Supplementary Material for
Random Graph Asymptotics for Treatment Effect Estimation in Two-Sided Markets

A Proof of Proposition 1

For the direct effect, from Assumption 4, we can Taylor expand $f_i\left(w_i, \frac{M_i}{N_i}\right)$ as follows:

$$f_i\left(w_i, \frac{M_i}{N_i}\right) = f_i(w_i, \pi) + f_i'(w_i, \pi)\left(\frac{M_i}{N_i} - \pi\right) + f_i''(w_i, \pi_i^*)\left(\frac{M_i}{N_i} - \pi\right)^2$$

for some π_i^* between π and $\frac{M_i}{N_i}$. Note that $\mathbb{E}\left[\left(\frac{M_i}{N_i} - \pi\right) \mid G, f(\cdot)\right] = 0$ and $\mathbb{E}\left[\left(\frac{M_i}{N_i} - \pi\right)^2 \mid G, f(\cdot)\right] = \pi(1-\pi)/N_i$. Therefore, the expectation conditional on the inference graph and the potential outcome functions is

$$\mathbb{E}\left[f_i\left(w_i, \frac{M_i}{N_i}\right) \mid G, f(\cdot)\right] = f_i(w_i, \pi) + \mathbb{E}[f_i''(W_i, \pi_i^*) \mid G, f(\cdot)]\pi(1-\pi)/N_i.$$

By (3), $|f_i''(W_i, \pi_i^*)| \leq B$, thus $|\mathbb{E}[f_i''(W_i, \pi_i^*) \mid G, f(\cdot)]\pi(1-\pi)/N_i| \leq B\pi(1-\pi)/N_i$. Therefore,

$$\bar{\tau}_{\mathrm{DIR}} = \frac{1}{n}\sum_i \mathbb{E}\left[f_i\left(1, \frac{M_i}{N_i}\right) - f_i\left(0, \frac{M_i}{N_i}\right) \mid G, f(\cdot)\right] = \frac{1}{n}\sum_i \left(f_i(1,\pi) - f_i(0,\pi)\right) + \mathcal{O}\left(\frac{B}{\min_i N_i}\right).$$

For the indirect effect, recall that the total effect is

$$\bar{\tau}_{\mathrm{TOT}}(\pi) = \frac{d}{d\pi}\bar{V}(\pi), \quad \bar{V}(\pi) = \frac{1}{n}\sum_i \mathbb{E}_\pi[Y_i | Y(\cdot)].$$

For any $\pi' \in (0,1)$, the Horvitz–Thompson estimate of $\bar{V}(\pi')$ is

$$\hat{V}(\pi') = \frac{1}{n}\sum_{i=1}^n Y_i \left(\frac{\pi'}{\pi}\right)^{M_i + W_i} \left(\frac{1-\pi'}{1-\pi}\right)^{(N_i - M_i) + (1 - W_i)}.$$

Thus by taking the derivative of $\hat{V}(\pi')$, we can define

$$\hat{\tau}^U_{\mathrm{TOT}}(\pi) = \left[\frac{d}{d\pi'}\hat{V}(\pi')\right]_{\pi'=\pi} = \frac{1}{n}\sum_i Y_i \left(\frac{M_i + W_i}{\pi} - \frac{N_i - M_i + 1 - W_i}{1-\pi}\right).$$

We can verify that this estimator is unbiased for $\bar{\tau}_{\mathrm{TOT}}(\pi)$ by following the line of argumentation in Section 4.1. It can be naturally decomposed into two parts:

$$\hat{\tau}^U_{\mathrm{TOT}} = \frac{1}{n}\sum_i Y_i \left(\frac{W_i}{\pi} - \frac{1 - W_i}{1-\pi}\right) + \frac{1}{n}\sum_i Y_i \left(\frac{M_i}{\pi} - \frac{N_i - M_i}{1-\pi}\right) = \hat{\tau}^{\mathrm{HT}}_{\mathrm{DIR}} + \hat{\tau}^U_{\mathrm{IND}}.$$

Recalling that $\hat{\tau}^{\mathrm{HT}}_{\mathrm{DIR}}$ is unbiased for $\bar{\tau}_{\mathrm{DIR}}$, we see that $\hat{\tau}^U_{\mathrm{IND}}$ is also unbiased for $\bar{\tau}_{\mathrm{IND}}$, thus

$$\bar{\tau}_{\mathrm{IND}} = \mathbb{E}[\hat{\tau}^{U}_{\mathrm{IND}} \mid G, f(\cdot)] = \frac{1}{n\pi(1-\pi)} \sum_i \mathbb{E}[Y_i \left(M_i - \pi N_i\right) \mid G, f(\cdot)].$$

