FINE-GRAINED DIFFERENTIABLE PHYSICS: A YARN- LEVEL MODEL FOR FABRICS
–SUPPLEMENTARY MATERIAL–

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A APPENDIX

All simulations are available in the accompanied videos:https://youtu.be/pCB8AD9R4Dk

A.1 TRAINING DETAILS

Our ground-truth data is simulated with a piece of cloth hanging at its two corners, blown by a wind with a constant magnitude (Figure 1). The simulation is conducted with a time step h = 0.001.

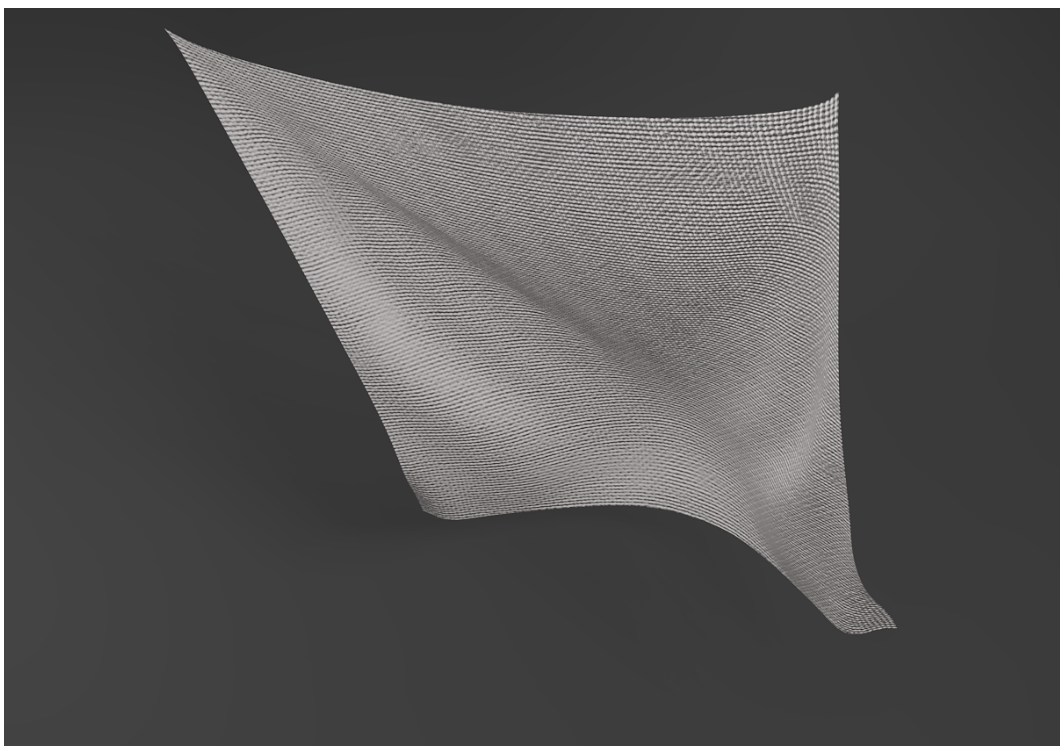

Figure 1: A piece of square cloth blown by constant magnitude wind.

In all experiments, we use Stochastic Gradient Descent and run 70 epochs for training, except in XXX-(1,3) where we trained our model for 90 epochs. The training is conducted on a machine with Intel(R) Xeon(R) Silver 4216 CPU, 187G memory, NVIDIA TITAN RTX graphics card on Linux. The main factors of training speed are the cloth size and the training data size. In our experiments,

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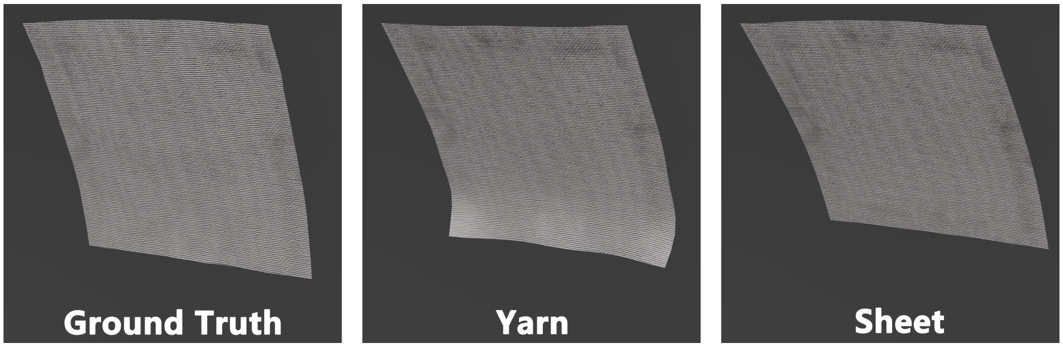

Figure 2: The visual learning results of the differentiable sheet-level simulator (Liang et al., 2019) and our model learns on the data generate by (Narain et al., 2012)

the training takes approximately 68, 133, and 328 seconds per epoch on a 17×17 cloth with training data containing 5, 10, and 25 frames respectively. The training per epoch takes approximately 13, 106, 328, and 1310 seconds with 25 training frames, on a 5×5, 10×10, 17×17, and 25×25 cloth respectively.

Additional experiments. Further, we also conduct comparisons on the data simulated under the same settings by a sheet-level simulator (Narain et al., 2012), which tends to be stiffer. This is to compare the performance when the ground-truth does not contain the same level of subtle dynamics. Since there is no Eulerian coordinates in the sheet-level simulation, we only use Lagragian coordinates in the loss function. The visual comparison is in Figure 2 and the prediction errors are shown in Table 1. Our model can learn comparable results on 5 frames, and better results on 10 and 25 frames. The slightly worse 5-frame result is mainly because the first 5 frames contain small dynamics and therefore is insufficient for our model to learn the overall stiffness of the cloth. However, when 10 and 25 frames are given, the learning is significantly improved and even outperforms Liang et al. (2019). Also, since there is no woven pattern information in the ground truth, we examine our model across the three woven patterns, all giving more accurate predictions. Overall, the comparisons show our model has higher prediction accuracy regardless the granularity of the underlying physics model.

Parameters. We induce prior knowledge to limit the parameter learning within valid ranges, so that the multi-solution problem, also met by existing methods, can be mitigated. All cloths we used are made of two types of yarns. We use the same range, $d \in [0.001, 0.003]$, $b \in [0.00005, 0.00018]$, $S \in [0, 1200]$ and $\mu \in [0, 1.0]$ for both yarns, where d, b, S and μ are the density, bending modulus, shear modulus and friction coefficient respectively. We use $s_1 \in [0, 800000]$ and $s_2 \in [0, 300000]$ for the stretching for both yarns. For other coefficients, we use $k_f = 1000$ and $d_f = 1000$ in the friction force, $c = 3$ and $\sigma = 0.6$ in the shear force, $k_c = 1$ in yarn-to-yarn collision in all experiments.

When training our model on the data generated by a sheet-level cloth simulator (Narain et al., 2012), we use a pure woven cloth made of one type of yarn. This is because it is not possible to specify multiple yarn behaviors in a sheet simulator, so we use a pure yarn cloth for generating the ground truth. The cloth parameters are from the 'white-dots-on-black' cloth in Wang et al. (2011) which is 100 percent polyester. To learn from it, we employ all three woven patterns in our model as there is no prior knowledge about the woven pattern of the 'white-dots-on-black' cloth. We also fix the friction coefficient $\mu = 0.5$ and impose the ranges on parameters shown in Table 5. Finally, we would like to point it out in real-world applications, information such as woven patterns and yarn materials are easily available so that the ranges of parameter values such as density, bending and stretching can be obtained. Although the knowledge of shearing and friction cannot be easily acquired, the ranges we use are general enough.

Note that in all experiments, the prior knowledge we induce is only a weak prior, i.e. using the same general ranges for multiple experiments across different woven patterns, so that the learning success still lies in our model's ability to infer the right parameter values.

Table 1: Testing errors ($\times 10^{-6}$) of our model and (Liang et al., 2019) trained on 5, 10 and 25 frames generated by (Narain et al., 2012).

fabrics/frames	5	10	25
Plain-(1,2)	6.702	1.167	0.496
Satin-(1,2)	7.972	1.225	0.624
Twill-(1,2)	8.218	1.772	0.776
(Liang et al., 2019)	4.098	4.752	1.716

Table 2: Learning cloth parameters with different initial values (part one).

Size	Shear S	Friction μ
5×5	1011.79 ± 6.12	0.39 ± 0.08
10×10	983.41 ± 6.84	0.44 ± 0.03
17×17	962.29 ± 8.99	0.47 ± 0.06

Parameter Initialization. The material estimation results are affected by initialization. To test if our model can learn stably, we report the mean and the standard deviation of multiple experiments with different parameter initial values. The initial values of the physical parameters are randomly selected from a range of $\pm 10\%$ of the average of the two yarns. For instance, in learning the stretch in Plain-(1,2), we only know the ranges of the stretching parameters Y1 and Y2 of Yarn1 and Yarn2 but not the exact values. Therefore, when initializing Y1 and Y2, we randomly sample values from a range of $\pm 10\%$ of the mean stretch stiffness of the Yarn1 and Yarn2, $[\text{mean(Y1, Y2)} \times 0.9, \text{mean(Y1, Y2)} \times 1.1]$ for initialization. The results of the 5 repetitions are shown Table 2 and Table 3. Given that the standard deviations are small, it shows that our model can stably learn reasonable parameter values.

Different Force Magnitude. To evaluate the influence of the wind force, we conduct experiments using 5N, 10N, and 15N wind force to blow a piece of 17×17 Plain-(1,2) cloth. The learning result is shown in the Table 4 which demonstrate wind force strength has ignorable influence on the learned parameters.

Table 4: Learning cloth physical parameters with different wind force.

Wind	Shear S	Friction μ	Yarn	Density	Stretch	Bend
5	947	0.402	1	1.969×10^{-3}	505421	1.323×10^{-4}
			2	2.440×10^{-3}	171304	1.034×10^{-4}
10	942	0.520	1	2.026×10^{-3}	494109	1.311×10^{-4}
			2	2.441×10^{-3}	168267	1.049×10^{-4}
15	934	0.586	1	2.029×10^{-3}	487918	1.341×10^{-4}
			2	2.437×10^{-3}	167601	1.066×10^{-4}

Table 3: Learning cloth parameters with different initial values (part two).

Size	Yarn	Density	Stretch	Bend
5×5	1	$1.98 \times 10^{-3} \pm 3.00 \times 10^{-5}$	498595 ± 8862	$1.37 \times 10^{-4} \pm 1.41 \times 10^{-6}$
	2	$2.45 \times 10^{-3} \pm 4.81 \times 10^{-5}$	186710 ± 3776	$1.11 \times 10^{-4} \pm 4.78 \times 10^{-6}$
10×10	1	$2.03 \times 10^{-3} \pm 5.04 \times 10^{-5}$	542375 ± 7099	$1.44 \times 10^{-4} \pm 2.08 \times 10^{-6}$
	2	$2.47 \times 10^{-3} \pm 4.73 \times 10^{-5}$	180032 ± 1848	$1.05 \times 10^{-4} \pm 8.18 \times 10^{-6}$
17×17	1	$2.00 \times 10^{-3} \pm 6.66 \times 10^{-5}$	519993 ± 3175	$1.43 \times 10^{-4} \pm 5.55 \times 10^{-6}$
	2	$2.45 \times 10^{-3} \pm 5.04 \times 10^{-5}$	176232 ± 1514	$1.19 \times 10^{-4} \pm 6.50 \times 10^{-6}$

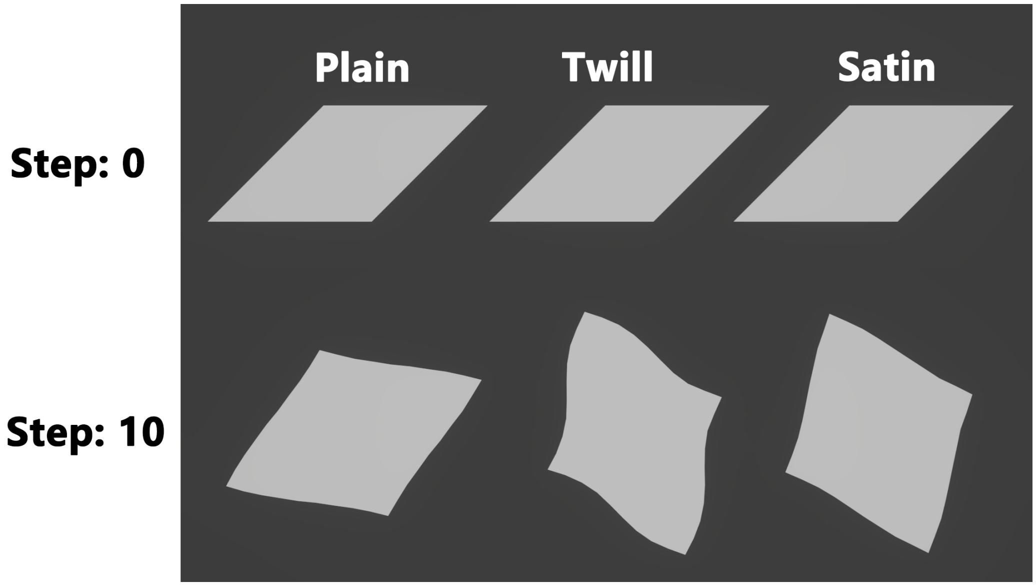

Figure 3: Three pieces of cloth woven in different patterns show different dynamics.

Table 5: Cloth parameters' initial values and ranges when ground-truth generated by sheet-level cloth simulator(Narain et al., 2012)

Name	Density(kg/m)	Stretch(N/m)	Bend(N/m)	Shear(N/m)
Value	0.004	1e6	0.0001	20000
Upper limit	0.008	2e6	0.0002	30000
Lower limit	0.001	0	0	0

Influence of Woven Patterns. The investigation on different woven patterns is crucial as they affect the cloth dynamics significantly. To show this, we conducted simulations of three pieces of cloths with the same parameters, but with different woven patterns. We shear three pieces of cloth then release them. The Figure 3 shows three pieces of cloth in the initial state and 10 steps later. There are obvious differences after merely 10 steps. This demonstrates woven patterns have considerable influences on the overall mechanical properties.

