A Appendix

B Limitations

Here we discuss key limitations of our work and areas that are not explored in this paper.

- Developed theoretical claims are for strongly convex loss functions. The globalization mechanism with cubic regularization can be anaylized for convex functions as well, but we do not consider non-convex objectives in this work.
- Our methods are analyzed in the regime when the exact local gradients and exact local Hessians of local loss functions are computed for all participating devices. We do not consider stochastic gradient or stochastic Hessian oracles of local loss functions in our analyses. However, when we use sketch-and-project operator we rely on Hessian-vector products which does not require full Hessian computations.

C Detailed Literature Review of Second-Order Methods

In this section we provide more detailed literature review of second-order methods. The comparison is made based on the most relevant prior works in the literature highlighting main differences over our work. The comparison is performed based on criterias including generality of the considered problem structure, assumptions made on the (local) loss functions, communication complexity per iteration, theoretical convergence guarantees and other aspects of the method.

- GIANT (Wang et al., 2018) and NL (Islamov et al., 2021) are not designed to handle a general finite sum problem. In contrast to our work, they only work with Generalized Linear models.
- Communication costs per iteration of DAN (Zhang et al., 2020a) and Quantized Newton (Alimisis et al., 2021) are significantly high which make them impractical.
- NL (Islamov et al., 2021) directly reveals local data in each iteration which breaks privacy preserving guarantees.
- The drawback of GIANT, DINGO is in their convergence rates which depend on the condition number of the problem. In some cases theoretical convergence guarantees are even worse than those of first-order methods. DANE (Shamir et al., 2014) and AIDE (Reddi et al., 2016) suffer from the same problem because those methods are first-order methods.
- The first drawback of FLECS (Agafonov et al., 2022b) is that SVD decomposition is needed in each step to perform a truncation which means that the computation cost of those methods can not be reduced. Besides, there is no good local theory for that method; the only convergence guarantee is derived under the assumption that the iterates remain close to the optimum. Next, convergence of FLECS and FLECS-SGD depend on the product of the condition number and truncation parameters. For example, using the same truncation parameters as in their experiments, the convergence guarantees are of the order $10^{22}\kappa$, where κ is the condition number. Finally, they need backtracking line search or other learning techniques with additional parameters to perform one step of the methods.
- Fib-IOS (Fabbro et al., 2022) introduces Newton-type method based on SVD decomposition which means that similarly to FLECS the computation cost can not be reduced. Besides, this approach is also restricted to rank-type compression only, and consequently does not support popular compression techniques such as Top-K or Rand-K. On top of that, Fib-IOS always requires backtracking line search technique to find appropriate stepsize.
- GIANT and DANE work well only in homogeneous setting while in practice the problem could be significantly heterogeneous. In our work we do not make any assumptions on heterogeneity of the problem.

- While convergence guarantees of FedNL (Safaryan et al., 2022) and Newton-3PC are the same (fast local linear/superlinear rates independent of the condition number) there are several points that make our work superior: (i) FedNL can be seen as a special case of Newton-3PC; (ii) we provide much wider compression mechanisms going beyond those proposed in (Safaryan et al., 2022); (iii) we propose two ways how to reduce computation costs (lazy aggregation and sketching) of Newton-3PC.

D More on Sketch-and-Project Mechanism

Sketch-and-project operator has been widely studied as a technique to solve linear systems (Richtárik & Takac, 2017; Gower & Richtárik, 2015), an application to first-order methods (Hanzely et al., 2018). Besides, Gower & Richtárik (2017) showed that various Quasi-Newton updates can be seen as a special case of sketch-and-project mechanism. Let us now describe it in more details. For this reason, we introduce an arbitrary twice differentiable function $\varphi : \mathbb{R}^d \to \mathbb{R}$. Our desire is to compute an approximation of its Hessian $\nabla^2\varphi$ at point x. Let $\mathbf{X}_0$ be the first approximation of $\nabla^2\varphi(x)$. Then we define a sequence $\{\mathbf{X}_k\}$ that approximates $\nabla^2\varphi(x)$ better and better as k goes to infinity solving the following optimization problem

$$\mathbf{X}_{k+1} := \operatorname{argmin}_{\mathbf{X}\in\mathbb{R}^{d\times d}} \frac{1}{2}\|\mathbf{X} - \mathbf{X}_k\|_{\mathrm{F}}^2 \quad \text{s.t.} \quad \mathbf{S}^\top \nabla^2\varphi(x) = \mathbf{S}^\top \mathbf{X}, \tag{14}$$

where $\mathbf{S} \in \mathbb{R}^{d\times\tau}$ is a random matrix drawn in i.i.d. fashion from a fixed distribution $\mathcal{D}$. To solve this problem we define a function $\operatorname{vec}(\mathbf{A}) = (\mathbf{A}_{11}, \ldots, \mathbf{A}_{d1}, \mathbf{A}_{12}, \ldots, \mathbf{A}_{d2}, \ldots, \mathbf{A}_{1d}, \ldots, \mathbf{A}_{dd})^\top$. Moreover, we need to define an extended sketch matrix $\tilde{\mathbf{S}}$ of the form

$$\tilde{\mathbf{S}} := \begin{pmatrix} \mathbf{S} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{S} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{S} \end{pmatrix} \in \mathbb{R}^{d^2\times d\tau}. \tag{15}$$

