VI. Supplementary Material: Contact Kinematics and Mechanics

A. Contact kinematics

1) Transformations: According to [4], transformations of a twist $\mathcal{V}$ and a wrench $\mathcal{F}$ between frames $\{\mathrm{i}\}$ and $\{\mathrm{j}\}$ are

$$\mathcal{V}^{\mathrm{i}} = [\mathrm{Ad}_{\mathbf{T}_{\mathrm{ij}}}]\mathcal{V}^{\mathrm{j}},\ \mathcal{F}^{\mathrm{i}} = [\mathrm{Ad}_{\mathbf{T}_{\mathrm{ji}}}]^{\intercal}\mathcal{F}^{\mathrm{j}}, \tag{5}$$

where $[\mathrm{Ad}_{\mathbf{T}_{\mathrm{ij}}}]$ is the adjoint representation of $\mathbf{T}_{\mathrm{ij}} \in SE(3)$. The matrix representation of the wrench $\mathcal{F} = (\mathbf{m}, \mathbf{f})$ is

$$\mathbf{W} := \begin{bmatrix} [\mathbf{m}] & [\mathbf{f}] \\ [\mathbf{f}] & \mathbf{0_{3\times3}} \end{bmatrix},$$

where $[\cdot]$ is the skew-symmetric matrix representation of a 3-vector. The transformations of the 6×6 stiffness matrix $\mathbf{K}$ and the 6×6 wrench representation $\mathbf{W}$ between frames $\{\mathrm{i}\}$ and $\{\mathrm{j}\}$ are

$$\mathbf{K}^{\mathrm{i}} = [\mathrm{Ad}_{\mathbf{T}_{\mathrm{ji}}}]^{\intercal}\mathbf{K}^{\mathrm{j}}[\mathrm{Ad}_{\mathbf{T}_{\mathrm{ji}}}],\ \mathbf{W}^{\mathrm{i}} = [\mathrm{Ad}_{\mathbf{T}_{\mathrm{ji}}}]^{\intercal}\mathbf{W}^{\mathrm{j}}[\mathrm{Ad}_{\mathbf{T}_{\mathrm{ji}}}]. \tag{6}$$

2) Time derivatives: The time derivative of $\mathbf{T}_{\mathrm{ij}}$ is

$$\dot{\mathbf{T}}_{\mathrm{ij}} = [\mathcal{V}^{\mathrm{i}}_{\mathrm{ij}}]\mathbf{T}_{\mathrm{ij}} \ \Rightarrow\ \dot{\mathbf{R}}_{\mathrm{ij}} = [\boldsymbol{\omega}^{\mathrm{i}}_{\mathrm{ij}}]\mathbf{R}_{\mathrm{ij}},\ \dot{\mathbf{p}}^{\mathrm{i}}_{\mathrm{ij}} = [\boldsymbol{\omega}^{\mathrm{i}}_{\mathrm{ij}}]\mathbf{p}^{\mathrm{i}}_{\mathrm{ij}} + \mathbf{v}^{\mathrm{i}}_{\mathrm{ij}} \tag{7}$$

The time derivative of $[\mathrm{Ad}_{\mathbf{T}_{\mathrm{ij}}}]$, using Equation (7), is

$$\frac{d}{dt}([\mathrm{Ad}_{\mathbf{T}_{\mathrm{ij}}}]) = -[\mathrm{Ad}_{\mathbf{T}_{\mathrm{ij}}}]\begin{bmatrix} [\boldsymbol{\omega}^{\mathrm{j}}_{\mathrm{ji}}] & \mathbf{0_{3\times3}} \\ [\mathbf{v}^{\mathrm{j}}_{\mathrm{ji}}] & [\boldsymbol{\omega}^{\mathrm{j}}_{\mathrm{ji}}] \end{bmatrix}. \tag{8}$$