Table 6: Testing errors ($\times 10^{-6}$) of our model (left) and (Liang et al., 2019) (right) trained on 5, 10 and 25 frames. Ground-truth generated by a yarn-level simulator (Cirio et al., 2014).

fabrics/frames	5	10	25	5	10	25
Plain-(1,2)	1.152×10^{-4}	1.068×10^{-4}	3.962×10^{-5}	1.462	0.7375	0.4124
Plain-(1,3)	1.516×10^{-4}	1.268×10^{-4}	3.555×10^{-5}	1.608	0.7906	0.4567
Plain-(2,3)	5.233×10^{-4}	1.291×10^{-4}	2.117×10^{-5}	1.952	0.5999	0.2294
Satin-(1,2)	1.134×10^{-4}	1.070×10^{-4}	4.285×10^{-5}	1.466	0.7405	0.4146
Satin-(1,3)	1.551×10^{-4}	1.355×10^{-4}	4.362×10^{-5}	1.624	0.8004	0.4445
Satin-(2,3)	6.254×10^{-4}	1.355×10^{-4}	4.413×10^{-5}	2.128	0.5949	0.2265
Twill-(1,2)	1.130×10^{-4}	1.068×10^{-4}	4.208×10^{-5}	1.472	0.7451	0.4160
Twill-(1,3)	1.550×10^{-4}	1.349×10^{-4}	4.200×10^{-5}	1.633	0.8059	0.4577
Twill-(2,3)	6.470×10^{-4}	1.352×10^{-4}	4.938×10^{-5}	2.181	0.5994	0.2278

Figure 4: The visual results of our model learning on different cloth sizes. From left to right: 5×5, 10×10, 17×17 and 25×25.

Table 7: Learned parameters by Bayesian Optimization on different kinds of fabrics.

Frames	Density	Stretch	Bend	Density	Stretch	Bend
5	2.483×10^{-3}	647270	0.636×10^{-4}	2.125×10^{-3}	270641	1.576×10^{-4}
10	2.176×10^{-3}	577235	0.798×10^{-4}	2.264×10^{-3}	217144	1.542×10^{-4}
25	2.328×10^{-3}	537434	1.687×10^{-4}	2.097×10^{-3}	249896	0.976×10^{-4}
5	2.202×10^{-3}	605289	1.403×10^{-4}	2.349×10^{-3}	272153	0.868×10^{-4}
10	1.669×10^{-3}	257877	1.582×10^{-4}	2.635×10^{-3}	268451	0.529×10^{-4}
25	1.454×10^{-3}	315715	1.213×10^{-4}	2.950×10^{-3}	23702	1.656×10^{-4}
5	2.514×10^{-3}	250093	1.611×10^{-4}	2.363×10^{-3}	20371	0.985×10^{-4}
10	2.964×10^{-3}	164021	0.524×10^{-4}	2.255×10^{-3}	49648	1.225×10^{-4}
25	2.414×10^{-3}	73734	0.890×10^{-4}	2.436×10^{-3}	267452	1.113×10^{-4}

A.2 Visual results

Here we show some snapshots of our model on cloths of different sizes in Figure 4. As expected, small cloths tend to show low dynamics and appear to be more 'rigid'. Bigger cloths tend to have more subtle dynamics such as wrinkles, even under the same external impact, i.e. gravity and wind with a constant magnitude. More visual results can be found in the supplementary video.

A.3 Yarn-level versus Sheet-level

A full comparison between our model and (Liang et al., 2019) is shown in Table 6, where a yarn-level simulator (Cirio et al., 2016) is used to generate the ground-truth. We exhaustively conduct comparisons using all combinations of yarns and woven patterns. We can see that our model is consistently better than (Liang et al., 2019) by large margins. Visually, we show snapshots in Figure 5. The sheet model results are in general more rigid and do not contain as much subtle dynamics as ours do, across different training frame numbers. Since 5, 10 and 25 frames contain different amounts of information on (subtle) motion dynamics, Figure 5 shows that there is a lack of granularity in the sheet model when capturing subtle dynamics compared with ours.

Further, we also show the plots on the data efficiency in Figure 6, under all 9 yarn-woven pattern combinations, across different amounts of training data. In all settings, our data efficiency is significantly higher. By extrapolation, it would take a large number of extra training frames for the sheet-level model to achieve similar accuracy. More comparisons are also available in the supplementary video.

A.4 Our model versus Bayesian Optimization

Table 8 shows the testing errors of the Bayesian Optimization. Although the MSE errors are small, the learned parameters are far from the ground truth (shown in the Table 7), which is somewhat surprising. After examining the results, we find that Bayesian Optimization suffers from the multi-solution problem so that it merely gives a set of working parameters instead of the true parameters. In other words, although the prediction error is low, physically speaking, the learned parameters are far from the true materials. This happens even when we use the same parameter ranges as in

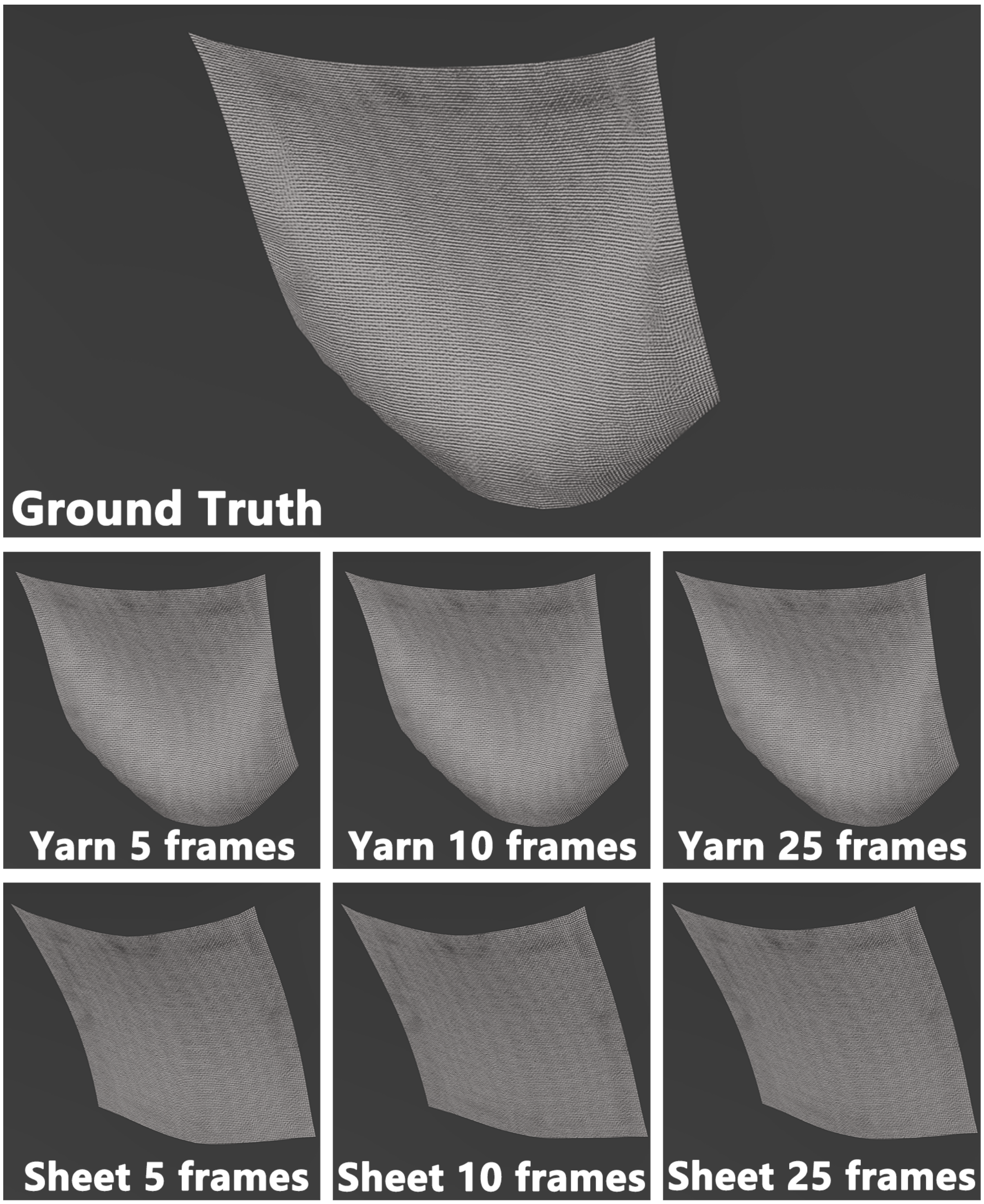

Figure 5: The visual results of Plain-(1, 2) ground-truth, our model, and sheet-level model trained with different number of frames. The snapshots are the $133th$ frame of the simulations after learning.

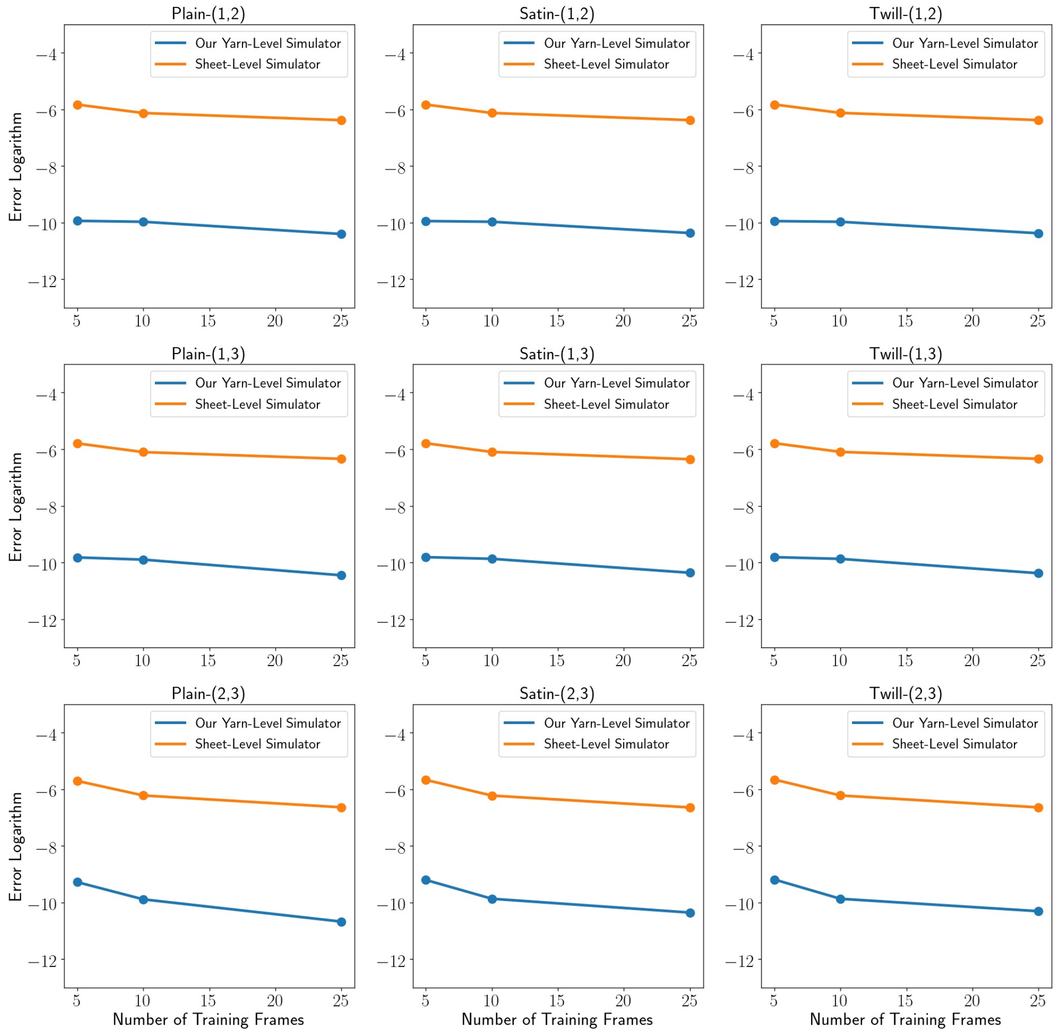

Figure 6: Prediction error logarithm vs training data.

our model. This is an intrinsic property of Bayesian optimization which is based on sampling, and therefore difficult to avoid during learning.

Table 8: Testing error ($\times 10^{-6}$) of Bayesian Optimization with yarn-level simulator (Cirio et al., 2016) learned on 5, 10, and 25 frames.

Fabrics/Frames	5	10	25
Plain-(1,2)	0.512	0.176	0.109
Plain-(1,3)	1.280	1.269	0.738
Plain-(2,3)	28.19	19.22	18.16

A.5 Control Experiment Setting

The control experiment scenario is illustrated in the Figure 7.

Figure 7: A square cloth is thrown from the table into the black box by four forces applied on the four corners of the cloth.

A.6 SIGNIFICANT ERROR IN VISUAL

We discussed the significance of the small error in physics-based simulation. Figure 8 and Figure 9 visually prove our explanations in the main paper: the error accumulates over time and increases with increasing cloth size.

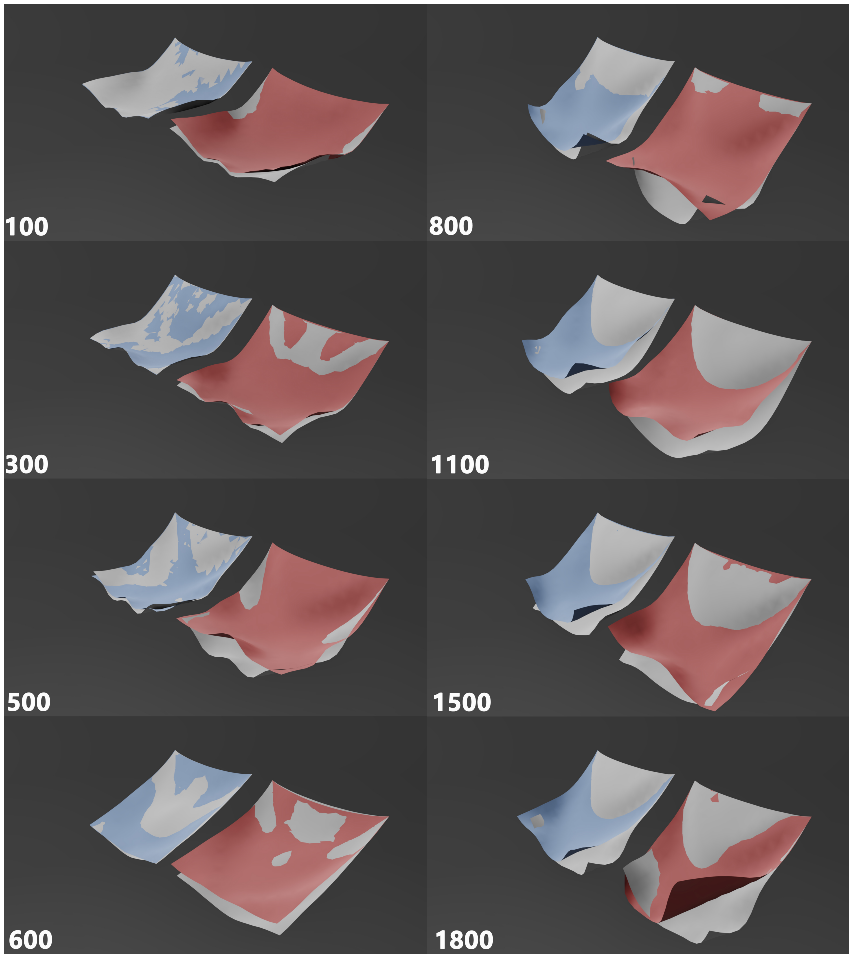

Figure 8: Visual differences in long simulations. The grey cloth is ground truth. The blue cloth and the red cloth are simulated with the parameters learned by our model and BO. The blue cloth shows smaller visual differences than the red one.