Using (15), we can reformulate (14) as follows

$$\operatorname{vec}(\mathbf{X}_{k+1}) = \operatorname{argmin}_{\mathbf{X}} \|\operatorname{vec}(\mathbf{X}) - \operatorname{vec}(\mathbf{X}_k)\|^2, \quad \tilde{\mathbf{S}}^\top \operatorname{vec}(\mathbf{X}) = \tilde{\mathbf{S}}^\top \operatorname{vec}(\nabla^2\varphi(x)). \tag{16}$$

The latter has an explicit solution (Hanzely et al., 2018) of the following form

$$\operatorname{vec}(\mathbf{X}_{k+1}) = \operatorname{vec}(\mathbf{X}_k) + \tilde{\mathbf{Z}}(\operatorname{vec}(\nabla^2\varphi(x)) - \operatorname{vec}(\mathbf{X}_k)), \tag{17}$$

where $\tilde{\mathbf{Z}} := \tilde{\mathbf{S}}(\tilde{\mathbf{S}}^\top\tilde{\mathbf{S}})^\dagger\tilde{\mathbf{S}}^\top$. It is easy to show that $\tilde{\mathbf{Z}}$ can be rewritten as follows

$$\tilde{\mathbf{Z}} = \begin{pmatrix} \mathbf{Z} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{Z} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{Z} \end{pmatrix}, \tag{18}$$

where $\mathbf{Z} := \mathbf{S}(\mathbf{S}^\top\mathbf{S})^\dagger\mathbf{S}^\top$ is a projection matrix onto the range of a sketch $\mathbf{S}$. Since it is not clear how to compute the process (17), we rewrite it explicitly as

$$\mathbf{X}_{k+1} = \mathbf{X}_k + \mathbf{S}(\mathbf{S}^\top\mathbf{S})^\dagger(\nabla^2\varphi(x) - \mathbf{X}_k).$$

The way in which the above formula is written resembles the update from (Safaryan et al., 2022).

E Deferred Proofs from Section 3 and New 3PC Compressors

E.1 Proof of Lemma 3.4: Adaptive Thresholding

Basically, we show two upper bounds for the error and combine them to get the expression for α. From the definition (5), we get

$$\|\mathcal{C}(\mathbf{X}) - \mathbf{X}\|_{\mathrm{F}}^2 = \sum_{j,l:|\mathbf{X}_{jl}|<\lambda\|\mathbf{X}\|_\infty} \mathbf{X}_{jl}^2 \leq d^2\lambda^2\|\mathbf{X}\|_\infty^2 \leq d^2\lambda^2\|\mathbf{X}\|_{\mathrm{F}}^2.$$

The second inequality is derived from the observation that at least on entry, the top one in magnitude, is selected always. Since the top entry is missing in the sum below, we imply that the average without the top one is smaller than the overall average.

$$\|\mathcal{C}(\mathbf{X})-\mathbf{X}\|_{\mathrm{F}}^2=\sum_{j,l:|\mathbf{X}_{jl}|<\lambda\|\mathbf{X}\|_\infty}\mathbf{X}_{jl}^2\leq\frac{d^2-1}{d^2}\sum_{j,l=1}^{d}\mathbf{X}_{jl}^2\leq\left(1-\frac{1}{d^2}\right)\|\mathbf{X}\|_{\mathrm{F}}^2.$$

E.2 Proof of Lemma 3.6: Sketch-and-Project

Note that since $\mathbf{Z}$ is a projection matrix, then it is symmetric and satisfies

$$\begin{aligned}\mathbf{Z}\mathbf{Z} &= \mathbf{S}(\mathbf{S}^\top\mathbf{S})^\dagger\mathbf{S}^\top\mathbf{S}(\mathbf{S}^\top\mathbf{S})^\dagger\mathbf{S}^\top\\ &= \mathbf{S}(\mathbf{S}^\top\mathbf{S})^\dagger\mathbf{S}^\top=\mathbf{Z},\end{aligned}$$

As a consequence, its eigenvalues are between 0 and 1. Assuming that $\mathbf{X}$ is symmetric we derive

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{Z}\mathbf{X}-\mathbf{X}\|_{\mathrm{F}}^2\right] &= \mathbb{E}\left[\|\mathbf{Z}\mathbf{X}\|_{\mathrm{F}}^2-2\left\langle\mathbf{Z}\mathbf{X},\mathbf{X}\right\rangle\right]+\|\mathbf{X}\|_{\mathrm{F}}^2\\ &= \mathbb{E}\left[\mathrm{Tr}(\mathbf{X}^\top\mathbf{Z}^\top\mathbf{Z}\mathbf{X})\right]-2\left\langle\mathbb{E}\left[\mathbf{Z}\right]\mathbf{X},\mathbf{X}\right\rangle+\|\mathbf{X}\|_{\mathrm{F}}^2\\ &= \mathbb{E}\left[\left\langle\mathbf{X},\mathbf{Z}\mathbf{X}\right\rangle\right]-2\left\langle\mathbb{E}\left[\mathbf{Z}\right]\mathbf{X},\mathbf{X}\right\rangle+\|\mathbf{X}\|_{\mathrm{F}}^2\\ &= \|\mathbf{X}\|_{\mathrm{F}}^2-\mathrm{Tr}(\mathbf{X}^\top\mathbb{E}\left[\mathbf{Z}\right]\mathbf{X})\\ &\leq (1-\lambda_{\min}^{+}(\mathbf{Z}))\|\mathbf{X}\|_{\mathrm{F}}^2.\end{aligned}$$

