Appendix: Future Gradient Descent for Adapting the Temporal Shifting Data Distribution in Online Recommendation Systems

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Extra Notation We introduce several new notations for the appendix. We use $\langle \cdot, \cdot \rangle$ to denote the inner product between two vectors and use $\circ$ to denote the entrywise product.

1 PROOF OF THEOREM 1

Proof. We start with a simple decomposition using the triangle inequality:

$$\|u_{w,t}(\theta_t)\| \leq \|u_{w,t}(\theta_t) - \bar{m}(\theta_t; t)\| + \|\bar{m}(\theta_t; t)\|.$$

By the termination condition of Algorithm **??**, we have $\|\bar{m}(\theta_t; t)\| \leq \delta$. Furthermore, it follows from (5) that

$$\|u_{w,t}(\theta_t) - \bar{m}(\theta_t; t)\| = \frac{1}{w}\|\nabla r_t(\theta_t) - m(\theta_t; t)\|.$$

Hence, we obtain

$$\|u_{w,t}(\theta_t)\|^2 \leq \left(\delta + \frac{1}{w}\|\nabla r_t(\theta_t) - m(\theta_t; t)\|\right)^2 \leq 2\delta^2 + \frac{2}{w^2}\|\nabla r_t(\theta_t) - m(\theta_t; t)\|^2. \tag{1}$$

This further implies that

$$\mathfrak{R}_w(T) = \frac{1}{T}\sum_{t=1}^{T}\|u_{w,t}(\theta_t)\|^2 \leq \frac{2}{w^2 T}\sum_{t=1}^{T}\|\nabla r_t(\theta_t) - m(\theta_t; t)\|^2 + 2\delta^2, \tag{2}$$

and the main result follows from the fact that $\|\nabla r_t(\theta_t) - m(\theta_t; t)\|^2 \leq \sup_\theta \|\nabla r_t(\theta) - m(\theta; t)\|^2$ for all $t \in [T]$. Furthermore, under the boundedness assumption, we have for all $t \in [T]$

$$\|\nabla r_t(\theta_t) - m(\theta_t; t)\|^2 \leq (\|\nabla r_t(\theta_t)\| + \|m(\theta_t; t)\|)^2 \leq 4M^2. \tag{3}$$

Hence, (**??**) also implies $\mathfrak{R}_w(T) \leq 8M^2/w^2 + 2\delta^2$, which leads to $\mathfrak{R}_w(T) = O(1/w^2)$ when $\delta = 1/w$. $\square$

2 DETAILS OF THE RESULT IN SECTION 4.4

Algorithm. Given θ_t, define $h_t(\phi) = \|\nabla r_t(\theta_t) - m(\theta_t; \phi, t)\|^2$ as a function of ϕ, where we view θ_t as a constant. Thus, if follows from that (**??**) that

$$\mathfrak{R}_w(T) \leq \frac{2}{w^2 T}\sum_{t=1}^{T} h_t(\phi_t) + 2\delta^2. \tag{4}$$

Accepted for the 38th Conference on Uncertainty in Artificial Intelligence (UAI 2022).

Algorithm 1 Generalized Future Gradient Descent for Smoothed Regret (simplified version for the theoretical study)

Input: The learning rate η, η_ϕ for updating the model parameter θ and ϕ.
Initialize $\phi_1 = [1/b, ..., 1/b]$.
for $t \in [T]$ **do**
  Deploy the prediction model f_{θ_t} with the parameter θ_t and collect the new dataset D_t.
  Construct the function $h_t(\phi) = \|\nabla r_t(\theta_t) - m(\theta_t; \phi, t)\|^2$
  $\phi_{t+1} = \frac{\phi_t \circ \exp(-\eta_\phi \nabla h_t(\phi_t))}{\|\phi_t \circ \exp(-\eta_\phi \nabla h_t(\phi_t))\|_1}$. ▷ One step of Exponentiated gradient descent from ϕ_t
  Initialize the model parameter θ_{t+1}.
  while $\|\bar{m}(\theta_{t+1}; \phi_{t+1}, t+1)\| \geq \delta$ **do**
    $\theta_{t+1} = \theta_{t+1} - \eta \bar{m}(\theta_{t+1}; \phi_{t+1}, t+1)$.
  end while
end for