B Differentiable Yarn-level Cloth Simulator

In this section, we give the full details of our model and mathematical derivation.

B.1 Intro yarn force models

Representing the cloth as in Figure 10, we employ an EoL discretization (Sueda et al., 2011) and denote the spatial positions of crossing nodes in Lagrangian coordinates and represent the contact sliding movement in Eulerian coordinates, $\mathbf{q}_i \equiv (\mathbf{x}_i, u_i, v_i)$ where $\mathbf{x}_i \in \mathbb{R}^3$ implies crossing node i's spatial position and (u_i, v_i) the node's position in the material frame. The two end points of yarns are taken as special crossing nodes as they do not contact with other yarns and therefore have no Eulerian terms, i.e. $\mathbf{q}_j \equiv \mathbf{x}_i$. Therefore, on a $r(rows) \times c(columns)$ cloth, there are $(r-2) \times (c-2)$ crossing nodes with five Degrees of Freedom (DoFs) and $2r + 2c - 4$ crossing nodes with three DoFs. Every two neighboring crossing nodes on the same warp/weft delimit a warp/weft segment. A warp segment whose two end points are $\mathbf{q}_0$ and $\mathbf{q}_1$ is denoted as $[\mathbf{q_0}, \mathbf{q_1}]$ and its position is

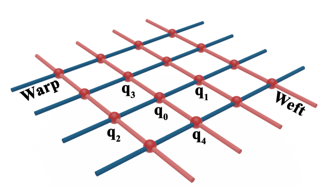

Figure 10: Blue and red rods denote warps and wefts respectively. $\mathbf{q}$s are the crossing nodes.

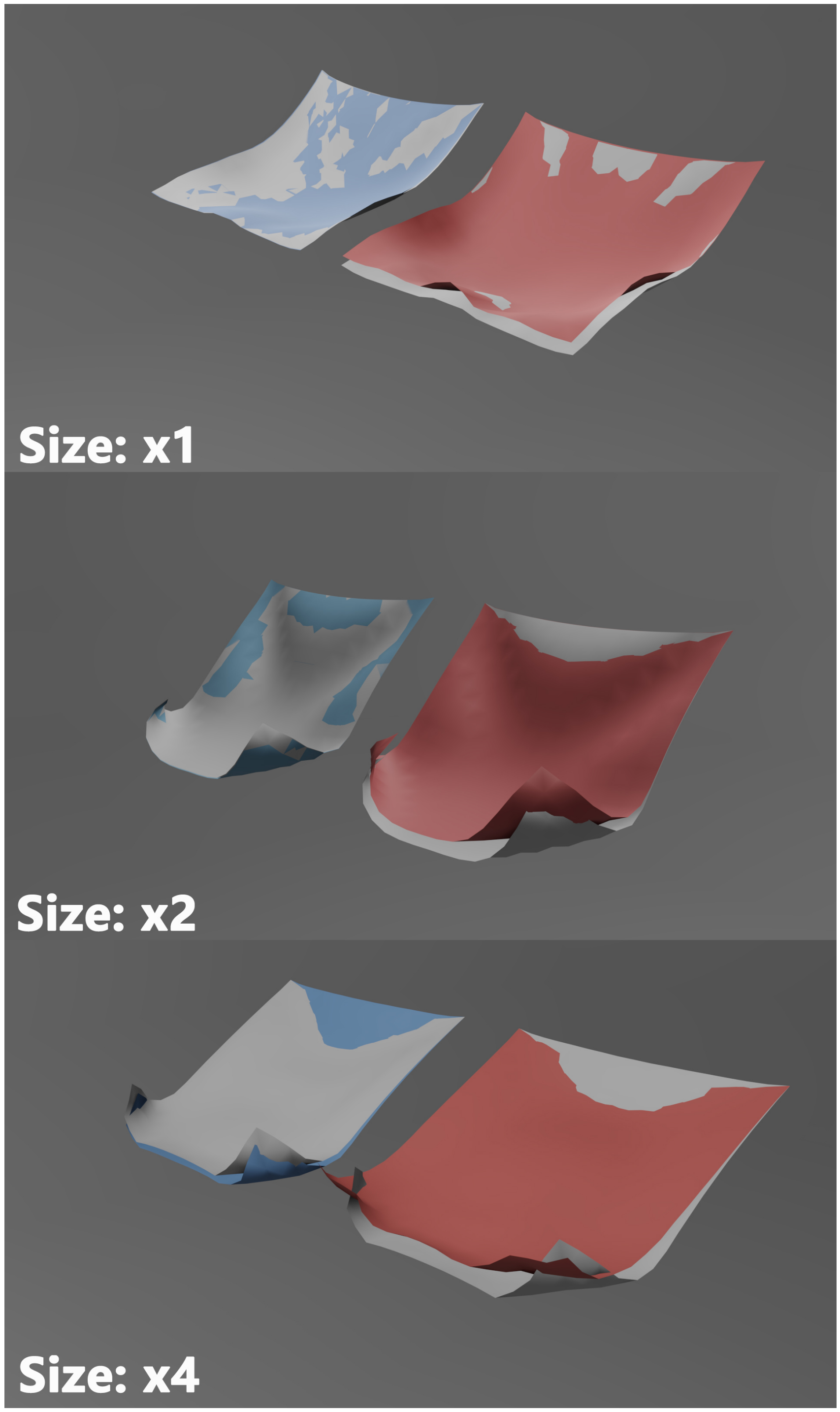

Figure 9: Visual differences on larger cloths and long simulation (500 steps). The grey cloth is ground truth. The blue cloth and the red cloth are simulated with the parameters learned by our model and BO. The blue cloth shows smaller differences than the red one.

$(\mathbf{x}_0, \mathbf{x}_1, u_0, u_1)$ (shown in 10). This way, a woven cloth is discretized into crossing nodes and segments which are the primitive units of the simulated cloth. Every segment is assumed to be straight so that linear interpolation can be employed on the segment, e.g. the spatial position of a point in the segment $[\mathbf{q_0}, \mathbf{q_1}]$ is $\mathbf{x}(u) = \frac{u-u_0}{\Delta u}\mathbf{x}_0 + \frac{u_1-u}{\Delta u}\mathbf{x}_1$, where u is the point's position in Eulerian coordinates and $\Delta u = u_1 - u_0$ is the length of the segment. We use L to denote the distances between neighbor yarns and R to denote the yarn radius.

B.2 SYSTEM EQUATION FOR SIMULATION

A cloth's state at time t, $\mathcal{S}_{(t)} = \{\mathcal{Q}_{(t)}, \dot{\mathcal{Q}}_{(t)}\}$, includes all its crossing nodes' positions $\mathcal{Q} = \{\mathbf{q}_i | i = 1, 2, \dots, N\}$ and velocities $\dot{\mathcal{Q}} = \{\dot{\mathbf{q}}_i | i = 1, 2, \dots, N\}$, where N is the number of crossing nodes. Knowing the states, then we can calculate the internal and external forces:

$$\mathbf{F} = \mathbf{M}\ddot{\mathbf{q}} = \frac{\partial T}{\partial \mathbf{q}} - \frac{\partial V}{\partial \mathbf{q}} - \dot{\mathbf{M}}\dot{\mathbf{q}} \tag{1}$$

where $\mathbf{q}$, $\dot{\mathbf{q}}$, and $\ddot{\mathbf{q}}$ are the nodes general position, velocity, and acceleration respectively, with a dimension $l = 3 \times r \times c + 2 \times (r-2) \times (c-2)$. $\mathbf{M} \in \mathbb{R}^{l \times l}$ is the general mass matrix. The model assumes mass is distributed homogeneously in one segment, so the mass matrix of a warp segment $[\mathbf{q}_0, \mathbf{q}_1]$ is

$$\mathbf{M}_{0,1} = \frac{1}{6}\Delta u \rho \begin{pmatrix} 2\mathbf{I}_3 & \mathbf{I}_3 & -2\mathbf{w} & -\mathbf{w} \\ \mathbf{I}_3 & 2\mathbf{I}_3 & -\mathbf{w} & -2\mathbf{w} \\ -2\mathbf{w}^\top & -\mathbf{w}^\top & 2\mathbf{w}^\top\mathbf{w} & \mathbf{w}^\top\mathbf{w} \\ -\mathbf{w}^\top & -2\mathbf{w}^\top & \mathbf{w}^\top\mathbf{w} & 2\mathbf{w}^\top\mathbf{w} \end{pmatrix} \tag{2}$$

where $\mathbf{w} = \frac{\mathbf{x}_1 - \mathbf{x}_0}{\Delta u}$, and ρ is yarn density. T and V are the kinetic and potential energy respectively. As the partial derivative of energy with respect to position is force, the right hand terms in Equation 1 are inertia, conservative forces, and part of the time derivative of $\mathbf{M}\dot{\mathbf{q}}$. Non-conservative forces are added to the right side of the equation.

We employ implicit Euler for stability in large steps (Baraff & Witkin, 1998). Given the acceleration $\ddot{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{F}$ and the change of speed over time step h, $\Delta\dot{\mathbf{q}}$ can be approximated by $\Delta\dot{\mathbf{q}} = h\mathbf{M}^{-1}\mathbf{F}_{(t+1)}$, where $\mathbf{F}_{(t+1)}$ is the force at $t+1$ that can be approximated by first-order Taylor expansion $\mathbf{F}_{(t+1)} = \mathbf{F}_{(t)} + \frac{\partial \mathbf{F}_{(t)}}{\partial \mathbf{q}}\Delta\mathbf{q} + \frac{\partial \mathbf{F}_{(t)}}{\partial \dot{\mathbf{q}}}\Delta\dot{\mathbf{q}}$, where $\mathbf{F}_{(t)}$ is the force at t which can be computed by Equation 1. Then node positions at $t+1$ are $\mathbf{q}_{(t+1)} = \mathbf{q}_{(t)} + h(\dot{\mathbf{q}}_{(t)} + \Delta\dot{\mathbf{q}})$. Finally, we have the system equation for simulation:

$$\left(\mathbf{M} - \frac{\partial \mathbf{F}_{(t)}}{\partial \mathbf{q}}h^2 - \frac{\partial \mathbf{F}_{(t)}}{\partial \dot{\mathbf{q}}}h\right)\dot{\mathbf{q}}_{(t+1)} = h\left(\mathbf{F}_{(t)} - \frac{\partial \mathbf{F}_{(t)}}{\partial \dot{\mathbf{q}}}\right) + \mathbf{M}\dot{\mathbf{q}}_{(t)} \tag{3}$$

To solve Equation 3, we explain every term including the general mass matrix $\mathbf{M}$ and every force contained in $\mathbf{F}_{(t)}$ below.

B.3 GENERAL MASS MATRIX

The mass matrix of a warp segment $[\mathbf{q}_0, \mathbf{q}_1]$ is

$$\mathbf{M}_{0,1} = \frac{1}{6}\Delta u \rho \begin{pmatrix} 2\mathbf{I}_3 & \mathbf{I}_3 & -2\mathbf{w} & -\mathbf{w} \\ \mathbf{I}_3 & 2\mathbf{I}_3 & -\mathbf{w} & -2\mathbf{w} \\ -2\mathbf{w}^\top & -\mathbf{w}^\top & 2\mathbf{w}^\top\mathbf{w} & \mathbf{w}^\top\mathbf{w} \\ -\mathbf{w}^\top & -2\mathbf{w}^\top & \mathbf{w}^\top\mathbf{w} & 2\mathbf{w}^\top\mathbf{w} \end{pmatrix} \tag{4}$$

where $\Delta u = u_1 - u_0$ is the distance between the two nodes in Eulerian coordinates, $\mathbf{w} = \frac{\mathbf{x}_1 - \mathbf{x}_0}{\Delta u}$, and ρ is yarn's linear density. The partial derivatives of general mass matrix with respect to nodes' position is

$$\begin{pmatrix} \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} & \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{M}_{0,1}}{\partial u_0} & \frac{\partial \mathbf{M}_{0,1}}{\partial u_1} \end{pmatrix}^\top \tag{5}$$

As $\mathbf{x}_0$ and $\mathbf{x}_1$ are vectors:

$$\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} = \begin{pmatrix} \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)}} \\ \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)}} \\ \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)}} \end{pmatrix} \text{ and } \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1} = \begin{pmatrix} \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(1)}} \\ \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(2)}} \\ \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(3)}} \end{pmatrix} \tag{6}$$

The component $\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)}}$ is

$$\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)}} = \frac{1}{6}\Delta u \rho \begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} & -\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} & -2\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} \\ -2\frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}} & -\frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} \\ -\frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}} & -2\frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} \end{pmatrix}$$

and $\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)}}$, $\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)}}$, $\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(1)}}$, $\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(2)}}$ and $\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(3)}}$ are in a similar form as $\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)}}$. In each term, we have:

$$\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} = -\begin{pmatrix} \frac{1}{\Delta u} \\ 0 \\ 0 \end{pmatrix}, \frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(2)}} = -\begin{pmatrix} 0 \\ \frac{1}{\Delta u} \\ 0 \end{pmatrix}, \text{and } \frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(3)}} = -\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\Delta u} \end{pmatrix}$$

$$\frac{\partial \mathbf{w}}{\partial \mathbf{x}_1^{(1)}} = \begin{pmatrix} \frac{1}{\Delta u} \\ 0 \\ 0 \end{pmatrix}, \frac{\partial \mathbf{w}}{\partial \mathbf{x}_1^{(2)}} = \begin{pmatrix} 0 \\ \frac{1}{\Delta u} \\ 0 \end{pmatrix}, \text{and } \frac{\partial \mathbf{w}}{\partial \mathbf{x}_1^{(3)}} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\Delta u} \end{pmatrix}$$

$$\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} = \frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}}\mathbf{w} + \mathbf{w}^\top \frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} = -2\frac{\mathbf{x}_1^{(1)} - \mathbf{x}_0^{(1)}}{\Delta u^2}$$

where $\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(2)}}$, $\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(3)}}$, $\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_1^{(1)}}$, $\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_1^{(2)}}$ and $\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_1^{(3)}}$ have a similar form as $\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)}}$.