For each finger, the spring stiffness represented in frame $\{\mathrm{f}_0\}$, $\mathbf{K}^{\mathrm{f}_0}_{\mathrm{spr}}$, is a constant matrix, so the time derivative of the wrench applied on the spring, when the spring has a small displacement (so $\dot{\mathcal{X}}^{\mathrm{f}_0}_{\mathrm{f}_0\mathrm{f}} \approx \mathcal{V}^{\mathrm{f}_0}_{\mathrm{f}_0\mathrm{f}}$), is

$$\dot{\mathcal{F}}^{\mathrm{f}_0}_{\mathrm{spr}} = \frac{d}{dt}(\mathbf{K}^{\mathrm{f}_0}_{\mathrm{spr}}\mathcal{X}^{\mathrm{f}_0}_{\mathrm{f}_0\mathrm{f}}) = \mathbf{K}^{\mathrm{f}_0}_{\mathrm{spr}}\dot{\mathcal{X}}^{\mathrm{f}_0}_{\mathrm{f}_0\mathrm{f}} \approx \mathbf{K}^{\mathrm{f}_0}_{\mathrm{spr}}\mathcal{V}^{\mathrm{f}_0}_{\mathrm{f}_0\mathrm{f}}. \tag{9}$$

The time derivative of a wrench $\dot{\mathcal{F}}^{\mathrm{i}}$ can be represented in another moving frame $\{\mathrm{j}\}$ using Equation (8) as

$$\dot{\mathcal{F}}^{\mathrm{i}} = \frac{d}{dt}([\mathrm{Ad}_{\mathbf{T}_{\mathrm{ji}}}]^{\intercal}\mathcal{F}^{\mathrm{j}}) = -\mathbf{W}^{\mathrm{i}}\mathcal{V}^{\mathrm{i}}_{\mathrm{ij}} + [\mathrm{Ad}_{\mathbf{T}_{\mathrm{ji}}}]^{\intercal}\dot{\mathcal{F}}^{\mathrm{j}}. \tag{10}$$

3) Contact kinematics: When a fingertip rolls, spins, or slides (or any combination) along the object's surface while maintaining contact with the object, the twist representing the object's motion relative to the fingertip is

$$\mathcal{V}^{\mathrm{l}_1}_{\mathrm{l}_1\mathrm{l}_2} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ v_x \\ v_y \\ v_z \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \rightarrow & \text{Rolling velocities} \\ \rightarrow & \text{Spinning velocity about } z\text{-axis} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \rightarrow & \text{Sliding velocities} \\ \equiv 0 & \text{(Impenetrability)} \end{matrix} \tag{11}$$

describing all five degrees of freedom of relative motion between the fingertip and the object.

Contact kinematics are derived in [3]. Defining a 6×2 matrix

$$\mathbf{G}_i := \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{K}_{\mathrm{cur,i}} \\ \mathbf{T}_{\mathrm{tor,i}} \\ \mathbf{I}_{2\times2} \\ \mathbf{0}_{1\times2} \end{bmatrix},\ i = 1, 2,$$

the results in [3] can be rewritten as

$$\begin{cases} \mathcal{V}^{\mathrm{c}_1}_{\mathrm{oc}_1} = \mathbf{G}_1\mathbf{M}_{\mathrm{met},1}\dot{\mathbf{u}}_1, \\ \mathcal{V}^{\mathrm{c}_2}_{\mathrm{fc}_2} = \mathbf{G}_2\mathbf{M}_{\mathrm{met},2}\dot{\mathbf{u}}_2, \\ \mathcal{V}^{\mathrm{c}_2}_{\mathrm{c}_2\mathrm{c}_1} = [0, 0, \dot{\phi}, 0, 0, 0]^{\intercal} \end{cases}, \tag{12}$$

where $\mathbf{K}_{\mathrm{cur,i}}$ is the 2×2 curvature form at a point on a surface, $\mathbf{T}_{\mathrm{tor,i}}$ is the 1×2 torsion form at a point on a surface, and $\mathbf{M}_{\mathrm{met,i}}$ is the metric 2×2 diagonal matrix at a point on a surface. Moreover, $\dot{\mathbf{u}}_1$ can be expressed by elements of $\mathcal{V}^{\mathrm{l}_1}_{\mathrm{l}_1\mathrm{l}_2}$ as

