Supplement for
"Efficient uniform approximation using Random Vector Functional Link networks"

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Here you may find parts of proofs omitted in the main article. The numbering herein employed is the same as in the main article.

We remind the reader that $K \subset \mathbb{R}^m$ is compact, $f : K \to \mathbb{R}$ is ℓ-Lipschitz, and

$$\begin{aligned}\tilde{f}(x) &= \rho\Big(|f(a)| - \ell|x-a|\Big)\operatorname{sg} f(a);\\ g(x) &= (2\pi)^{-m}\int F(v)\exp(i\langle v, x\rangle - |v|^2/2\lambda^2)\Psi(v/\lambda)\,dv;\\ h(x) &= (2\pi)^{-m/2}\lambda^m\mathbb{E}\Big(|F(\lambda n)|\Psi(n)\Big[|n| > \vartheta\Big]c(\lambda n, x)\Big).\end{aligned}$$

Furthermore, $H_1, \ldots, H_p$ are iid copies of $G(\mathscr{w}, \mathscr{b})\rho(\langle \mathscr{w}, \diamondsuit\rangle + \mathscr{b}))$ defined in Lemma 4 of the main paper.

Lemma 1. *$\tilde{f}$ is a compactly supported ℓ-Lipschitz extension of f.*

Proof. All that remains to be shown is that $\tilde{f}$ is ℓ-Lipschitz; the rest was shown in the main paper.

Let $x_1, x_2 \in \mathbb{R}^m$. Following [4] suppose WLOG that $|\tilde{f}(x_1)| \geqslant |\tilde{f}(x_2)|$. In other words,

$$\rho\Big(|f(a_1)| - \ell|x_1 - a_1|\Big) \geqslant \rho\Big(|f(a_2)| - \ell|x_2 - a_2|\Big).$$

If $f(a_1)f(a_2) > 0$, then

$$
\begin{aligned}
|\tilde{f}(x_1) - \tilde{f}(x_2)| &= \Big|\rho\Big(|f(a_1)| - \ell|x_1 - a_1|\Big) - \rho\Big(|f(a_2)| - \ell|x_2 - a_2|\Big)\Big| \\
&= \rho\Big(|f(a_1)| - \ell|x_1 - a_1|\Big) - \rho\Big(|f(a_2)| - \ell|x_2 - a_2|\Big) \qquad (1) \\
&\leqslant \rho\Big(|f(a_1)| - \ell|x_1 - a_1|\Big) - \rho\Big(|f(a_1)| - \ell|x_2 - a_1|\Big) \\
&\leqslant \Big|\rho\Big(|f(a_1)| - \ell|x_1 - a_1|\Big) - \rho\Big(|f(a_1)| - \ell|x_2 - a_1|\Big)\Big| \\
&\leqslant \Big|\ \Big(|f(a_1)| - \ell|x_1 - a_1|\Big) - \ \Big(|f(a_1)| - \ell|x_2 - a_1|\Big)\Big| \\
&= \ell\Big||x_2 - a_1| - |x_1 - a_1|\Big| \leqslant \ell|x_1 - x_2|,
\end{aligned}
$$

since ρ is evidently nondecreasing and 1-Lipschitz.

If $f(a_1)f(a_2) \leqslant 0$, however, we must distinguish three further cases.

If additionally $|\tilde{f}(x_2)| > 0$, i.e., $|f(a_2)| - \ell|x_2 - a_2| > 0$, then

$$
\begin{aligned}
|\tilde{f}(x_1) - \tilde{f}(x_2)| &= \rho\Big(|f(a_1)| - \ell|x_1 - a_1|\Big) + \rho\Big(|f(a_2)| - \ell|x_2 - a_2|\Big) \\
&= \ \Big(|f(a_1)| - \ell|x_1 - a_1|\Big) + \ \Big(|f(a_2)| - \ell|x_2 - a_2|\Big) \\
&= |f(a_1)| + |f(a_2)| - \ell|x_2 - a_2| - \ell|x_1 - a_1| \\
&= |f(a_1) - f(a_2)| - \ell|x_2 - a_2| - \ell|x_1 - a_1| \\
&\leqslant \ell|a_1 - a_2| - \ell|x_2 - a_2| - \ell|x_1 - a_1| \\
&\leqslant \ell|x_2 - a_1| - \ell|x_1 - a_1| \leqslant \ell|x_1 - x_2|.
\end{aligned}
$$

