Relationships between the Phase Retrieval Problem and Permutation Invariant Embeddings

***Abstract*—This paper discusses the connection between the phase retrieval problem and permutation invariant embeddings. We show that the real phase retrieval problem for $\mathbb{R}^d/O(1)$ is equivalent to Euclidean embeddings of the quotient space $\mathbb{R}^{2\times d}/S_2$ performed by the sorting encoder introduced in an earlier work. In addition, this relationship provides us with inversion algorithms of the orbits induced by the group of permutation matrices.**

I. Introduction

The phase retrieval problem has a long and illustrious history involving several Nobel prizes along the way. The issue of reconstruction from magnitude of frame coefficients is related to a significant number of problems that appear in separate areas of science and engineering. Here is an incomplete list of some of these applications and reference papers: *crystallography* [1], [2], [3]; ptychography [4], [5]; source separation and inverse problems [6], [7]; optical data processing [8]; mutually unbiased bases [9], [10], quantum state tomography [11], [12]; low-rank matrix completion problem [13], [14]; tensor algebra and systems of multivariate polynomial equations [15], [16], [17]; signal generating models [18], [19], bandlimited functions [20], [21], radar ambiguity problem [22], [23], learning and scattering networks [24], [25], [26].

In [27], this problem was shown to be a special form of the following setup. Let H denote a real or complex vector space and let $A = \{a_i\}_{i\in I}$ be a frame for H. The phase retrieval problem asks whether the map $H \ni x \mapsto \alpha_A(x) = \{|\langle x, a_i \rangle|\}_{i\ni I} \in l^2(I)$ determines x uniquely up to a unimodular scalar.

In this paper we focus on the finite dimensional real case of this problem (see also [28]), namely when $H = \mathbb{R}^d$. In this case, a frame $\mathcal{A} = \{a_1, \ldots, a_D\} \subset \mathbb{R}^d$ is simply a spanning set. The group $O(1) = \{-1, +1\}$ acts on H by scalar multiplication. Let $\hat{H} = H/O(1)$ denote the quotient space induced by this action, where the equivalence classes (orbits) are

$$[x] = \{x, -x\} \ , \ \text{for } x \neq 0 \ , \ [x] = \{0\} \ , \ \text{for } x = 0.$$

The analysis operator for this frame is

$$T_A : H \to \mathbb{R}^D \quad , \quad T_A(x) = (\langle x, a_k \rangle)_{k=1}^D. \quad (1)$$

The relevant nonlinear map α_A is given by taking the absolute value of entries of T_A:

$$\alpha_A : H \to \mathbb{R}^m \ , \ \alpha_A(x) = (|\langle x, a_k \rangle|)_{1\le k\le D}. \quad (2)$$

Notice α_A produces a well-defined map on $\hat{H}$, which, with a slight abuse, but for simplicity of notation, will be denoted also by α_A. Thus $\alpha_A(\hat{x}) = \alpha_A(x)$.

Another customary notation that is often employed: a frame is given either as an indexed set of vectors, $\mathcal{A} = \{a_1, \ldots, a_D\}$, or through the columns of a $d\times D$ matrix A. The matrix notation is not canonical, but this is not an issue here. We always identify $H = \mathbb{R}^d$ with its columns vector representation in its canonical basis.

Definition 1. *We say that (the columns of a matrix) $A \in \mathbb{R}^{d\times D}$ form/is a* phase retrievable frame, *if $\alpha_A : \widehat{\mathbb{R}}^d \to \mathbb{R}^D$, $\alpha_A(x) = (|\langle x, a_k \rangle|)_{k=1}^D$ is an injective map (on the quotient space).*