Thus, our goal is to minimize $\sum_{t=1}^T h_t(\phi_t)$ in an *online* manner, since we can only access $h_t(\phi_t)$ after ϕ_t is chosen. To achieve this, we use the classic exponentiated gradient method to update ϕ_t. Specifically, for any $\phi = [a_1, \ldots, a_b] \in S_b$, define the negative potential function $\psi(\phi) = \sum_{i=1}^b a_i \log a_i$ and its Bregman divergence

$$\mathcal{B}_\psi(\phi; \phi') = \psi(\phi) - \psi(\phi') - \langle \nabla \psi(\phi'), \phi - \phi' \rangle = \sum_{i=1}^b a_i \log \frac{a_i}{a_i'}.$$

Then ϕ_{t+1} is given by

$$\phi_{t+1} = \arg\min_{\phi \in S_b} \left(\langle \nabla h_t, \phi \rangle + \frac{1}{\eta_\phi} \mathcal{B}_\psi(\phi; \phi_t) \right) = \frac{\phi_t \circ \exp(-\eta_\phi \nabla h_t(\phi_t))}{\|\phi_t \circ \exp(-\eta_\phi \nabla h_t(\phi_t))\|_1},$$

where η_ϕ is the learning rate. See Section 6.6 in **?** for the derivation of the last equality. Intuitively, $\frac{1}{\eta_\phi}\mathcal{B}_\psi(\phi; \phi_t)$ stabilizes the algorithm by ensuring that ϕ_{t+1} remains close to ϕ_t.

This simplified version of FGD is summarized in Algorithm **??**. Note that when updating ϕ, we only use the last recommendation model θ_t.

Lemma 1. *Suppose that we have $\|\nabla r_t(\theta)\| \leq M$ for all $\theta \in \Theta$ and t. Then $\|\nabla h_t(\phi)\|_\infty \leq 8M^2$ for all $\phi \in S_b$.*

Proof. By definition, we have

$$h_t(\phi) = \|\nabla r_t(\theta_t) - \sum_{i=1}^b a_i \nabla r_{t-i}(\theta_t)\|^2 = \|\sum_{i=1}^b a_i (\nabla r_t(\theta_t) - \nabla r_{t-i}(\theta_t))\|^2,$$

where we used the fact that $\sum_{i=1}^b a_i = 1$. Direct computation shows that

$$\begin{aligned}
\left|\frac{\partial h_t}{\partial a_i}(\phi)\right| &= 2\left|\left\langle \nabla r_t(\theta_t) - \nabla r_{t-i}(\theta_t), \sum_{j=1}^b a_j (\nabla r_t(\theta_t) - \nabla r_{t-j}(\theta_t)) \right\rangle\right| && (5)\\
&\leq 2\|\nabla r_t(\theta_t) - \nabla r_{t-i}(\theta_t)\| \left\| \sum_{j=1}^b a_j (\nabla r_t(\theta_t) - \nabla r_{t-j}(\theta_t)) \right\| && (6)\\
&\leq 2(\|\nabla r_t(\theta_t)\| + \|\nabla r_{t-i}(\theta_t)\|) \left(\sum_{j=1}^b a_j (\|\nabla r_t(\theta_t)\| + \|\nabla r_{t-j}(\theta_t)\|) \right) && (7)\\
&\leq 8M^2, && (8)
\end{aligned}$$

where we used Cauchy-Schwarz inequality in (**??**), the triangle inequality in (**??**) and the boundedness of the gradients in (**??**). Hence, we conclude that $\|\nabla h_t(\phi)\|_\infty \leq 8M^2$. □

Proof of Theorem 2. Now we proceed to the proof of Theorem 2. This is a standard result in the online learning literature (see, e.g., **?**). For completeness, we present the proof below.

