\nabla_y\Phi^t(x_k^t,y_k^t) - \theta w_{k-1}^t + \theta\nabla_y\Phi^t(x_{k-1}^t,y_{k-1}^t)\|^2 \\
&\quad\le 2(1+\theta)^2\|w_k^t - \nabla_y\Phi^t(x_k^t,y_k^t)\|^2 + 2\theta^2\|w_{k-1}^t - \nabla_y\Phi^t(x_{k-1}^t,y_{k-1}^t)\|^2.
\end{aligned}
\tag{127}
$$

First, $x_{-1}^t = x_0^t$, $y_{-1}^t = y_0^t$ and (59) imply that $s_0^t = \nabla_y\Phi^t(x_0^t,y_0^t)$, and recall that in Algorithm 3, we set $\tilde{s}_0^t = w_0^t$; hence,

$$
\|\tilde{s}_0^t - s_0^t\|^2 = \|w_0^t - \nabla_y\Phi^t(x_0^t,y_0^t)\|^2,
$$

and eq. (127) holds for $k \ge 0$ with $w_{-1}^t \triangleq \nabla_y\Phi^t(x_0^t,y_0^t)$. Then, Lemma 24 implies that

$$
\begin{aligned}
&\sum_{k=0}^{N-1}\rho^{-k}\mathbb{E}\left[\|\tilde{s}_k^t - s_k^t\|^2\right] \\
&\le 2(1+\theta)^2 \sum_{\substack{k\in\{1,\dots,N-1\}\\ \text{s.t. } \bmod(k,q)\neq 0}} \rho^{-k} \sum_{i=(n_k-1)q+1}^{k} \Big(\frac{2L_{yx}^2}{b'_y}\mathbb{E}\left[\|x_i^t - x_{i-1}^t\|^2\right] + \frac{2L_{yy}^2}{b'_y}\mathbb{E}\left[\|y_i^t - y_{i-1}^t\|^2\right]\Big) \\
&\quad + \frac{2\theta^2}{\rho} \sum_{\substack{k\in\{1,\dots,N-2\}\\ \text{s.t. } \bmod(k,q)\neq 0}} \rho^{-k} \sum_{i=(n_k-1)q+1}^{k} \Big(\frac{2L_{yx}^2}{b'_y}\mathbb{E}\left[\|x_i^t - x_{i-1}^t\|^2\right] + \frac{2L_{yy}^2}{b'_y}\mathbb{E}\left[\|y_i^t - y_{i-1}^t\|^2\right]\Big) \\
&\quad + 2(1+2\theta+2\theta^2)\frac{\delta_y^2}{b}\sum_{k=0}^{N-1}\rho^{-k} \\
&\le \frac{2}{\rho}(1+2\theta+2\theta^2) \sum_{\substack{k\in\{1,\dots,N-1\}\\ \text{s.t. } \bmod(k,q)\neq 0}} \rho^{-k} \sum_{i=(n_k-1)q+1}^{k} \Big(\frac{2L_{yx}^2}{b'_y}\mathbb{E}\left[\|x_i^t - x_{i-1}^t\|^2\right] + \frac{2L_{yy}^2}{b'_y}\mathbb{E}\left[\|y_i^t - y_{i-1}^t\|^2\right]\Big) \\
&\quad + 2(1+2\theta+2\theta^2)\frac{\delta_y^2}{b}\sum_{k=0}^{N-1}\rho^{-k},
\end{aligned}
\tag{128}
$$

where the first inequality follows from Lemma 24; in the last inequality we used $\rho \leq 1$ and combined the two sums. Next, as in eq. (126), we can further obtain that

$$\begin{aligned}
&\sum_{k=0}^{N-1} \rho^{-k}\mathbb{E}\left[\|\tilde{s}_k^t - s_k^t\|^2\right] \\
&\leq 2(1+2\theta+2\theta^2)\sum_{k=0}^{N-1}\rho^{-k}\cdot\frac{1}{1-\rho}(\rho^{-q+1}-1)\big(\frac{2L_{yx}^2}{b'_y}\mathbb{E}\left[\|x_k^t - x_{k-1}^t\|^2\right] + \frac{2L_{yy}^2}{b'_y}\mathbb{E}\left[\|y_k^t - y_{k-1}^t\|^2\right]\big) \\
&+ 2(1+2\theta+2\theta^2)\frac{\delta_y^2}{b}\sum_{k=0}^{N-1}\rho^{-k}.
\end{aligned} \tag{129}$$

Now we can bound **part 2** in eq. (124) using eq. (126) and eq. (129). In addition, Given $(\bar{x}_N^t, \bar{y}_N^t)$, the point $(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t)) \triangleq \operatorname{argmax}_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \mathcal{L}^t(\bar{x}_N^t, y) - \mathcal{L}^t(x, \bar{y}_N^t)$ uniquely exists. We will use the fact that

$$\mathcal{G}^t(\bar{x}_N^t, \bar{y}_N^t) = \sup_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \mathcal{L}^t(\bar{x}_N^t, y) - \mathcal{L}^t(x, \bar{y}_N^t) = \mathcal{L}^t(\bar{x}_N^t, y_*(\bar{x}_N^t)) - \mathcal{L}^t(x_*^t(\bar{y}_N^t), \bar{y}_N^t)$$

to complete the proof.

Recall that we defined $D_N^t(x,y) = \frac{1}{2\rho}(\frac{1}{\tau} - \mu_x)\|x_N^t - x\|^2 + \frac{1}{2}\Big(\frac{1}{\rho\sigma} - \alpha\Big)\|y_N^t - y\|^2$; first, we substitute $(x,y) = (x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t))$ into (124), and then add $\rho^{-N+1}D_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t))$ to both sides of (124). Finally, taking the expectation of the new inequality, and then using eq. (125), eq. (126) and eq. (129) to bound **part 1** and **part 2**, we obtain

$$\begin{aligned}
&\mathbb{E}\big[K_N(\rho)\mathcal{G}^t(\bar{x}_N^t, \bar{y}_N^t) + \rho^{-N+1}D_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t))\big] \\
&\qquad \leq \mathbb{E}\left[\hat{U}_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t))\right] + \Big(\frac{\delta_x^2}{2\pi_x b} + (1+2\theta+2\theta^2)\frac{\delta_y^2}{\pi_y b}\Big)\sum_{k=0}^{N-1}\rho^{-k},
\end{aligned} \tag{130}$$

where $\hat{U}_N^t(x,y)$ is defined as

$$\begin{aligned}
&\hat{U}_N^t(x,y) \triangleq \sum_{k=0}^{N-1}\rho^{-k}\Big(\Gamma_{k+1}^t + \Lambda_k^t(x,y) - \Sigma_{k+1}^t(x,y) \\
&+ |1-\frac{\theta}{\rho}|\ S_{k+1}^t(x,y) + \frac{\pi_x}{2}\|x_{k+1}^t - x\|^2 + \frac{\pi_y}{2}\|y_{k+1}^t - y\|^2\Big) \\
&+ \sum_{k=0}^{N-1}\frac{\rho^{-k+1}}{1-\rho}(\rho^{-q+1}-1)\left(\left(\frac{{L'_{xx}}^2}{\pi_x b'_x} + \frac{2(1+2\theta+2\theta^2)\rho^{-1}L_{yx}^2}{\pi_y b'_y}\right)\|x_k^t - x_{k-1}^t\|^2 + \frac{L_{xy}^2}{\pi_x b'_x}\|y_{k+1}^t - y_k^t\|^2\right) \\
&+ \sum_{k=0}^{N-1}\frac{\rho^{-k+1}}{1-\rho}(\rho^{-q+1}-1)\frac{2(1+2\theta+2\theta^2)\rho^{-1}L_{yy}^2}{\pi_y b'_y}\|y_k^t - y_{k-1}^t\|^2 - \rho^{-N+1}(-D_N^t(x,y) - \frac{\theta}{\rho}S_N^t(x,y)).
\end{aligned} \tag{131}$$

The remaining part of the analysis directly follows from the arguments we used in the proof of Lemma 12. We can analyze $\hat{U}_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t))$ through writing it as a telescoping sum. After adding and subtracting $\frac{\alpha}{2}\|y_{k+1}^t - y_k^t\|^2$, and rearranging the terms, we get

$$\begin{aligned}
\hat{U}_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t)) =& \frac{1}{2}\sum_{k=0}^{N-1}\rho^{-k}\Big({\xi_k^*}^\top \hat{A}\xi_k^* - {\xi_{k+1}^*}^\top \hat{B}\xi_{k+1}^*\Big) \\
&- \rho^{-N+1}(-D_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t)) - \frac{\theta}{\rho}S_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t))) \\
=& \frac{1}{2}{\xi_0^*}^\top \hat{A}\xi_0^* - \frac{1}{2}\sum_{k=1}^{N-1}\rho^{-k+1}[{\xi_k^*}^\top(\hat{B} - \tfrac{1}{\rho}\hat{A})\xi_k^*] \\
&- \rho^{-N+1}(\frac{1}{2}{\xi_N^*}^\top \hat{B}\xi_N^* - D_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t)) - \frac{\theta}{\rho}S_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t))),
\end{aligned} \tag{132}$$

where $\xi_k^* \in \mathbb{R}^5$ is defined for $k \geq 0$ as follows: $\xi_k^* \triangleq \begin{pmatrix} \|x_k^t - x_*^t(\bar{y}_N^t)\| \\ \|y_k^t - y_*(\bar{y}_N^t)\| \\ \|x_k^t - x_{k-1}^t\| \\ \|y_k^t - y_{k-1}^t\| \\ \|y_{k+1}^t - y_k^t\| \end{pmatrix}$ such that $x_{-1}^t = x_0^t$, $y_{-1}^t = y_0^t$; and $\hat{A}, \hat{B} \in \mathbb{R}^{5\times 5}$ are defined as:

$$\hat{A} \triangleq \begin{pmatrix} \frac{1}{\tau} - \mu_x & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sigma} & 0 & 0 & 0 \\ 0 & 0 & \rho L_x' & 0 & \theta L_{yx} \\ 0 & 0 & 0 & \rho L_y^+ & \theta L_{yy} \\ 0 & 0 & \theta L_{yx} & \theta L_{yy} & -\alpha \end{pmatrix},$$

$$\hat{B} \triangleq \begin{pmatrix} \frac{1}{\tau} - \pi_x & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sigma} + \mu_y - \pi_y & -|1-\frac{\theta}{\rho}|\, L_{yx} & -|1-\frac{\theta}{\rho}|\, L_{yy} & 0 \\ 0 & -|1-\frac{\theta}{\rho}|\, L_{yx} & \frac{1}{\tau} - L_{xx}' & 0 & 0 \\ 0 & -|1-\frac{\theta}{\rho}|\, L_{yy} & 0 & \frac{1}{\sigma} - \alpha - L_y^- & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$\begin{aligned} L_x' &\triangleq \frac{2}{1-\rho}(\rho^{-q+1}-1)\Big(\frac{{L_{xx}'}^2}{\pi_x b_x'} + \frac{2(1+2\theta+2\theta^2)\rho^{-1}L_{yx}^2}{\pi_y b_y'}\Big), \\ L_y^+ &\triangleq \frac{2}{1-\rho}(\rho^{-q+1}-1)\frac{2(1+2\theta+2\theta^2)\rho^{-1}L_{yy}^2}{\pi_y b_y'}, \\ L_y^- &\triangleq \frac{2\rho}{1-\rho}(\rho^{-q+1}-1)\frac{L_{xy}^2}{\pi_x b_x'}. \end{aligned}$$

