A supplementary for the paper Falconn++: A Locality-sensitive Filtering Approach for Approximate Nearest Neighbor Search

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1 Proof of Theorem 1

Theorem 1. *Given sufficiently large D random vectors and $c > 1$, the filtering mechanism described in the paper has the following properties:*

- *If* $\|\mathbf{x} - \mathbf{q}\| \leq r$, *then* $\mathbf{Pr}\left[\mathbf{x} \textit{ is not filtered}\right] \geq q_1 = 1/2\,;$
- *If* $\|\mathbf{y} - \mathbf{q}\| \geq cr$, *then* $\mathbf{Pr}\left[\mathbf{y} \textit{ is not filtered}\right] \leq q_2 = \frac{1}{\sqrt{2\pi}\gamma} \exp(-\gamma^2/2) < q_1$ *where* $\gamma = \frac{cr(1-1/c^2)}{\sqrt{4-c^2r^2}} \cdot \sqrt{2 \ln D}\,.$

Proof. We first show the two properties for the case $\|\mathbf{x} - \mathbf{q}\| = r, \|\mathbf{y} - \mathbf{q}\| = cr$ by analyzing the tail of Gaussian random variables $X = \mathbf{x}^\top \mathbf{r}_1 \sim N(\mu_1, \sigma_1^2)$ and $Y = \mathbf{y}^\top \mathbf{r}_1 \sim N(\mu_2, \sigma_2^2)$, where

$$\mu_1 = \mathbf{x}^\top \mathbf{q} \sqrt{2 \ln D} = (1 - r^2/2) \sqrt{2 \ln D}, \sigma_1^2 = 1 - \left(1 - r^2/2\right)^2 \,,$$
$$\mu_2 = \mathbf{y}^\top \mathbf{q} \sqrt{2 \ln D} = (1 - c^2r^2/2) \sqrt{2 \ln D}, \sigma_2^2 = 1 - \left(1 - c^2r^2/2\right)^2 \,.$$

We use the classic tail bound of normal random variables. If $Z \sim N(0, 1)$, then for any $a > 0$,

$$\mathbf{Pr}\left[Z \geq a\right] \leq \frac{1}{a\sqrt{2\pi}} e^{-a^2/2} \,.$$

We define $\Delta\mu = \mu_1 - \mu_2 > 0$ and set the threshold $t = \mu_1 = (1 - r^2/2)\sqrt{2 \ln D}$. Since $X \sim N(\mu_1, \sigma_1^2)$ and $t = \mu_1$, $\mathbf{Pr}\left[X \geq t\right] = 1/2 = q_1$. Applying the tail bound on $Y \sim N(\mu_2, \sigma_2^2)$,

$$\mathbf{Pr}\left[Y \geq t\right] = \mathbf{Pr}\left[\frac{Y - \mu_2}{\sigma_2} \geq \frac{\Delta\mu}{\sigma_2}\right] \leq \frac{1}{\sqrt{2\pi}(\Delta\mu)/\sigma_2} \exp\left(-\frac{(\Delta\mu)^2}{2\sigma_2^2}\right) = \frac{1}{\sqrt{2\pi}\gamma} \exp(-\gamma^2/2) = q_2 \,,$$

where $\gamma = \Delta\mu/\sigma_2$. Since $\Delta\mu = \frac{c^2r^2}{2}(1 - \frac{1}{c^2})\sqrt{2 \ln D}$ and $\sigma_2^2 = c^2r^2 \left(1 - \frac{c^2r^2}{4}\right)$, we have $\gamma = \frac{cr(1-1/c^2)}{\sqrt{4-c^2r^2}} \cdot \sqrt{2 \ln D}$.