Proof. As ψ is λ-strongly convex with $\lambda = 1$, we have

$$\mathcal{B}_\psi(\phi; \phi') \geq \frac{1}{2}\|\phi - \phi'\|_1^2. \tag{9}$$

Throughout the proof, we slightly abuse the notation by writing $\eta_\phi = \eta$ and $\nabla h_t = \nabla h_t(\phi_t)$ for simplicity. Notice that by our update rule ϕ_{t+1} is given by

$$\phi_{t+1} = \arg\min_{\phi \in S_b} \left(\eta \langle \nabla h_t, \phi \rangle + \mathcal{B}_\psi(\phi; \phi_t)\right).$$

From the first-order optimality condition, we get for any $\phi \in S_b$,

$$\begin{aligned}
& \langle \eta \nabla h_t + \nabla\psi(\phi_{t+1}) - \nabla\psi(\phi_t), \phi_{t+1} - \phi \rangle \leq 0 \\
\Leftrightarrow \quad & \eta \langle \nabla h_t, \phi_t - \phi \rangle \leq \eta \langle \nabla h_t, \phi_t - \phi_{t+1} \rangle + \langle \nabla\psi(\phi_{t+1}) - \nabla\psi(\phi_t), \phi - \phi_{t+1} \rangle \\
\Leftrightarrow \quad & \eta \langle \nabla h_t, \phi_t - \phi \rangle \leq \eta \langle \nabla h_t, \phi_t - \phi_{t+1} \rangle - \mathcal{B}_\psi(\phi; \phi_{t+1}) + \mathcal{B}_\psi(\phi; \phi_t) - \mathcal{B}_\psi(\phi_{t+1}; \phi_t),
\end{aligned}$$

where we used the three-point equality (**?**) in the last inequality. Furthermore,

$$\begin{aligned}
\eta \langle \nabla h_t, \phi_t - \phi_{t+1} \rangle - \mathcal{B}_\psi(\phi; \phi_{t+1}) &\leq \eta \|\nabla h_t\|_\infty \|\phi_t - \phi_{t+1}\|_1 - \frac{1}{2}\|\phi_t - \phi_{t+1}\|_1^2 \\
&\leq \frac{\eta^2}{2}\|\nabla h_t\|_\infty^2 + \frac{1}{2}\|\phi_t - \phi_{t+1}\|_1^2 - \frac{1}{2}\|\phi_t - \phi_{t+1}\|_1^2 \\
&= \frac{\eta^2}{2}\|\nabla h_t\|_\infty^2.
\end{aligned}$$

Combining these two bounds, we have

$$\eta \left\langle \nabla h_t, \phi_t - \phi \right\rangle \leq \mathcal{B}_\psi(\phi; \phi_t) - \mathcal{B}_\psi(\phi; \phi_{t+1}) + \frac{\eta^2}{2}\|\nabla h_t\|_\infty^2.$$

Since $h_t(\phi)$ is convex in ϕ, we have $h_t(\phi_t) - h_t(\phi) \leq \langle \nabla h_t, \phi_t - \phi \rangle$ for any $\phi \in S_b$. By telescoping, we obtain

$$\begin{aligned}
\sum_{t=1}^T (h_t(\phi_t) - h_t(\phi)) &\leq \sum_{t=1}^T \langle \nabla h_t, \phi_t - \phi \rangle \\
&\leq \frac{1}{\eta} \sum_{t=1}^T \left[\mathcal{B}_\psi(\phi; \phi_t) - \mathcal{B}_\psi(\phi; \phi_{t+1}) + \frac{\eta^2}{2}\|\nabla h_t\|_\infty^2\right] \\
&= \frac{1}{\eta}(\mathcal{B}_\psi(\phi; \phi_1) - \mathcal{B}_\psi(\phi; \phi_{T+1})) + \frac{\eta}{2}\sum_{t=1}^T \|\nabla h_t\|_\infty^2 \\
&\leq \frac{1}{\eta}\log b + 32\eta M^4 T.
\end{aligned}$$