Using the same argument as in the proof of Lemma 17, and noticing that L_y' in eq. (113) can be written as $L_y' = L_y^+ + L_y^-$, one can show that eq. (113) holds if and only if $\hat{B} - \frac{1}{\rho}\hat{A} \succeq 0$. Therefore, it follows from (132) that

$$\begin{aligned} \hat{U}_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t)) \leq& \frac{1}{2}{\xi_0^*}^\top \hat{A}\xi_0^* \\ &- \rho^{-N+1}\Big(\tfrac{1}{2}{\xi_N^*}^\top \hat{B}\xi_N^* - D_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t)) - \frac{\theta}{\rho}S_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t))\Big), \end{aligned}$$

holds w.p. 1. Furthermore, define

$$G''' \triangleq \begin{pmatrix} \frac{1}{\sigma}(1-\frac{1}{\rho}) + \mu_y - \pi_y + \alpha & (-|1-\frac{\theta}{\rho}| - \frac{\theta}{\rho})L_{yx} & (-|1-\frac{\theta}{\rho}| - \frac{\theta}{\rho})L_{yy} \\ (-|1-\frac{\theta}{\rho}| - \frac{\theta}{\rho})L_{yx} & \frac{1}{\tau} - L_{xx}' & 0 \\ (-|1-\frac{\theta}{\rho}| - \frac{\theta}{\rho})L_{yy} & 0 & \frac{1}{\sigma} - \alpha - L_y^- \end{pmatrix},$$

and recall that $D_N^t(x,y) = \frac{1}{2\rho}(\frac{1}{\tau} - \mu_x)\|x_N^t - x\|^2 + \frac{1}{2}\Big(\frac{1}{\rho\sigma} - \alpha\Big)\|y_N^t - y\|^2$. Using a similar argument as in the proof of Lemma 18, we can show that eq. (113) implies

$$\begin{aligned} &\frac{1}{2}{\xi_N^*}^\top \hat{B}\xi_N^* - D_N^t\Big(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t)\Big) - \frac{\theta}{\rho}S_N^t\Big(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t)\Big) \\ &= \frac{1}{2}{\xi_N^*}^\top \begin{pmatrix} \frac{1}{\tau}(1-\frac{1}{\rho}) + \frac{\mu_x}{\rho} - \pi_x & \mathbf{0}_{1\times 3} & 0 \\ \mathbf{0}_{3\times 1} & G''' & \mathbf{0}_{3\times 1} \\ 0 & \mathbf{0}_{1\times 3} & 0 \end{pmatrix} \xi_N^* \geq 0. \end{aligned}$$

Finally, since $x_{-1}^t = x_0^t$, $y_{-1}^t = y_0^t$, we have

$$\frac{1}{2}{\xi_0^*}^\top \hat{A}\xi_0^* \leq \Big(\frac{1}{2\tau} - \frac{\mu_x}{2}\Big)\|x_*^t(\bar{y}_N^t) - x_0^t\|^2 + \frac{1}{2\sigma}\|y_*(\bar{x}_N^t) - y_0^t\|^2.$$

Therefore, we obtain that

$$\hat{U}_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t)) \le \Big(\frac{1}{2\tau} - \frac{\mu_x}{2}\Big)\|x_*^t(\bar{y}_N^t) - x_0^t\|^2 + \frac{1}{2\sigma}\|y_*(\bar{x}_N^t) - y_0^t\|^2, \text{ holds w.p. 1.}$$

Now, we are ready to show the desired result of Lemma 23. Since $D_N^t(x_*^t(\bar{y}_N^t), y_*(\bar{x}_N^t)) \ge 0$, it follows from (130) that

$$\begin{aligned} K_N(\rho)\mathbb{E}\big[\mathcal{G}^t(\bar{x}_N^t, \bar{y}_N^t)\big] \le &\mathbb{E}\Big[\left(\frac{1}{2\tau} - \frac{\mu_x}{2}\right)\|x_*^t(\bar{y}_N^t) - x_0^t\|^2 + \frac{1}{2\sigma}\|y_*(\bar{x}_N^t) - y_0^t\|^2\Big] \\ &+ K_N(\rho)\Big(\frac{\delta_x^2}{2\pi_x b} + (1+2\theta+2\theta^2)\frac{\delta_y^2}{\pi_y b}\Big). \end{aligned}$$

Then dividing both side by $K_N(\rho)$ completes the proof. □

F.4 A particular parameter choice

We employ the matrix inequality (MI) in eq. (113) to describe the admissible set of `VR-SAPD` parameters that guarantee convergence. Next, in Lemma 26, we compute a particular solution to it by exploiting its structure.

Lemma 26. *For any $\mu_x > 0$, let $L'_{xx} = L_{xx} + \gamma + \mu_x$. Let $\theta \in (0, 1]$ and $\tau, \sigma > 0$ be chosen as*

$$\theta = 1, \quad \tau = \frac{1}{L_{yx} + L'_{xx} + L'_x}, \quad \sigma = \frac{1}{2L_{yy} + L_{yx} + L'_y}, \tag{133}$$

where L'_x and L'_y are defined in Lemma 23. Then $\{\tau, \sigma, \theta, \alpha, \rho, \pi_x, \pi_y\}$ is a solution to (113) *for $\rho = 1$, $\pi_x = \mu_x$, $\pi_y = \mu_y$ and $\alpha = L_{yx} + L_{yy}$.*

Proof. Define $M_1 \triangleq \begin{pmatrix} \frac{1}{\tau} - L'_{xx} - L'_x & 0 & -L_{yx} \\ 0 & 0 & 0 \\ -L_{yx} & 0 & L_{yx} \end{pmatrix}$ and $M_2 \triangleq \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sigma} - \alpha - L'_y & -L_{yy} \\ 0 & -L_{yy} & L_{yy} \end{pmatrix}$. Our choice of $\{\rho, \pi_x, \pi_y, \alpha\}$ implies that (113) holds whenever

$$M_1 + M_2 = \begin{pmatrix} \frac{1}{\tau} - L'_{xx} - L'_x & 0 & -L_{yx} \\ 0 & \frac{1}{\sigma} - \alpha - L'_y & -L_{yy} \\ -L_{yx} & -L_{yy} & L_{yx} + L_{yy} \end{pmatrix} \succeq \mathbf{0}.$$

Our choice of $\alpha = L_{yx} + L_{yy}$, and $\tau, \sigma > 0$ as in (133) implies that $M_1 \succeq 0$ and $M_2 \succeq 0$. Thus, $M_1 + M_2 \succeq 0$. □

Next, based on this lemma, we will give an explicit parameter choice for Algorithm 3.

F.5 Proof of Theorem 4

Proof. Lemma 26 implies that our choice of $\{\tau, \sigma, \theta, \alpha, \rho, \pi_x, \pi_y\}$ ensures that eq. (113) holds. For the outer loop, if we set N as in (11) and

$$p_1 = \frac{1}{16},\ p_2 = \frac{19}{32},\ p_3 = \frac{11}{32},\ \beta_1 = \frac{4}{5},\ \beta_2 = \frac{1}{2},\ \zeta = 32, \tag{134}$$

all assumptions of Theorem 8 are satisfied, i.e., both the inequality system in (21) and $N \ge (1 + \zeta)M^{\texttt{VR}}$ hold. Specifically, $M^{\texttt{VR}} = 2\max\{\frac{1}{\gamma\tau} - 1,\ \frac{1}{\mu_y\sigma}\}$; thus, $N \ge (1+\zeta)M^{\texttt{VR}}$ trivially holds by our choice of N in (11). The proof of eq. (21) holding for parameters in (134) follows directly from the proof of Theorem 1.

Since all assumptions of Theorem 8 are satisfied for $\mu_x = \gamma$, $\{\tau, \sigma, \theta\}$ as in (133), N and b as in eq. (11) and other parameters chosen as in eq. (134), if we substitute $\mu_x = \gamma$ and the specific parameter values given in eq. (134) into eq. (22) with $\Xi_{\tau,\sigma,\theta} = \Xi^{\texttt{VR}} = \frac{\delta_x^2}{2\gamma b} + 5\frac{\delta_y^2}{\mu_y b}$, it follows that

$$\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[\|\nabla\phi_\lambda(x_0^t)\|^2\right] \le 48\gamma\left(\frac{1}{T+1}\mathcal{G}(x_0^0, y_0^0) + \frac{\delta_x^2}{2\gamma b} + \frac{5\delta_y^2}{\mu_y b}\right). \tag{135}$$

Thus, for any $\epsilon > 0$, the right side of the above inequality can be bounded by ϵ^2 when

$$\frac{48\gamma}{T+1}\mathcal{G}(x_0^0, y_0^0) \leq \frac{\epsilon^2}{6}, \quad \frac{24\delta_x^2}{b} \leq \frac{\epsilon^2}{6}, \quad \frac{240\gamma\delta_y^2}{\mu_y b} \leq \frac{2\epsilon^2}{3}. \tag{136}$$

Our choice of b in (11) and $T \geq 288\mathcal{G}(x_0^0, y_0^0)\frac{\gamma}{\epsilon^2}$ ensures that all the inequalities in (136) hold. Moreover, our choice of N and $\{\tau, \sigma, \theta\}$in (11) and $\rho = 1$ together with the definitions of L'_x and L'_y in Lemma 23 implies (12). Furthermore, it follows from the statement of Algorithm 3 that the total computation complexity is $T(Nb/q + N(b'_x + b'_y))$, which completes the proof. □

G Proof of Theorem 5 and preliminary technical results

Recall the definition of $\hat{\mathcal{L}}$ given in eq. (13). For any $x \in \mathcal{X}$, define $\phi(x) \triangleq \max_{y\in\mathcal{Y}} \mathcal{L}(x, y)$ and $\hat{\phi}(x) \triangleq \max_{y\in\mathcal{Y}} \hat{\mathcal{L}}(x, y)$; moreover, let $\phi_\lambda(\cdot)$ and $\hat{\phi}_\lambda(\cdot)$ be respective Moreau envelopes for some $\lambda \in (0, \gamma^{-1})$.