Since $\Delta\mu/\sigma_2$ is monotonic with respect to c, further points has a higher probability of being discarded. Therefore, the second property holds for any far away point $\mathbf{y}$, i.e. $\|\mathbf{y} - \mathbf{q}\| \geq cr$. The first property holds for any close point $\mathbf{x}$, i.e. $\|\mathbf{x} - \mathbf{q}\| \leq r$, since their projection value onto $\mathbf{r}_1$ follows a Gaussian distribution with mean $\mu \geq \mu_1$. □

2 Falconn++ vs. theoretical LSF frameworks

Figure 1 shows the recall-speed comparison between Falconn++ and recent theoretical LSF frameworks [2, 3]. All 3 data sets use $L = 100$, $\alpha = \{0.1, 0.5\}$, $iProbes = 1$, and the centering trick. We

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do not apply the limit scaling trick to ensure that both Falconn++ and the theoretical LSF approaches share the same number of points in a table. We use $D = \{128, 256, 256\}$ for NYTimes, Glove200, and Glove300. With 2 LSH functions, each table of both approaches has the same $4D^2$ buckets.

Given α, Falconn++ simply keeps αB points in a bucket of size B whose absolute dot products to the corresponding vector $\mathbf{r}_i$ are the largest. To ensure that the table has αn points, theoretical LSF computes the global threshold t_u such that it keeps $\mathbf{x}$ in the bucket corresponding to $\mathbf{r}_i$ with probability $\alpha/4D^2$. Since $\mathbf{x}^\top \mathbf{r}_i \sim N(0, 1)$, we use the `inverseCDF(.)` of a normal distribution to compute t_u such that $\mathbf{Pr}\left[\mathbf{x}^\top \mathbf{r}_i \geq t_u\right]^2 = \alpha/4D^2$. Given this setting, the theoretical LSF with pre-computed t_u has a similar number of indexed points as Falconn++.

Figure 1 shows superior performance of Falconn++ compared to the theoretical LSF with $\alpha = \{0.1, 0.5\}$ on 3 data sets. Note that NYTimes has the center vector $\mathbf{c} = \mathbf{0}$, hence does not need centering. Figure 1(b) and (c) show that Falconn++ with centering trick can even improve the performance on Glove200 and Glove300, whereas theoretical LSF significantly decreases the performance with centering trick. This is because the LSF mechanism of Falconn++ can work on the general inner product (after centering the data points) while the theoretical LSF mechanism works on a unit sphere.

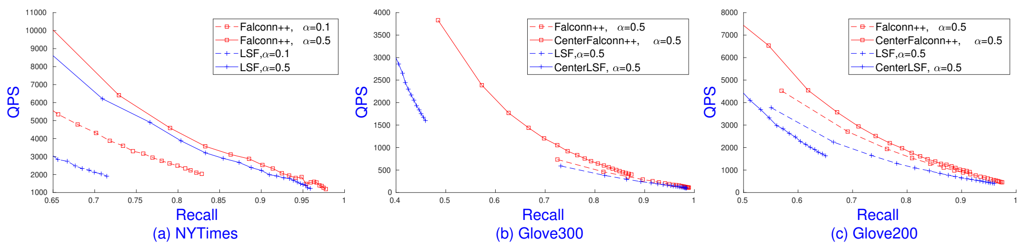


Figure 1: The recall-speed comparison between Falconn++ and theoretical LSF frameworks.

3 A heuristic to select parameter values for Falconn++

This section will present a heuristic to select parameters of Falconn++, including number of random vectors D, number of tables L, scale factor α, and number of indexing probing $iProbes$.

Since Falconn++ uses 2 LSH functions, the number of buckets is $4D^2$. We apply the limit scaling trick to keep $max(k, \alpha B/iProbes)$ points in any bucket. Since we expect to see approximately k near neighbors in a bucket, this trick prevents scaling small buckets that might contain top-k nearest neighbors. When applying $iProbes$, we expect the number of points in a table, i.e. $n \cdot iProbes$, will be distributed equally into $4D^2$ buckets. Hence, each bucket has $n \cdot iProbes/4D^2$ points in expectation.