$$\dot{\mathbf{u}}_1 = \mathbf{M}^{-1}_{\mathrm{met},1}(\mathbf{K}_{\mathrm{cur},1} + \widetilde{\mathbf{K}}_{\mathrm{cur},2})^{-1}(\begin{bmatrix} \omega_y \\ -\omega_x \end{bmatrix} + \widetilde{\mathbf{K}}_{\mathrm{cur},2}\begin{bmatrix} v_x \\ v_y \end{bmatrix}), \tag{13}$$

where

$$\widetilde{\mathbf{K}}_{\mathrm{cur},2} := \mathbf{R}_{\phi}\mathbf{K}_{\mathrm{cur},2}\mathbf{R}_{\phi},\ \mathbf{R}_{\phi} := \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}.$$

When rolling, $v_x = v_y = v_z = 0$. Substituting this into Equations (13) and (12) yields

$$\mathcal{V}^{\mathrm{c}_1}_{\mathrm{oc}_1} = \mathbf{P}_1\mathcal{V}^{\mathrm{l}_1}_{\mathrm{l}_1\mathrm{l}_2},\ \mathbf{P}_1 := \mathbf{G}_1(\mathbf{K}_{\mathrm{cur},1} + \widetilde{\mathbf{K}}_{\mathrm{cur},2})^{-1}\mathbf{S}. \tag{14}$$

B. Contact mechanics

A rigid object with a known geometry is in the grasp of n identical compliant fingers. The object is in point contact with each fingertip and rolls or spins on the fingertip's surface without sliding. The forward mechanics problem can be described as: given (1) the current configuration of the object and fingers in wrench balance, (2) the rate of change of the external wrench on the object, and (3) the velocities of all finger anchors, find the quasistatically-consistent object and fingertip twists.

1) Rolling constraints: For each finger, the kinematics of the finger anchor, fingertip, and object indicates

$$\begin{aligned} \mathcal{V}_{\mathrm{f}_0\mathrm{f}} &= \cancelto{\mathbf{0}}{\mathcal{V}_{\mathrm{f}_0\mathrm{a}}} + \mathcal{V}_{\mathrm{aw}} + \mathcal{V}_{\mathrm{wo}} + \mathcal{V}_{\mathrm{oc}_1} + \mathcal{V}_{\mathrm{c}_1\mathrm{c}_2} + \mathcal{V}_{\mathrm{c}_2\mathrm{f}} \\ &= \mathcal{V}_{\mathrm{wo}} - \mathcal{V}_{\mathrm{wa}} + [\mathrm{Ad}_{\mathbf{T}_{\mathrm{wc}_1}}]\mathcal{V}^{\mathrm{c}_1}_{\mathrm{oc}_1} - [\mathrm{Ad}_{\mathbf{T}_{\mathrm{wc}_2}}](\mathcal{V}^{\mathrm{c}_2}_{\mathrm{fc}_2} + \mathcal{V}^{\mathrm{c}_2}_{\mathrm{c}_2\mathrm{c}_1}), \end{aligned} \tag{15}$$

where $\mathcal{V}_{\mathrm{f}_0\mathrm{f}}$ is the fingertip displacement rate. For each contact, the rolling constraint indicates there is no relative linear velocity between the object and fingertip:

$$\begin{aligned} \mathbf{v}^{\mathrm{l}_1}_{\mathrm{l}_1\mathrm{l}_2} = \begin{bmatrix} \mathbf{0_{3\times3}} & \mathbf{I_{3\times3}} \end{bmatrix}\mathcal{V}^{\mathrm{l}_1}_{\mathrm{l}_1\mathrm{l}_2} &= \mathbf{0_{3\times1}} \\ \begin{bmatrix} \mathbf{0_{3\times3}} & \mathbf{I_{3\times3}} \end{bmatrix}\cancelto{\mathbf{I}}{[\mathrm{Ad}_{\mathbf{T}_{\mathrm{l}_1\mathrm{c}_1}}]}[\mathrm{Ad}_{\mathbf{T}_{\mathrm{c}_1\mathrm{w}}}]\mathcal{V}_{\mathrm{l}_1\mathrm{l}_2} &= \mathbf{0_{3\times1}} \\ \begin{bmatrix} [\mathbf{p}^{\mathrm{c}_1}_{\mathrm{c}_1\mathrm{w}}]\mathbf{R}_{\mathrm{c}_1\mathrm{w}} & \mathbf{R}_{\mathrm{c}_1\mathrm{w}} \end{bmatrix}(\cancelto{\mathbf{0}}{\mathcal{V}_{\mathrm{l}_1\mathrm{o}}} + \mathcal{V}_{\mathrm{ow}} + \mathcal{V}_{\mathrm{wf}} + \cancelto{\mathbf{0}}{\mathcal{V}_{\mathrm{fl}_2}}) &= \mathbf{0_{3\times1}}. \end{aligned}$$

Rewriting in the world frame $\{\mathrm{w}\}$ yields

$$\begin{bmatrix} [\mathbf{p}_{\mathrm{wc}_{1,i}}] & -\mathbf{I}_{3\times3} \end{bmatrix}(\mathcal{V}_{\mathrm{wf},i} - \mathcal{V}_{\mathrm{wo}}) = \mathbf{0}_{3\times1}\ \forall i = 1, \ldots, n. \tag{16}$$

2) Force equilibrium: Each fingertip is in force equilibrium under the quasistatic assumption

$$-\mathcal{F}_{\mathrm{con}} - \mathcal{F}_{\mathrm{spr}} = \mathbf{0}_{6\times1} \ \Leftrightarrow\ -\mathbf{W}_{\mathrm{con}} - \mathbf{W}_{\mathrm{spr}} = \mathbf{0}_{6\times6} \tag{17}$$

where $\mathcal{F}_{\text{con}}$ is the contact wrench exerted by the fingertip on the object. Taking the time derivative of $\mathcal{F}^{c_1}_{\text{con}}$ and substituting Equations (9), (10) and (17) yields

$$\begin{aligned}
&\dot{\mathcal{F}}^{c_1}_{\text{con}} = -\dot{\mathcal{F}}^{c_1}_{\text{spr}} \\
&= -\mathbf{W}^{c_1}_{\text{con}}(\mathcal{V}^{c_1}_{c_1\text{o}} + \mathcal{V}^{c_1}_{\text{ow}} + \mathcal{V}^{c_1}_{\text{wa}} + \cancelto{\mathbf{0}}{\mathcal{V}^{c_1}_{\text{af}_0}}) - [\mathrm{Ad}_{\mathbf{T}_{\text{f}_0c_1}}]^\intercal \dot{\mathcal{F}}^{\text{f}_0}_{\text{spr}} \\
&= \mathbf{W}^{c_1}_{\text{con}}\mathcal{V}^{c_1}_{\text{o}c_1} + \mathbf{W}^{c_1}_{\text{con}}[\mathrm{Ad}_{\mathbf{T}_{c_1\text{w}}}](\mathcal{V}_{\text{wo}} - \mathcal{V}_{\text{wa}}) - [\mathrm{Ad}_{\mathbf{T}_{\text{f}_0c_1}}]^\intercal \mathbf{K}^{\text{f}_0}_{\text{spr}}\mathcal{V}^{\text{f}_0}_{\text{f}_0\text{f}}.
\end{aligned} \tag{18}$$