where we used Lemma **??**, $\mathcal{B}_\psi(\phi; \phi_{T+1}) \geq 0$ and $\mathcal{B}_\psi(\phi; \phi_1) = \psi(\phi) + \log b \leq \log b$ in the last inequality. Choosing $\eta = c\sqrt{(\log b)/(TM^4)}$ with some constant $c > 0$ leads to

$$\sum_{t=1}^T [h_t(\phi_t) - h_t(\phi)] \leq O(M^2 \sqrt{T \log b}). \tag{10}$$

Algorithm 2 Generalized Future Gradient Descent for Smoothed Loss

Input: The learning rate η, η_ϕ for updating the model parameter θ and ϕ. The initial trajectory buffer B.
for $t \in [T]$ **do**
  Deploy the prediction model f_{θ_t} with parameter θ_t. Then collect the new dataset D_t.
  Initialize the parameter of MFGG ϕ_{t+1}. ▷ Initialization of ϕ_{t+1} is user-specific.
  for Inner loop iteration $k \in K$ **do** ▷ Update the meta network.
    $\phi_{t+1} \leftarrow \phi_{t+1} - \eta_\phi \sum_{\theta \in B} \nabla_\phi \|m(\theta; \phi_{t+1}, t) - \nabla r_t(\theta)\|^2$. ▷ May replace with the mini-batch version.
  end for
  Initialize the trajectory buffer $B = \emptyset$ and model parameter θ_{t+1}. ▷ Initialization scheme of θ_{t+1} is specified by user.
  while $\|\bar{m}(\theta_{t+1}; \phi_{t+1}, t+1)\| \geq \delta$ **do** ▷ Alternatively, we may run gradient descent with a fixed number of iterations.
    $\theta_{t+1} \leftarrow \theta_{t+1} - \eta m(\theta_{t+1}; \phi_{t+1}, t+1)$. ▷ May replace with the mini-batch version.
    $B \leftarrow B \cup \{\theta_{t+1}\}$ ▷ Alternatively, we may update the trajectory buffer B every a few iterations.
  end while
end for

Note that (**??**) holds for any $\phi \in S_b$. In particular, we can set $\phi = \phi^*$ defined by $\phi^* = \arg\min_{\phi \in S_b} \sum_{t=1}^T h_t(\phi)$. Therefore,

$$\begin{aligned}
\sum_{t=1}^T h_t(\phi_t) &\leq \sum_{t=1}^T h_t(\phi^*) + O(M^2 \sqrt{T \log b}) \\
&= \min_{\phi \in S_b} \sum_{t=1}^T \|\nabla r_t(\theta_t) - m(\theta_t; \phi, t)\|^2 + O(M^2 \sqrt{T \log b}) \\
&\leq \min_{\phi \in S_b} \sum_{t=1}^T \sup_\theta \|\nabla r_t(\theta) - m(\theta; \phi, t)\|^2 + O(M^2 \sqrt{T \log b}) = \min_{m \in \mathcal{M}} Q[T; m] + O(M^2 \sqrt{T \log b}).
\end{aligned}$$

We thus conclude from (**??**) that

$$\mathfrak{R}_w(T) \leq \frac{2}{w^2 T} (\min_{m \in \mathcal{M}} Q[T; m] + O(M^2 \sqrt{T \log b})) + 2\delta^2.$$

□

3 A PRACTICAL GENERALIZED FGD ALGORITHM.

Compared with FGD in Algorithm 2, we use a smoothed version of MFGG $\bar{m}$ for training, which is due to the consideration of minimizing a smoothed loss in (2). For completeness, we also summarize the practical algorithm of the generalized version of FGD in Algorithm **??**.

References

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