We note that we would not want to use large $iProbes$ since the bucket will tend to keep the points closest to the random vector $\mathbf{r}_i$, and therefore degrades the performance. Falconn++ with a large $iProbes$ works similarly to the theoretical LSF framework [2, 3] which keeps the point $\mathbf{x}$ in the bucket corresponding to $\mathbf{r}_i$ such that $\mathbf{x}^\top \mathbf{r}_i \geq t_u$ for a given threshold t_u. LSF frameworks need to use a large t_u so that a bucket will contain a small number of points to ensure the querying performance. Figure 1 shows Falconn++ with a *local* threshold t adaptive to the data in each bucket, outperforms the theoretical LSF frameworks that use a *global* t_u for all buckets.

The heuristic idea is that we select $iProbes$ and D such that the bucket size has roughly k points in expectation by setting $k \approx n \cdot iProbes/4D^2$. For instance, on Glove200: $n = 1M$, $D = 256$, $k = 20$, each table has $4D^2 = 2^{18}$ buckets. The setting $iProbes = 3, D = 256$ leads to $1M \cdot 3/2^{18} = 2^4 = 16 < k = 20$ points in a bucket in expectation.

Falconn++ needs a sufficiently large D to maintain the LSF property. Since we deal with high dimensional data set with large d, $D \approx 2^{\lceil \log_2 d \rceil}$ is sufficient. Falconn++ with larger values of D and $iProbes$ requires larger memory footprint but achieves higher recall-speed tradeoffs, as can be seen in Figure 2.

On NYTimes with $n = 300K$, we set $L = 500$, $D = \{128, 256\}$, $iProbes = \{10, 40\}$. On Glove200 with $n = 1M$, we set $L = 350$, $D = \{256, 512\}$, $iProbes = \{3, 10\}$. On Glove300 with

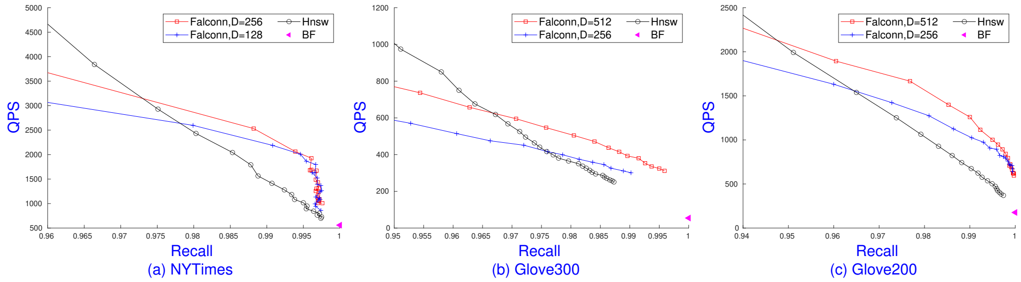


Figure 2: The recall-speed comparison between Falconn++ and HNSW with various D.

$n = 2M$, we set $L = 900$, $D = \{256, 512\}$, $iProbes = \{1, 4\}$. On all 3 data sets, the first setting of D and $iProbes$ lead to similar memory footprints of HNSW. The second setting increases the indexing space to approximately 4 times since we double D.

Regarding α, given the scaling limit trick, we set $\alpha = 0.01$ to reduce large buckets without affecting the performance. We observe that $\alpha = \{0.01, \ldots, 0.1\}$ gives the best performance without dramatically changing the indexing size.

4 Comparison between Falconn++ and HnswLib on different top-k values on Glove200

Figure 3 shows the recall-speed tradeoffs between Falconn++ and HNSW on several values of $k = \{1, 5, 10, \ldots, 100\}$ on Glove200 with $L = 350, D = 256, iProbes = 3, \alpha = 0.01$. Since we apply the limit scaling trick to keep $max(k, \alpha B/iProbes)$ points in any bucket, Falconn++ does not work well on small $k = \{1, 5, 10\}$, compared to HNSW in Figure 3(a). This is due to the fact that many high quality candidates in a bucket are filtered away with $\alpha = 0.01$. However, Falconn++ can beat HNSW for larger k, i.e. at recall ratio of 0.95 for $k \geq 60$ and at recall ratio of 0.96 for $20 \leq k \leq 50$ in Figure 3(b) and (c).