Further substituting Equation (15) and transforming using Equations (5) and (6) yields

$$\begin{aligned}
\dot{\mathcal{F}}^{c_1}_{\text{con.}} =& [\mathrm{Ad}_{\mathbf{T}_{\text{wc}_1}}]^\intercal((\mathbf{K}_{\text{spr}} - \mathbf{W}_{\text{con}})(\mathcal{V}_{\text{wa}} - \mathcal{V}_{\text{wo}} - \mathcal{V}_{\text{oc}_1}) \\
&+ \mathbf{K}_{\text{spr}}(\mathcal{V}_{\text{fw}} + \mathcal{V}_{\text{wo}} + \mathcal{V}_{\text{oc}_1})) \\
=& [\mathrm{Ad}_{\mathbf{T}_{\text{wc}_1}}]^\intercal((\mathbf{K}_{\text{spr}} - \mathbf{W}_{\text{con}})\mathcal{V}_{\text{wa}} - \mathbf{K}_{\text{spr}}\mathcal{V}_{\text{wf}} \\
&+ \mathbf{W}_{\text{con}}(\mathcal{V}_{\text{wo}} + \mathcal{V}_{\text{oc}_1})),
\end{aligned} \tag{19}$$

where $\mathcal{V}_{\text{oc}_1}$ can be expressed using Equation (14) as

$$\begin{aligned}
\mathcal{V}_{\text{oc}_1} &= [\mathrm{Ad}_{\mathbf{T}_{\text{wc}_1}}]\mathbf{P}_1\mathcal{V}^{l_1}_{l_1l_2} \\
&= [\mathrm{Ad}_{\mathbf{T}_{\text{wc}_1}}]\mathbf{P}_1\cancelto{\mathbf{I}}{[\mathrm{Ad}_{\mathbf{T}_{l_1c_1}}]}[\mathrm{Ad}_{\mathbf{T}_{c_1\text{w}}}](\cancelto{\mathbf{0}}{\mathcal{V}_{l_1\text{o}}} + \mathcal{V}_{\text{ow}} + \mathcal{V}_{\text{wf}} + \cancelto{\mathbf{0}}{\mathcal{V}_{\text{f}l_2}}) \\
&= \widetilde{\mathbf{P}}_1(\mathcal{V}_{\text{wf}} - \mathcal{V}_{\text{wo}}),
\end{aligned} \tag{20}$$

where $\widetilde{\mathbf{P}}_1 := [\mathrm{Ad}_{\mathbf{T}_{\text{wc}_1}}]\mathbf{P}_1[\mathrm{Ad}_{\mathbf{T}_{\text{wc}_1}}]^{-1}$. On the other hand, $\dot{\mathbf{m}}^{c_1}_{\text{con}} \equiv \mathbf{0}_{3\times1}$. Using Equations (19) and (20) yields

$$\begin{aligned}
&\begin{bmatrix}-\mathbf{I}_{3\times3} & [\mathbf{p}_{\text{wc}_{1,i}}]\end{bmatrix}((\mathbf{K}_{\text{spr},i} - \mathbf{W}_{\text{con},i}\widetilde{\mathbf{P}}_{1,i})\mathcal{V}_{\text{wf},i} \\
&+ \mathbf{W}_{\text{con},i}(\widetilde{\mathbf{P}}_{1,i} - \mathbf{I}_{6\times6})\mathcal{V}_{\text{wo}} - (\mathbf{K}_{\text{spr},i} - \mathbf{W}_{\text{con},i})\mathcal{V}_{\text{wa},i}) \\
&= \mathbf{0}_{3\times1}\ \forall i = 1,\dots,n.
\end{aligned} \tag{21}$$

Moreover, the wrench equilibrium of the object under the quasistatic assumption is

$$\sum_{i=1}^{n}\mathcal{F}_{\text{con},i} = \mathcal{F}_{\text{ext}},$$

where $\mathcal{F}_{\text{ext}}$ is the wrench applied by the environment to the object. Taking the derivative and substituting Equations (10) and (19) yields

$$\sum_{i=1}^{n}(\mathbf{K}_{\text{spr},i}\mathcal{V}_{\text{wf},i}) = \sum_{i=1}^{n}((\mathbf{K}_{\text{spr},i} - \mathbf{W}_{\text{con},i})\mathcal{V}_{\text{wa},i}) - \dot{\mathcal{F}}_{\text{ext}}. \tag{22}$$