We first show that one can obtain an ϵ-stationary point for the WCMC problem in the form of (1) such that $f(\cdot) = 0$, $\mu_y = 0$ and $\mathcal{D}_y < \infty$ by computing an ϵ-stationary point for eq. (13) with $\hat{\mu}_y = \Theta(\epsilon^2/(\gamma\mathcal{D}_y^2))$. Indeed, in Lemma 27 below, we extend [24, Corollary A.8] from g being an indicator function of a closed convex set to a closed convex function.

Lemma 27. *Under the premise of Theorem 5, for some fixed $\hat{\mu}_y = \Theta(\epsilon^2/(\gamma\mathcal{D}_y^2))$, let $x_\epsilon \in \mathcal{X}$ be such that $\|\nabla\hat{\phi}(x_\epsilon)\| \leq \epsilon/(2\sqrt{6})$, where $\hat{\phi}(x) \triangleq \max_{y\in\mathcal{Y}} \hat{\mathcal{L}}(x, y)$. Then, x_ϵ is an ϵ-stationary point of $\phi(\cdot)$, i.e., $\|\nabla\phi_\lambda(x_\epsilon)\| \leq \epsilon$ for $\lambda \in (0, \gamma^{-1})$, where $\phi(x) \triangleq \max_{y\in\mathcal{Y}} \mathcal{L}(x, y)$.*

Proof. Below We state some useful relations that will be used later in the proof. Since $f(\cdot) = 0$, eq. (13) implies that for all $(x, y) \in \mathcal{X} \times \mathbf{dom}\, g$,

$$\nabla_x\mathcal{L}(x, y) = \nabla_x\hat{\mathcal{L}}(x, y), \quad \|\nabla_y\Phi(x, y) - \nabla_y\hat{\Phi}(x, y)\| \leq \hat{\mu}_y\mathcal{D}_y. \tag{137}$$

We define $\hat{y}_*(\cdot) \triangleq \mathrm{argmax}_{y\in\mathcal{Y}} \hat{\mathcal{L}}(\cdot, y)$. It follows that from Lemma 11 that

$$\hat{y}_*(x_\epsilon) = \mathbf{prox}_{\alpha g}\big(\hat{y}_*(x_\epsilon) + \alpha\nabla_y\hat{\Phi}(x_\epsilon, \hat{y}_*(x_\epsilon))\big). \tag{138}$$

for any $\alpha > 0$. We are now ready for the proof of Lemma 27.

Let $y^+ \triangleq \mathbf{prox}_{\alpha g}\big(\hat{y}_*(x_\epsilon) + \alpha\nabla_y\Phi(x_\epsilon, \hat{y}_*(x_\epsilon))\big)$, then we have

$$\begin{aligned} &\|y^+ - \hat{y}_*(x_\epsilon)\| \\ =&\|\mathbf{prox}_{\alpha g}\big(\hat{y}_*(x_\epsilon) + \alpha\nabla_y\Phi(x_\epsilon, \hat{y}_*(x_\epsilon))\big) - \mathbf{prox}_{\alpha g}\big(\hat{y}_*(x_\epsilon) + \alpha\nabla_y\hat{\Phi}(x_\epsilon, \hat{y}_*(x_\epsilon))\big)\| \\ \leq&\alpha\|\nabla_y\Phi(x_\epsilon, \hat{y}_*(x_\epsilon)) - \nabla_y\hat{\Phi}(x_\epsilon, \hat{y}_*(x_\epsilon))\| \leq \alpha\hat{\mu}_y\mathcal{D}_y \end{aligned} \tag{139}$$

where the first equality is by eq. (138); the second inequality is by $\|\mathbf{prox}_{\alpha g}(y_1) - \mathbf{prox}_{\alpha g}(y_2)\| \leq \|y_1 - y_2\|$ for all $y_1, y_2 \in \mathcal{Y}$ and eq. (137). Moreover, using Assumption 2 and the above inequalities, we have

$$\begin{aligned} &\|\nabla_x\mathcal{L}(x_\epsilon, y^+)\| \leq \|\nabla_x\mathcal{L}(x_\epsilon, y^+) - \nabla\hat{\phi}(x_\epsilon)\| + \|\nabla\hat{\phi}(x_\epsilon)\| \\ \leq&\|\nabla_x\mathcal{L}(x_\epsilon, y^+) - \nabla_x\mathcal{L}(x_\epsilon, \hat{y}_*(x_\epsilon))\| + \frac{\epsilon}{2\sqrt{6}} \leq L_{xy}\alpha\hat{\mu}_y D_y + \frac{\epsilon}{2\sqrt{6}}, \end{aligned}$$

where the second inequality follows from Danskin's theorem and the fact that $\|\nabla\hat{\phi}(x_\epsilon)\| \leq \epsilon/(2\sqrt{6})$; finally, the last inequality use Assumption 2 and (139). Thus, using $(a + b)^2 \leq 2(a^2 + b^2)$ for any $a, b \in \mathbb{R}$, we get

$$\|\nabla_x\mathcal{L}(x_\epsilon, y^+)\|^2 \leq \frac{\epsilon^2}{12} + 2L_{xy}^2\alpha^2\hat{\mu}_y^2 D_y^2. \tag{140}$$

Later in the proof, eq. (139) and eq. (140) will be useful when we further analyze y^+.

Recall that our ultimate goal is to show that $\|\nabla\phi_\lambda(x_\epsilon)\| \leq \epsilon$. Now, for some arbitrary $\mu_x > 0$, consider $\mathbf{prox}_{\lambda\phi}(x_\epsilon) = \mathrm{argmin}_{v\in\mathcal{X}}\, \phi(v) + \frac{1}{2\lambda}\|v - x_\epsilon\|^2$, where $\lambda = (\mu_x + \gamma)^{-1}$. It follows from Lemma 1 that

$$\|\nabla\phi_\lambda(x_\epsilon)\|^2 = \frac{1}{\lambda^2}\|x_\epsilon - \mathbf{prox}_{\lambda\phi}(x_\epsilon)\|^2.$$

Since $\lambda = (\mu_x + \gamma)^{-1}$, $\phi(\cdot) + \frac{1}{2\lambda}\|\cdot - x_\epsilon\|^2$ is μ_x-strongly convex with the unique minimizer $\mathbf{prox}_{\lambda\phi}(x_\epsilon)$; therefore,

$$\begin{aligned}
&\max_{y\in\mathcal{Y}} \mathcal{L}(x_\epsilon, y) - \max_{y\in\mathcal{Y}} \mathcal{L}(\mathbf{prox}_{\lambda\phi}(x_\epsilon), y) - \frac{1}{2\lambda}\|\mathbf{prox}_{\lambda\phi}(x_\epsilon) - x_\epsilon\|^2 \\
&= \phi(x_\epsilon) - \phi(\mathbf{prox}_{\lambda\phi}(x_\epsilon)) - \frac{1}{2\lambda}\|\mathbf{prox}_{\lambda\phi}(x_\epsilon) - x_\epsilon\|^2 \\
&\geq \frac{\mu_x}{2}\|x_\epsilon - \mathbf{prox}_{\lambda\phi}(x_\epsilon)\|^2 = \lambda^2\frac{\mu_x}{2}\|\nabla\phi_\lambda(x_\epsilon)\|^2.
\end{aligned} \tag{141}$$

In the following analysis, we will continue to polish the upper bound on $\|\nabla\phi_\lambda(x_\epsilon)\|^2$ on the left hand side of eq. (141). Indeed,

$$\begin{aligned}
&\max_{y\in\mathcal{Y}} \mathcal{L}(x_\epsilon, y) - \max_{y\in\mathcal{Y}} \mathcal{L}(\mathbf{prox}_{\lambda\phi}(x_\epsilon), y) - \frac{1}{2\lambda}\|\mathbf{prox}_{\lambda\phi}(x_\epsilon) - x_\epsilon\|^2 \\
&= \max_{y\in\mathcal{Y}} \mathcal{L}(x_\epsilon, y) - \mathcal{L}(x_\epsilon, y^+) + \mathcal{L}(x_\epsilon, y^+) - \max_{y\in\mathcal{Y}} \mathcal{L}(\mathbf{prox}_{\lambda\phi}(x_\epsilon), y) - \frac{1}{2\lambda}\|\mathbf{prox}_{\lambda\phi}(x_\epsilon) - x_\epsilon\|^2 \\
&\leq \max_{y\in\mathcal{Y}} \mathcal{L}(x_\epsilon, y) - \mathcal{L}(x_\epsilon, y^+) + \mathcal{L}(x_\epsilon, y^+) - \mathcal{L}(\mathbf{prox}_{\lambda\phi}(x_\epsilon), y^+) - \frac{1}{2\lambda}\|\mathbf{prox}_{\lambda\phi}(x_\epsilon) - x_\epsilon\|^2 \\
&\leq \max_{y\in\mathcal{Y}} \mathcal{L}(x_\epsilon, y) - \mathcal{L}(x_\epsilon, y^+) + \|x_\epsilon - \mathbf{prox}_{\lambda\phi}(x_\epsilon)\|\|\nabla_x\mathcal{L}(x_\epsilon, y^+)\| - \frac{\mu_x}{2}\|x_\epsilon - \mathbf{prox}_{\lambda\phi}(x_\epsilon)\|^2 \\
&\leq \max_{y\in\mathcal{Y}} \mathcal{L}(x_\epsilon, y) - \mathcal{L}(x_\epsilon, y^+) + \frac{\|\nabla_x\mathcal{L}(x_\epsilon, y^+)\|^2}{2\mu_x},
\end{aligned} \tag{142}$$

where the second inequality follows from the μ_x-strongly convexity of $\mathcal{L}(\cdot, y^+) + \frac{1}{2\lambda}\|\cdot - x_\epsilon\|^2$ and Cauchy-Schwarz inequality. Next, we continue to derive an appropriate upper bound on $\max_{y\in\mathcal{Y}} \mathcal{L}(x_\epsilon, y) - \mathcal{L}(x_\epsilon, y^+)$. Recall that $y^+ = \mathbf{prox}_{\alpha g}\big(\hat{y}_*(x_\epsilon) + \alpha\nabla_y\Phi(x_\epsilon, \hat{y}_*(x_\epsilon))\big)$; hence, the first-order optimality condition yields that

$$-\frac{1}{\alpha}\left(y^+ - \hat{y}_*(x_\epsilon) - \alpha\nabla_y\Phi(x_\epsilon, \hat{y}_*(x_\epsilon))\right) \in \partial g(y^+).$$