Unsurprisingly, we can find that

$$\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)}} = -\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(1)}}, \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)}} = -\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(2)}} \text{ and } \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)}} = -\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(3)}}$$

After deriving the partial derivatives of $\mathbf{M}_{0,1}$ with respect to the Lagrangian coordinates, we give its partial derivatives with respect to Eulerian coordinates:

$$\begin{aligned} \frac{\partial \mathbf{M}_{0,1}}{\partial u_0} = &-\frac{1}{6}\rho \begin{pmatrix} 2\mathbf{I}_3 & \mathbf{I}_3 & -2\mathbf{w} & -\mathbf{w} \\ \mathbf{I}_3 & 2\mathbf{I}_3 & -\mathbf{w} & -2\mathbf{w} \\ -2\mathbf{w}^\top & -\mathbf{w}^\top & 2\mathbf{w}^\top\mathbf{w} & \mathbf{w}^\top\mathbf{w} \\ -\mathbf{w}^\top & -2\mathbf{w}^\top & \mathbf{w}^\top\mathbf{w} & 2\mathbf{w}^\top\mathbf{w} \end{pmatrix} \\ &+ \frac{1}{6}\Delta u \rho \begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial \mathbf{w}}{\partial u_0} & -\frac{\partial \mathbf{w}}{\partial u_0} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial \mathbf{w}}{\partial u_0} & -2\frac{\partial \mathbf{w}}{\partial u_0} \\ -2\frac{\partial \mathbf{w}^\top}{\partial u_0} & -\frac{\partial \mathbf{w}^\top}{\partial u_0} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} \\ -\frac{\partial \mathbf{w}^\top}{\partial u_0} & -2\frac{\partial \mathbf{w}^\top}{\partial u_0} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} \end{pmatrix} \end{aligned} \tag{7}$$

where $\frac{\partial \mathbf{M}_{0,1}}{\partial u_1}$ has a similar form as $\frac{\partial \mathbf{M}_{0,1}}{\partial u_0}$ and:

$$\frac{\partial \mathbf{w}}{\partial u_0} = \frac{\mathbf{x}_1 - \mathbf{x}_0}{\Delta u^2} = \frac{\mathbf{w}}{\Delta u} \text{ and } \frac{\partial \mathbf{w}}{\partial u_1} = -\frac{\mathbf{x}_1 - \mathbf{x}_0}{\Delta u^2} = -\frac{\mathbf{w}}{\Delta u}$$

$$\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} = \frac{\partial \mathbf{w}^\top}{\partial u_0}\mathbf{w} + \mathbf{w}^\top \frac{\partial \mathbf{w}}{\partial u_0} = 2\frac{\mathbf{w}^\top \mathbf{w}}{\Delta u}$$

$$\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_1} = \frac{\partial \mathbf{w}^\top}{\partial u_1}\mathbf{w} + \mathbf{w}^\top \frac{\partial \mathbf{w}}{\partial u_1} = -2\frac{\mathbf{w}^\top \mathbf{w}}{\Delta u}$$

Likewise, we can find

$$\frac{\partial \mathbf{M}_{0,1}}{\partial u_0} = -\frac{\partial \mathbf{M}_{0,1}}{\partial u_1}$$

So far, we have given the full details of $\mathbf{M}_{0,1}$' s partial derivatives with respect to positions in Equation 5. Now we give its time derivative:

$$\dot{\mathbf{M}}_{0,1} = \frac{1}{6}\rho(\dot{u}_1 - \dot{u}_0)\begin{pmatrix} 2\mathbf{I}_3 & \mathbf{I}_3 & -2\mathbf{w} & -\mathbf{w} \\ \mathbf{I}_3 & 2\mathbf{I}_3 & -\mathbf{w} & -2\mathbf{w} \\ -2\mathbf{w}^\top & -\mathbf{w}^\top & 2\mathbf{w}^\top\mathbf{w} & \mathbf{w}^\top\mathbf{w} \\ -\mathbf{w}^\top & -2\mathbf{w}^\top & \mathbf{w}^\top\mathbf{w} & 2\mathbf{w}^\top\mathbf{w} \end{pmatrix} + \frac{1}{6}\rho\Delta u \begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial \mathbf{w}}{\partial t} & -\frac{\partial \mathbf{w}}{\partial t} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial \mathbf{w}}{\partial t} & -2\frac{\partial \mathbf{w}}{\partial t} \\ -2\frac{\partial \mathbf{w}^\top}{\partial t} & -\frac{\partial \mathbf{w}^\top}{\partial t} & 2\frac{\partial \mathbf{w}^\top\mathbf{w}}{\partial t} & \frac{\partial \mathbf{w}^\top\mathbf{w}}{\partial t} \\ -\frac{\partial \mathbf{w}^\top}{\partial t} & -2\frac{\partial \mathbf{w}^\top}{\partial t} & \frac{\partial \mathbf{w}^\top\mathbf{w}}{\partial t} & 2\frac{\partial \mathbf{w}^\top\mathbf{w}}{\partial t} \end{pmatrix} \tag{8}$$

where

$$\frac{\partial \mathbf{w}}{\partial t} = \frac{\partial}{\partial t}\frac{\mathbf{x}_1 - \mathbf{x}_0}{\Delta u} = \frac{(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_0)\Delta u - (\mathbf{x}_1 - \mathbf{x}_0)(\dot{u}_1 - \dot{u}_0)}{\Delta u^2}$$

$$\frac{\partial \mathbf{w}^\top\mathbf{w}}{\partial t} = \frac{\partial \mathbf{w}^\top}{\partial t}\mathbf{w} + \mathbf{w}^\top\frac{\partial \mathbf{w}}{\partial t}$$

In addition, the derivatives of $\dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}$ with respect to the nodes' positions are:

$$\frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial \mathbf{x}_0} = \begin{pmatrix} \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \mathbf{x}_0^{(1)}}\dot{\mathbf{q}}_{0,1} \\ \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \mathbf{x}_0^{(2)}}\dot{\mathbf{q}}_{0,1} \\ \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \mathbf{x}_0^{(3)}}\dot{\mathbf{q}}_{0,1} \end{pmatrix}, \frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial \mathbf{x}_1} = \begin{pmatrix} \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \mathbf{x}_1^{(1)}}\dot{\mathbf{q}}_{0,1} \\ \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \mathbf{x}_1^{(2)}}\dot{\mathbf{q}}_{0,1} \\ \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \mathbf{x}_1^{(3)}}\dot{\mathbf{q}}_{0,1} \end{pmatrix}$$

$$\frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial u_0} = \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial u_0}\dot{\mathbf{q}}_{0,1}, \frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial u_1} = \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial u_1}\dot{\mathbf{q}}_{0,1} \tag{9}$$

The components in Equation 9 are:

$$\frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \mathbf{x}_0^{(1)}} = \frac{1}{6}\rho(\dot{u}_1 - \dot{u}_0)\begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} & -\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} & -2\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} \\ -2\frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}} & -\frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}} & 2\frac{\partial \mathbf{w}^\top\mathbf{w}}{\partial \mathbf{x}_0^{(1)}} & \frac{\partial \mathbf{w}^\top\mathbf{w}}{\partial \mathbf{x}_0^{(1)}} \\ -\frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}} & -2\frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}} & \frac{\partial \mathbf{w}^\top\mathbf{w}}{\partial \mathbf{x}_0^{(1)}} & 2\frac{\partial \mathbf{w}^\top\mathbf{w}}{\partial \mathbf{x}_0^{(1)}} \end{pmatrix} + \frac{1}{6}\rho\Delta u\begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial^2 \mathbf{w}}{\partial t\partial \mathbf{x}_0^{(1)}} & -\frac{\partial^2 \mathbf{w}}{\partial t\partial \mathbf{x}_0^{(1)}} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial^2 \mathbf{w}}{\partial t\partial \mathbf{x}_0^{(1)}} & -2\frac{\partial^2 \mathbf{w}}{\partial t\partial \mathbf{x}_0^{(1)}} \\ -2\frac{\partial^2 \mathbf{w}^\top}{\partial t\partial \mathbf{x}_0^{(1)}} & -\frac{\partial^2 \mathbf{w}^\top}{\partial t\partial \mathbf{x}_0^{(1)}} & 2\frac{\partial^2 \mathbf{w}^\top\mathbf{w}}{\partial t\partial \mathbf{x}_0^{(1)}} & \frac{\partial^2 \mathbf{w}^\top\mathbf{w}}{\partial t\partial \mathbf{x}_0^{(1)}} \\ -\frac{\partial^2 \mathbf{w}^\top}{\partial t\partial \mathbf{x}_0^{(1)}} & -2\frac{\partial^2 \mathbf{w}^\top}{\partial t\partial \mathbf{x}_0^{(1)}} & \frac{\partial^2 \mathbf{w}^\top\mathbf{w}}{\partial t\partial \mathbf{x}_0^{(1)}} & 2\frac{\partial^2 \mathbf{w}^\top\mathbf{w}}{\partial t\partial \mathbf{x}_0^{(1)}} \end{pmatrix}$$

and

$$\frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial u_0} = \frac{1}{6}\rho(\dot{u}_1 - \dot{u}_0)\begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial \mathbf{w}}{\partial u_0} & -\frac{\partial \mathbf{w}}{\partial u_0} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial \mathbf{w}}{\partial u_0} & -2\frac{\partial \mathbf{w}}{\partial u_0} \\ -2\frac{\partial \mathbf{w}^\top}{\partial u_0} & -\frac{\partial \mathbf{w}^\top}{\partial u_0} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} \\ -\frac{\partial \mathbf{w}^\top}{\partial u_0} & -2\frac{\partial \mathbf{w}^\top}{\partial u_0} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} \end{pmatrix}$$
$$- \frac{1}{6}\rho\begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial \mathbf{w}}{\partial t} & -\frac{\partial \mathbf{w}}{\partial t} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial \mathbf{w}}{\partial t} & -2\frac{\partial \mathbf{w}}{\partial t} \\ -2\frac{\partial \mathbf{w}^\top}{\partial t} & -\frac{\partial \mathbf{w}^\top}{\partial t} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial t} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial t} \\ -\frac{\partial \mathbf{w}^\top}{\partial t} & -2\frac{\partial \mathbf{w}^\top}{\partial t} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial t} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial t} \end{pmatrix}$$
$$+ \frac{1}{6}\Delta u\rho\begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial^2 \mathbf{w}}{\partial t\partial u_0} & -\frac{\partial^2 \mathbf{w}}{\partial t\partial u_0} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial^2 \mathbf{w}}{\partial t\partial u_0} & -2\frac{\partial^2 \mathbf{w}}{\partial t\partial u_0} \\ -2\frac{\partial^2 \mathbf{w}^\top}{\partial t\partial u_0} & -\frac{\partial^2 \mathbf{w}^\top}{\partial t\partial u_0} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t\partial u_0} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t\partial u_0} \\ -\frac{\partial^2 \mathbf{w}^\top}{\partial t\partial u_0} & -2\frac{\partial^2 \mathbf{w}^\top}{\partial t\partial u_0} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t\partial u_0} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t\partial u_0} \end{pmatrix}$$

where

$$\frac{\partial^2 \mathbf{w}}{\partial t \partial \mathbf{x}_0^{(1)}} = \begin{pmatrix} \frac{\dot{u}_1 - \dot{u}_0}{\Delta u^2} \\ 0 \\ 0 \end{pmatrix}, \frac{\partial^2 \mathbf{w}}{\partial t \partial \mathbf{x}_0^{(2)}} = \begin{pmatrix} 0 \\ \frac{\dot{u}_1 - \dot{u}_0}{\Delta u^2} \\ 0 \end{pmatrix}, \text{and } \frac{\partial^2 \mathbf{w}}{\partial t \partial \mathbf{x}_0^{(2)}} = \begin{pmatrix} 0 \\ 0 \\ \frac{\dot{u}_1 - \dot{u}_0}{\Delta u^2} \end{pmatrix}$$

$$\frac{\partial^2 \mathbf{w}}{\partial t \partial \mathbf{x}_1^{(1)}} = -\begin{pmatrix} \frac{\dot{u}_1 - \dot{u}_0}{\Delta u^2} \\ 0 \\ 0 \end{pmatrix}, \frac{\partial^2 \mathbf{w}}{\partial t \partial \mathbf{x}_1^{(2)}} = -\begin{pmatrix} 0 \\ \frac{\dot{u}_1 - \dot{u}_0}{\Delta u^2} \\ 0 \end{pmatrix}, \text{and } \frac{\partial^2 \mathbf{w}}{\partial t \partial \mathbf{x}_1^{(2)}} = -\begin{pmatrix} 0 \\ 0 \\ \frac{\dot{u}_1 - \dot{u}_0}{\Delta u^2} \end{pmatrix}$$

$$\frac{\partial^2 \mathbf{w}}{\partial t \partial u_0} = \frac{\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_1}{(u_1 - u_0)^2} - \frac{2(\mathbf{x}_1 - \mathbf{x}_0)(\dot{u}_1 - \dot{u}_0)}{(u_1 - u_0)^3}$$
$$\frac{\partial^2 \mathbf{w}}{\partial t \partial u_1} = -\frac{\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_1}{(u_1 - u_0)^2} + \frac{2(\mathbf{x}_1 - \mathbf{x}_0)(\dot{u}_1 - \dot{u}_0)}{(u_1 - u_0)^3}$$

$$\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t \partial \mathbf{x}_0^{(1)}} = \frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_1^{(1)}}\frac{\partial \mathbf{w}}{\partial t} + \mathbf{w}^\top \frac{\partial^2 \mathbf{w}}{\partial t \partial \mathbf{x}_1^{(1)}} + \frac{\partial^2 \mathbf{w}^\top}{\partial t \partial \mathbf{x}_1^{(1)}}\mathbf{w} + \frac{\partial \mathbf{w}^\top}{\partial t}\frac{\partial \mathbf{w}}{\partial \mathbf{x}_1^{(1)}}$$

$$\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t \partial u_0} = \frac{\partial \mathbf{w}^\top}{\partial u_0}\frac{\partial \mathbf{w}}{\partial t} + \mathbf{w}^\top \frac{\partial^2 \mathbf{w}}{\partial t \partial u_0} + \frac{\partial^2 \mathbf{w}^\top}{\partial t \partial u_0}\mathbf{w} + \frac{\partial \mathbf{w}^\top}{\partial t}\frac{\partial \mathbf{w}}{\partial u_0}$$