3) General solution: Stacking Equations (16), (21), and (22) yields

$$\mathbf{\Omega}_{\text{f\&o}}\mathbf{V}_{\text{f\&o}} = \boldsymbol{\beta}, \tag{23}$$

where $\mathbf{V}_{\text{f\&o}}$ consists of the object and fingertip velocities

$$\mathbf{V}_{\text{f\&o}} := \begin{bmatrix}\mathcal{V}_{\text{wf},1} \\ \vdots \\ \mathcal{V}_{\text{wf},n} \\ \mathcal{V}_{\text{wo}}\end{bmatrix},$$

while

$$\begin{cases}
\mathbf{\Omega}_{\text{f\&o}} := \begin{bmatrix}
\begin{bmatrix}\mathbf{A}_1 \\ \mathbf{D}_1\end{bmatrix} & \cdots & \mathbf{0}_{6\times6} & \begin{bmatrix}\mathbf{B}_1 \\ -\mathbf{D}_1\end{bmatrix} \\
\vdots & \ddots & \vdots & \vdots \\
\mathbf{0}_{6\times6} & \cdots & \begin{bmatrix}\mathbf{A}_n \\ \mathbf{D}_n\end{bmatrix} & \begin{bmatrix}\mathbf{B}_n \\ -\mathbf{D}_n\end{bmatrix} \\
\mathbf{K}_{\text{spr},1} & \cdots & \mathbf{K}_{\text{spr},n} & \mathbf{0}_{6\times6}
\end{bmatrix} \\
\boldsymbol{\beta} := \begin{bmatrix}
\begin{bmatrix}\mathbf{C}_1\mathcal{V}_{\text{wa},1} \\ \mathbf{0}_{3\times1}\end{bmatrix} \\
\vdots \\
\begin{bmatrix}\mathbf{C}_n\mathcal{V}_{\text{wa},n} \\ \mathbf{0}_{3\times1}\end{bmatrix} \\
\sum_{i=1}^{n}((\mathbf{K}_{\text{spr},i} - \mathbf{W}_{\text{con},i})\mathcal{V}_{\text{wa},i}) - \dot{\mathcal{F}}_{\text{ext}}
\end{bmatrix}
\end{cases}$$

are both known, where

$$\begin{cases}
\mathbf{A}_i := \begin{bmatrix}-\mathbf{I}_{3\times3} & [\mathbf{p}_{\text{wc}_{1,i}}]\end{bmatrix}(\mathbf{K}_{\text{spr},i} - \mathbf{W}_{\text{con},i}\widetilde{\mathbf{P}}_{1,i}) \\
\mathbf{B}_i := \begin{bmatrix}-\mathbf{I}_{3\times3} & [\mathbf{p}_{\text{wc}_{1,i}}]\end{bmatrix}\mathbf{W}_{\text{con},i}(\widetilde{\mathbf{P}}_{1,i} - \mathbf{I}_{6\times6}) \\
\mathbf{C}_i := \begin{bmatrix}-\mathbf{I}_{3\times3} & [\mathbf{p}_{\text{wc}_{1,i}}]\end{bmatrix}(\mathbf{K}_{\text{spr},i} - \mathbf{W}_{\text{con},i}) \\
\mathbf{D}_i := [[\mathbf{p}_{\text{wc}_{1,i}}], -\mathbf{I}_{3\times3}]
\end{cases}$$

for $\forall i = 1,\dots,n$. The rank of $\mathbf{\Omega}_{\text{f\&o}}$ depends on the configuration and the contact wrenches. For the forward mechanics problem, if $\mathrm{rank}(\mathbf{\Omega}_{\text{f\&o}}) = 6n + 6$, then $\mathbf{\Omega}_{\text{f\&o}}$ is invertible and Equation (23) gives a unique solution $\mathbf{V}_{\text{f\&o}} = \mathbf{\Omega}^{-1}_{\text{f\&o}}\boldsymbol{\beta}$. However, multiple solutions exist when $\mathbf{\Omega}_{\text{f\&o}}$ is not invertible, indicating that the problem in those singular configurations and contact wrenches cannot be solved under the current assumptions. Dynamics or a higher-order analysis would be needed to resolve such cases.