Therefore, for any $y \in \mathcal{Y}$, we have that

$$g(y) - g(y^+) \geq \langle y - y^+, -\frac{1}{\alpha}\left(y^+ - \hat{y}_*(x_\epsilon) - \alpha\nabla_y\Phi(x_\epsilon, \hat{y}_*(x_\epsilon))\right)\rangle,$$

which is equivalent to

$$g(y^+) - g(y) \leq \frac{1}{\alpha}\langle y - y^+, y^+ - \hat{y}_*(x_\epsilon)\rangle - \langle\nabla_y\Phi(x_\epsilon, \hat{y}_*(x_\epsilon)),\ y - y^+\rangle. \tag{143}$$

Now, we ready to provide a useful upper bound on $\max_{y\in\mathcal{Y}} \mathcal{L}(x_\epsilon, y) - \mathcal{L}(x_\epsilon, y^+)$. Indeed, given any $\tilde{y} \in \mathrm{argmax}_{y\in\mathcal{Y}} \mathcal{L}(x_\epsilon, y)$, we have

$$\begin{aligned}
&\max_{y\in\mathcal{Y}} \mathcal{L}(x_\epsilon, y) - \mathcal{L}(x_\epsilon, y^+) = \mathcal{L}(x_\epsilon, \tilde{y}) - \mathcal{L}(x_\epsilon, \hat{y}_*(x_\epsilon)) + \mathcal{L}(x_\epsilon, \hat{y}_*(x_\epsilon)) - \mathcal{L}(x_\epsilon, y^+) \\
&= \underbrace{\Phi(x_\epsilon, \tilde{y}) - \Phi(x_\epsilon, \hat{y}_*(x_\epsilon))}_{\textbf{part 1}} - g(\tilde{y}) + g(\hat{y}_*(x_\epsilon)) + \underbrace{\Phi(x_\epsilon, \hat{y}_*(x_\epsilon)) - \Phi(x_\epsilon, y^+)}_{\textbf{part 2}} - g(\hat{y}_*(x_\epsilon)) + g(y^+) \\
&\leq \langle\nabla_y\Phi(x_\epsilon, \hat{y}_*(x_\epsilon)), \tilde{y} - \hat{y}_*(x_\epsilon)\rangle - g(\tilde{y}) + g(y^+) \\
&\quad + \langle\nabla_y\Phi(x_\epsilon, \hat{y}_*(x_\epsilon)), \hat{y}_*(x_\epsilon) - y^+\rangle + \frac{L_{yy}}{2}\|\hat{y}_*(x_\epsilon) - y^+\|^2 \\
&= \langle\nabla_y\Phi(x_\epsilon, \hat{y}_*(x_\epsilon)), \tilde{y} - y^+\rangle - g(\tilde{y}) + g(y^+) + \frac{L_{yy}}{2}\|\hat{y}_*(x_\epsilon) - y^+\|^2 \\
&\leq \frac{1}{\alpha}\langle\tilde{y} - y^+, y^+ - \hat{y}_*(x_\epsilon)\rangle + \frac{L_{yy}}{2}\|\hat{y}_*(x_\epsilon) - y^+\|^2 \\
&= -\frac{L_{yy}}{2}\|\hat{y}_*(x_\epsilon) - y^+\|^2 + L_{yy}\langle\tilde{y} - \hat{y}_*(x_\epsilon), y^+ - \hat{y}_*(x_\epsilon)\rangle \\
&\leq L_{yy}\mathcal{D}_{\mathcal{Y}}\|y^+ - \hat{y}_*(x_\epsilon)\|,
\end{aligned} \tag{144}$$

where in the first inequality, we use concavity and smoothness of $\Phi(x_\epsilon, \cdot)$ for **part 1** and **part 2**, respectively; in the second inequality, we use eq. (143); in the last equality, we set $\alpha = L_{yy}^{-1}$; and in the last inequality, we use Cauchy-Schwarz inequality and the fact that $\sup_{y_1,y_2\in\mathbf{dom}\, g}\|y_1 - y_2\| \leq \mathcal{D}_\mathcal{Y}$. Next, if we use eq. (144) within eq. (142), it follows that

$$\begin{aligned}
&\max_{y\in\mathcal{Y}}\mathcal{L}(x_\epsilon, y) - \max_{y\in\mathcal{Y}}\mathcal{L}(\mathbf{prox}_{\lambda\phi}(x_\epsilon), y) - \frac{1}{2\lambda}\|\mathbf{prox}_{\lambda\phi}(x_\epsilon) - x_\epsilon\|^2 \\
&\leq L_{yy}\mathcal{D}_\mathcal{Y}\|y^+ - \hat{y}_*(x_\epsilon)\| + \frac{\|\nabla_x\mathcal{L}(x_\epsilon, y^+)\|^2}{2\mu_x} \\
&\leq \hat{\mu}_y\mathcal{D}_y^2 + \frac{\epsilon^2}{24\mu_x} + \frac{L_{xy}^2}{L_{yy}^2}\cdot\frac{\hat{\mu}_y^2}{\mu_x}\cdot D_y^2,
\end{aligned} \tag{145}$$

where the last inequality follows from eq. (139) and eq. (140) with $\alpha = L_{yy}^{-1}$.

Finally, if we use eq. (145) within eq. (141) and substitute $\lambda = (\gamma + \mu_x)^{-1}$, it follows that

$$\frac{\mu_x}{2(\gamma+\mu_x)^2}\|\nabla\phi_\lambda(x_\epsilon)\|^2 \leq \hat{\mu}_y\mathcal{D}_y^2 + \frac{\epsilon^2}{24\mu_x} + \frac{L_{xy}^2}{L_{yy}^2}\cdot\frac{\hat{\mu}_y^2}{\mu_x}\cdot D_y^2. \tag{146}$$

Thus, choosing the free parameter $\mu_x = \gamma$ implies that

$$\|\nabla\phi_\lambda(x_\epsilon)\|^2 \leq 8\gamma\hat{\mu}_y\mathcal{D}_y^2 + \frac{\epsilon^2}{3} + 8\frac{L_{xy}^2}{L_{yy}^2}\cdot\hat{\mu}_y^2\cdot D_y^2. \tag{147}$$

Thus, we get $\|\nabla\phi_\lambda(x_\epsilon)\| \leq \epsilon$ for $\hat{\mu}_y = \min\left\{\frac{\epsilon^2}{24\gamma\mathcal{D}_y^2},\ \frac{L_{yy}}{L_{xy}}\cdot\frac{\epsilon}{2\sqrt{6}\mathcal{D}_y}\right\}$. □

G.1 Proof of Theorem 5

Proof. To get a worst-case complexity, as in the previous sections, let

$$L \triangleq \max\{L_{xy}, L_{yx}, L_{xx}, L_{yy}\},\ \delta \triangleq \max\{\delta_x, \delta_y\},\ \gamma = L.$$

Assumption 2 implies that $\nabla_y\hat{\Phi}$ and $\nabla_x\hat{\Phi}$ are Lipschitz such that for all $x, x' \in \mathcal{X}$ and $y, y' \in \mathbf{dom}\, g$,

$$\begin{aligned}
\|\nabla_y\hat{\Phi}(x,y) - \nabla_y\hat{\Phi}(x',y')\| &\leq L_{yx}\|x - x'\| + \hat{L}_{yy}\|y - y'\|, \\
\|\nabla_x\hat{\Phi}(x,y) - \nabla_x\hat{\Phi}(x',y')\| &\leq L_{xx}\|x - x'\| + L_{xy}\|y - y'\|,
\end{aligned}$$

where $\hat{L}_{yy} = L_{yy} + \hat{\mu}_y$. Therefore, the proof immediately follows from Lemma 27 and Theorem 1, considering `SAPD+` with `VR-flag` = **false** is applied on (13) with $\hat{\mu}_y = \min\left\{\frac{\epsilon^2}{24\gamma\mathcal{D}_y^2},\ \frac{L_{yy}}{L_{xy}}\cdot\frac{\epsilon}{2\sqrt{6}\mathcal{D}_y}\right\}$. □

H Details of fair classification example

In the experiment of fair classification, $\{(\mathbf{a}_i, b_i)\}_{i=1}^n$ denotes the (data,label) pairs of the labeled image data set. $a_i \in \mathbb{R}^{d_1\times d_2\times c}$, and b_i is a label associated with one of the K-classes, i.e., $b_i \in \mathcal{C} \triangleq \{\ C_j\}_{j=1}^K$ with $K \leq n$. We employ the classifier

$$h(\cdot\ ; \mathbf{x}) : \mathbf{a}_i \in \mathbb{R}^{d_1\times d_2\times c} \to \mathbf{p}_i \in \mathbb{R}^K,$$

where $\mathbf{p}_i = (p_{ij})_{j=1}^K$ s.t. $\sum_{j=1}^K p_{ij} = 1$ and $p_{ij} \geq 0$ for $j = 1, 2, .., K$, and $\mathbf{x}$ is the parameters of the classifier. Specifically, $h(\cdot; \mathbf{x})$ is a CNN with the structure as follows:

$$[input] \to [conv - elu - maxpool] \times 3 \to [fc - elu] \times 2 \to [softmax]$$

where *exponential linear unit* (elu) [8] is the smoothed variant of *rectified linear units* (relu) activation function. Furthermore, given the input $\{(\mathbf{a}_i, b_i)\}_{i=1}^n$ and the output $\{\mathbf{p}_i\}_{i=1}^n$, the loss functions $\{l_j\}_{j=1}^K$ used in eq. (15) are

$$\ell_j(\{(\mathbf{a}_i, b_i)\}_{i=1}^n; \mathbf{x}) = -\frac{1}{N_j}\sum_{i=1}^n \log(p_{ij})\mathbf{1}_{C_j}(b_i)$$

where N_j is the number of data with label C_j, i.e., $N_j = \sum_{i=1}^n \mathbf{1}_{C_j}(b_i)$ and

$$\mathbf{1}_{C_j}(b_i) = \begin{cases} 1 & \text{if } b_i = C_j \\ 0 & \text{o.w.} \end{cases}$$

and p_{ij} is the j-th element of $\mathbf{p}_i$, and $\mathbf{p}_i = h(\mathbf{a}_i; \mathbf{x})$.

