A PROOFS

A.1 PROOF OF PROPOSITION 1

Proof. (α Property): For any $\mathcal{F}, \mathcal{G} \in \mathscr{B}$ such that $\mathcal{G} \subseteq \mathcal{F}$, let $h \in \mathbb{A}_r(\mathcal{F})$. Thus, $r(h) \leq r(g)$ for all $g \in \mathcal{F}$. Assume that $h \in \mathcal{G}$ but $h \notin \mathbb{A}_r(\mathcal{G})$. This implies that there exists another hypothesis $f \in \mathcal{G}$ for which $r(f) < r(h)$. However, since f is also in $\mathcal{F}$, it contradicts with $r(h) \leq r(g)$ for all $g \in \mathcal{F}$. Hence, h must also be in $\mathbb{A}_r(\mathcal{G})$. ($\beta$ Property): For any $h, g \in \mathcal{G}$, let $h, g \in \mathbb{A}_r(\mathcal{G})$. Assume that $h \in \mathbb{A}_r(\mathcal{F})$ but $g \notin \mathbb{A}_r(\mathcal{F})$. This implies that $r(h) \leq r(f)$ for all $f \in \mathcal{F}$ and $r(g) > r(h)$, which contradicts the fact that $g \in \mathbb{A}_r(\mathcal{G})$. Hence, g must also be in $\mathbb{A}_r(\mathcal{F})$. This implies that $\mathbb{A}_r(\mathcal{G}) \subseteq \mathbb{A}_r(\mathcal{F})$. $\square$

A.2 PROOF OF PROPOSITION 2

Proof. Since $\mathbb{A}(\{f,g\}) \neq \emptyset$, $\succeq_{\mathbb{A}}$ is complete. If $f \succeq_{\mathbb{A}} g$ and $g \succeq_{\mathbb{A}} h$, then $f \in \mathbb{A}(\{f,g\})$ and $g \in \mathbb{A}(\{g,h\})$. By β Property, if $g \in \mathbb{A}(\{f,g,h\})$, then $f \in \mathbb{A}(\{f,g,h\})$. Also, if $h \in \mathbb{A}(\{f,g,h\})$, then $g \in \mathbb{A}(\{f,g,h\})$. Hence, we have $f \in \mathbb{A}(\{f,g,h\})$ in any case. By α Property, $f \in \mathbb{A}(\{f,h\})$ and $f \succeq_{\mathbb{A}} h$, which shows that $\succeq_{\mathbb{A}}$ is transitive. Next, we show that $\mathbb{A}(\mathcal{H}) = \mathbb{A}_{\succeq_{\mathbb{A}}}(\mathcal{H})$ for every $\mathcal{H} \in \mathscr{B}$. Assume that $f \in \mathbb{A}(\mathcal{H})$. By α Property, we have for every $g \in \mathcal{H}$ that $f \in \mathbb{A}(\{f,g\})$. This implies that $f \succeq_{\mathbb{A}} g$ and thus $f \in \mathbb{A}_{\succeq_{\mathbb{A}}}(\mathcal{H})$. Now, let us assume that $f \neq g$, $f \in \mathbb{A}_{\succeq_{\mathbb{A}}}(\mathcal{H})$, and $g \in \mathbb{A}(\mathcal{H})$. Then, $f \in \mathbb{A}(\{f,g\})$ and by β Property, $f \in \mathbb{A}(\mathcal{H})$, which completes the proof. $\square$

A.3 PROOF OF THEOREM 1 AND COROLLARY 1

This section provides a proof of Theorem 1, which relies heavily on the insights from the original proof of Arrow's General Possibility Theorem [3] and its simplification in Sen [21, pp. 286].

Let $\mathcal{E}$ be a set of environments. Crucial to the proof is the idea of a set $\mathcal{E}$ being "decisive".

Definition 3 (Decisiveness)**.** *A set of environments $\mathcal{E}$ is said to be* locally decisive *over a pair of hypotheses f, g if $r_e(f) < r_e(g)$ for all $e \in \mathcal{E}$ implies that $\{f\} = \mathbb{A}(\{f,g\})$. It is said to be* globally decisive *if it is locally decisive over every pairs of hypotheses.*

The following two intermediate results provide basic properties of decisive set of environments $\mathcal{E}$.