If additionally $|\tilde{f}(x_1)| > 0$ and $|f(a_2)| - \ell|x_2 - a_2| \leqslant 0$, that is, $\tilde{f}(x_2)$ is zero, then

$$
\begin{aligned}
|\tilde{f}(x_1) - \tilde{f}(x_2)| &= |\tilde{f}(x_1)| = |\tilde{f}(x_1)| - |\tilde{f}(x_2)| \\
&= \rho\Big(|f(a_1)| - \ell|x_1 - a_1|\Big) - \rho\Big(|f(a_2)| - \ell|x_2 - a_2|\Big),
\end{aligned}
$$

which is (1). Lastly, if both $\tilde{f}(x_1)$ and $\tilde{f}(x_2)$ are zero, then $|\tilde{f}(x_1) - \tilde{f}(x_2)| = 0$, whence the desired Lipschitz continuity of $\tilde{f}$ readily follows.

□

Lemma 2. $\big\|\tilde{f} - g\big\|_\infty \leqslant \dfrac{\ell}{\lambda}\Big(2 - 2^{1/3}am^{-2/3}\Big)\sqrt{m}$.

Proof. Using various substitutions yields that

$$
\begin{aligned}
(2\pi)^{-m}\int F(v)\exp(i\langle v,x\rangle-|v|^2/2\lambda^2)\Psi(v/\lambda)\,dv &= \\
(2\pi)^{-m}\iint \tilde f(u)\exp(-i\langle v,u\rangle)\exp(i\langle v,x\rangle-|v|^2/2\lambda^2)\Psi(v/\lambda)\,du\,dv &= \\
(2\pi)^{-m}\iint \tilde f(u)\exp(i\langle v,x-u\rangle-|v|^2/2\lambda^2)\Psi(v/\lambda)\,dv\,du &= \\
(2\pi)^{-m}\int \tilde f(x-t)\int\exp(i\langle v,t\rangle-|v|^2/2\lambda^2)\Psi(v/\lambda)\,dv\,dt &= \\
(2\pi)^{-m}\int \tilde f(x-s/\lambda)\int\exp(i\langle w,s\rangle-|w|^2/2)\Psi(w)\,dw\,dt &= \\
\int \tilde f(x-s/\lambda)(\delta_Z*\psi)(s)\,ds&,
\end{aligned}
$$

where the second equality follows readily from Fubini's theorem because $\tilde f$ and Ψ are compactly supported. In the main article it was shown that

$$\big\|\tilde f-g\big\|_\infty\leqslant\frac{\ell}{\lambda}\int|s|(\delta_Z*\psi)(s)\,ds,$$

so all that remains to be shown is that $\int|s|(\delta_Z*\psi)(s)\,ds\leqslant\left(2-2^{1/3}am^{-2/3}\right)\sqrt{m}$.

Since ψ is a pdf, $\psi=\delta_X$ for some random variable X. Thus,

$$\int|s|(\delta_Z*\psi)(s)\,ds=\int|s|(\delta_Z*\delta_X)(s)\,ds=\mathbb{E}|Z+X|\leqslant\mathbb{E}|Z|+\mathbb{E}|X|.$$

Since $|Z|$ is chi distributed with m degrees of freedom, Wendel's inequality [6] yields that

$$\mathbb{E}|Z|=\frac{\Gamma((m+1)/2)}{\Gamma(m/2)}\sqrt{2}\leqslant\sqrt{m}.$$

By Jensen's inequality and [3, thm 5.1],

$$\mathbb{E}|X|\leqslant\sqrt{\int|x|^2\psi(x)\,dx}=2j_\nu/\sqrt{m}.$$

Indeed, as follows from the scaling property of the Fourier transform,

$$\int |x|^2 \mathscr{F}^{-1}\Big\{(\omega*\omega)\Big(\Diamond/\sqrt{m}\Big)\Big\}(x)\,dx =$$
$$m^{m/2}\int |x|^2\mathscr{F}^{-1}\{\omega*\omega\}\Big(x\sqrt{m}\Big)\,dx =$$
$$\frac{1}{m}\int |u|^2\mathscr{F}^{-1}\{\omega*\omega\}(u)\,du = 4j_\nu^2/m.$$