I Additional analyses on the related work

In some of the existing work on WCSC problems, particularly [17, 16, 26, 37], except for $\kappa_y = L/\mu_y$, the individual effects of L or μ_y are not explicitly stated in the final complexity bounds. To better compare existing bounds with ours, it is necessary to state the complexity bound dependence on L and μ_y. For example, Huang *et al.* [17, 16] assume that $\frac{1}{\mu_y} \leq L$, that is equivalent to $L \geq \sqrt{\kappa_y}$; however, a constant factor depending on L was ignored in their oracle complexity result. Moreover, Huang *et al.*[17] employ a different convergence metric and claim that they obtain a competitive result. It turns out that their convergence metric is scaled by an algorithmic constant and when their results are converted into GNP metric, i.e., $\|\nabla\phi(\cdot)\|$, this constant adversely affects their complexity bounds. A similar issue with the claimed complexity bounds also exists in [16], where the complexity bound are computed after the objective function is rescaled. In this section, to provide a fair comparison,

- we give an explicit oracle complexity bound for the related works in [17, 16, 26, 37];
- we discuss those parts in their analysis that are not convincing, and try our best to provide the corrected and optimized complexity bounds based on their analysis.

Without loss of generality, for the sake of easier comparison, we consider the *smooth minimax* problems, i.e., $\min_{x\in\mathcal{X}} \max_{y\in\mathcal{Y}} \mathcal{L}(x, y) = \Phi(x, y)$. We first fix the notation to unify the discussion for the WCSC setting, i.e., $\mathcal{L}(x, y)$ is weakly convex in x and strongly concave in y.

Recall that $\phi(x) \triangleq \max_{y\in\mathcal{Y}} \mathcal{L}(x, y)$; thus, $\phi(\cdot)$ is differentiable and we use $\|\nabla\phi(\cdot)\|$ as the convergence metric. In addition, we let $\phi_* \triangleq \inf_{x\in\mathcal{X}} \phi(x)$ and recall that $y_*(\cdot) = \operatorname{argmax}_{y\in\mathcal{Y}} \mathcal{L}(\cdot, y)$. Moreover, for simplicity of the notation, we consider the worst-case complexity bounds using L, i.e.,

$$L = \max\{L_{xy}, L_{yx}, L_{xx}, L_{yy}\}, \quad \kappa_y = \frac{L}{\mu_y}, \quad \delta = \max\{\delta_x, \delta_y\}, \quad \gamma = L. \tag{148}$$

I.1 Revisit of [17, Theorem 1]

In this section, we provide the oracle complexity of Huang *et al.*[17, Theorem 1] using the metric $\|\nabla\phi(\cdot)\|$ for the Stochastic Mirror Descent Algorithm (`SMDA`), stated in [17, algorithm 1]. Let τ, σ be the primal and dual stepsizes, respectively, η be the momentum parameter, b be the large batchsize, and u be convexity modulus of the Bregman distance generating function. We also list our notational convention in table 2 for reader's convenience.

Below, we restate the convergence result of `SMDA` for the class of Bregman distance functions such that $D_t(x, x') \triangleq (x - x')^\top H_t(x - x')/2$ for some $H_t \succ 0$ –this class of Bregman functions are used for all the numerical experiments reported in [17].

Theorem 9. *[17, Thoerem 1] Suppose Assumptions 1, 2, 3 hold with $f(\cdot) = g(\cdot) = 0$. Let $\{x_t, y_t\}_{t=1}^T$ be generated by `SMDA`, stated in [17, Algorithm 1], employing a stochastic first-order oracle to sample stochastic partial derivatives. For parameters chosen as $\eta \in (0, 1]$, $\tau \in (0, \min\{\frac{3u}{4L(1+\kappa_y)}, \frac{9\eta u\mu_y\sigma}{800\kappa_y^2}, \frac{2\eta\mu_y u\sigma}{25L^2}\}]$ and $\sigma \in (0, \frac{1}{6L}]$, let $\eta_t = \eta$, $\tau_t = \tau$ and $\sigma_t = \sigma$ for $t \geq 0$. Then, for any given initial point (x_0, y_0), it holds that*

$$\frac{1}{T}\sum_{t=1}^T \mathbb{E}[\|\mathbf{G}^t\|] \leq \frac{4\sqrt{2(\phi(x_0) - \phi_*)}}{\sqrt{3T\tau u}} + \frac{4\sqrt{2}\Delta_0}{\sqrt{3T\tau u}} + \frac{10\delta}{\sqrt{3bu}} + \frac{20\delta\sqrt{\eta\sigma}}{3\sqrt{\tau u\mu_y b}}, \tag{149}$$

where $\phi(x) = \max_y \mathcal{L}(x, y)$, $\phi_* = \inf_{x\in\mathcal{X}} \phi(x)$, $\Delta_0 = \|y_0 - y_*(x_0)\|$, $y_*(x_0) = \operatorname{argmax}_{y\in\mathcal{Y}} \mathcal{L}(x_0, y)$, $\mathbf{G}^t = H_t^{-1}\nabla\phi(x_t)$, and H_t is a diagonal matrix such that $H_t \succeq u\mathbf{I}$ for all $t \geq 1$ and $u > 0$.

Notation in [17]	Notation in our paper	Meaning
γ	τ	primal stepsize
λ	σ	dual stepsize
L_f	L	Lipschitz constant as in (148)
μ	μ_y	concavity modulus of $\mathcal{L}(x,\cdot)$
κ	κ_y	condition number
σ	δ	variance bound for the SFO
b_1	b'	small batch size for VR methods
ρ	u	convexity modulus of Bregman distance generating function

Table 2: Important notation for [17] and this paper.
Table notes. (1) SFO: stochastic first-order oracle. (2) u is only used in the analysis provided in this section.

Remark 11. *When $f(\cdot) = g(\cdot) = 0$, it follows from the update rules and the definition of Bregman distance function in [17, eq.(12-13), eq.(22-23)] that*

$$\mathbf{G}^t = H_t^{-1}\nabla\phi(x_t),$$

where H_t is a diagonal matrix such that $H_t \succeq u\mathbf{I}$. Note that

$$\mathbf{G}^t = \nabla\phi(x_t) \iff H_t = \mathbf{I}.$$

We noticed that the authors chose the value of u to improve their bounds; but, without addressing its effect on $\mathbf{G}^t$. More precisely, they still use $\|\mathbf{G}^t\|$ as the convergence metric and compare their complexity results with those papers using $\|\nabla\phi(x_t)\|$ as the convergence metric.

In the following corollary, we will provide the optimal complexity for `SMDA` based the result in eq. (149), i.e., [17, Thoerem 1].

Corollary 1. *Suppose Assumptions 1, 2, 3 hold with $f(\cdot) = g(\cdot) = 0$, and $\frac{1}{\mu_y} \le L$ hold*[8]*. Consider the setting of Theorem 9, then `SMDA` [17, Algorithm 1] can generate x_ϵ such that $\mathbb{E}\left[\|\nabla\phi(x_\epsilon)\|\right] \le \epsilon$ by requiring at most $\mathcal{O}(\frac{\kappa_y^5\delta^2}{\mu_y^2\epsilon^4})$ stochastic first-order oracle calls.*

Proof. Recall that $H_t \succeq u\mathbf{I}$, $\mathbf{G}^t = H_t^{-1}\nabla\phi(x_t)$ and H_t is a diagonal matrix; therefore, we can obtain a tight upper bound on $\mathbb{E}[\|\nabla\phi(x_t)\|]$ using eq. (149) as follows:

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\|\nabla\phi(x_t)\|] \le \frac{4\sqrt{2(\phi(x_0)-\phi_*)}}{\sqrt{3T}}\sqrt{\frac{u}{\tau}} + \frac{4\sqrt{2}\Delta_0}{\sqrt{3T}}\sqrt{\frac{u}{\tau}} + \frac{10\delta}{\sqrt{3b}} + \frac{20\delta\sqrt{\eta\sigma}}{3\sqrt{\mu_y b}}\sqrt{\frac{u}{\tau}}. \quad (150)$$

If we use their parameter choices, i.e., $\eta \in (0, 1]$,

$$\sigma = \mathcal{O}\Big(\frac{1}{L}\Big), \quad \tau = u\ \min\Big\{\frac{3}{4L(1+\kappa_y)}, \frac{9\eta\mu_y\sigma}{800\kappa_y^2}, \frac{2\eta\mu_y\sigma}{25L^2}\Big\}, \quad u = \mathcal{O}(L^\nu), \quad (151)$$

for some free parameter $\nu \ge 0$, then we get

$$\frac{u}{\tau} = \max\Big\{\frac{4L(1+\kappa_y)}{3}, \frac{800\kappa_y^2}{9\eta\mu_y\sigma}, \frac{25L^2}{2\eta\mu_y\sigma}\Big\} = \Omega(\kappa_y^3), \quad (152)$$

where the second term leads to κ_y^3. It is essential to note that τ choice in (151) implies that u/τ ratio is independent of u; hence, the parameter u indeed does not affect the bound on the right-hand-side of eq. (150). Therefore, contrary to what is suggested in [17], choosing different values for u through picking different $\nu \ge 0$ values indeed is not useful for proving tighter bounds in GNP metric $\|\nabla\phi(x_k)\|$ in this simple scenario using their parameter choices.

[8]The assumption $\frac{1}{\mu_y} \le L$ is also made in [17].