The derivatives of $\dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}$ with respect to the nodes' velocities are:

$$\frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0} = \begin{pmatrix} \frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(1)}} \\ \frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(2)}} \\ \frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(3)}} \end{pmatrix} = \begin{pmatrix} \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(1)}}\dot{\mathbf{q}}_{0,1} + \dot{\mathbf{M}}_{0,1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(1)}} \\ \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(2)}}\dot{\mathbf{q}}_{0,1} + \dot{\mathbf{M}}_{0,1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(2)}} \\ \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(3)}}\dot{\mathbf{q}}_{0,1} + \dot{\mathbf{M}}_{0,1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(3)}} \end{pmatrix}$$

$$\frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1} = \begin{pmatrix} \frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(1)}} \\ \frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(2)}} \\ \frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(3)}} \end{pmatrix} = \begin{pmatrix} \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(1)}}\dot{\mathbf{q}}_{0,1} + \dot{\mathbf{M}}_{0,1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(1)}} \\ \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(2)}}\dot{\mathbf{q}}_{0,1} + \dot{\mathbf{M}}_{0,1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(2)}} \\ \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(3)}}\dot{\mathbf{q}}_{0,1} + \dot{\mathbf{M}}_{0,1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(3)}} \end{pmatrix}$$

$$\frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial \dot{u}_0} = \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \dot{u}_0}\dot{\mathbf{q}}_{0,1} + \dot{\mathbf{M}}_{0,1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{u}_0}$$

$$\frac{\partial \dot{\mathbf{M}}_{0,1}\dot{\mathbf{q}}_{0,1}}{\partial \dot{u}_1} = \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \dot{u}_1}\dot{\mathbf{q}}_{0,1} + \dot{\mathbf{M}}_{0,1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{u}_1} \tag{10}$$

where

$$\frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(1)}} = \frac{1}{6}\Delta u \rho \begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial^2 \mathbf{w}}{\partial t \partial \mathbf{x}_0^{(1)}} & -\frac{\partial^2 \mathbf{w}}{\partial t \partial \mathbf{x}_0^{(1)}} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial^2 \mathbf{w}}{\partial t \partial \mathbf{x}_0^{(1)}} & -2\frac{\partial^2 \mathbf{w}}{\partial t \partial \mathbf{x}_0^{(1)}} \\ -2\frac{\partial^2 \mathbf{w}^\top}{\partial t \partial \mathbf{x}_0^{(1)}} & -\frac{\partial^2 \mathbf{w}^\top}{\partial t \partial \mathbf{x}_0^{(1)}} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t \partial \mathbf{x}_0^{(1)}} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t \partial \mathbf{x}_0^{(1)}} \\ -\frac{\partial^2 \mathbf{w}^\top}{\partial t \partial \mathbf{x}_0^{(1)}} & -2\frac{\partial^2 \mathbf{w}^\top}{\partial t \partial \mathbf{x}_0^{(1)}} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t \partial \mathbf{x}_0^{(1)}} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t \partial \mathbf{x}_0^{(1)}} \end{pmatrix}$$

$$\begin{aligned} \frac{\partial \dot{\mathbf{M}}_{0,1}}{\partial \dot{u}_0} = & -\frac{1}{6}\rho \begin{pmatrix} 2\mathbf{I}_3 & \mathbf{I}_3 & -2\mathbf{w} & -\mathbf{w} \\ \mathbf{I}_3 & 2\mathbf{I}_3 & -\mathbf{w} & -2\mathbf{w} \\ -2\mathbf{w}^\top & -\mathbf{w}^\top & 2\mathbf{w}^\top\mathbf{w} & \mathbf{w}^\top\mathbf{w} \\ -\mathbf{w}^\top & -2\mathbf{w}^\top & \mathbf{w}^\top\mathbf{w} & 2\mathbf{w}^\top\mathbf{w} \end{pmatrix} \\ & + \frac{1}{6}\Delta u \rho \begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial^2 \mathbf{w}}{\partial t \partial \dot{u}_0} & -\frac{\partial^2 \mathbf{w}}{\partial t \partial \dot{u}_0} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial^2 \mathbf{w}}{\partial t \partial \dot{u}_0} & -2\frac{\partial^2 \mathbf{w}}{\partial t \partial \dot{u}_0} \\ -2\frac{\partial^2 \mathbf{w}^\top}{\partial t \partial \dot{u}_0} & -\frac{\partial^2 \mathbf{w}^\top}{\partial t \partial \dot{u}_0} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t \partial \dot{u}_0} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t \partial \dot{u}_0} \\ -\frac{\partial^2 \mathbf{w}^\top}{\partial t \partial \dot{u}_0} & -2\frac{\partial^2 \mathbf{w}^\top}{\partial t \partial \dot{u}_0} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t \partial \dot{u}_0} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial t \partial \dot{u}_0} \end{pmatrix} \end{aligned}$$

$$\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(1)}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial u_0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

B.4 INERTIA

Kinetic energy is computed segment-wise, e.g. for a segment $[\mathbf{q_0}, \mathbf{q_1}]$:

$$T_{0,1} = \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \mathbf{M}_{0,1}\dot{\mathbf{q}}_{0,1} = \frac{1}{2}\begin{pmatrix} \dot{\mathbf{x}}_0^\top & \dot{\mathbf{x}}_1^\top & \dot{u}_0 & \dot{u}_1 \end{pmatrix} \mathbf{M}_{0,1} \begin{pmatrix} \dot{\mathbf{x}}_0 \\ \dot{\mathbf{x}}_1 \\ \dot{u}_0 \\ \dot{u}_1 \end{pmatrix} \tag{11}$$

Its derivatives with respect to each node's position is the node's inertia:

$$\frac{\partial T_{0,1}}{\partial \mathbf{q}_{0,1}} = \begin{pmatrix} \mathbf{F}_{\mathbf{x}_0} \\ \mathbf{F}_{\mathbf{x}_1} \\ \mathbf{F}_{u_0} \\ \mathbf{F}_{u_1} \end{pmatrix} \tag{12}$$

$$\begin{aligned} \mathbf{F}_{\mathbf{x}_0} &= \frac{\partial T_{0,1}}{\partial \mathbf{x}_0} = \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0}\dot{\mathbf{q}}_{0,1} \\ \mathbf{F}_{\mathbf{x}_1} &= \frac{\partial T_{0,1}}{\partial \mathbf{x}_1} = \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1}\dot{\mathbf{q}}_{0,1} \\ F_{u_0} &= \frac{\partial T_{0,1}}{\partial u_0} = \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial u_0}\dot{\mathbf{q}}_{0,1} \\ F_{u_1} &= \frac{\partial T_{0,1}}{\partial u_1} = \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial u_1}\dot{\mathbf{q}}_{0,1} \end{aligned}$$

where $\mathbf{F}_{x_0}$ and $\mathbf{F}_{u_0}$ are the inertia of $\mathbf{q}_0$ in Lagrangian and Eulerian coordinates respectively. Similarly, $\mathbf{F}_{x_1}$ and $\mathbf{F}_{u_1}$ are the inertia of $\mathbf{q}_1$. The derivative of the forces with respect to positions are:

$$\frac{\partial^2 T_{0,1}}{\partial \mathbf{q}_{0,1} \partial \mathbf{q}_{0,1}} = \begin{pmatrix} \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0} & \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_0} & \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_1} \\ \frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} & \frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_0} & \frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_1} \\ \frac{\partial \mathbf{F}_{u_0}}{\partial \mathbf{x}_0} & \frac{\partial \mathbf{F}_{u_0}}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{F}_{u_0}}{\partial u_0} & \frac{\partial \mathbf{F}_{u_0}}{\partial u_1} \\ \frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_0} & \frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{F}_{u_1}}{\partial u_0} & \frac{\partial \mathbf{F}_{u_1}}{\partial u_1} \end{pmatrix} \tag{13}$$

The derivative of the force in Lagrangian coordinates with respect to Lagrangian coordinates is

$$\begin{aligned} \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0} &= \frac{1}{2} \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0 \partial \mathbf{x}_0} \dot{\mathbf{q}}_{0,1} \\ &= \frac{1}{2} \begin{pmatrix} \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_0^{(1)}} \dot{\mathbf{q}}_{0,1} & \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_0^{(2)}} \dot{\mathbf{q}}_{0,1} & \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_0^{(3)}} \dot{\mathbf{q}}_{0,1} \\ \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_0^{(1)}} \dot{\mathbf{q}}_{0,1} & \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_0^{(2)}} \dot{\mathbf{q}}_{0,1} & \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_0^{(3)}} \dot{\mathbf{q}}_{0,1} \\ \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_0^{(1)}} \dot{\mathbf{q}}_{0,1} & \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_0^{(2)}} \dot{\mathbf{q}}_{0,1} & \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_0^{(3)}} \dot{\mathbf{q}}_{0,1} \end{pmatrix} \end{aligned}$$

$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_1}$, $\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0}$ and $\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1}$ are in similar forms as $\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0}$. Also, the derivative of the force in Lagrangian coordinate with respect to Eulerian coordinates is:

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_0} = \frac{1}{2} \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0 \partial u_0} \dot{\mathbf{q}}_{0,1}$$

$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_1}$, $\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_0}$ and $\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_1}$ are in similar forms as $\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_0}$. Correspondingly, the derivative of the force in Eulerian coordinates with respect to Lagrangian coordinates is:

$$\frac{\partial \mathbf{F}_{u_0}}{\partial \mathbf{x}_0} = \frac{1}{2} \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial u_0 \partial \mathbf{x}_0} \dot{\mathbf{q}}_{0,1}$$

$\frac{\partial \mathbf{F}_{u_0}}{\partial \mathbf{x}_1}$, $\frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_0}$ and $\frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_1}$ are in similar forms as $\frac{\partial \mathbf{F}_{u_0}}{\partial \mathbf{x}_0}$. The derivative of the force in Eulerian coordinates with respect to Eulerian coordinates is:

$$\frac{\partial \mathbf{F}_{u_0}}{\partial u_0} = \frac{1}{2} \dot{\mathbf{q}}_{0,1}^\top \frac{\partial^2 \mathbf{M}_{0,1}}{\partial u_0 \partial u_0} \dot{\mathbf{q}}_{0,1}$$

$\frac{\partial \mathbf{F}_{u_0}}{\partial u_1}$, $\frac{\partial \mathbf{F}_{u_1}}{\partial u_0}$ and $\frac{\partial \mathbf{F}_{u_1}}{\partial u_1}$ are in similar forms as $\frac{\partial \mathbf{F}_{u_0}}{\partial u_0}$.

Specially, the entries in $\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0}$ are:

$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_0^{(1)}} = \frac{1}{6} \Delta u \rho \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_0^{(1)}} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_0^{(1)}} \\ \mathbf{0} & \mathbf{0} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_0^{(1)}} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_0^{(1)}} \end{pmatrix}$$

$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_0^{(2)}} = \frac{1}{6} \Delta u \rho \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_0^{(2)}} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_0^{(2)}} \\ \mathbf{0} & \mathbf{0} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_0^{(2)}} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_0^{(2)}} \end{pmatrix}$$

$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_0^{(3)}} = \frac{1}{6} \Delta u \rho \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_0^{(3)}} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_0^{(3)}} \\ \mathbf{0} & \mathbf{0} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_0^{(3)}} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_0^{(3)}} \end{pmatrix}$$

where
$$\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_0^{(1)}} = \frac{2}{\Delta u^2}, \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_0^{(2)}} = \frac{2}{\Delta u^2} \text{,and } \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_0^{(3)}} = \frac{2}{\Delta u^2}$$
The other components are
$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_0^{(2)}} = \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_0^{(3)}} = \mathbf{0}$$
$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_0^{(1)}} = \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_0^{(3)}} = \mathbf{0}$$
$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_0^{(1)}} = \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_0^{(2)}} = \mathbf{0}$$
Moreover, as
$$\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_1^{(1)}} = -\frac{2}{\Delta u^2}, \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_1^{(2)}} = -\frac{2}{\Delta u^2} \text{,and } \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_1^{(3)}} = -\frac{2}{\Delta u^2}$$
$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_1^{(2)}} = \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_1^{(3)}} = \mathbf{0}$$
$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_1^{(1)}} = \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_1^{(3)}} = \mathbf{0}$$
$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_1^{(1)}} = \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_1^{(2)}} = \mathbf{0}$$
We can find that
$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_0^{(1)}} = -\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial \mathbf{x}_1^{(1)}}$$
$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_0^{(2)}} = -\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)} \partial \mathbf{x}_1^{(2)}}$$
$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_0^{(3)}} = -\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)} \partial \mathbf{x}_1^{(3)}}$$
Therefore,
$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_1}$$
$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} = -\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} = \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0}$$
To compute the derivatives of the forces in Lagrangian coordinates with respect to Eulerian coordinates, we need to compute:
$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0 \partial u_0} = \begin{pmatrix} \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial u_0} \\ \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)} \partial u_0} \\ \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)} \partial u_0} \end{pmatrix} \text{ and } \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0 \partial u_1} = \begin{pmatrix} \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial u_1} \\ \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(2)} \partial u_1} \\ \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(3)} \partial u_1} \end{pmatrix}$$
$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_1 \partial u_0} = \begin{pmatrix} \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(1)} \partial u_0} \\ \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(2)} \partial u_0} \\ \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(3)} \partial u_0} \end{pmatrix} \text{ and } \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_1 \partial u_1} = \begin{pmatrix} \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(1)} \partial u_1} \\ \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(2)} \partial u_1} \\ \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_1^{(3)} \partial u_1} \end{pmatrix}$$

Take one component $\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial u_0}$ as example:

$$\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0^{(1)} \partial u_0} = -\frac{1}{6}\begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} & -\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} & -2\frac{\partial \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} \\ -2\frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}} & -\frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} \\ -\frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}} & -2\frac{\partial \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)}} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)}} \end{pmatrix}$$
$$+\frac{1}{6}\Delta u \rho \begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial u_0} & -\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial u_0} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial u_0} & -2\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial u_0} \\ -2\frac{\partial^2 \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)} \partial u_0} & -\frac{\partial^2 \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)} \partial u_0} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial u_0} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial u_0} \\ -\frac{\partial^2 \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)} \partial u_0} & -2\frac{\partial^2 \mathbf{w}^\top}{\partial \mathbf{x}_0^{(1)} \partial u_0} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial u_0} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial u_0} \end{pmatrix}$$

in which

$$\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial u_0} = -\begin{pmatrix} \frac{1}{\Delta u^2} \\ 0 \\ 0 \end{pmatrix},\ \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_0^{(2)} \partial u_0} = -\begin{pmatrix} 0 \\ \frac{1}{\Delta u^2} \\ 0 \end{pmatrix},\ \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_0^{(3)} \partial u_0} = -\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\Delta u^2} \end{pmatrix}$$

$$\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_0^{(1)} \partial u_1} = \begin{pmatrix} \frac{1}{\Delta u^2} \\ 0 \\ 0 \end{pmatrix},\ \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_0^{(2)} \partial u_1} = \begin{pmatrix} 0 \\ \frac{1}{\Delta u^2} \\ 0 \end{pmatrix},\ \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_0^{(3)} \partial u_1} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\Delta u^2} \end{pmatrix}$$

$$\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_1^{(1)} \partial u_0} = -\begin{pmatrix} \frac{1}{\Delta u^2} \\ 0 \\ 0 \end{pmatrix},\ \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_1^{(2)} \partial u_0} = -\begin{pmatrix} 0 \\ \frac{1}{\Delta u^2} \\ 0 \end{pmatrix},\ \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_1^{(3)} \partial u_0} = -\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\Delta u^2} \end{pmatrix}$$

$$\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_1^{(1)} \partial u_0} = \begin{pmatrix} \frac{1}{\Delta u^2} \\ 0 \\ 0 \end{pmatrix},\ \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_1^{(2)} \partial u_0} = \begin{pmatrix} 0 \\ \frac{1}{\Delta u^2} \\ 0 \end{pmatrix},\ \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}_1^{(3)} \partial u_0} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\Delta u^2} \end{pmatrix}$$