By Taylor expanding f_i, we can rewrite $\bar{\tau}_{\mathrm{IND}}$ as

$$\begin{aligned}
\bar{\tau}_{\mathrm{IND}} =& \frac{1}{n\pi(1-\pi)} \sum_i \mathbb{E}\left[\left(f_i'(W_i,\pi)\left(\frac{M_i}{N_i}-\pi\right) + \frac{1}{2} f_i''(W_i,\pi_i^*)\left(\frac{M_i}{N_i}-\pi\right)^2\right)(M_i - \pi N_i) \mid G, f(\cdot)\right] \\
=& \frac{1}{n\pi(1-\pi)} \sum_i \mathbb{E}\left[f_i'(W_i,\pi) \mid G, f(\cdot)\right] \mathbb{E}\left[\frac{(M_i-\pi N_i)^2}{N_i} \mid G, f(\cdot)\right] \\
&+ \frac{1}{2n\pi(1-\pi)} \sum_i \mathbb{E}\left[f_i''(W_i,\pi_i^*)\frac{(M_i-\pi N_i)^3}{N_i^2} \mid G, f(\cdot)\right] \\
=& D_1 + D_2,
\end{aligned}$$

where $D_1 = \frac{1}{n}\sum_i(\pi f_i'(1,\pi) + (1-\pi)f_i'(0,\pi))$. For D_2, note that

$$\mathbb{E}\left[\left(f_i''(W_i,\pi_i^*)\frac{(M_i-\pi N_i)^3}{N_i^2}\right)^2 \mid G, f(\cdot)\right] \leq \frac{B^2\mathbb{E}[(M_i-\pi N_i)^6 \mid G, f(\cdot)]}{N_i^4}.$$

Since each neighbor independently receives treatment with probability π, M_i follows a binomial distribution: $M_i \sim \mathrm{Binomial}(N_i, \pi)$. For a binomial random variable M_i, the central moments $\mathbb{E}[(M_i - \mathbb{E}[M_i])^k]$ generally scale as:

$$\mathbb{E}[(M_i - \mathbb{E}[M_i])^k] = \mathcal{O}(N_i^{k/2}) \quad \text{for } k \geq 2.$$

So we get $\mathbb{E}[(M_i - \pi N_i)^6] = \mathcal{O}(N_i^3)$. Hence $D_2 = \mathcal{O}\left(\frac{B}{\sqrt{\min_i N_i}}\right)$, and

$$\bar{\tau}_{\mathrm{IND}} = \frac{1}{n}\sum_i \left(\pi f_i'(1,\pi) + (1-\pi)f_i'(0,\pi)\right) + \mathcal{O}\left(\frac{B}{\sqrt{\min_i N_i}}\right).$$

Furthermore, by plugging in $f_k(w,x)$ we obtained in (2), we get the results in Proposition 1.

B Proof of Theorem 2

By Lemma 15 (Li & Wager, 2022) and the assumption that $\liminf \log\rho_N/\log N > -1$, we have $\min_i N_i = \Omega(N\rho_N)$, and

$$\hat{\tau}^{\mathrm{HT}}_{\mathrm{DIR}} - \bar{\tau}_{\mathrm{DIR}} = \frac{1}{N}\sum_i \left(\frac{f_i(1,\pi)}{\pi} + \frac{f_i(0,\pi)}{1-\pi} + \sum_{j\neq i}\frac{E_{ij}}{N_j}(f_j'(1,\pi) - f_j'(0,\pi))\right)(W_i - \pi) + o_p\left(\frac{B}{\sqrt{N}}\right).$$

Define $R_i = \frac{f_i(1,\pi)}{\pi} + \frac{f_i(0,\pi)}{1-\pi}$. By Assumption 4, $|f_j'(1,\pi) - f_j'(0,\pi)| \leq 2B$. For a fixed j, given U_j, $E_{jk} \overset{\text{i.i.d.}}{\sim} \mathrm{Bernoulli}(g_N(U_j))$. Thus we can rewrite as follow:

$$\sum_{j\neq i}\frac{E_{ij}}{N_j}(f_j'(1,\pi) - f_j'(0,\pi)) = \sum_{j\neq i}\frac{E_{ij}(f_j'(1,\pi) - f_j'(0,\pi))}{(N-1)g_N(U_j)} - \sum_{j\neq i}\frac{E_{ij}(f_j'(1,\pi) - f_j'(0,\pi))(N_j - (N-1)g_N(U_j))}{(N-1)g_N(U_j)N_j}.$$