In a different line of works [29], [30], [31], [32] it was recognized that the phase retrieval problem is a special case of Euclidean representations of metric spaces of orbits defined by certain unitary group actions on Hilbert spaces. Specifically, the setup is as follows. Let V denote a Hilbert space, and let G be a group acting unitarily on V. Let $\hat{V} = V/G$ denote the metric space of orbits, where the quotient space is induced by the equivalence relation $x, y \in V$, $x \sim y$ iff $y = g.x$, for some $g \in G$. Here $g.x$ represents the action of the group element $g \in G$ on vector x. For the purposes of this paper we specialize to the finite dimensional real case, $V = \mathbb{R}^{n\times d}$ and $G = S_n$, is the group of $n\times n$ permutation matrices acting on V by *left multiplication*. Other cases are discussed in aforementioned papers. In particular, in [30] the authors have shown a deep connection to graph deep learning problems. In [31], the authors linked this framework to certain graph matching problems and more. The bi-Lipschitz Euclidean embedding problem for the finite dimensional case is as follows. Given V/G, construct a map $\beta : V \to \mathbb{R}^m$ so that, (i) $\beta(g.x) = \beta(x)$ for all $g \in G$, $x \in V$, and (ii) for some $0 < A \leq B < \infty$, and for all $x, y \in V$,

$$A\, d(x, y) \leq \|\beta(x) - \beta(y)\| \leq B\, d(x, y) \quad (3)$$

where $d(x, y) = \inf_{g\in G} \|x - g.y\|_V$ is the *natural metric* on the quotient space $\hat{V}$.

In [30] the following embedding was introduced. Let $A \in \mathbb{R}^{d\times D}$ be a fixed matrix (termed as *key*) whose columns are denoted by $a_1, \ldots, a_D$. The induced encoder $\beta_A : V \to \mathbb{R}^{n\times D}$ is defined by

$$\beta_A(X) = \downarrow(XA) = \begin{bmatrix} \Pi_1 X a_1 & \cdots & \Pi_D X a_D \end{bmatrix} \quad (4)$$

where $\Pi_k \in S_n$ is the permutation matrix that sorts monotone decreasing the vector Xa_k. It was shown in [30] that, for D large enough, β_A provides a bi-Lipschitz Euclidean embedding of $\hat{V}$. This motivates the following definition.

Definition 2. *We say that* $A \in \mathbb{R}^{d\times D}$ *is a* universal key *for* $\mathbb{R}^{n\times d}$ *if* $\beta_A : \widehat{\mathbb{R}^{n\times d}} \to \mathbb{R}^{n\times D}$, $\beta_A(X) = \downarrow(XA)$ *is an injective map (on the quotient space).*

The purpose of this paper is to show the equivalence between the real phase retrieval problem, specifically the embedding α_A, and the permutation invariant embedding β_A defined above, in the special case $n = 2$.

II. Main Results

Recall the Hilbert spaces $H = \mathbb{R}^d$ and $V = \mathbb{R}^{2\times d}$. For $A \in \mathbb{R}^{d\times D}$ recall also the encoders $\alpha_A : \hat{H} \to \mathbb{R}^D$ and $\beta_A : \hat{V} \to \mathbb{R}^{2\times D}$ given respectively by $\alpha_A(x) = (|\langle x, a_k\rangle|)_{k\in[D]}$, and $\beta_A(X) = \downarrow(XA)$. Our main result reads as follows.

Theorem 3. *In the case $n = 2$, the following are equivalent.*

1) *α_A is injective, hence the columns of A form a phase retrievable frame;*
2) *β_A is injective, hence A is a universal key.*

Remark 4. *Perhaps it is not surprising that, if an equivalence between the phase retrieval problem and permutation invariant representations is possible, then this should occur for $n = 2$. This statement is suggested by the observation that $O(1)$ is isomorphic with S_2, the group of the 2×2 permutation matrices. What is surprising that that, in fact, the two embeddings are intimately related, as the proof and corollaries show.*

Proof of Theorem 3. Let $X \in V = \mathbb{R}^{2\times d}$. Denote by $x_1, x_2 \in \mathbb{R}^d$ its two rows transposed, that is

$$X = \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix}.$$

Notice that, for each $k \in [D]$, the k^{th} column of $\beta_A(X)$ is given by

$$\downarrow(Xa_k) = \begin{bmatrix} max(\langle x_1, a_k\rangle, \langle x_2, a_k\rangle) \\ min(\langle x_1, a_k\rangle, \langle x_2, a_k\rangle) \end{bmatrix}.$$