We now claim that $2j_\nu/\sqrt{m} < \sqrt{m} - 2^{1/3}am^{-1/6}$. In [5], it has been proven that

$$j_\nu < \nu - a(\nu/2)^{1/3} + \tfrac{3}{20}a^2(\nu/2)^{-1/3}$$

for all $\nu > 0$. If $m \geqslant 3$, then $\nu = m/2 - 1 > 0$, so

$$\begin{aligned} 2j_\nu/\sqrt{m} &< \sqrt{m} - 2/\sqrt{m} - 2^{2/3}a(m/2-1)^{1/3}/\sqrt{m} + \tfrac{6}{20}a^2(\nu/2)^{-1/3}/\sqrt{m} \\ &< \sqrt{m} - 2^{2/3}a(m/2)^{1/3}/\sqrt{m} + (\tfrac{6}{20}a^2(\nu/2)^{-1/3} - 2)/\sqrt{m} \\ &= \sqrt{m} - 2^{1/3}am^{-1/6} + (\tfrac{6}{20}a^2(\nu/2)^{-1/3} - 2)/\sqrt{m}. \end{aligned}$$

Since $(\nu/2)^{-1/3}$ is clearly decreasing in m and $m = 3 \Rightarrow \frac{6}{20}a^2(\nu/2)^{-1/3} < 2$, as can be verified numerically, the last term will always be negative and can thus be discarded to yield our claim for all $m \geqslant 3$. One can verify the cases $m = 1, 2$ numerically. Putting everything together yields the desideratum. □

Lemma 3. $\big\|g - h\big\|_\infty \leqslant \dfrac{2\ell R}{\sqrt{\pi m}} V_m\left\{\dfrac{R\vartheta\lambda}{\sqrt{2\pi/e}}\right\}^m.$

Proof. In the main article we has already shown that

$$\big\|g-h\big\|_\infty \leqslant (2\pi)^{-m/2}\lambda^m\big\|\tilde{f}\big\|_1\mathbb{P}\Big\{|\boldsymbol{n}| \leqslant \vartheta\sqrt{m}\Big\},$$

so all that remains is to bound $\big\|\tilde{f}\big\|_1$ and $(2\pi)^{-m/2}\lambda^m\mathbb{P}\Big\{|\boldsymbol{n}| \leqslant \vartheta\sqrt{m}\Big\}$.

Starting with the former, since $|\tilde{f}(x)| = \rho\Big(|f(a)| - \ell|x-a|\Big)$,

$$\big\|\tilde{f}\big\|_1 = \int_{\tilde{K}}|\tilde{f}| = \int_K |\tilde{f}| + \int_{\tilde{K}-K}|\tilde{f}| \leqslant M|K| + \int_{|u|\leqslant M/\ell}\Big(M - \ell|u|\Big)\,du.$$

Now,

$$\int\limits_{|u|\leqslant M/\ell}\Big(M-\ell|u|\Big)\,du = V_m\int\limits_0^{M/\ell}(M-\ell y)y^{m-1}\,dy = \frac{V_m M^{m+1}}{\ell^m m(m+1)}.$$

In conjunction with the previous display, this yields that

$$\big\|\tilde{f}\big\|_1 \leqslant M\Big(|K| + V_m\frac{(M/\ell)^m}{m(m+1)}\Big) \leqslant M\Big(V_m R^m + V_m (M/\ell)^m\Big) \leqslant 2\ell R V_m R^m,$$

because $M \leqslant \ell R$. Indeed, since K has circumradius R, it follows that $\operatorname{diam}(K) \leqslant 2R$, so $2M = \max f - \min f \leqslant \ell \operatorname{diam}(K) = 2\ell R$ because f is ℓ-Lipschitz.