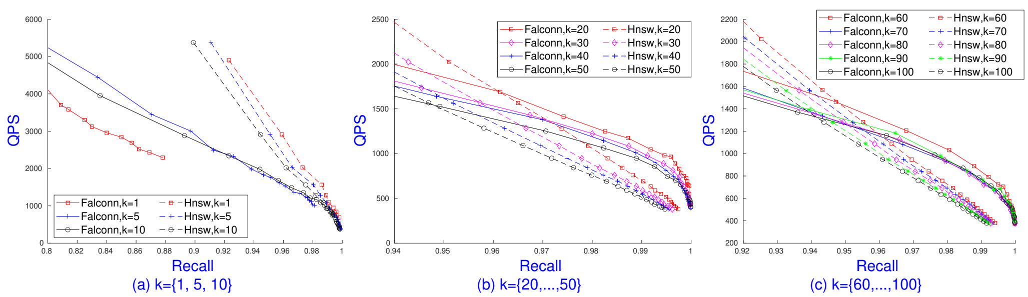


Figure 3: The recall-speed comparison between Falconn++ and Hnsw on different k with the scaling limit $max(k, \alpha B/iProbes)$.

To deal with small k, we set the limit scaling to $max(\kappa, \alpha B/iProbes)$ where $\kappa = 20$ to maintain enough high quality candidates in a bucket without affecting indexing time and space (see in Table 1 for $k = \{1, 5, 10\}$). Figure 4 shows that Falconn++ with the setting of $max(20, \alpha B/iProbes)$ is competitive with HNSW at recall ratio of 0.93 for $k = 1$, and recall ratio of 0.96 for $k = \{5, 10\}$ given the same indexing size.

5 Comparison between Falconn++ and other state-of-the-art ANNS solvers

This section will give a comprehensive comparison between Falconn++ with other state-of-the-art ANNS solvers, including ScaNN [4], Faiss [5], and coCEOs [7] on high search recall regimes on three real-world data sets, including NYTimes, Glove200, and Glove300. The detailed data sets are

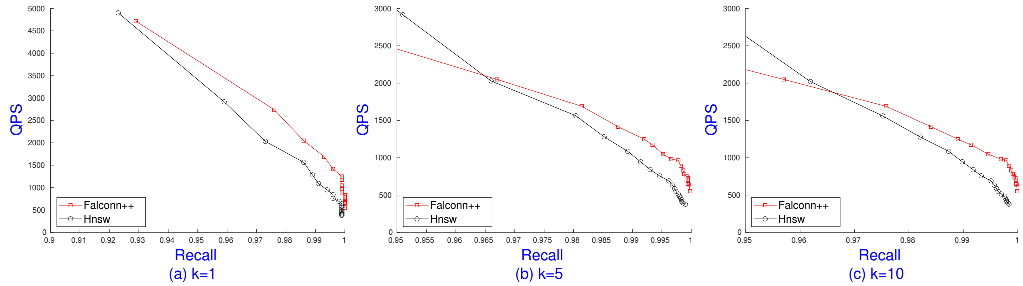


Figure 4: The recall-speed comparison between Falconn++ and Hnsw on different k with the scaling limit $max(20, \alpha B/iProbes)$.

on Table 3. For ScaNN, we use the latest version 1.2.6 released on 29 April, 2022. [1] For FAISS, we use the latest version Faiss-CPU 1.7.2 released on 11 January, 2022. [2] For coCEOs, we use the latest released source code. [3] We note that ScaNN does not support multi-threading while Falconn++, FAISS and coCEOs do though their thread-scaling is not perfect.

Parameter settings of Falconn++. Since Falconn++ uses 2 concatenating cross-polytope LSH functions and D random projections, there are $4D^2$ number of buckets in a hash table. Since we focus on $k = 20$, we set $D = 2^b$ where $b \approx \lceil \log_2(n/k) \rceil / 2$ to expect that each bucket has roughly k points. Hence, we use $D = \{128, 256, 256\}$ for NYTimes, Glove300, and Glove200, respectively. This setting corresponds to $\{2^{16}, 2^{18}, 2^{18}\}$ buckets in a hash table on three data sets. Note that the setting of D is proportional to the size of the data sets. The hash function is evaluated in $\mathcal{O}(D \log D)$ time, so it does not dominate the query time. Furthermore, these values of D are large enough to ensure the asymptotic CEOs property.

