A PROOFS

Proof of Lemma 5.

Proof. We construct $\hat{p}$ inductively as follows. For a product node, we have $\hat{p}(t) = \hat{p}_0(t_0)\hat{p}_1(t_1)$, and so

$$\begin{aligned}
p(x) &= \sum_{t \in \{0,1\}^n} \hat{p}(t)(-1)^{\langle t,x \rangle} \\
&= \sum_{t \in \{0,1\}^n} \hat{p}_0(t_0)\hat{p}_1(t_1)(-1)^{\langle t,x \rangle} \\
&= \left(\sum_{t_0 \in \{0,1\}^n} \hat{p}_0(t_0)(-1)^{\langle t_0,x_0 \rangle} \right) \cdot \left(\sum_{t_1 \in \{0,1\}^n} \hat{p}_1(t_1)(-1)^{\langle t_1,x_1 \rangle} \right) \\
&= p_0(x_0)p_1(x_1)
\end{aligned}$$

where the first equality follows from definition, the second from the hypothesis, the third from algebra, and the final from definition. For a sum node, we have $\hat{p}(t) = \sum_i w_i \hat{p}_i(t)$, and so

$$\begin{aligned}
p(x) &= \sum_{t \in \{0,1\}^n} \hat{p}(t)(-1)^{\langle t,x \rangle} \\
&= \sum_{t \in \{0,1\}^n} \left(\sum_i w_i \hat{p}_i(t) \right) (-1)^{\langle t,x \rangle} \\
&= \sum_i w_i \sum_{t \in \{0,1\}^n} \hat{p}_i(t)(-1)^{\langle t,x \rangle} \\
&= \sum_i w_i p_i(x)
\end{aligned}$$

where the equalities follow, respectively, from definition, assumption, commutativity of addition, and definition.

For leaf nodes, it suffices to consider only univariate leaves that are children of sums; for any leaf a child of a product node, add a sum node with weight 1 between them. Then, for a univariate child of a sum node with scope the singleton $\{i\}$, we have either $p(x_i) = c$, and so

$$\begin{aligned}
\hat{p}(t_i) &= \sum_{S \subseteq [n]} p(x_S) \prod_{i \in S} (1 - 2t_i) \\
&= 2^{-n}(c + c(1 - 2x_i)) = 2^{-n+1}c(1 - x_i)
\end{aligned}$$

or $p(x_i) = x_i$, in which case

$$\hat{p}(t_i) = \sum_{S \subseteq [n]} p(x_S) \prod_{i \in S} (1 - 2t_i) = 2^{-n}(1 - 2x_i). \tag{12}$$

□