As for the latter, $|\mathfrak{n}|$ has a chi distribution with m degrees of freedom, so the cdf of $|\mathfrak{n}|$ may be expressed as $P(m/2, \diamondsuit^2/2)$ [2, §8.2(i)]. If $x \geqslant 0$, then [2, (8.6.3)]

$$P(a,x) = \frac{x^a}{\Gamma(a)}\int\limits_0^\infty \exp(-at - xe^{-t})\,dt \leqslant \frac{x^a}{\Gamma(a)}\int\limits_0^\infty \exp(-at)\,dt = \frac{x^a}{a\Gamma(a)}.$$

Additionally $a\Gamma(a) > (a/e)^a\sqrt{2\pi a}$ for all $a > 0$ [2, (5.6.1)]. Ergo,

$$\begin{aligned}(2\pi)^{-m/2}\lambda^m\mathbb{P}\Big\{|\mathfrak{n}| \leqslant \vartheta\sqrt{m}\Big\} &= (2\pi)^{-m/2}\lambda^m P(m/2, m\vartheta^2/2)\\ &< (2\pi)^{-m/2}\lambda^m\frac{1}{\sqrt{\pi m}}\left(\frac{m\vartheta^2/2}{m/2e}\right)^{m/2}\\ &= \frac{1}{\sqrt{\pi m}}\left(\frac{\vartheta\lambda}{\sqrt{2\pi/e}}\right)^m.\end{aligned}$$

Multiplying the obtained bounds readily yields the desideratum. □

Lemma 4. $h = \mathbb{E}(G(\mathfrak{w}, \mathfrak{b})\rho(\langle\mathfrak{w}, \diamondsuit\rangle + \mathfrak{b}))$ *on* K, *where*

- $G(w,b) = -2\sigma R\sqrt{m}\Lambda^2(2\pi)^{-m/2}\lambda^m|F(\Lambda w)|\Psi(w/\sigma)$ $\Big[|w| \geqslant \vartheta\sigma\sqrt{m}\Big]\cos(\Lambda b - \arg F(\Lambda w))$*;*
- $\mathfrak{b}$ *being uniformly distributed on* $\Big[-\sigma R\sqrt{m}, \sigma R\sqrt{m}\Big]$*;*
- $\mathfrak{w} \sim N(0, \sigma I_m)$.

Proof. Letting $N \sim N(0, \sigma I_m)$ and $\varphi(v) = \arg F(v)$ allows us to write

$$(2\pi)^{-m/2}\lambda^m \mathbb{E}\Big(|F(\lambda \boldsymbol{n})|\Psi(\boldsymbol{n})\Big[|\boldsymbol{n}| \geqslant \vartheta\sqrt{m}\Big]c(\boldsymbol{n}, x)\Big) =$$

$$(2\pi)^{-m/2}\lambda^m \int |F(\lambda u)|\Psi(u)\Big[|u| \geqslant \vartheta\sqrt{m}\Big]c(\lambda u, x)\delta_Z(u)\, du =$$

$$(2\pi)^{-m/2}\lambda^m \int |F(\Lambda w)|\Psi(w/\sigma)\Big[|w| \geqslant \vartheta\sigma\sqrt{m}\Big]c(\Lambda w, x)\delta_N(w)\, dw =$$

$$\int \Big((2\pi)^{-m/2}\lambda^m |F(\Lambda w)|\Big[|w| \geqslant \vartheta\sigma\sqrt{m}\Big]\Psi(w/\sigma)\Big)$$

$$\cos(\varphi(\Lambda w) + \Lambda\langle w, x\rangle)\Big[|\langle w, x\rangle| \leqslant \sigma R\sqrt{m}\Big]\delta_N(w)\, dw,$$

since $|\langle w, x\rangle| \leqslant |w|.|x|$ and the support of Ψ is $\Big\{w \in \mathbb{R}^m : |w| \leqslant \sqrt{m}\Big\}$.