Lemma 1. *If a set of environments $\mathcal{E}$ is decisive over any pair $\{f,g\}$, then $\mathcal{E}$ is globally decisive.*

Proof. Let $\{p,q\}$ be any other pair of hypotheses that is different from $\{f,g\}$. Assume that in every environment e in $\mathcal{E}$, $r_e(p) < r_e(f)$, $r_e(f) < r_e(g)$, and $r_e(g) < r_e(q)$. For all other environments e' not in $\mathcal{E}$, we assume that $r_{e'}(p) < r_{e'}(f)$ and $r_{e'}(g) < r_{e'}(q)$ and leave the remaining relations unspecified. By PO condition, $\{p\} = \mathbb{A}(\{p,f\})$ and $\{g\} = \mathbb{A}(\{q,g\})$. By the decisiveness of $\mathcal{E}$ over $\{f,g\}$, we have $\{f\} = \mathbb{A}(\{f,g\})$. Then, it follows from the transitivity implied by Proposition 2 that $\{p\} = \mathbb{A}(\{p,q\})$. By IIH condition, this must be related only to the relation between p and q. Since we have only specified information in $\mathcal{E}$, $\mathcal{E}$ must be decisive over $\{p,q\}$ and for all other pairs. Hence, $\mathcal{E}$ is globally decisive. $\square$

Lemma 2. *If a set of environments $\mathcal{E}$ consists of more than one element and is decisive, then some proper subset of $\mathcal{E}$ is also decisive.*

Proof. Since there are at least two environments, we can partition $\mathcal{E}$ into two subsets $\mathcal{E}_1$ and $\mathcal{E}_2$. Assume that $r_e(f) < r_e(g)$ and $r_e(f) < r_e(h)$ in every environment $e \in \mathcal{E}_1$ with the relation between g and h unspecified. Let $r_{e'}(f) < r_{e'}(g)$ and $r_{e'}(h) < r_{e'}(g)$ in every environment $e' \in \mathcal{E}_2$. By the decisiveness of $\mathcal{E}$, we have $\{f\} = \mathbb{A}(\{f,g\})$. Now, if h is at least as good as f for some environments over $\{h,f\}$, then we must have $\{h\} = \mathbb{A}(\{h,g\})$ for that configuration. Since we do not specify relation over $\{g,h\}$ other than those in $\mathcal{E}_2$, and $r_{e'}(h) < r_{e'}(g)$ in $\mathcal{E}_2$, $\mathcal{E}_2$ is decisive over $\{g,h\}$. By Lemma 1, $\mathcal{E}_2$ must be globally decisive. That is, some proper subset of $\mathcal{E}$ is indeed decisive for that particular case. To avoid this possibility, we must remove the assumption that h is at least as good as f. But then f must be better than h. However, no environment has this relation over $\{f,h\}$ other than those in $\mathcal{E}_1$ where f is better than h. Clearly, $\mathcal{E}_1$ is decisive over $\{f,h\}$. Thus, by Lemma 1, $\mathcal{E}_1$ is globally decisive. So either E_1 or E_2 must be decisive. $\square$

An important observation from the proofs of Lemma 1 and Lemma 2 is that we rely only on the relative rankings of hypotheses. We are now in a position to prove Theorem 1.

Proof of Theorem 1. Consider any two risk profiles $\mathbf{r} = (r_1, \ldots, r_n)$ and $\mathbf{r}^* = (r_1^*, \ldots, r_n^*)$ such that for any f, g and for all $i \in [n]$, $r_i(f) < r_i(g) \Leftrightarrow r_i^*(f) < r_i^*(g)$. For every pair $\{f, g\}$, there exists a positive affine transformation $\{\varphi_i\}$ applied to $\mathbf{r}^*$ such that

$$r_i'(f) = \varphi_i(r_i^*(f)) = r_i(f) \quad \text{and} \quad r_i'(g) = \varphi_i(r_i^*(g)) = r_i(g) \;\; \text{for all} \;\; i \in [n].$$

By IIH condition, $\{f\} = \mathbb{A}_{\mathbf{r}}(\{f, g\})$ if and only if $\{f\} = \mathbb{A}_{\mathbf{r}'}(\{f, g\})$ and by IR condition, $\{f\} = \mathbb{A}_{\mathbf{r}'}(\{f, g\})$ if and only if $\{f\} = \mathbb{A}_{\mathbf{r}^*}(\{f, g\})$. Since this holds pair by pair, clearly $\mathbb{A}_{\mathbf{r}}(\mathcal{H}) = \mathbb{A}_{\mathbf{r}'}(\mathcal{H})$ for all $\mathcal{H} \in \mathscr{B}$. As a result, what matters when comparing any two hypotheses is the relative ranking between them. Next, by PO condition, the set of all environments $\mathcal{E}$ is decisive. By Lemma 2, some proper subset of $\mathcal{E}$ must also be decisive. Given that smaller subset of environments, some proper subset of it must also be decisive, and so on. Since the number of environments is finite, the set will eventually contain just a single environment that is decisive. However, this violates CI condition, resulting in the impossibility. $\square$

Corollary 1 follows by omitting the last step in the proof of Theorem 1.