Note eq. (150) can be simplified as

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\|\nabla\phi(x_t)\|] \le \mathcal{O}\Big(\sqrt{\frac{\kappa_y^3(\phi(x_0)-\phi_*)}{T}} + \frac{\delta}{\sqrt{b}} + \frac{\delta\kappa_y^2}{\sqrt{b}L}\Big).$$

Thus, for any $\epsilon > 0$, to find point x_t such that $\mathbb{E}[\|\nabla\phi(x_t)\|] \le \epsilon$, one should choose $t \ge T$ for

$$T = \mathcal{O}\Big(\frac{\kappa_y^3}{\epsilon^2}(\phi(x_0)-\phi_*)\Big), \quad b = \mathcal{O}\Big(\frac{\kappa_y^4\delta^2}{L^2\epsilon^2}\Big),$$

which leads to the oracles complexity of

$$2bT = \mathcal{O}\Big(\frac{\kappa_y^7\delta^2}{L^2\epsilon^4}\Big) = \mathcal{O}\Big(\frac{\kappa_y^5\delta^2}{\mu_y^2\epsilon^4}\Big).$$

□

I.2 Revisit of [17, Theorem 3]

In this section, we provide the oracle complexity of Huang *et al.* [17, Theorem 3] using the metric $\|\nabla\phi(\cdot)\|$ for the Stochastic Mirror Descent Algorithm with variance reduction (`SMDA-VR`), stated in [17, algorithm 2]. Let τ, σ be the primal and dual stepsizes, respectively, η be the momentum parameter, b be the large batchsize, b' be the small batchsize, q be the period for sampling large batch size (i.e., once every q batches is large), and u be the strongly-convex constant of the Bregman distance generating function. We also list our notational convention in table 2 for reader's convenience.

Below, as we did in the previous section for `SMDA`, we restate the convergence result of `SMDA-VR` for the class of Bregman distance functions such that $D_t(x,x') = (x-x')^\top H_t(x-x')/2$ for some $H_t \succ 0$ –this class of Bregman functions are used for all the numerical experiments reported in [17].

Theorem 10. *[17, Thoerem 3] Suppose Assumptions 1, 2, 3 hold with $f(\cdot) = g(\cdot) = 0$. Let $\{x_t, y_t\}_{t=1}^T$ be generated by `SMDA-VR`, stated in [17, Algorithm 2], employing a stochastic first-order oracle to sample stochastic partial derivatives. For parameters chosen as $\eta \in (0,1]$, $\tau = (0,\ \min\Big\{\frac{3u}{4L(1+\kappa_y)}, \frac{\eta\mu_y\sigma u}{38L^2}, \frac{3u}{19L^2\eta}, \frac{u\eta}{8}, \frac{9u\eta\mu_y\sigma}{400\kappa_y^2},\Big\}\Big]$ and $\sigma \in (0,\ \min\Big\{\frac{1}{6L}, \frac{9\mu_y}{100\eta^2L^2}\Big\}\Big]$, let $\eta_t = \eta$, $\tau_t = \tau$ and $\sigma_t = \sigma$ for $t \ge 0$ and $b' = q$. Then, for any given initial point (x_0, y_0), we have*

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\|\mathbf{G}^t\|] \le \frac{4\sqrt{2(\phi(x_0)-\phi_*)}}{\sqrt{3T\tau u}} + \frac{4\sqrt{2}\Delta_0}{\sqrt{3T\tau u}} + \frac{2\sqrt{2}\delta}{\sqrt{\tau u\eta b}L}. \tag{153}$$

where $\phi(x) = \max_y \mathcal{L}(x,y)$, $\phi_* = \inf_{x\in\mathcal{X}}\phi(x)$, $\Delta_0 = \|y_0 - y_*(x_0)\|$, $y_*(x_0) = \operatorname{argmax}_{y\in\mathcal{Y}}\mathcal{L}(x_0,y)$, $\mathbf{G}^t = H_t^{-1}\nabla\phi(x_t)$, and H_t is a diagonal matrix such that $H_t \succeq u\mathbf{I}$ for some $u > 0$.

In the following corollary, we will provide the optimal complexity for `SMDA-VR` based the result in eq. (153), i.e., [17, Thoerem 3].

Corollary 2. *Suppose Assumptions 1, 2, 3 hold with $f(\cdot) = g(\cdot) = 0$, and $\frac{1}{\mu_y} \le L$ hold[9]. Consider the setting of Theorem 10, then `SMDA-VR` [17, Algorithm 2] can generate x_ϵ such that $\mathbb{E}[\|\nabla\phi(x_\epsilon)\|] \le \epsilon$ by requiring at most $\mathcal{O}(\frac{\kappa_y^5\delta^2}{\mu_y\epsilon^3})$ stochastic first-order oracle calls.*

Proof. Recall that $H_t \succeq u\mathbf{I}$, $\mathbf{G}^t = H_t^{-1}\nabla\phi(x_t)$ and H_t is a diagonal matrix; therefore, we can obtain a tight upper bound on $\mathbb{E}[\|\nabla\phi(x_t)\|]$ using eq. (153) as follows:

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\|\nabla\phi(x_t)\|] \le \frac{4\sqrt{2(\phi(x_0)-\phi_*)}}{\sqrt{3T}}\sqrt{\frac{u}{\tau}} + \frac{4\sqrt{2}\Delta_0}{\sqrt{3T}}\sqrt{\frac{u}{\tau}} + \frac{2\sqrt{2}\delta}{\sqrt{\eta b}L}\sqrt{\frac{u}{\tau}}. \tag{154}$$

If we use their parameter choices, i.e., $\eta \in (0,1]$,

$$\sigma = \mathcal{O}\Big(\frac{1}{\kappa_y L}\Big),\ \tau = u\ \min\Big\{\frac{3}{4L(1+\kappa_y)}, \frac{\eta\mu_y\sigma}{38L^2}, \frac{3}{19L^2\eta}, \frac{\eta}{8}, \frac{9\eta\mu_y\sigma}{400\kappa_y^2}\Big\},\ u = \mathcal{O}(L^{1+\nu}), \tag{155}$$

[9]The assumption $\frac{1}{\mu_y} \le L$ is also made in [17].

for some free design parameter $\nu \geq 0$, then we get

$$\frac{u}{\tau} = \max\Big\{\frac{4L(1+\kappa_y)}{3}, \frac{38L^2}{\eta\mu_y\sigma}, \frac{19L^2\eta}{3}, \frac{8}{\eta}, \frac{400\kappa_y^2}{9\eta\mu_y\sigma}\Big\} = \Omega(\kappa_y^4),$$

where the last term leads to κ_y^4. It is essential to note that τ choice in (155) implies that u/τ ratio is independent of u; hence, the parameter u indeed does not affect the bound on the right-hand-side of eq. (154). Therefore, contrary to what is suggested in [17], for the simple scenarios considered here choosing different values for u through picking different $\nu \geq 0$ values indeed is not useful for proving tighter bounds in GNP metric $\|\nabla\phi(x_k)\|$ with their parameter choices.

Note that eq. (154) can be simplified as

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\|\nabla\phi(x_t)\|] \leq \mathcal{O}\Big(\sqrt{\frac{\kappa_y^4(\phi(x_0)-\phi_*)}{T}} + \delta\frac{\kappa_y^2}{\sqrt{b}L}\Big).$$

Thus, for any $\epsilon > 0$, to find point x_t such that $\mathbb{E}[\|\nabla\phi(x_t)\|] \leq \epsilon$, one should choose $t \geq T$ for

$$T = \mathcal{O}\Big(\frac{\kappa_y^4(\phi(x_0)-\phi_*)}{\epsilon^2}\Big), \quad b = \mathcal{O}\Big(\frac{\kappa_y^4\delta^2}{L^2\epsilon^2}\Big),$$

which leads to the oracle complexity of

$$4b'T + 2bT/q = \mathcal{O}\Big(\frac{b'\kappa_y^4}{\epsilon^2} + \frac{\kappa_y^8\delta^2}{L^2\epsilon^4}/q\Big).$$

Since their parameter choice requires $b' = q$, to optimize the above bound, we let $b' = q = \mathcal{O}\left(\frac{\kappa_y^2}{L\epsilon}\right)$, which leads to

$$\mathcal{O}\left(\frac{\kappa_y^6\delta^2}{L\epsilon^3}\right) = \mathcal{O}\left(\frac{\kappa_y^5\delta^2}{\mu_y\epsilon^3}\right).$$

□

I.3 Revisit of [26, Theorem 1]

Recall that $\phi(x) = \max_y \mathcal{L}(x,y)$ and $\phi_* = \inf_{x\in\mathcal{X}}\phi(x)$. In this paper, the total oracle complexity to find point x_ϵ such that $\mathbb{E}[\|\nabla\phi(x_\epsilon)\|] \leq \epsilon$ is given by

$$\mathcal{O}(\kappa_y^2\epsilon^{-2}\log(\kappa_y/\epsilon)) + \mathcal{O}(T/q\cdot b) + \mathcal{O}(T\cdot b'\cdot m) \tag{156}$$

where[10]

$$T = \Big\lceil\frac{100\kappa_y L\Delta_f}{9\epsilon^2}\Big\rceil,\ q = \lceil\epsilon^{-1}\rceil,\ b = \lceil\frac{2250}{19}\delta^2\kappa_y^2\epsilon^{-2}\rceil,\ b' = \Big\lceil\frac{3687}{76}\kappa_y q\Big\rceil,\ m = \lceil 1024\kappa_y\rceil. \tag{157}$$

Given an arbitrary initial point x_0, let y_0 be obtained by inexactly solving $\max_y \mathcal{L}(x_0,y)$, and they define $\Delta_f = \mathcal{L}(x_0,y_0) - \frac{134\epsilon^2}{\kappa_y L} - \phi_*$. In (157), the other parameters are defined as follows: b is the large batchsize, b' is the small batchsize, q is the period such that once every q outer iterations, `SREDA` calls for a large batchsize, T is the number of the outer iterations and m is the number of the inner iterations –each outer iteration requires m inner iterations and each inner iteration calls for a small batchsize. Then eq. (156) becomes $\mathcal{O}(\frac{L\kappa_y^3}{\epsilon^3})$.

I.4 Revisit of [37, Theorem 1]

Recall that $\phi(x) = \max_y \mathcal{L}(x,y)$ and $\phi_* = \inf_{x\in\mathcal{X}}\phi(x)$. In this paper, the precise parameter selection for [37, Theorem 1] is provided in [37, Theorem 3] of the supplementary material. Using these parameter choice implies that the total oracle complexity to find point x_ϵ such that $\mathbb{E}[\|\nabla\phi(x_\epsilon)\|] \leq \epsilon$ is given by

$$T\cdot b'\cdot m + \Big\lceil\frac{T}{q}\Big\rceil\cdot b + T_0, \tag{158}$$

[10] In [26], there is a typo in the choice of $b = \lceil\frac{2250}{19}\delta^2\kappa_y^{-2}\epsilon^2\rceil$. Here, we provide the correct one.

for an arbitrary initial point x_0, where the number of outer iterations, T, the number of the inner iterations per each outer iteration, m, are set as follows:

$$T = \max\left\{\frac{3345\kappa_y}{\epsilon^2},\ 6600(1+\kappa_y)L\frac{(\phi(x_0)-\phi_*)}{\epsilon^2}\right\}, \quad b = \frac{9366\delta^2\kappa_y^2}{\epsilon^2},$$
$$b' = \frac{\kappa_y}{\epsilon}, \quad m = 52\kappa_y - 1, \quad q = \frac{2}{13(1+\kappa_y)}\frac{\kappa_y}{\epsilon}, \quad T_0 = \mathcal{O}(\kappa_y \log(\kappa_y)).$$

Above b is the large batchsize, b' is the small batchsize, q is the period such that once every q outer iterations, a large batch size is sampled rather than a small batch size. Then eq. (158) becomes $\mathcal{O}(\frac{L\kappa_y^3}{\epsilon^3})$.