The *key observation* are the following relationships between *min*, *max*, and the absolute value $|\cdot|$:

$$\begin{aligned} |u - v| &= max(u, v) - min(u, v) \\ u + v &= max(u, v) + min(u, v) \\ max(u, v) &= \frac{1}{2}(u + v + |u - v|) \\ min(u, v) &= \frac{1}{2}(u + v - |u - v|) \end{aligned}$$

In particular, these show that:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \beta_A(X) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \downarrow(XA) =$$
$$= \begin{bmatrix} |\langle x_1 - x_2, a_1\rangle|, & \ldots & , |\langle x_1 - x2, a_D\rangle| \\ \langle x_1 + x_2, a_1\rangle, & \ldots & , \langle x_1 - x_2, a_D\rangle \end{bmatrix}$$
$$= \begin{bmatrix} (\alpha_A(x_1 - x_2))^T \\ (T_A(x_1 + x_2))^T \end{bmatrix}$$

Where, T_A was introduced in equation (1).

(1) $\to$ (2) : Suppose that β_A is injective. Let $x, y \in \mathbb{R}^d$ such that $\alpha_A(x) = \alpha_A(y)$, i.e.

$|\langle x, a_k \rangle| = |\langle y, a_k \rangle|$, $\forall k \in [D]$. Let $X = \begin{bmatrix} x^T \\ -x^T \end{bmatrix}$ and $Y = \begin{bmatrix} y^T \\ -y^T \end{bmatrix}$. Then,

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \beta_A(X) = \begin{bmatrix} \alpha_A(2x)^T \\ T_A(0)^T \end{bmatrix} = 2 \begin{bmatrix} \alpha_A(2x)^T \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \beta_A(X) = \begin{bmatrix} \alpha_A(2y)^T \\ T_A(0)^T \end{bmatrix} = 2 \begin{bmatrix} \alpha_A(2y)^T \\ 0 \end{bmatrix}$$

Thus $\beta_A(X) = \beta_A(Y)$. Since β_A is assumed injective, it follows that $X = Y$ or $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Y$. So, $x = y$ or $x = -y$. We conclude that $[x] = [y]$, so α_A is injective.

(2) $\rightarrow$ (1) : Suppose that α_A is injective. Let Let $X = \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix}$ and $Y = \begin{bmatrix} y_1^T \\ y_2^T \end{bmatrix}$, such that $\beta_A(X) = \beta_A(Y)$. Then

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \beta_A(X) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \beta_A(Y)$$

$$\implies \begin{bmatrix} (\alpha_A(x_1 - x_2))^T \\ T_A(x_1 + x_2)^T \end{bmatrix} = \begin{bmatrix} (\alpha_A(y_1 - y_2))^T \\ T_A(y_1 + y_2)^T \end{bmatrix}.$$

But now, $\alpha_A(x_1 - x_2) = \alpha_A(y_1 - y_2)) \implies x_1 - x_2 = y_1 - y_2$ or $x_1 - x_2 = y_2 - y_1$ and

$$T_A(x_1+x_2) = T_A(y_1+y_2)^T \implies x_1+x_2 = y_1+y_2$$

Thus we have that

$$\left\{ \begin{array}{ccc} x_1 & = & y_1 \\ x_2 & = & y_2 \end{array} \right\} \text{ or } \left\{ \begin{array}{ccc} x_1 & = & y_2 \\ x_2 & = & y_1 \end{array} \right\}$$

Either case means

$$\iff X = Y \text{ or } X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Y$$

$$\iff [X] = [Y]$$

So, β_A is injective. □

A few immediate corollaries of this result are as follows.

Corollary 5. *If β_A is injective, then $D \geq 2d - 1$.*

Corollary 6. *If $D = 2d - 1$, then β_A is injective if and only if A is a full spark frame.*

Both results follow necessary and sufficient conditions established in, e.g. [27]. Recall that a frame in $\mathbb{R}^d$ is said full spark if any subset of d vectors is linearly independent (hence basis).