We note that Falconn++ with the heuristics of centering the data and limit scaling make the bucket size smaller and more balancing. We observe that different small values of α does not change the size of Falconn++ index. Hence, to maximize the performance of Falconn++, we set $\alpha = 0.01$. For the

[1] https://github.com/google-research/google-research/tree/master/scann
[2] https://github.com/facebookresearch/faiss
[3] https://github.com/NinhPham/MIPS

Table 1: Hnsw takes **13.7 mins** to build 5.4GB indexing space. Falconn++ takes **1.1 mins** and needs different memory footprints dependent on k. For $k \leq 10$, we use $max(20, \alpha B/iProbes)$. For $k \geq 20$, we use $max(k, \alpha B/iProbes)$.

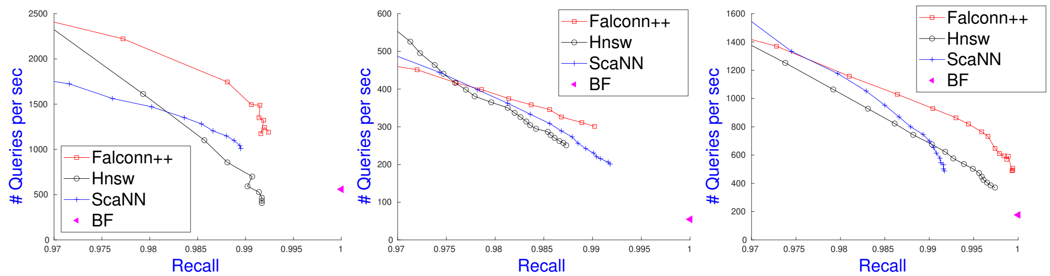
k	1	5	10	20	30	40	50	60 – 100
Falconn++	5.3GB	5.3GB	5.3GB	5.3GB	5.8GB	6.0GB	6.1GB	6.2GB

Table 2: Indexing space and time comparison between Falconn++ and HNSW on 3 data sets.

Algorithms	NYTimes		Glove300		Glove200	
	Space	Time	Space	Time	Space	Time
Hnsw	2.5 GB	7.8 mins	10.9 GB	26.7 mins	5.4 GB	13.7 mins
Falconn++	2.7 GB	**0.6 mins**	10.8 GB	**5.4 mins**	5.3 GB	**1.1 mins**

Table 3: Data sets.

	NYTimes	Glove300	Glove200
n	290,000	2,196,017	1,183,514
d	256	300	200



(a) NYTimes (b) Glove300 (c) Glove200

Figure 5: The recall-speed comparison between Falconn++, HNSW, and ScaNN on 3 data sets.

sake of comparison, we first select optimal parameter settings for HNSW [6] to achieve high search recall ratios given a reasonable query time. Based on the size of HNSW's index, we tune the number of hash tables L for Falconn++ to ensure that Falconn++ shares a similar indexing size with HNSW but builds significantly faster, as can be seen in Table 2. In particular, we use $L = 500, iProbes = 10$ for NYTimes, $L = 900, iProbes = 1$ for Glove300, and $L = 350, iProbes = 3$ for Glove200. Since the characteristics of the data sets are different, it uses different values of $iProbes$.

Parameter settings of HNSW. We first fix $ef\_index = 200$ and increase M from 32 to 1024 to get the best recall-speed tradeoff. Then, we choose $M = \{1024, 512, 512\}$ for NYTimes, Glove300, and Glove200, respectively. We observe that changing ef_index while building the index does not improve the recall-speed tradeoff. We vary $ef\_query = \{100, \ldots, 2000\}$ to get the recall ratios and running time.