I.5 Revisit of [16, Theorem 12]

In this section, we provide the oracle complexity of [16, Theorem 12] using the metric $\|\nabla\phi(\cdot)\|$ for the Accelerated first-order Momentum Descent Ascent (ACC-MDA) algorithm, stated in [16, algorithm 3]. Let τ, σ be the primal and dual stepsizes, respectively, $\{\eta_t\}$ be the momentum parameter sequence, and b be the batchsize. We also list our notational convention in table 3 for reader's convenience.

Notations in [16]	Notations in our paper	Meaning
γ	τ	primal stepsize
λ	σ	dual stepsize
L_f	L	Lipschitz constant as in (148)
L_g	$L(1+\kappa_y)$	L-smooth constant of $\phi(x)$
τ	μ_y	concavity modulus of $\mathcal{L}(x,\cdot)$

Table 3: Important notations for [16] and this paper.

Below, we restate the convergence result of ACC-MDA reported in [16].

Theorem 11. *[16, Thoerem 12] Suppose Assumptions 1, 2, 3 hold with $f(\cdot) = g(\cdot) = 0$. Let $\{x_t, y_t\}_{t=1}^T$ be generated by ACC-MDA algorithm, stated in [16, Algorithm 3], when applied to the* smooth minimax *problem $\min_{x\in\mathcal{X}}\max_{y\in\mathcal{Y}}\mathcal{L}(x,y) = \Phi(x,y)$. For some given $p > 0$, let $\eta_t = \frac{p}{(\psi+t)^{1/3}}$ for all $t \geq 0$, $\tau \in (0,\ \min\{\frac{\sigma\mu_y}{2L}\sqrt{\frac{2b}{8\sigma^2+75\kappa_y^2 b}}, \frac{\psi^{1/3}}{2L(1+\kappa_y)p}\}]$ and $\sigma \in (0, \min\{\frac{1}{6L}, \frac{27b\mu_y}{16}\}]$ such that $\psi \geq \max\{2, p^3, (c_1p)^3, (c_2p)^3\}$ for some $c_1 \geq \frac{2}{3p^3} + \frac{9\mu_y^2}{4}$ and $c_2 \geq \frac{2}{3p^3} + \frac{75L^2}{2}$. Then for any given x_0, we have*

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\|\nabla\phi(x_t)\|] \leq \frac{\sqrt{2M''}\psi^{1/6}}{T^{1/2}} + \frac{\sqrt{2M''}}{T^{1/3}}, \tag{159}$$

where $\phi(x) = \max_y \mathcal{L}(x,y)$, $\Delta_0 = \|y_0 - y_*(x_0)\|^2$, $y_*(x_0) = \operatorname{argmax}_{y\in\mathcal{Y}}\mathcal{L}(x_0,y)$, $M'' = \frac{\phi(x_0)-\phi_*}{\tau p} + \frac{9L^2\Delta_0}{p\sigma\mu_y} + \frac{2\psi^{1/3}\delta^2}{b\mu_y^2p^2} + \frac{2(c_1^2+c_2^2)\delta^2p^2}{b\mu_y^2}\ln(\psi+T)$, and $\phi_* = \inf_{x\in\mathcal{X}}\phi(x)$.

Remark 12. *[16, Remarks 13 and 14] When $b = \mathcal{O}(\kappa_y^\nu)$ for $\nu > 0$ and $\kappa_y^\nu \leq \frac{8}{81L\mu_y}$, they claim that they can obtain the gradient complexity of $\tilde{\mathcal{O}}(\kappa_y^3\epsilon^{-3})$ if $\nu = 3$, and $\tilde{\mathcal{O}}(\kappa_y^{2.5}\epsilon^{-3})$ if $\nu = 4$. However, for the assumption $\kappa_y^\nu \leq \frac{8}{81L\mu_y}$ to hold in general, one needs to rescale the original objective function $\mathcal{L}(x,y)$ with some $s \in (0,1]$ to define*

$$\mathcal{L}_s(x,y) \triangleq s\cdot\mathcal{L}(x,y). \tag{160}$$

Then the Lipschitz constant of $\nabla\mathcal{L}_s$, strongly concavity modulus of $\mathcal{L}(x,\cdot)$ for any $x \in \mathcal{X}$ and the variance bound of the stochastic oracle for $\nabla_x\mathcal{L}_s$ and $\nabla_y\mathcal{L}_s$ can be written as sL, $s\mu_y$, and $s^2\delta^2$, respectively. We notice that the effect of scaling $\mathcal{L}$ on the problem parameters is not discussed in [16] and eq. (159) *is directly used to derive the convergence result assuming $\kappa_y^\nu \leq \frac{8}{81L\mu_y}$. As a consequence, the complexity results of $\tilde{\mathcal{O}}(\kappa_y^3\epsilon^{-3})$, $\tilde{\mathcal{O}}(\kappa_y^{2.5}\epsilon^{-3})$ do not hold without loss of generality unless the original function $\mathcal{L}$ satisfies the restrictive assumption of $\kappa_y^\nu \leq \frac{8}{81L\mu_y}$.*

In the following discussion, we analyze the effect of scaling $\mathcal{L}$ on the complexity bounds whenever $\kappa_y^\nu \leq \frac{8}{81 L \mu_y}$ is not satisfied for the original objective function $\mathcal{L}$, and we provide complexity bounds holding without loss of generality that are optimized by choosing $\nu > 0$ properly. Now, consider implementing ACC-MDA on an appropriately scaled problem $\min_x \max_y \mathcal{L}_s(x, y)$ where $\mathcal{L}_s$ is defined in (160). Let

$$L_s \triangleq sL, \quad \mu_s \triangleq s\mu_y, \quad \delta_s \triangleq s\delta. \tag{161}$$

Note that the condition numbers of $\mathcal{L}_s$ and $\mathcal{L}$ are the same, and are equal to κ_y, i.e., $\kappa_y = \frac{L}{\mu_y} = \frac{L_s}{\mu_s}$. In the upcoming discussion, suppose that $s \in (0, 1]$ is chosen such that $\kappa_y^\nu \leq \frac{8}{81 L_s \mu_s}$.

To facilitate the complexity analysis and make the upcoming discussion easier, first we restate Theorem 11 for the function $\mathcal{L}_s$, where we used the relation $\nabla \phi$ and the derivative of $\max_y \mathcal{L}_s(\cdot, y)$; indeed, the derivative of $\max_y \mathcal{L}_s(\cdot, y)$ is equal to $s\nabla\phi(\cdot)$, where $\phi(x) = \max_y \mathcal{L}(x, y)$.

Theorem 12. *[16, Thoerem 12] Suppose Assumptions 1, 2, 3 hold with $f(\cdot) = g(\cdot) = 0$. Let $\{x_t, y_t\}_{t=1}^T$ be generated by ACC-MDA algorithm, stated in [16, Algorithm 3], when applied to the* smooth minimax *problem $\min_{x\in\mathcal{X}} \max_{y\in\mathcal{Y}} \mathcal{L}_s(x, y) = s \cdot \mathcal{L}(x, y)$. For some given $p \geq 0$, let $\eta_t = \frac{p}{(\psi+t)^{1/3}}$ for all $t \geq 0$, $\tau \in (0, \min\{\frac{\sigma\mu_s}{2L_s}\sqrt{\frac{2b}{8\sigma^2+75\kappa_y^2 b}}, \frac{\psi^{1/3}}{2L_s(1+\kappa_y)p}\}]$ and $\sigma \in (0, \min\{\frac{1}{6L_s}, \frac{27b\mu_s}{16}\}]$ such that $\psi \geq \max\{2, p^3, (c_1'p)^3, (c_2'p)^3\}$ for some $c_1' \geq \frac{2}{3p^3} + \frac{9\mu_s^2}{4}$ and $c_2' \geq \frac{2}{3p^3} + \frac{75L_s^2}{2}$. Then for any given x_0, we have*

$$\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[\|\nabla\phi(x_t)\|] \leq \frac{1}{s}\left(\frac{\sqrt{2M_s''}\psi^{1/6}}{T^{1/2}} + \frac{\sqrt{2M_s''}}{T^{1/3}}\right), \tag{162}$$

where $\phi(x) = \max_y \mathcal{L}(x, y)$, $\Delta_0 = \|y_0 - y_*(x_0)\|^2$, $y_*(x_0) = \operatorname{argmax}_{y\in\mathcal{Y}} \mathcal{L}(x_0, y)$, $M_s'' = \frac{s(\phi(x_0)-\phi_*)}{\tau p} + \frac{9L_s^2\Delta_0}{p\sigma\mu_s} + \frac{2\psi^{1/3}\delta_s^2}{b\mu_s^2 p^2} + \frac{2({c_1'}^2+{c_2'}^2)\delta_s^2 p^2}{b\mu_s^2}\ln(\psi + T)$, and $\phi_* = \inf_{x\in\mathcal{X}} \phi(x)$.

Next, following the analysis in [16, Remarks 10 and 13], we provide a particular parameter choice for ACC-MDA so that it is applicable to the setting of Theorem 12.

