$$(1)$$

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> $with(VectorCalculus):$

> $with(LinearAlgebra):$

>

> $with(PolynomialTools):$

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> #####

> $ALFstep\_z := (z0, v0, h, t) \rightarrow z0 + h \cdot f\left(z0 + \frac{h}{2} \cdot v0, t + \frac{h}{2}\right)$

$$ALFstep\_z := (z0, v0, h, t) \mapsto z0 + h \cdot f\left(z0 + h \cdot 2^{-1} \cdot v0, t + h \cdot 2^{-1}\right) \quad (2)$$

> $ALFstep\_v := (z0, v0, h, t) \rightarrow 2 \cdot f\left(z0 + \frac{h}{2} \cdot v0, t + \frac{h}{2}\right) - v0$

$$ALFstep\_v := (z0, v0, h, t) \mapsto 2 \cdot f\left(z0 + h \cdot 2^{-1} \cdot v0, t + h \cdot 2^{-1}\right) + (-v0) \quad (3)$$

>

> $ALF2step\_z := (z0, v0, h, t) \rightarrow ALFstep\_z\left(ALFstep\_z\left(z0, v0, \frac{h}{2}, t\right), ALFstep\_v\left(z0, v0, \frac{h}{2}, t\right), \frac{h}{2}, t + \frac{h}{2}\right)$

$$ALF2step\_z := (z0, v0, h, t) \mapsto ALFstep\_z\left(ALFstep\_z\left(z0, v0, h \cdot 2^{-1}, t\right), ALFstep\_v\left(z0, v0, h \cdot 2^{-1}, t\right), h \cdot 2^{-1}, t + h \cdot 2^{-1}\right) \quad (4)$$

> $ALF2step\_v := (z0, v0, h, t) \rightarrow ALFstep\_v\left(ALFstep\_z\left(z0, v0, \frac{h}{2}, t\right), ALFstep\_v\left(z0, v0, \frac{h}{2}, t\right), \frac{h}{2}, t + \frac{h}{2}\right)$

$$ALF2step\_v := (z0, v0, h, t) \mapsto ALFstep\_v\left(ALFstep\_z\left(z0, v0, h \cdot 2^{-1}, t\right), ALFstep\_v\left(z0, v0, h \cdot 2^{-1}, t\right), h \cdot 2^{-1}, t + h \cdot 2^{-1}\right) \quad (5)$$

> $z0 := Vector(3, symbol = z)$

$$z0 := Vector(3, symbol = z) \quad (6)$$

> ####### (z1, v1) obtained by one step of ALF2

> $z1 := subs(v0 = f(z0, t), collect(simplify(ALF2step\_z(z0, v0, h, t)), h))$

$$z1 := \left(\frac{f\left(z0 + \frac{h\, f(z0, t)}{4}, t + \frac{h}{4}\right)}{2} + \frac{f\left(z0 + h\, f\left(z0 + \frac{h\, f(z0, t)}{4}, t + \frac{h}{4}\right) - \frac{h\, f(z0, t)}{4}, t + \frac{3\, h}{4}\right)}{2}\right) h + z0 \quad (7)$$

> $v1 := collect(simplify(ALF2step\_v(z0, v0, h, t)), h)$

$$v1 := 2f\left(z0 + h f\left(z0 + \frac{h\, v0}{4}, t + \frac{h}{4}\right) - \frac{h\, v0}{4}, t + \frac{3h}{4}\right) - 2f\left(z0 + \frac{h\, v0}{4}, t + \frac{h}{4}\right) + v0 \quad \textbf{(8)}$$

> *#################### Tayloer series when initialized at (z0,f(z0,t0)) coincide with the series for the exact from given in the Appendix A of the paper up to* $\mathrm{O}(h^3)$

> $Taylorv1 := h \to collect(simplify(subs(v0 = f(z0, t), v1)), h)$

$$Taylorv1 := h \mapsto collect(simplify(subs(v0 = f(z0, t), v1)), h) \quad \textbf{(9)}$$

> $simplify(taylor(Taylorv1(h), h = 0, 3))$

$$f(z0, t) + (\mathrm{D}_1(f)(z0, t)\, f(z0, t) + \mathrm{D}_2(f)(z0, t))\, h + \left(\frac{f(z0, t)^2\, \mathrm{D}_{1,1}(f)(z0, t)}{2} + \frac{(\mathrm{D}_1(f)(z0, t)^2 + 2\,\mathrm{D}_{1,2}(f)(z0, t))\, f(z0, t)}{2} + \frac{\mathrm{D}_1(f)(z0, t)\, \mathrm{D}_2(f)(z0, t)}{2} + \frac{\mathrm{D}_{2,2}(f)(z0, t)}{2}\right) h^2 + \mathrm{O}(h^3) \quad \textbf{(10)}$$

> $approx := h \to simplify(collect(subs(v0 = f(z0, t), z1), h)) :$

> $approx := simplify(\ taylor(approx(h), h = 0, 3))$

$$approx := z0 + f(z0, t)\, h + \left(\frac{\mathrm{D}_1(f)(z0, t)\, f(z0, t)}{2} + \frac{\mathrm{D}_2(f)(z0, t)}{2}\right) h^2 + \mathrm{O}(h^3) \quad \textbf{(11)}$$

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> *######### Computation of the the error between v1 and f(z1,t2) when initialized by v0 = f(z0,t0)*

>

> $ErrV1 := collect(\ simplify(v1 - f(z1, t + h)), h) :$

> $approxErrV1 := h \to collect(simplify(subs(v0 = f(z0, t)\ , ErrV1)), h) :$

> $simplify(taylor(approxErrV1(h), h = 0, 3))$

$$\mathrm{O}(h^3) \quad \textbf{(12)}$$

>

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> *######### Computation of the the error between v1 and f(z1,t2) when initialized by v0 = f (z0,t0)+const*$\cdot\mathrm{O}(h^3)$

>

> $z1 := subs(v0 = f(z0, t) + \mathrm{alpha}\cdot h^3, collect(simplify(Q(z0, v0, h, t)), h)) :$

> $v1 := subs(v0 = f(z0, t) + \mathrm{alpha}\cdot h^3, collect(simplify(P(z0, v0, h, t)), h)) :$

>

>

> $ErrV1 := collect(\ simplify(v1 - f(z1, t + h)), h) :$

> $approxErrV1 := h \rightarrow collect\left(simplify\left(subs\left(v0 = f(z0, t) + \text{alpha}\cdot h^3, ErrV1\right)\right), h\right):$

> $simplify(taylor(approxErrV1(h), h = 0, 3))$

$$\mathrm{O}\left(h^3\right) \tag{13}$$

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(14)

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