Remark 7. *Assume $D = 2d - 1$. Note the embedding dimension for $\hat{V} = \widehat{\mathbb{R}^{2\times d}}$ is $m = 2(2d-1) = 4d-2 = 2\,dim(V) - 2$. In particular this shows the minimal dimension of bi-Lipschitz Euclidean embeddings may be smaller than twice the intrinsic dimension of the Hilbert space where the group acts on. Both papers [30] and [31] present (bi)Lipschitz embeddings into $\mathbb{R}^{2\,dim(V)}$.*

Remark 8. *As was derived in the proof, α_A, β_A and T_A are intimately related:*

$$\beta_A\left(\begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_A(x_1 - x_2)^T \\ T_A(x_1 + x_2)^T \end{bmatrix} \tag{5}$$

In particular, any algorithm for solving the phase retrieval problem solves also the inversion problem for β_A. Let $\omega_A : \mathbb{R}^D \to \mathbb{R}^d$ denote a left inverse of α_A on the metric space $\widehat{\mathbb{R}^d}$. This means $\omega_A(\alpha_A(x)) \sim x$ in $\mathbb{R}^d/O(1)$. Denote by $T_A^\dagger$ a left inverse of the analysis operator (e.g., the synthesis operator associated to the canonical dual frame). Thus $T_A^\dagger T_A = I_d$. Then an inverse for β_A is:

$$\beta_A^{-1}(Y) = \frac{1}{2}\begin{bmatrix} T_A^\dagger(y_2) + \omega_A(y_1) \\ T_A^\dagger(y_2) - \omega_A(y_1) \end{bmatrix} \tag{6}$$

where $Y = \begin{bmatrix} y_1^T \\ y_2^T \end{bmatrix}$.

Remark 9. *Equations (6) suggest a lower dimensional embedding than β_A. Specifically, first we compute the average $y_1 = \frac{1}{2}(x_1 + x_2)$ which is of size $\mathbb{R}^d$, and then encode the difference $x_1 - x_2$ using α_A, $y_2 = \alpha_A(x_1 - x_2)$. We obtain the following modified encoder, $\tilde{\beta}_A : \mathbb{R}^{2\times d} \to \mathbb{R}^{d+D}$:*

$$\tilde{\beta}_A(x) = \begin{bmatrix} \frac{1}{2}(x_1 + x_2)^T & \alpha_A(x_1 - x_2)^T \end{bmatrix}. \tag{7}$$

With the ω_A left inverse of α_A, the inverse of $\tilde{\beta}_A$ is given by:

$$\tilde{\beta}_A^{-1}(Y) = \begin{bmatrix} y_1 + \frac{1}{2}\omega_A(y_2) \\ y_1 - \frac{1}{2}\omega_A(y_2) \end{bmatrix}$$

where $y_1 = Y(1:d)$ and $y_2 = Y(d+1:d+D)$. In the case when $D = D_{min} = 2d-1$, the minimal embedding dimension is $m = d + D = 3d - 1$ (instead of $4d-2$ or $4d = 2\,dim(V)$).

III. Conclusion

In this paper we analyzed two representation problems, one arising in the phase retrieval problem and the other one in the context of permutation invariant representations. We showed that the real phase retrieval problem in a finite dimensional vector space H is entirely equivalent to the permutation invariant representations for the space $V = \mathbb{R}^{2\times dim(H)}$. Our analysis proved that phase retrievability is equivalent to the universal key property in the case of encoding $2 \times d$ matrices. This result is derived based on the lattice space structure $(\mathbb{R}, +, min, max)$. It is still an open problem to understand the relationship between α_A and β_A in the case $n > 2$. A related problem is the implementation of the sorting operator using a neural network that has ReLU as activation function (or, even the absolute value $|\cdot|$). Efficient implementations of such operator may yield novel relationships between α_A and β_A, in the case $n \geq 3$.

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