For the first term, define

$$\mathbb{E}\left[\frac{E_{ij}(f_j'(1,\pi) - f_j'(0,\pi))}{g_N(U_j)}\middle| U_i\right] = Q_{N,i}$$

Then we have

$$\mathbb{E}\left[\left(\sum_{j\neq i}\frac{E_{ij}(f_j'(1,\pi)-f_j'(0,\pi))}{(N-1)g_N(U_j)}-Q_{N,i}\right)^2\right]=\frac{1}{N-1}\mathbb{E}\left[\left(\frac{E_{ij}(f_j'(1,\pi)-f_j'(0,\pi))}{g_N(U_j)}-Q_{N,i}\right)^2\right]$$
$$\leq\frac{1}{N-1}\mathbb{E}\left[\left(\frac{E_{ij}(f_j'(1,\pi)-f_j'(0,\pi))}{g_N(U_j)}\right)^2\right]=\mathcal{O}\left(\frac{B^2}{(N-1)\rho_N}\right).$$

Therefore, the first term can be well approximated by $Q_{N,i}$ as $N\to\infty$.

For the second term, $(N_j-E_{ij})\sim\text{Binomial}(N-2,g_N(U_j))$ conditional on U and $f(\cdot)$. Thus

$$\frac{E_{ij}(f_j'(1,\pi)-f_j'(0,\pi))(N_j-(N-1)g_N(U_j))}{g_N(U_j)N_j}=\frac{E_{ij}(f_j'(1,\pi)-f_j'(0,\pi))((N_j-E_{ij}+1)-(N-1)g_N(U_j))}{g_N(U_j)(N_j-E_{ij}+1)},$$

$$\mathbb{E}\left[\frac{(N_j-E_{ij}+1)-(N-1)g_N(U_j)}{g_N(U_j)(N_j-E_{ij}+1)}\middle|U,f(\cdot)\right]=\frac{1}{g_N(U_j)}-\mathbb{E}\left[\frac{N-1}{N_j-E_{ij}+1}\middle|U,f(\cdot)\right]$$
$$=\frac{(1-g_N(U_j))^N}{g_N(U_j)}\leq\frac{(1-c_l\rho_N)^N}{c_l\rho_N}\leq e^{-CN\rho_N}$$

for some constant C. By following the lines of argumentation in Lemma 15-17 (Li & Wager, 2022), we have

$$\mathbb{E}\left[\left(\sum_{j\neq i}\frac{E_{ij}}{N_j}\left(f_j'(1,\pi)-f_j'(0,\pi)-Q_{N,i}\right)\right)^2\right]\leq\frac{CB^2}{N\rho_N}.$$

Thus the Horvitz-Thompson estimator can be written as

$$\hat{\tau}_{\text{DIR}}^{\text{HT}}=\bar{\tau}_{\text{DIR}}+\frac{1}{N}\sum_i(R_i+Q_{N,i})(W_i-\pi)+o_p\left(\frac{B}{\sqrt{N}}\right).$$

Define

$$Q_i=\mathbb{E}\left[\frac{G(U_i,U_j)(f_j'(1,\pi)-f_j'(0,\pi))}{g(U_j)}\middle|U_i\right].$$

Using the dominated convergence theorem, the asymptotic behavior of $Q_{N,i}$ will essentially mirror that of Q_i. Now we compute the variance of $\hat{\tau}_{\text{DIR}}^{\text{HT}}-\bar{\tau}_{\text{DIR}}$ to get the central limit theorem:

$$\mathbb{E}[(R_i+Q_{N,i})(W_i-\pi)]^2=\mathbb{E}\left[(R_i+Q_{N,i})^2\right]\mathbb{E}\left[(W_i-\pi)^2\right]=\pi(1-\pi)\mathbb{E}\left[(R_i+Q_{N,i})^2\right]$$

$$\to\pi(1-\pi)\mathbb{E}\left[(R_i+Q_i)^2\right],$$

Then

$$\sqrt{n}(\hat{\tau}_{\text{DIR}}^{\text{HT}}-\bar{\tau}_{\text{DIR}})\xrightarrow{d}\mathcal{N}\left(0,\pi(1-\pi)\mathbb{E}\left[(R_i+Q_i)^2\right]\right).$$