A Full Proof of Propositions and Theorems

A.1 Proof of Proposition 3.1

Proposition 3.1. $I - \kappa \mathbf{f}\mathbf{f}^\top$ *is an elementary matrix, and its eigen values are* $\{1, 1, 1, ..., 1 - \kappa \mathbf{f}^\top \mathbf{f}\}$.

Proof. First, the definition of elementary matrix (Horn & Johnson, 2012) is given: Assume $u, v \in C^n, \kappa \in C$, then $E(u, v, \kappa) = I - \kappa uv^\top$ is an elementary matrix, where C^n is the vector space and C is the set of real numbers.

Then we let $u = \mathbf{f}, v = \mathbf{f}$, obviously $E(\mathbf{f}, \mathbf{f}, \kappa) = I - \kappa \mathbf{f}\mathbf{f}^\top$ is an elementary matrix.

Regarding eigenvalues, an elementary matrix $E(u, v, \kappa)$ has the following properties (Horn & Johnson, 2012):

Proposition A.1. *The eigenvalues of* $E(u, v, \kappa)$ *are* $\{1, 1, 1, ..., 1 - \kappa v^\top u\}$.

Based on A.1, the eigenvalues of $I - \kappa \mathbf{f}\mathbf{f}^\top$ can be formulated as $\{1, 1, 1, ..., 1 - \kappa \mathbf{f}^\top \mathbf{f}\}$.

A.2 Proof of Theorem 3.2

Theorem 3.2. *D(C)CE provides a unique solution to the equilibrium solution problem and the solution exists when* $\kappa < \frac{1}{\mathbf{f}^\top \mathbf{f}}$.

Proof. When the objective function of the optimization problem is convex quadratic and the constraints are affine, the problem is called quadratic programming, and there is a unique solution to the quadratic programming problem (Boyd & Vandenberghe, 2004).

The objective function of D(C)CE, $-\frac{1}{2}\sigma^\top(I - \kappa \mathbf{f}\mathbf{f}^\top)\sigma$, is a quadratic. To make it a convex quadratic, the matrix $I - \kappa \mathbf{f}\mathbf{f}^\top$ must be positive definite. A square matrix is a positive definite matrix if all its eigenvalues are positive (Strang, 2022). Based on Proposition 3.1, if $1 - \kappa \mathbf{f}^\top \mathbf{f} > 0$ or $\kappa < \frac{1}{\mathbf{f}^\top \mathbf{f}}$, then all eigenvalues of $I - \kappa \mathbf{f}\mathbf{f}^\top$ are positive. In addition, the constraints of Diverse (C)CE are all linear affine. To sum up, when $\kappa < \frac{1}{\mathbf{f}^\top \mathbf{f}}$, there is a unique solution to Diverse CCE.

A.3 Proof of Theorem 3.3

Theorem 3.3. *The closed form solution* σ^* *to D(C)CE is formulated as:*

$$\text{General support: } \sigma^* = F^{-1}CA^\top \lambda_1^* + F^{-1}C\lambda_2^* - D,$$

$$\text{Full support: } \sigma^* = F^{-1}CA^\top \lambda_1^* - D,$$

where $F = I - \kappa \mathbf{f}\mathbf{f}^\top$, $F^{-1} = I - \frac{\kappa}{\kappa \mathbf{f}^\top \mathbf{f} - 1}\mathbf{f}\mathbf{f}^\top$, $D^\top = \frac{\mathbf{1}^\top F^{-1}}{\mathbf{1}^\top F^{-1}\mathbf{1}}$, $C = \mathbf{1}D^\top - I$ and $\mathbf{1}$ is a vector of ones.

Proof. We start with the primal Lagrangian form:

$$L = -\frac{1}{2}\sigma^\top(I - \kappa \mathbf{f}\mathbf{f}^\top)\sigma + \lambda_1^\top(A\sigma - \epsilon) - \lambda_2^\top \sigma + v(\mathbf{1}^\top \sigma - 1),$$

Then we take derivatives with respect to the primal variables σ, and make them equal to zero.

$$\frac{\partial L}{\partial \sigma} = F\sigma + A^\top \lambda_1 - \lambda_2 + v\mathbf{1} = 0 \Rightarrow$$

$$\sigma^* = -F^{-1}(A^\top \lambda_1 - \lambda_2 + v\mathbf{1})$$

Then σ^* can be substituted into the Lagrangian function.

$$\begin{aligned}L = &-\frac{1}{2}(\lambda_1^\top AF^{-1}A^\top\lambda_1 - \lambda_1^\top AF^{-1}\lambda_2 + \lambda_1^\top AF^{-1}v\mathbf{1} - \lambda_2^\top F^{-1}A^\top\lambda_1 \\ &+ \lambda_2^\top F^{-1}\lambda_2 - \lambda_2^\top F^{-1}v\mathbf{1} + \mathbf{1}^\top vF^{-1}v\mathbf{1} + \mathbf{1}^\top vF^{-1}A^\top\lambda_1 - \mathbf{1}^\top vF^{-1}\lambda_2) \\ &- \lambda_1^\top\epsilon - v\end{aligned}$$