Now, using the fundamental theorem of calculus and integration by parts,

$$\begin{aligned}
\cos(\varphi + \Lambda z)\Big[|z| \leqslant B\Big] &= -\Lambda \int_{-\infty}^{z} \sin(\varphi + \Lambda y)\Big[|y| \leqslant B\Big]\, dy \\
&= -\Lambda \int \sin(\varphi + \Lambda y)\Big[|y| \leqslant B\Big]\dot{\rho}(z - y)\, dy \\
&= -\Lambda^2 \int \cos(\varphi + \Lambda y)\Big[|y| \leqslant B\Big]\rho(z - y)\, dy \\
&= -\Lambda^2 \int \cos(\varphi - \Lambda b)\rho(z + b)\Big[|b| \leqslant B\Big]\, db.
\end{aligned}$$

Upon plugging back in, this yields that

$$h(x) = \iint G(w, b)\rho(\langle w, x\rangle + b)\frac{1}{2\sigma R\sqrt{m}}\Big[|b| \leqslant \sigma R\sqrt{m}\Big]\delta_N(w)\, db\, dw,$$

which is the expanded form of desired expectation. $\square$

Lemma 5. *Let* $N_n = \frac{1}{n}\sum_{p=1}^{n} H_p$ *and* $t > 0$. *Then*

$$\mathbb{P}\Big\{\big\|N_n - \mathbb{E}(H)\big\|_K > t\Big\} \leqslant$$

$$2\exp\Bigg(-\frac{n}{2}\bigg(\frac{t}{2R^2\sqrt{m}(2\pi)^{-m/2}\lambda^{m+1}(1 + 1/\vartheta)\ell|\tilde{K}|}\bigg)^2\Bigg).$$

Proof. We first prove the existence of the measurable selector X_n.

To that end, let (Ω, Σ) be the measurable space underlying $\mathbb{P}$. We define

$$\xi : \Omega \times K \ni (\omega, x) \mapsto \big|N_n(x)(\omega) - \mathbb{E}(H(x))\big|;$$
$$\Xi(\omega) = \operatorname*{argmax}_{x \in K} \xi(\omega, x) = \Big\{x \in K : \xi(\omega, x) - \max_{u \in K} \xi(\omega, u) = 0\Big\} \subset K.$$

Note that $H(\diamondsuit)(\omega)$ and $\mathbb{E}(H(\diamondsuit))$ are Lipschitz continuous for all $\omega \in \Omega$, and

$$\Big\{\max_{u \in K} \xi(\diamondsuit, u) \leqslant c\Big\} = \bigcap_{u \in K} \{\xi(\diamondsuit, u) \leqslant c\} = \bigcap_{q \in S} \{\xi(\diamondsuit, q) \leqslant c\},$$

where S is a countably dense subset of K, whence $\max_{u \in K} \xi(\diamondsuit, u)$ is measurable. Ergo,

$$\Omega \times K \ni (\omega, x) \mapsto \xi(\omega, x) - \max_{u \in K} \xi(\omega, u)$$

is a Carathéodory function [1, def. 4.50]. Since K is a compact subset of $\mathbb{R}^m$, it follows that Ξ is a (weakly) [1, lem. 18.2] measurable [1, cor. 18.8] correspondence [1, def. 17.1] with nonempty closed values from a measurable space into a Polish space. Thus Ξ admits a measurable selector [1, thm 18.13], that is, there exists a random variable

$$X_n : \Omega \ni \omega \mapsto X_n(\omega) \in \Xi(\omega)$$

such that $\big|N_n(X_n) - \mathbb{E}(H(X_n))\big| = \big\|N_n - \mathbb{E}(H)\big\|_K$.

In order to apply Hoeffding's inequality to

$$\mathbb{P}\Big\{\big\|N_n - \mathbb{E}(H)\big\|_K > t\Big\} = \mathbb{P}\Big\{\big|N_n(X_n) - \mathbb{E}(H(X_n))\big| > t\Big\},$$

we need to bound $H(X_n)$ a.s., which we can do as follows.