It should be easy to compute $\frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0 \partial u_0}, \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_0 \partial u_1}, \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_1 \partial u_0}, \frac{\partial^2 \mathbf{M}_{0,1}}{\partial \mathbf{x}_1 \partial u_1}$ and then compute $\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_0}, \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_1}, \frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_0}, \frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_1}$. The derivatives of the forces in Eulerain coordinates with respect to Lagrangian coordinates are the transpose of the forces in Lagrangian coordinates with respect to Eulerian coordinates:

$$\frac{\partial \mathbf{F}_{u_0}}{\partial \mathbf{x}_0} = \left(\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_0}\right)^\top \quad \frac{\partial \mathbf{F}_{u_0}}{\partial \mathbf{x}_1} = \left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_0}\right)^\top$$

$$\frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_0} = \left(\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_1}\right)^\top \quad \frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_1} = \left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_1}\right)^\top$$

To compute the derivatives of the forces in Eluerian coordinates with respect to Eulerian coordinates, we need to compute: $\frac{\partial^2 \mathbf{M}_{0,1}}{\partial u_0 \partial u_0}$, $\frac{\partial^2 \mathbf{M}_{0,1}}{\partial u_0 \partial u_1}$, $\frac{\partial^2 \mathbf{M}_{0,1}}{\partial u_1 \partial u_0}$, and $\frac{\partial^2 \mathbf{M}_{0,1}}{\partial u_1 \partial u_1}$. For example,

$$
\begin{aligned}
\frac{\partial^2 \mathbf{M}_{0,1}}{\partial u_0 \partial u_0} = &- \frac{1}{6}\rho \begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial \mathbf{w}}{\partial u_0} & -\frac{\partial \mathbf{w}}{\partial u_0} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial \mathbf{w}}{\partial u_0} & -2\frac{\partial \mathbf{w}}{\partial u_0} \\ -2\frac{\partial \mathbf{w}^\top}{\partial u_0} & -\frac{\partial \mathbf{w}^\top}{\partial u_0} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} \\ -\frac{\partial \mathbf{w}^\top}{\partial u_0} & -2\frac{\partial \mathbf{w}^\top}{\partial u_0} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} \end{pmatrix} \\
&- \frac{1}{6}\rho \begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial \mathbf{w}}{\partial u_0} & -\frac{\partial \mathbf{w}}{\partial u_0} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial \mathbf{w}}{\partial u_0} & -2\frac{\partial \mathbf{w}}{\partial u_0} \\ -2\frac{\partial \mathbf{w}^\top}{\partial u_0} & -\frac{\partial \mathbf{w}^\top}{\partial u_0} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} \\ -\frac{\partial \mathbf{w}^\top}{\partial u_0} & -2\frac{\partial \mathbf{w}^\top}{\partial u_0} & \frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} & 2\frac{\partial \mathbf{w}^\top \mathbf{w}}{\partial u_0} \end{pmatrix} \\
&+ \frac{1}{6}\Delta u \rho \begin{pmatrix} \mathbf{0} & \mathbf{0} & -2\frac{\partial^2 \mathbf{w}}{\partial u_0 \partial u_0} & -\frac{\partial^2 \mathbf{w}}{\partial u_0 \partial u_0} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial^2 \mathbf{w}}{\partial u_0 \partial u_0} & -2\frac{\partial^2 \mathbf{w}}{\partial u_0 \partial u_0} \\ -2\frac{\partial^2 \mathbf{w}^\top}{\partial u_0 \partial u_0} & -\frac{\partial^2 \mathbf{w}^\top}{\partial u_0 \partial u_0} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial u_0 \partial u_0} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial u_0 \partial u_0} \\ -\frac{\partial^2 \mathbf{w}^\top}{\partial u_0 \partial u_0} & -2\frac{\partial^2 \mathbf{w}^\top}{\partial u_0 \partial u_0} & \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial u_0 \partial u_0} & 2\frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial u_0 \partial u_0} \end{pmatrix}
\end{aligned}
$$

in which

$$\frac{\partial^2 \mathbf{w}}{\partial u_0 \partial u_0} = 2\frac{\mathbf{w}}{\Delta u^2} \text{ and } \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial u_0 \partial u_0} = 6\frac{\mathbf{w}^\top \mathbf{w}}{\Delta u^2}$$

Similarly, we can compute

$$\frac{\partial^2 \mathbf{w}}{\partial u_0 \partial u_1} = -2\frac{\mathbf{w}}{\Delta u^2} \quad \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial u_0 \partial u_1} = -6\frac{\mathbf{w}^\top \mathbf{w}}{\Delta u^2}$$

$$\frac{\partial^2 \mathbf{w}}{\partial u_1 \partial u_0} = -2\frac{\mathbf{w}}{\Delta u^2} \quad \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial u_1 \partial u_0} = -6\frac{\mathbf{w}^\top \mathbf{w}}{\Delta u^2}$$

$$\frac{\partial^2 \mathbf{w}}{\partial u_1 \partial u_1} = 2\frac{\mathbf{w}}{\Delta u^2} \quad \frac{\partial^2 \mathbf{w}^\top \mathbf{w}}{\partial u_1 \partial u_1} = 6\frac{\mathbf{w}^\top \mathbf{w}}{\Delta u^2}$$

to compute $\frac{\partial^2 \mathbf{M}_{0,1}}{\partial u_0 \partial u_1}$, $\frac{\partial^2 \mathbf{M}_{0,1}}{\partial u_1 \partial u_0}$, and $\frac{\partial^2 \mathbf{M}_{0,1}}{\partial u_1 \partial u_1}$. The derivatives of the inertia with respect to nodes' velocities are:

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \dot{\mathbf{x}}_0} = \begin{pmatrix} \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{\mathbf{x}}_0^{(1)}} \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} \dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} \frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(1)}} \\ \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{\mathbf{x}}_0^{(2)}} \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} \dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} \frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(2)}} \\ \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{\mathbf{x}}_0^{(3)}} \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} \dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} \frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(3)}} \end{pmatrix}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \dot{\mathbf{x}}_1} = \begin{pmatrix} \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{\mathbf{x}}_1^{(1)}} \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} \dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} \frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(1)}} \\ \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{\mathbf{x}}_1^{(2)}} \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} \dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} \frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(2)}} \\ \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{\mathbf{x}}_1^{(3)}} \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} \dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0} \frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(3)}} \end{pmatrix}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \dot{\mathbf{x}}_0} = \begin{pmatrix} \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{\mathbf{x}}_0^{(1)}} \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1} \dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1} \frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(1)}} \\ \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{\mathbf{x}}_0^{(2)}} \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1} \dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1} \frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(2)}} \\ \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{\mathbf{x}}_0^{(3)}} \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1} \dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top \frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1} \frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_0^{(3)}} \end{pmatrix}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \dot{\mathbf{x}}_1} = \begin{pmatrix} \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{\mathbf{x}}_1^{(1)}}\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1}\dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(1)}} \\ \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{\mathbf{x}}_1^{(2)}}\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1}\dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(2)}} \\ \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{\mathbf{x}}_1^{(3)}}\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1}\dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{\mathbf{x}}_1^{(3)}} \end{pmatrix}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \dot{u}_0} = \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{u}_0}\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0}\dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{u}_0}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \dot{u}_0} = \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{u}_0}\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1}\dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{u}_0}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \dot{u}_1} = \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{u}_1}\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0}\dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_0}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{u}_1}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \dot{u}_1} = \frac{1}{2}\frac{\partial \dot{\mathbf{q}}_{0,1}^\top}{\partial \dot{u}_1}\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1}\dot{\mathbf{q}}_{0,1} + \frac{1}{2}\dot{\mathbf{q}}_{0,1}^\top\frac{\partial \mathbf{M}_{0,1}}{\partial \mathbf{x}_1}\frac{\partial \dot{\mathbf{q}}_{0,1}}{\partial \dot{u}_1}$$

B.5 STRETCHING

Stretch force resists length changes of segments (with the rest length $\|\mathbf{w}\| = 1$). Therefore, the stretching energy is generated when the length changes. We compute the energy of segment $[\mathbf{q_0}, \mathbf{q_1}]$, in a similar way as (Loock et al., 2001; Spillmann & Teschner, 2007) :

$$V_{0,1} = \frac{1}{2} Y \pi R^2 \Delta u (\|\mathbf{w}\| - 1)^2 \tag{14}$$

where Y is yarn's elastic modulus and R is yarns' radius. The stretching forces at the two nodes are:

$$\mathbf{F}_{x_1} = -\mathbf{F}_{x_0} = -\frac{\partial V_{0,1}}{\partial \mathbf{x}_1} = -Y\pi R^2(\|\mathbf{w}\| - 1)\mathbf{d}_{0,1} \tag{15}$$

$$\mathbf{F}_{u_1} = -\mathbf{F}_{u_0} = -\frac{\partial V_{0,1}}{\partial u_1} = \frac{1}{2} Y\pi R^2(\|\mathbf{w}\|^2 - 1) \tag{16}$$

where $\mathbf{d}_{0,1}$ is the unit vector points from $\mathbf{q}_0$ to $\mathbf{q}_1$, $\mathbf{d}_{0,1} = \frac{\mathbf{x}_1 - \mathbf{x}_0}{\|\mathbf{x}_1 - \mathbf{x}_0\|}$. The derivatives of the stretching forces with respect to nodes' positions are:

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} = \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_1} = Y\pi R^2 (\frac{1}{l_1}\mathbf{P}_{0,1} - \frac{1}{\Delta u}\mathbf{I}) \tag{17}$$

$$\frac{\partial F_{u_1}}{\partial u_1} = \frac{\partial F_{u_0}}{\partial u_0} = -\frac{\partial F_{u_1}}{\partial u_0} = -\frac{\partial F_{u_0}}{\partial u_1} = -Y\pi R^2 \frac{\|\mathbf{w}\|^2}{\Delta u} \tag{18}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_1} = \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_1} = Y\pi R^2 \frac{\|\mathbf{w}\|^2}{\Delta u}\mathbf{d}_{0,1} \tag{19}$$

$$\frac{\partial F_{u_1}}{\partial \mathbf{x}_1} = \frac{\partial F_{u_0}}{\partial \mathbf{x}_0} = -\frac{\partial F_{u_1}}{\partial \mathbf{x}_0} = -\frac{\partial F_{u_0}}{\partial \mathbf{x}_1} = \frac{Y\pi R^2}{\Delta u}\mathbf{w}^\top \tag{20}$$

where $\mathbf{P}_{0,1} = \mathbf{I}_3 - \mathbf{d}_{0,1}\mathbf{d}_{0,1}^\top$

B.6 BENDING

We adopt the discrete differential geometry method (Sullivan, 2008) to define the curvature at the common crossing node of two adjacent segments. Bending energy is defined as the integration of bending energy density along the two segments. The bending energy on the two connected warp segments $[\mathbf{q}_2, \mathbf{q}_0]$ and $[\mathbf{q}_0, \mathbf{q}_1]$ is

$$V_{2,0,1} = B\pi R^2 \frac{\theta^2}{u_1 - u_2} \tag{21}$$

where B is yarn bending modulus and $\theta = \arcsin(-\mathbf{d}_{0,1}^\top \mathbf{d}_{0,2})$ is the angle between the two segments. Its derivatives with respective to the node position are the bending forces:

$$\mathbf{F}_{\mathbf{x}_1} = -\frac{2B\pi R^2\theta}{l_1(u_1 - u_2)\sin\theta}\mathbf{P}_{0,1}\mathbf{d}_{\mathbf{0},\mathbf{2}} \tag{22}$$

$$\mathbf{F}_{\mathbf{x}_2} = -\frac{(2B\pi R^2\theta)}{l_2(u_1 - u_2)\sin\theta}\mathbf{P}_{0,2}\mathbf{d}_{\mathbf{0},\mathbf{1}} \tag{23}$$

$$\mathbf{F}_{\mathbf{x}_0} = -(\mathbf{F}_{\mathbf{x}_1} + \mathbf{F}_{\mathbf{x}_2}) \tag{24}$$

$$\mathbf{F}_{u_1} = -\mathbf{F}_{u_2} = \frac{2B\pi R^2\theta^2}{(u_1 - u_2)^2} \tag{25}$$

$$\mathbf{F}_{u_0} = 0 \tag{26}$$

The derivatives of the bending forces with respected to the nodes' position are

$$\begin{aligned}\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} =& \frac{2B\pi R^2}{l_1^2(u_1 - u_0)\sin\theta}\bigg(\theta\bigg(\mathbf{P}_{0,1}\mathbf{d}_{0,2}\mathbf{d}_{0,1}^\top + \frac{\cos\theta}{\sin^2\theta}\mathbf{P}_{0,1}\mathbf{d}_{0,2}\mathbf{d}_{0,2}^\top\mathbf{P}_{0,1} + \cos\theta\mathbf{P}_{0,1} \\ &+ \mathbf{d}_{0,1}\mathbf{d}_{0,2}^\top\mathbf{P}_{0,1}\bigg) - \frac{1}{\sin\theta}\mathbf{P}_{0,1}\mathbf{d}_{0,2}\mathbf{d}_{0,2}^\top\mathbf{P}_{0,1}\bigg)\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_2} =& \frac{2B\pi R^2}{l_2^2(u_1 - u_0)\sin\theta}\bigg(\theta\bigg(\mathbf{P}_{0,2}\mathbf{d}_{0,1}\mathbf{d}_{0,2}^\top + \frac{\cos\theta}{\sin^2\theta}\mathbf{P}_{0,2}\mathbf{d}_{0,1}\mathbf{d}_{0,1}^\top\mathbf{P}_{0,2} + \cos\theta\mathbf{P}_{0,2} \\ &+ \mathbf{d}_{0,2}\mathbf{d}_{0,1}^\top\mathbf{P}_{0,2}\bigg) - \frac{1}{\sin\theta}\mathbf{P}_{0,2}\mathbf{d}_{0,1}\mathbf{d}_{0,1}^\top\mathbf{P}_{0,2}\bigg)\end{aligned} \tag{28}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_2} = -\frac{2B\pi R^2}{l_2 l_1(u_1 - u_2)\sin\theta}\bigg(\theta\bigg(\mathbf{P}_{0,1} - \frac{\cos\theta}{\sin^2\theta}\mathbf{P}_{0,1}\mathbf{d}_{0,2}\mathbf{d}_{0,1}^\top\bigg) + \frac{1}{\sin\theta}\mathbf{P}_{0,1}\mathbf{d}_{0,2}\mathbf{d}_{0,1}^\top\bigg)\mathbf{P}_{0,2} \tag{29}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_1} = -\frac{2B\pi R^2}{l_1 l_2(u_1 - u_2)\sin\theta}\bigg(\theta\bigg(\mathbf{P}_{0,2} - \frac{\cos\theta}{\sin^2\theta}\mathbf{P}_{0,2}\mathbf{d}_{0,1}\mathbf{d}_{0,2}^\top\bigg) + \frac{1}{\sin\theta}\mathbf{P}_{0,2}\mathbf{d}_{0,1}\mathbf{d}_{0,2}^\top\bigg)\mathbf{P}_{0,1} \tag{30}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} = -\bigg(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_2}\bigg) \tag{31}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_0} = -\bigg(\frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_2}\bigg) \tag{32}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_1} = -\bigg(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_1}\bigg) \tag{33}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_2} = -\bigg(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_2} + \frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_2}\bigg) \tag{34}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} + \frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_0}\right) \tag{35}$$