We take derivatives with respect to v and then make the derivatives equal to zero.

$$\frac{\partial L}{\partial v} = -\frac{1}{2}(\lambda_1^\top AF^{-1}\mathbf{1} - \lambda_2^\top F^{-1}\mathbf{1} + \mathbf{1}^\top F^{-1}A^\top\lambda_1 - \mathbf{1}^\top F^{-1}\lambda_2 + 2\mathbf{1}^\top F^{-1}\mathbf{1}v) - 1 = 0 \Rightarrow$$

$$v^* = -\frac{\mathbf{1}^\top F^{-1}A^\top\lambda_1 - \mathbf{1}^\top F^{-1}\lambda_2 + 1}{\mathbf{1}^\top F^{-1}\mathbf{1}}$$

We then substitute v^* into the Lagrangian function L.

$$\begin{aligned}L =& \frac{1}{2}\lambda_1^\top AF^{-1}(\frac{\mathbf{1}\mathbf{1}^\top F^{-1}}{\mathbf{1}^\top F^{-1}\mathbf{1}} - I)A^\top\lambda_1 + \frac{\mathbf{1}^\top F^{-1}A^\top\lambda_1}{\mathbf{1}^\top F^{-1}\mathbf{1}} \\ &+ \frac{1}{2}\lambda_2^\top F^{-1}(\frac{\mathbf{1}\mathbf{1}^\top F^{-1}}{\mathbf{1}^\top F^{-1}\mathbf{1}} - I)A^\top\lambda_2 - \frac{\mathbf{1}^\top F^{-1}\lambda_2}{\mathbf{1}^\top F^{-1}\mathbf{1}} \\ &- \lambda_1^\top AF^{-1}(\frac{\mathbf{1}\mathbf{1}^\top F^{-1}}{\mathbf{1}^\top F^{-1}\mathbf{1}} - I)\lambda_2 - \lambda_1^\top\epsilon + \frac{1}{2\times\mathbf{1}^\top F^{-1}\mathbf{1}}\end{aligned}$$

We let $F = I - \kappa\mathbf{f}\mathbf{f}^\top$, $F^{-1} = I - \frac{\kappa}{\kappa\mathbf{f}^\top\mathbf{f}-1}\mathbf{f}\mathbf{f}^\top$, $D^\top = \frac{\mathbf{1}^\top F^{-1}}{\mathbf{1}^\top F^{-1}\mathbf{1}}$ and $C = \mathbf{1}D^\top - I$. The Lagrangian function can be simply formulated.

$$\begin{aligned}L =& \frac{1}{2}\lambda_1^\top AF^{-1}CA^\top\lambda_1 + D^\top A^\top\lambda_1 \\ &+ \frac{1}{2}\lambda_2^\top F^{-1}CA^\top\lambda_2 - D^\top\lambda_2 \\ &- \lambda_1^\top AF^{-1}C\lambda_2 - \epsilon^\top\lambda_1 + \frac{1}{2\times\mathbf{1}^\top F^{-1}\mathbf{1}}\end{aligned}$$

Next we substitute v^* into σ^* to get the closed form solution:

$$\begin{aligned}\sigma^* =& -F^{-1}(A^\top\lambda_1 - \lambda_2 + v^*\mathbf{1}) \\ =& F^{-1}CA^\top\lambda_1^* - F^{-1}C\lambda_2^* - D.\end{aligned}$$

For the case where σ has full support, $\lambda_2 = 0$ holds, because any $\sigma \geq 0$ constraint has no effect. Then the closed form solution in full-support cases is

$$\sigma^* = F^{-1}CA^\top\lambda_1^* - D.$$

A.4 Proof of Theorem 3.4

Theorem 3.4. *There is an ϵ such that a full-support D(C)CE solution exists. Specifically, the relationship between ϵ and types of solutions is:*

1. *A uniform solution b always exists when* $\max(Ab) \leq \epsilon$.
2. *The existing solution is non-uniform when* $\epsilon < \max(Ab)$.

Proof. Recalling the optimization objective $-\frac{1}{2}\sigma^\top(I - \kappa \mathbf{ff}^\top)\sigma$, it is obvious that when $\sigma = b$, the optimization objective reaches its maximum value. Additionally, σ must satisfy the linear constraint $A\sigma \leq \epsilon$. Therefore, if $\max(Ab) > \epsilon$, then σ cannot be uniform. On the contrary, when $\max(Ab) \leq \epsilon$, a uniform solution b must exist.