$$\begin{aligned}
&|H(X_n)| = |G(\mathfrak{w}, \mathfrak{b})\rho(\langle \mathfrak{w}, X_n\rangle + \mathfrak{b})| \leqslant \\
&2\sigma R\sqrt{m}\Lambda^2(2\pi)^{-m/2}\lambda^m|F(\Lambda w)|.|\Psi(w/\sigma)|\Big(\langle \mathfrak{w}, X_n\rangle + \sigma R\sqrt{m}\Big)\Big[|w| \geqslant \vartheta\sigma\sqrt{m}\Big] \leqslant \\
&2\sigma R\sqrt{m}\Lambda(2\pi)^{-m/2}\lambda^m\Big(\Lambda|F(\Lambda\mathfrak{w})|\Big)R(1 + 1/\vartheta)|\mathfrak{w}| \leqslant \\
&2R^2\sqrt{m}(2\pi)^{-m/2}\lambda^{m+1}(1 + 1/\vartheta)\Big\||\Lambda\diamondsuit|.|F(\Lambda\diamondsuit)|\Big\|_\infty = \\
&2R^2\sqrt{m}(2\pi)^{-m/2}\lambda^{m+1}(1 + 1/\vartheta)\Big\||\diamondsuit|.|F(\diamondsuit)|\Big\|_\infty,
\end{aligned}$$

because $|\Psi| \leqslant 1$ and $X_n(\omega) \in K$. Now, using the fact that $\left\|\mathscr{F}\{\Diamond\}\right\|_\infty \leqslant \left\|\Diamond\right\|_1$ and utilizing Minkowski's integral inequality twice yields that

$$\begin{aligned}\left\| |\Diamond|.|F(\Diamond)| \right\|_\infty &= \left\| |\{\Diamond_p F(\Diamond)\}_{p=1}^m| \right\|_\infty \leqslant \left| \left\{ \left\|\Diamond_p F(\Diamond)\right\|_\infty \right\}_{p=1}^m \right| \\ &= \left| \left\{ \left\|\mathscr{F}\{\partial_p \tilde{f}\}\right\|_\infty \right\}_{p=1}^m \right| \leqslant \left| \left\{ \left\|\partial_p \tilde{f}\right\|_1 \right\}_{p=1}^m \right| \\ &\leqslant \left\| |\{\partial_p \tilde{f}\}_{p=1}^m| \right\|_1 = \left\|\nabla \tilde{f}\right\|_1 \leqslant \ell |\tilde{K}|,\end{aligned}$$

so $H(X_n)$ is (a.s.) bounded by $2R^2\sqrt{m}(2\pi)^{-m/2}\lambda^{m+1}(1+1/\vartheta)\ell|\tilde{K}|$. Here, $|\{\Diamond_p\}_{p=1}^m|$ denotes the Euclidean norm of the vector $\{\Diamond_p\}_{p=1}^m \in \mathbb{R}^m$.

Above we implicitly used Rademacher's theorem to conclude that $\nabla \tilde{f}$ exists a.e. We also used that $\left\|\nabla \tilde{f}\right\|_\infty \leqslant \ell$, which can be seen as follows.

It suffices to show that $|\nabla \tilde{f}(x)| \leqslant \ell$ for all $x \in \tilde{K}$ for which $\nabla \tilde{f}(x)$ exists. Suppose $x \in \tilde{K}$ is such that $\nabla \tilde{f}(x)$ exists. If $\nabla \tilde{f}(x) = 0$, there is nothing to prove, so we may additionally assume that $\nabla \tilde{f}(x) \neq 0$. For almost every such x it holds that

$$\begin{aligned}\lim_{h\to 0}\left| \frac{|\tilde{f}(x+hu) - \tilde{f}(x)|}{h} - |\nabla \tilde{f}(x)| \right| &\leqslant \lim_{h\to 0}\left| \frac{\tilde{f}(x+hu) - \tilde{f}(x)}{h} - |\nabla \tilde{f}(x)| \right| \\ &= \lim_{h\to 0} \frac{|\tilde{f}(x+hu) - \tilde{f}(x) - \langle hu, \nabla \tilde{f}(x)\rangle|}{|hu|} = 0,\end{aligned}$$

where $u = \dfrac{\nabla \tilde{f}(x)}{|\nabla \tilde{f}(x)|}$ and $h > 0$. As such,

$$|\nabla \tilde{f}(x)| = \lim_{h\to 0} \frac{|\tilde{f}(x+hu) - \tilde{f}(x)|}{h} \leqslant \ell,$$

because u is a unit vector.

The claim of the lemma then follows from directly applying Hoeffding's inequality. $\square$

References

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