$$\frac{\partial F_{u_1}}{\partial u_1} = \frac{\partial F_{u_2}}{\partial u_2} = \frac{\partial F_{u_1}}{\partial u_2} = \frac{\partial F_{u_2}}{\partial u_1} = -\frac{2B\pi R^2\theta^2}{(u_1-u_2)^2} \tag{36}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_1} = -\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_2} = \frac{2B\pi R^2\theta}{l_1(u_1-u_2)^2\sin\theta}\mathbf{P}_{0,1}\mathbf{d}_{0,2} \tag{37}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial u_1} = -\frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial u_2} = \frac{2B\pi R^2\theta}{l_2(u_1-u_2)^2\sin\theta}\mathbf{P}_{0,2}\mathbf{d}_{0,1} \tag{38}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_1} = -\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_2} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial u_1}\right) \tag{39}$$

$$\frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_1} = -\frac{\partial \mathbf{F}_{u_2}}{\partial \mathbf{x}_1} = \frac{2B\pi R^2\theta}{l_1(u_1-u_2)^2\sin\theta}\mathbf{d}_{0,2}^\top\mathbf{P}_{0,1} \tag{40}$$

$$\frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_2} = -\frac{\partial \mathbf{F}_{u_2}}{\partial \mathbf{x}_2} = \frac{2B\pi R^2\theta}{l_2(u_1-u_2)^2\sin\theta}\mathbf{d}_{0,1}^\top\mathbf{P}_{0,2} \tag{41}$$

$$\frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_0} = -\frac{\partial \mathbf{F}_{u_2}}{\partial \mathbf{x}_0} = -\left(\frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_2}\right) \tag{42}$$

B.7 SLIDE FRICTION

The slide friction at a crossing node $\mathbf{q}_0$ along warp u direction is

$$F_{Slide} = -\Big(\frac{k_f\delta u - K(\delta u)\mu F_n}{2}K(\mu F_n - F_u) + \frac{k_f\delta u + K(\delta u)\mu F_n}{2}\Big) - d_f\dot{u}_0 \tag{43}$$

The derivative of friction force with respect to node position in Eulerian coordinate is

$$\begin{aligned}\frac{\partial F_{Slide}}{\partial u_0} =& -\frac{k_f - ((1-\tanh^2\delta u)\mu F_n + \tanh\delta u\mu\frac{\partial F_n}{\partial u_0})}{2}\tanh\left(\mu F_n - F_u\right)\\ &-\frac{k_f\delta u - \tanh\delta u\mu F_n}{2}(1-\tanh^2\left(\mu F_n - F_u\right))\left(\frac{\partial F_u}{\partial u_0} - \mu\frac{\partial F_n}{\partial u_0}\right)\\ &-\frac{k_f + (1-\tanh^2\delta u)\mu F_n + \tanh\delta u\mu\frac{\partial F_n}{\partial u_0}}{2}\end{aligned} \tag{44}$$

The derivative of friction force with respect to node velocity in Eulerian coordinate is

$$\frac{\partial F_{Slide}}{\partial \dot{u}_0} = \frac{k_f\delta u - \tanh\delta u\mu F_n}{2}(1-\tanh^2(\mu F_n - F_u))\frac{\partial F_u}{\partial \dot{u}_0} - d_f \tag{45}$$

B.8 SHEARING

The potential energy over the segments $[\mathbf{q}_0, \mathbf{q}_1]$ and $[\mathbf{q}_0, \mathbf{q}_3]$ caused by shearing deformation is

$$V_{1,0,3} = \frac{1}{2}k_sL(\phi - \bar{\phi})^2 \tag{46}$$

$$k_s = \frac{1}{2}(F_n+1)SR^2\left((1+\gamma^c) + (1-\gamma^c)\tanh\left(\frac{\bar{\phi}^5(\phi-\phi_l)}{(\phi(\phi-\phi_l)(\phi-\bar{\phi}))^2 + \bar{\phi}^4\sigma^2}\right)\right)$$

The shear forces at those crossing nodes are

$$\mathbf{F}_{\mathbf{x}_1} = -\frac{\partial V_{1,0,3}}{\partial \mathbf{x}_1} = -\frac{1}{2}\frac{\partial k_s}{\partial \mathbf{x}_1}L(\phi-\bar{\phi})^2 + \frac{k_sL(\phi-\bar{\phi})}{l_1\sin\phi}\mathbf{P}_{0,1}\mathbf{d}_{0,3} \tag{47}$$

$$\mathbf{F}_{\mathbf{x}_3} = -\frac{\partial V_{1,0,3}}{\partial \mathbf{x}_3} = -\frac{1}{2}\frac{\partial k_s}{\partial \mathbf{x}_3}L(\phi-\bar{\phi})^2 + \frac{k_s L(\phi-\bar{\phi})}{l_3 \sin\phi}\mathbf{P}_{0,3}\mathbf{d}_{0,1} \tag{48}$$

$$\mathbf{F}_{\mathbf{x}_0} = -(\mathbf{F}_{\mathbf{x}_1} + \mathbf{F}_{\mathbf{x}_3}) \tag{49}$$

For the sake of simplicity, we define:

$$g(\phi) = \frac{\bar{\phi}^5(\phi-\phi_l)}{(\phi(\phi-\phi_l)(\phi-\bar{\phi}))^2 + \bar{\phi}^4\sigma^2}$$

$$f(\phi) = \tanh g(\phi)$$

The numerator and denominator of $g(\phi)$ are

$$g_{num}(\phi) = \bar{\phi}^5(\phi-\phi_l)$$

and

$$g_{den}(\phi) = (\phi(\phi-\phi_l)(\phi-\bar{\phi}))^2 + \bar{\phi}^4\sigma^2$$

Then, we have:

$$\frac{\partial k_s}{\partial \mathbf{x}_3} = \frac{1}{2}(F_n+1)SR^2\left(c\gamma^{c-1}\frac{\partial \gamma}{\partial \mathbf{x}_3} - c\gamma^{c-1}\frac{\partial \gamma}{\partial \mathbf{x}_3}f(\phi) + (1-\gamma^c)(1-f(\phi)^2)\frac{\partial g(\phi)}{\partial \mathbf{x}_3}\right)$$

$$\frac{\partial k_s}{\partial \mathbf{x}_1} = \frac{1}{2}(F_n+1)SR^2\left(c\gamma^{c-1}\frac{\partial \gamma}{\partial \mathbf{x}_1} - c\gamma^{c-1}\frac{\partial \gamma}{\partial \mathbf{x}_1}f(\phi) + (1-\gamma^c)(1-f(\phi)^2)\frac{\partial g(\phi)}{\partial \mathbf{x}_1}\right)$$

where

$$\frac{\partial \gamma}{\partial \mathbf{x}_1} = -\frac{L}{R}\cos\frac{\phi}{2}\frac{\partial \phi}{\partial \mathbf{x}_1}, \frac{\partial \gamma}{\partial \mathbf{x}_3} = -\frac{L}{R}\cos\frac{\phi}{2}\frac{\partial \phi}{\partial \mathbf{x}_3},$$

$$\frac{\partial g(\phi)}{\partial \mathbf{x}_1} = \frac{\frac{\partial g_{num}(\phi)}{\partial \mathbf{x}_1}g_{den}(\phi) - g_{num}(\phi)\frac{\partial g_{den}(\phi)}{\partial \mathbf{x}_1}}{g^2_{den}(\phi)},$$

$$\frac{\partial g(\phi)}{\partial \mathbf{x}_3} = \frac{\frac{\partial g_{num}(\phi)}{\partial \mathbf{x}_3}g_{den}(\phi) - g_{num}(\phi)\frac{\partial g_{den}(\phi)}{\partial \mathbf{x}_3}}{g^2_{den}(\phi)}.$$

The terms $\frac{\partial g_{num}(\phi)}{\partial \mathbf{x}_1}$, $\frac{\partial g_{den}(\phi)}{\partial \mathbf{x}_1}$, $\frac{\partial g_{num}(\phi)}{\partial \mathbf{x}_3}$, and $\frac{\partial g_{den}(\phi)}{\partial \mathbf{x}_3}$ are:

$$\frac{\partial g_{num}(\phi)}{\partial \mathbf{x}_1} = \bar{\phi}^5\frac{\partial \phi}{\partial \mathbf{x}_1} = -\bar{\phi}^5\frac{\mathbf{P}_{0,1}\mathbf{d}_{0,3}}{l_1\sin\phi}$$

$$\frac{\partial g_{num}(\phi)}{\partial \mathbf{x}_3} = \bar{\phi}^5\frac{\partial \phi}{\partial \mathbf{x}_3} = -\bar{\phi}^5\frac{\mathbf{P}_{0,3}\mathbf{d}_{0,1}}{l_3\sin\phi}$$

$$\frac{\partial g_{den}(\phi)}{\partial \mathbf{x}_1} = 2(\phi(\phi-\phi_l)(\phi-\bar{\phi}))\left(\frac{\partial \phi}{\partial \mathbf{x}_1}(\phi-\phi_l)(\phi-\bar{\phi}) + \phi\frac{\partial \phi}{\partial \mathbf{x}_1}(\phi-\bar{\phi}) + \phi(\phi-\phi_l)\frac{\partial \phi}{\partial \mathbf{x}_1}\right)$$

$$\frac{\partial g_{den}(\phi)}{\partial \mathbf{x}_3} = 2(\phi(\phi-\phi_l)(\phi-\bar{\phi}))\left(\frac{\partial (\phi)}{\partial \mathbf{x}_3}(\phi-\phi_l)(\phi-\bar{\phi}) + \phi\frac{\partial \phi}{\partial \mathbf{x}_3}(\phi-\bar{\phi}) + \phi(\phi-\phi_l)\frac{\partial \phi}{\partial \mathbf{x}_3}\right)$$

The derivatives of the shear forces with respect to the nodes' positions in Lagrangian coordinate are:

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} = -\frac{1}{2}\frac{\partial^2 k_s}{\partial \mathbf{x}_1\mathbf{x}_1}L(\phi-\bar{\phi})^2 - L(\phi-\bar{\phi})\frac{\partial k_s}{\partial \mathbf{x}_1}\frac{\partial \phi}{\partial \mathbf{x}_1} + \frac{\partial}{\partial \mathbf{x}_1}\frac{k_s L(\phi-\bar{\phi})}{l_1\sin\phi}\mathbf{P}_{0,1}\mathbf{d}_{0,3}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_3} = -\frac{1}{2}\frac{\partial^2 k_s}{\partial \mathbf{x}_3\mathbf{x}_3}L(\phi-\bar{\phi})^2 - L(\phi-\bar{\phi})\frac{\partial k_s}{\partial \mathbf{x}_3}\frac{\partial \phi}{\partial \mathbf{x}_3} + \frac{\partial}{\partial \mathbf{x}_3}\frac{k_s L(\phi-\bar{\phi})}{l_3\sin\phi}\mathbf{P}_{0,3}\mathbf{d}_{0,1}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_3} = -\frac{1}{2}\frac{\partial^2 k_s}{\partial \mathbf{x}_1\mathbf{x}_3}L(\phi-\bar{\phi})^2 - L(\phi-\bar{\phi})\frac{\partial k_s}{\partial \mathbf{x}_1}\frac{\partial \phi}{\partial \mathbf{x}_3} - L(\phi-\bar{\phi})\frac{\partial \phi}{\partial \mathbf{x}_1}\frac{\partial k_s}{\partial \mathbf{x}_3} + \frac{\partial}{\partial \mathbf{x}_3}\frac{k_s L(\phi-\bar{\phi})}{l_1\sin\phi}\mathbf{P}_{0,1}\mathbf{d}_{0,3}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_1} = -\frac{1}{2}\frac{\partial^2 k_s}{\partial \mathbf{x}_3\mathbf{x}_1}L(\phi-\bar{\phi})^2 - L(\phi-\bar{\phi})\frac{\partial k_s}{\partial \mathbf{x}_3}\frac{\partial \phi}{\partial \mathbf{x}_1} - L(\phi-\bar{\phi})\frac{\partial \phi}{\partial \mathbf{x}_3}\frac{\partial k_s}{\partial \mathbf{x}_1} + \frac{\partial}{\partial \mathbf{x}_1}\frac{k_s L(\phi-\bar{\phi})}{l_3 \sin\phi}\mathbf{P}_{0,3}\mathbf{d}_{0,1}$$

where

$$\frac{\partial}{\partial \mathbf{x}_1}\frac{k_s L(\phi-\bar{\phi})}{l_1 \sin\phi}\mathbf{P}_{0,1}\mathbf{d}_{0,3} = \frac{k_s L}{l_1^2 \sin\phi}\Bigg(\Big(\phi-\bar{\phi}\Big)\Bigg(-\mathbf{P}_{0,1}\mathbf{d}_{0,3}\mathbf{d}_{0,1}^\top + \frac{\cos\phi}{\sin^2\phi}\mathbf{P}_{0,1}\mathbf{d}_{0,3}\mathbf{d}_{0,3}^\top\mathbf{P}_{0,1} - \cos\phi\mathbf{P}_{0,1} - \mathbf{d}_{0,1}\mathbf{d}_{0,3}^\top\mathbf{P}_{0,1}\Bigg) - \frac{1}{\sin\phi}\mathbf{P}_{0,1}\mathbf{d}_{0,3}\mathbf{d}_{0,3}^\top\mathbf{P}_{0,1}\Bigg)$$