Parameter settings of ScaNN. We used the suggested parameter provided in ScaNN's GitHub. We use *all* points to train ScaNN model with $num\_leaves = 1000$ and $score\_ah(2, anisotropic\_quantization\_threshold = 0.2)$. For querying, we use $pre\_reorder\_num\_neighbors = 500$ and vary $leaves\_to\_search \in \{50, \ldots, 1000\}$ to get the recall ratios and running time.

Parameter settings of FAISS. We compare with Faisee.IndexIVFPQ and set the sub-quantizers $m = d, nlist = 1000$, and 8 bits for each centroid. We again use *all* points to train FAISS. We observe that $m < d$ or increasing $nlist$ returns lower recall-speed tradeoffs. We vary $probe \in \{50, \ldots, 1000\}$ to get the recall ratios and running time.

Parameter settings of coCEOs. We use $D = 1024$ and $SamplingSize = n$, $s_0 = 20$, and vary the number of candidates from 10,000 to 100,000 to get the recall ratios and running time.

Comparison of recall-speed tradeoffs. Figure 5 shows that Falconn++, though lacking many important optimized routines, achieves higher recall-speed tradeoffs when $recall > 0.97$ compared to both HNSW and ScaNN on all three data sets. We emphasize that the speed of ScaNN and HNSW comes from several optimized routines, including pre-fetching instructions, SIMD in-register lookup tables [1] for faster distance computation, and optimized multi-threading primitives. Compared to HNSW and ScaNN, both FalconnLib and Falconn++ simply use the Eigen library to support SIMD vectorization for computing inner products.

Figure 6 and 7 shows that Falconn++ achieves higher recall-speed tradeoffs than both FAISS and coCEOs over a wide range of recall ratios. Since coCEOs is designed for maximum inner product search, its performance is inferior to other ANNS solvers for angular distance.

References

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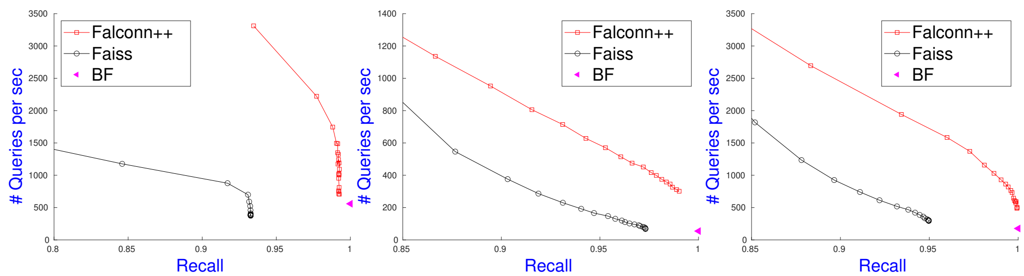
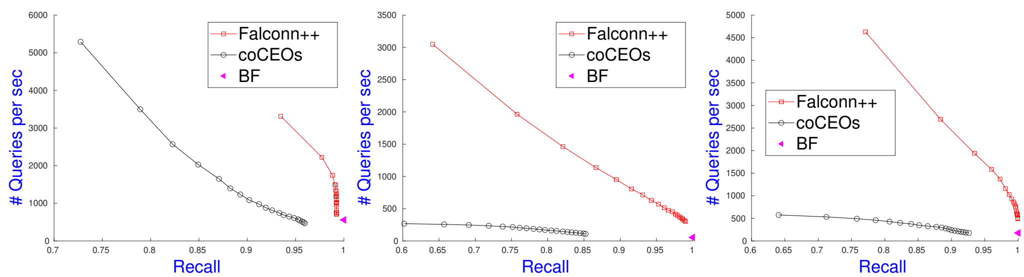


(a) NYTimes (b) Glove300 (c) Glove200

Figure 6: The recall-speed comparison between Falconn++ and FAISS on 3 data sets.



(a) NYTimes (b) Glove300 (c) Glove200

Figure 7: The recall-speed comparison between Falconn++ and coCEOs on 3 data sets.

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