Lemma 28. *Under the premise of Theorem 12. Suppose $\kappa_y^\nu \leq \frac{8}{81 L_s\mu_s}$, $b = \kappa_y^\nu$ for some $\nu > 0$, and*

$$\frac{\sigma\mu_s}{2L_s}\sqrt{\frac{2b}{8\sigma^2 + 75\kappa_y^2 b}} \leq \frac{\psi^{1/3}}{2L_s(1+\kappa_y)p}. \tag{163}$$

If $\sigma = \min\{\frac{1}{6L_s}, \frac{27b\mu_s}{16}\}$ and $\tau = \min\{\frac{\sigma\mu_s}{2L_s}\sqrt{\frac{2b}{8\sigma^2+75\kappa_y^2 b}}, \frac{\psi^{1/3}}{2L_s(1+\kappa_y)p}\}$, then $\psi = \Theta(\max\{1, L_s^6\})$ satisfies the condition in Theorem 12 and

$$\sigma = \Theta(b\mu_s), \quad \tau^{-1} = \Theta\Big(\frac{\kappa_y^3}{bL_s}\Big). \tag{164}$$

Remark 13. *The conditions $\kappa_y^\nu \leq \frac{8}{81L_s\mu_s}$ and eq.* (163)*, and the choice $b = \kappa_y^\nu$ are as suggested in [16].*

Proof. Since $\kappa_y^\nu \leq \frac{8}{81L_s\mu_s}$, we have $\sigma = \frac{27b\mu_s}{16} = \Theta(b\mu_s)$. Furthermore, eq. (163) implies that we can simplify τ as

$$\tau = \frac{\sigma\mu_s}{2L_s}\sqrt{\frac{2b}{8\sigma^2 + 75\kappa_y^2 b}}.$$

Then it follows that,

$$\tau^{-1} = \Theta\Big(\frac{\kappa_y}{b\mu_s}\sqrt{b\mu_s^2 + \kappa_y^2}\Big) = \Theta\big(\frac{\kappa_y^2}{bL_s}(\kappa_y^{\nu/2}\mu_y + \kappa_y)\big) = \Theta\Big(\frac{\kappa_y^2}{b}\Big(\kappa_y^{\nu/2-1} + \frac{\kappa_y}{L_s}\Big)\Big) = \Theta\Big(\frac{\kappa_y^3}{bL_s}\Big),$$

where we use the relation $\kappa_y^\nu \leq \frac{8}{81L_s\mu_s} \Rightarrow L_s^2 \leq \frac{8}{81}\kappa_y^{1-\nu}$ for the last equality. Next, from eq. (163) and the requirement on ψ in Theorem 12, a sufficient condition ψ is

$$\psi \geq \max\left\{2,\ p^3,\ (c_1'p)^3,\ (c_2'p)^3,\ \left(\sigma\mu_s(1+\kappa_y)p\sqrt{\frac{2b}{8\sigma^2 + 75\kappa_y^2 b}}\right)^3\right\}, \tag{165}$$

Now we consider the components of $\max$ operator in (165). Note positive constant p can be chosen independent of other problem parameters, e.g., $p = 1$. Furthermore, the requirement on c_1', c_2' can be satisfied for

$$c_1' = \Theta(\mu_s^2), \quad c_2' = \Theta(L_s^2). \tag{166}$$

Finally, using $\sigma = \frac{27b\mu_s}{16}$ together with $b = \kappa_y^\nu$ yields that

$$\sigma\mu_s(1+\kappa_y)p\sqrt{\frac{2b}{8\sigma^2 + 75\kappa_y^2 b}} = \Theta\Big(\kappa_y b\mu_s^2\sqrt{\frac{1}{b\mu_s^2 + \kappa_y^2}}\Big) = \Theta\Big(L_s^2\kappa_y^{\nu-1}\sqrt{\frac{1}{\kappa_y^{\nu-2}L_s^2 + \kappa_y^2}}\Big)$$
$$= \Theta\Big(L_s^2\kappa_y^{\nu-1}\sqrt{\frac{1}{\kappa_y^2}}\Big) \le \Theta(1),$$

where we use the relation $\kappa_y^\nu \le \frac{8}{81L_s\mu_s} \Rightarrow L_s^2 \le \frac{8}{81}\kappa_y^{1-\nu}$ for the last equality and the last inequality. Therefore, using the above relations within eq. (165), we observe that one can set

$$\psi^{1/3} = \Theta(\max\{1, L_s^2\}), \tag{167}$$

which completes the proof. □

Next, we will use the parameters in Lemma 28 to provide an optimized complexity for `ACC-MDA` [16, Algorithm 12] to generate x_ϵ such that $\mathbb{E}\left[\|\nabla\phi(x_\epsilon)\|\right] \le \epsilon$.

Corollary 3. *Suppose Assumptions 1, 2, 3 hold with $f(\cdot) = g(\cdot) = 0$, and $\kappa_y^\nu > \frac{8}{81L\mu_y}$ for the original function $\mathcal{L}$. Running `ACC-MDA` on $\min_x \max_y \mathcal{L}_s(x,y)$ for*

$$s = \frac{2\sqrt{2}}{9}\frac{1}{L}\kappa_y^{(1-\nu)/2}, \tag{168}$$

and $b = \kappa_y^\nu$, one can generate x_ϵ such that $\mathbb{E}\left[\|\nabla\phi(x_\epsilon)\|\right] \le \epsilon$ requiring at most $\tilde{O}(\frac{L^{1.5}\kappa_y^{3.5}}{\epsilon^3})$ stochastic first-order oracle calls.

Proof. It follows from Theorem 12 that that

$$\frac{1}{T}\sum_{t=1}^T \mathbb{E}[\|\nabla\phi(x_t)\|] \le \frac{\sqrt{2M_s''}}{s}\Big(\frac{\psi^{1/6}}{T^{1/2}} + \frac{1}{T^{1/3}}\Big). \tag{169}$$

Based on Lemma 28, let $\psi = \Theta(\max\{1, L_s^6\})$; thus, $\frac{\psi^{1/6}}{T^{1/2}} \le \frac{1}{T^{1/3}}$ when T is large enough. Therefore, for all sufficiently small $\epsilon > 0$, $\frac{1}{T^{1/3}} \le \frac{s}{\sqrt{2M_s''}} \cdot \frac{\epsilon}{2}$ implies that $\frac{\psi^{1/6}}{T^{1/2}} \le \frac{s}{\sqrt{2M_s''}} \cdot \frac{\epsilon}{2}$, and we get

$$\frac{1}{s}\frac{\sqrt{2M_s''}}{T^{1/3}} \le \frac{\epsilon}{2} \implies \min_{t\in\{0,\ldots,T\}} \mathbb{E}[\|\nabla\phi(x_t)\|] = \epsilon. \tag{170}$$

Moreover, note that eq. (168) implies $\kappa_y \le \frac{8}{81L_s\mu_s}$; thus, we can choose τ, σ, b as in Lemma 28 which satisfy

$$\sigma = \Theta(b\mu_s), \quad \tau^{-1} = \Theta\big(\frac{\kappa_y^3}{bL_s}\big), \quad b = \kappa_y^\nu.$$

Recall that c_1', c_2' chosen as in (166) and ψ chosen as in (167) satisfy all the required conditions in Theorem 12; hence, $M_s'' = \frac{s(\phi(x_0)-\phi_*)}{\tau p} + \frac{9L_s^2\Delta_0}{p\sigma\mu_s} + \frac{2\psi^{1/3}\delta_s^2}{b\mu_s^2p^2} + \frac{2(c_1'^2+c_2'^2)\delta_s^2p^2}{b\mu_s^2}\ln(\psi+T)$ implies that

$$\begin{aligned}\frac{1}{s^2}M_s'' &= \Theta\Big(\frac{\kappa_y^3}{sbL_s} + \frac{\kappa_y^2}{s^2b} + \frac{\delta_s^2}{s^2b\mu_s^2}\max\{1, L_s^2\} + \frac{\kappa_y^2L_s^2\delta_s^2}{s^2b}\ln(\psi+T)\Big)\\ &= \tilde{\Theta}\Big(\frac{\kappa_y^3}{s^2bL} + \frac{\kappa_y^2}{s^2b} + \frac{\delta^2}{s^2b\mu_y^2}\max\{1, s^2L^2\} + \frac{s^2\kappa_y^2L^2\delta^2}{b}\Big).\end{aligned} \tag{171}$$

Moreover, to satisfy eq. (170), one needs to choose $T \geq \Theta((\frac{1}{s^2}M_s'')^{3/2}\frac{1}{\epsilon^3})$. Since the total oracle complexity is bT, we obtain that

$$bT \geq \tilde{\Theta}\left(\frac{1}{\epsilon^3}\Big(\frac{\kappa_y^{9/2-\nu/2}}{s^3L^{3/2}} + \frac{\kappa_y^{3-\nu/2}}{s^3} + \frac{\delta^3\kappa^{-\nu/2}}{s^3\mu_y^3}\max\{1, s^3L^3\} + s^3\kappa_y^{3-\nu/2}L^3\delta^3\Big)\right). \tag{172}$$

From eq. (168), i.e., $s^2 = \frac{8}{81}\frac{1}{L^2}\kappa_y^{1-\nu}$, it follows that

$$bT \geq \tilde{\Theta}\left(\frac{1}{\epsilon^3}\big(L^{3/2}\kappa_y^{\nu+3} + L^3\kappa_y^{\nu+3/2} + \delta^3\kappa^{\nu+3/2}\max\{1, \kappa_y^{\frac{3-3\nu}{2}}\} + \kappa_y^{9/2-2\nu}\delta^3\big)\right). \tag{173}$$

When $\nu \geq 1$, we have

$$bT \geq \tilde{\Theta}\big(L^{3/2}\kappa_y^{\nu+3} + L^3\kappa_y^{\nu+3/2} + \delta^3\kappa_y^{3-\nu/2} + \kappa_y^{9/2-2\nu}\delta^3\big),$$

the optimal value is achieved at $\nu = 1$ and $bT \geq \Theta(\frac{L^{1.5}\kappa_y^4}{\epsilon^3})$; when $\nu < 1$, we have

$$bT \geq \tilde{\Theta}\big(L^{3/2}\kappa_y^{\nu+3} + L^3\kappa_y^{\nu+3/2} + \delta^3\kappa_y^{\nu+3/2} + \kappa_y^{9/2-2\nu}\delta^3\big),$$

the optimal value is achieved at $\nu = \frac{1}{2}$ and $bT \geq \tilde{\Theta}(\frac{L^{1.5}\kappa_y^{3.5}}{\epsilon^3})$, which completes the proof. □

Remark 14. *In [16], Huang* et al. *claims the oracle complexity of $\tilde{\mathcal{O}}(\kappa_y^3\epsilon^{-3})$ for $\nu = 3$, and $\tilde{\mathcal{O}}(\kappa_y^{2.5}\epsilon^{-3})$ for $\nu = 4$. However, our analysis leading to eq.* (173) *demonstrates that the complexities would be $\tilde{\mathcal{O}}(\frac{L^{1.5}\kappa_y^6}{\epsilon^3})$ for $\nu = 3$ and $\tilde{\mathcal{O}}(\frac{L^{1.5}\kappa_y^7}{\epsilon^3})$ for $\nu = 4$.*