$$\frac{\partial}{\partial \mathbf{x}_3}\frac{k_s L(\phi-\bar{\phi})}{l_3 \sin\phi}\mathbf{P}_{0,3}\mathbf{d}_{0,1} = \frac{k_s L}{l_3^2 \sin\phi}\Bigg(\Big(\phi-\bar{\phi}\Big)\Bigg(-\mathbf{P}_{0,3}\mathbf{d}_{0,1}\mathbf{d}_{0,3}^\top + \frac{\cos\phi}{\sin^2\phi}\mathbf{P}_{0,3}\mathbf{d}_{0,1}\mathbf{d}_{0,1}^\top\mathbf{P}_{0,3} - \cos\phi\mathbf{P}_{0,3} - \mathbf{d}_{0,3}\mathbf{d}_{0,1}^\top\mathbf{P}_{0,3}\Bigg) - \frac{1}{\sin\phi}\mathbf{P}_{0,3}\mathbf{d}_{0,1}\mathbf{d}_{0,1}^\top\mathbf{P}_{0,3}\Bigg)$$

$$\frac{\partial}{\partial \mathbf{x}_3}\frac{k_s L(\phi-\bar{\phi})}{l_1 \sin\phi}\mathbf{P}_{0,1}\mathbf{d}_{0,3} = \frac{k_s L}{l_3 l_1 \sin\phi}\Bigg(\Big(\phi-\bar{\phi}\Big)\Bigg(\frac{\cos\phi}{\sin^2\phi}\mathbf{P}_{0,1}\mathbf{d}_{0,3}\mathbf{d}_{0,1}^\top\mathbf{P}_{0,3} + \mathbf{P}_{0,1}\mathbf{P}_{0,3}\Bigg) - \frac{1}{\sin\phi}\mathbf{P}_{0,1}\mathbf{d}_{0,3}\mathbf{d}_{0,1}^\top\mathbf{P}_{0,3}\Bigg)$$

$$\frac{\partial}{\partial \mathbf{x}_1}\frac{k_s L(\phi-\bar{\phi})}{l_3 \sin\phi}\mathbf{P}_{0,3}\mathbf{d}_{0,1} = \frac{k_s L}{l_1 l_3 \sin\phi}\Bigg(\Big(\phi-\bar{\phi}\Big)\Bigg(\frac{\cos\phi}{\sin^2\phi}\mathbf{P}_{0,3}\mathbf{d}_{0,1}\mathbf{d}_{0,3}^\top\mathbf{P}_{0,1} + \mathbf{P}_{0,3}\mathbf{P}_{0,1}\Bigg) - \frac{1}{\sin\phi}\mathbf{P}_{0,3}\mathbf{d}_{0,1}\mathbf{d}_{0,3}^\top\mathbf{P}_{0,1}\Bigg)$$

Moreover, the other terms are

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_3}\right), \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_0} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_3}\right)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_1} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_1}\right), \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_3} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_3} + \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_3}\right)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} + \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_0}\right)$$

B.9 Yarn-to-yarn collision

$$V_{0,1} = \frac{1}{2}k_c L \mathrm{ReLU}(d-\Delta u)^2 \tag{50}$$

The yarn-to-yarn collision forces are:

$$F_{u_0} = -\frac{\partial V_{0,1}}{\partial u_0} = k_c L(\Delta u - d) \tag{51}$$

$$F_{u_1} = -\frac{\partial V_{0,1}}{\partial u_1} = -k_c L(\Delta u - d) \tag{52}$$

The derivatives of the forces with respect to the nodes' position in Eulerian coordiates:

$$\frac{\partial F_{u_0}}{\partial u_0} = \frac{\partial F_{u_1}}{\partial u_1} = -\frac{\partial F_{u_0}}{\partial u_1} = -\frac{\partial F_{u_1}}{\partial u_0} = -k_c L \tag{53}$$

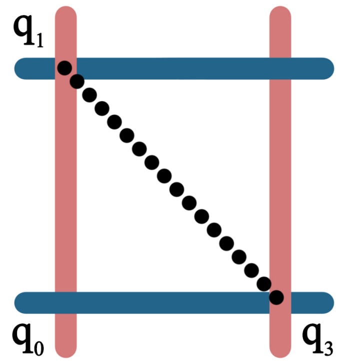

Figure 11: Treat a square hold in 4 segments as two triangles.

B.10 GRAVITY

We define a gravitational energy which is computed segment-wise. To a warp segment $[\mathbf{q}_0, \mathbf{q}_1]$, it gravitational energy is defined as

$$V_{0,1} = \rho \Delta u \mathbf{g}^\top \frac{\mathbf{x}_0 + \mathbf{x}_1}{2} \tag{54}$$

where $\mathbf{g} \in \mathbb{R}_3$ is the gravity of earth which is approximately set to $(0, 0, 9.8)$. The gravity at the nodes are

$$\mathbf{F}_{\mathbf{x}_0} = -\frac{\partial V_{0,1}}{\partial \mathbf{x}_0} = -\frac{1}{2}\rho \mathbf{g} \Delta u \tag{55}$$

$$\mathbf{F}_{\mathbf{x}_1} = -\frac{\partial V_{0,1}}{\partial \mathbf{x}_1} = -\frac{1}{2}\rho \mathbf{g} \Delta u \tag{56}$$

$$\mathbf{F}_{u_0} = -\frac{\partial V_{0,1}}{\partial u_0} = \frac{1}{2}\rho \mathbf{g}^\top (\mathbf{x}_1 + \mathbf{x}_0) \tag{57}$$

$$\mathbf{F}_{u_1} = -\frac{\partial V_{0,1}}{\partial u_1} = -\frac{1}{2}\rho \mathbf{g}^\top (\mathbf{x}_1 + \mathbf{x}_0) \tag{58}$$

The derivative of the force with respect to the nodes' position are:

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_0} = \frac{1}{2}\rho \mathbf{g} \quad \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_1} = -\frac{1}{2}\rho \mathbf{g} \tag{59}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_0} = \frac{1}{2}\rho \mathbf{g} \quad \frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_1} = -\frac{1}{2}\rho \mathbf{g} \tag{60}$$

$$\frac{\partial \mathbf{F}_{u_0}}{\partial \mathbf{x}_1} = \frac{1}{2}\rho \mathbf{g}^\top \quad \frac{\partial \mathbf{F}_{u_0}}{\partial \mathbf{x}_0} = \frac{1}{2}\rho \mathbf{g}^\top \tag{61}$$

$$\frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_1} = -\frac{1}{2}\rho \mathbf{g}^\top \quad \frac{\partial \mathbf{F}_{u_1}}{\partial \mathbf{x}_0} = -\frac{1}{2}\rho \mathbf{g}^\top \tag{62}$$

B.11 WIND FORCE

To apply wind force to the surface of the cloth, we need to compute an area-based force. Every square composed of four segments can be split into two triangles when computing wind force(shown in 11). The wind force has three properties affecting its influence on the cloth: velocity $\mathbf{v}_w$, density ρ_w, and drag d_w. $\mathbf{v}_w = (0, 5, 0)$, density $\rho_w = 2$, and drag $d_w = 0.5$. The wind force imposed on a triangle face $[\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_3]$ is:

$$\mathbf{F}_w = \rho_w a |v_n| v_n \mathbf{n}_f + d_w \mathbf{v}_t \tag{63}$$

where a is face area, $\mathbf{n}_f$ is face normal, and

$$v_n = \mathbf{n}_f \left(\mathbf{v}_w - \frac{\dot{\mathbf{x}}_0 + \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_3}{3} \right),$$

$$\mathbf{v}_t = \frac{\dot{\mathbf{x}}_0 + \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_3}{3} - v_n \mathbf{n}_f.$$

The forces on the nodes are

$$\mathbf{F}_{\mathbf{x}_0} = \mathbf{F}_{\mathbf{x}_1} = \mathbf{F}_{\mathbf{x}_3} = \frac{1}{3}\mathbf{F}_w. \tag{64}$$

B.12 Collision Response

We adopt a collision handling method originally designed for triangular meshes stored in bounding volume hierarchy (Tang et al., 2010) where continuous collision detection (CCD) can detect edge-edge and vertex-face collision. The detected vertices, edges, and faces are grouped into non-rigid impact zones (Harmon et al., 2008) for computing collision response. We treat collision response as a constrained optimization problem to prevent penetrations (Liang et al., 2019):

$$\begin{aligned} \underset{z}{\text{minimize}} \quad & \frac{1}{2}(\mathbf{x}_{colli} - \mathbf{x})^\top \mathbf{W} (\mathbf{x}_{colli} - \mathbf{x}) \\ \text{subject to} \quad & \mathbf{G}\mathbf{x}_{colli} + \mathbf{h} \leq \mathbf{0} \end{aligned}$$

where $\mathbf{W}$ is a weight matrix, $\mathbf{x}$ is the Lagrangian part of $\mathbf{q}$, $\mathbf{x}_{colli}$ is the updated $\mathbf{x}$ where no collision can be detected. G and h are constraint parameters. We assume neither self-collision nor cloth-object collision can generate considerable yarn-sliding motions, so we exclude the Eulerian terms.

C Derivatives of the Simulator

Now we have a fully differentiable cloth simulator. We then compute the loss $\mathcal{L}$ that indicates the difference between the predicted and ground truth cloth states. The loss gradients with respect to the parameters $\frac{\partial \mathcal{L}}{\partial w}$ can help learn the right physics parameters via back-propagation. For simplicity, we use $\mathbf{A}\dot{\mathbf{q}} = \mathbf{b}$ to represent Equation 3. The differential of $\mathbf{A}\dot{\mathbf{q}} = \mathbf{b}$ is (Magnus & Neudecker, 2019):

$$\mathbf{A}\mathsf{d}\dot{\mathbf{q}} = \mathsf{d}\mathbf{b} - \mathsf{d}\mathbf{A}\dot{\mathbf{q}} \tag{65}$$

We can form the Jacobians of $\dot{\mathbf{q}}$ with respect to $\mathbf{A}$ or $\mathbf{b}$ with Equation 65. For example, to compute the $\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{A}}$, we need to set $\mathsf{d}\mathbf{A} = \mathbf{I}$ and $\mathsf{d}\mathbf{b} = \mathbf{0}$, then solve the equation and the result is $\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{A}}$. As pointed out by Amos & Kolter (2017), it is unnecessary to explicitly compute these Jacobians in back-propagation. We want to compute the product of the vector passed from back-propagation, $\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}$ and the Jacobians of $\dot{\mathbf{q}}$, i.e. $\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{A}}$ and $\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{b}}$. Assume $\mathbf{A} \in \mathbb{R}^{3\times 3}$, $\dot{\mathbf{q}} \in \mathbb{R}^3$, and $\mathbf{b} \in \mathbb{R}^3$, then

$$\frac{\partial \mathcal{L}}{\partial \mathbf{b}} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{b}} = \left(\begin{pmatrix} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_1} & \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_2} & \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_3} \end{pmatrix} \begin{pmatrix} \frac{\partial \dot{\mathbf{q}}_1}{\partial \mathbf{b}_1} & \frac{\partial \dot{\mathbf{q}}_1}{\partial \mathbf{b}_2} & \frac{\partial \dot{\mathbf{q}}_1}{\partial \mathbf{b}_3} \\ \frac{\partial \dot{\mathbf{q}}_2}{\partial \mathbf{b}_1} & \frac{\partial \dot{\mathbf{q}}_2}{\partial \mathbf{b}_2} & \frac{\partial \dot{\mathbf{q}}_2}{\partial \mathbf{b}_3} \\ \frac{\partial \dot{\mathbf{q}}_3}{\partial \mathbf{b}_1} & \frac{\partial \dot{\mathbf{q}}_3}{\partial \mathbf{b}_2} & \frac{\partial \dot{\mathbf{q}}_3}{\partial \mathbf{b}_3} \end{pmatrix} \right)^\top \tag{66}$$

As

$$\frac{\partial \dot{\mathbf{q}}_1}{\partial \mathbf{b}_1} = \frac{\partial \left(\mathbf{A}^{-1}\right)_{1,1}\mathbf{b}_1 + (\mathbf{A}^{-1})_{1,1}\mathbf{b}_2 + (\mathbf{A}^{-1})_{1,1}\mathbf{b}_3}{\partial \mathbf{b}_1} = \mathbf{A}^{-1}_{1,1}$$

and similarly for $\frac{\partial \dot{\mathbf{q}}_i}{\partial \mathbf{b}_j}$, Equation 66 can be represented as:

$$\left(\begin{pmatrix} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_1} & \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_2} & \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_3} \end{pmatrix} \begin{pmatrix} (\mathbf{A}^{-1})_{1,1} & (\mathbf{A}^{-1})_{1,2} & (\mathbf{A}^{-1})_{1,3} \\ (\mathbf{A}^{-1})_{2,1} & (\mathbf{A}^{-1})_{2,2} & (\mathbf{A}^{-1})_{2,3} \\ (\mathbf{A}^{-1})_{3,1} & (\mathbf{A}^{-1})_{3,2} & (\mathbf{A}^{-1})_{3,3} \end{pmatrix} \right)^\top = (\mathbf{A}^{-1})^\top \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \tag{67}$$

After computing $\frac{\partial \mathcal{L}}{\partial \mathbf{b}}$, we need to compute $\frac{\partial \mathcal{L}}{\partial \mathbf{A}}$. The $\mathbf{b}$ in Equation 65 can be set to 0 because it is irrelevant when computing $\frac{\partial \mathcal{L}}{\partial \mathbf{A}}$. Then we have

$$\mathbf{A}\mathsf{d}\dot{\mathbf{q}} = -\mathsf{d}\mathbf{A}\dot{\mathbf{q}} \tag{68}$$

The derivative of $\dot{\mathbf{q}}$ with respect to $\mathbf{A}_{i,j}$, the entry in the ith row and jth column of the matrix $\mathbf{A}$, is

$$\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{A}_{i,j}} = \mathbf{A}^{-1} \begin{pmatrix} \mathbf{0} \\ -\dot{\mathbf{q}}_j \\ \mathbf{0} \end{pmatrix} \tag{69}$$

According to chain rule,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{A}_{i,j}} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{A}_{i,j}} = \frac{\partial \mathcal{L}}{\partial \mathbf{b}}^{\top} \mathbf{A}\mathbf{A}^{-1} \begin{pmatrix} \mathbf{0} \\ -\dot{\mathbf{q}}_j \\ \mathbf{0} \end{pmatrix} = -\left(\frac{\partial \mathcal{L}}{\partial \mathbf{b}}\right)_i \dot{\mathbf{q}}_j \tag{70}$$

The more general form is

$$\frac{\partial \mathcal{L}}{\partial \mathbf{A}} = -\frac{\partial \mathcal{L}}{\partial \mathbf{b}} \dot{\mathbf{q}}^{\top} \tag{71}$$

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