Note that we can force $\mathbf{X}$ to be symmetric in the same way as it was done in (Qian et al., 2022) by using symmetrization operator $[\mathbf{X}]_s=\frac{1}{2}(\mathbf{X}+\mathbf{X}^\top)$ which does not change the theory.

E.3 Proof of Lemma 3.10: Compressed Bernoulli AGgregation (CBAG)

As it was mentioned CBAG has two independent sources of randomness: Bernoulli aggregation and possible random contractive compression. To show that CBAG is a 3PC mechanism, we consider these randomness one by one and upper bound the error as follows:

$$\begin{aligned}\mathbb{E}\left[\|\mathcal{C}_{\mathbf{H},\mathbf{Y}}(\mathbf{X})-\mathbf{X}\|^2\right] &= (1-p)\|\mathbf{H}-\mathbf{X}\|^2+p\mathbb{E}\left[\|\mathcal{C}(\mathbf{X}-\mathbf{H})-(\mathbf{X}-\mathbf{H})\|^2\right]\\ &\leq (1-p)\|\mathbf{X}-\mathbf{H}\|^2+p(1-\alpha)\|\mathbf{X}-\mathbf{H}\|^2\\ &= (1-p\alpha)\|\mathbf{X}-\mathbf{H}\|^2\\ &\leq (1-p\alpha)(1+s)\|\mathbf{H}-\mathbf{Y}\|^2+(1-p\alpha)(1+1/s)\|\mathbf{X}-\mathbf{Y}\|^2.\end{aligned}$$

E.4 New 3PC: Adaptive Top-K

Assume that in our framework we are restricted by the number of floats we can send from clients to the server. For example, each client is able to broadcast $d_0\leq d^2$ floats to the server. Besides, we want to use Top-K compression operator with adaptive K, but due to the aforementioned restrictions we should control how K evolves. Let $K_{\mathbf{H},\mathbf{Y}}$ be such that

$$K_{\mathbf{H},\mathbf{Y}}=\min\left\{\left\lceil\frac{\|\mathbf{Y}-\mathbf{H}\|_{\mathrm{F}}^2}{\|\mathbf{X}-\mathbf{H}\|_{\mathrm{F}}^2}d^2\right\rceil,d_0\right\}.$$

We introduce the following compression operator

$$\mathcal{C}_{\mathbf{H},\mathbf{Y}}(\mathbf{X}):=\mathbf{H}+\text{Top-}K_{\mathbf{H},\mathbf{Y}}\left(\mathbf{X}-\mathbf{H}\right). \tag{19}$$

The next lemma shows that the described compressor satisfy (3).

Lemma E.1. *The compressor* $\mathcal{C}_{\mathbf{Y},\mathbf{H}}$ (19) *satisfy* (3) *with*

$$A=\frac{d_0}{2d^2},\quad B=\max\left\{\left(1-\frac{d_0}{d^2}\right)\left(\frac{2d^2}{d_0}-1\right),3\right\}.$$

Proof. Recall that if $\mathcal{C}$ is a Top-K compressor, then for all $\mathbf{X} \in \mathbb{R}^{d\times d}$

$$\|\mathcal{C}(\mathbf{X}) - \mathbf{X}\|_{\mathrm{F}}^2 \leq \left(1 - \frac{K}{d^2}\right) \|\mathbf{X}\|_{\mathrm{F}}^2,$$

Using this property we get in the case when $K_{\mathbf{Y},\mathbf{H}} = d_0$

$$\begin{aligned}
\|\mathcal{C}_{\mathbf{H},\mathbf{Y}}(\mathbf{X}) - \mathbf{X}\|_{\mathrm{F}}^2 &= \|\mathbf{H} + \text{Top-}K_{\mathbf{H},\mathbf{Y}}(\mathbf{X} - \mathbf{H}) - \mathbf{X}\|_{\mathrm{F}}^2 \\
&\leq \left(1 - \frac{d_0}{d^2}\right) \|\mathbf{H} - \mathbf{X}\|_{\mathrm{F}}^2 \\
&\leq \left(1 - \frac{d_0}{2d^2}\right) \|\mathbf{H} - \mathbf{Y}\|_{\mathrm{F}}^2 + \left(1 - \frac{d_0}{d^2}\right) \frac{2d^2 - d_0}{d_0} \|\mathbf{Y} - \mathbf{X}\|_{\mathrm{F}}^2.
\end{aligned}$$

If $K_{\mathbf{H},\mathbf{Y}} = \left\lceil \frac{\|\mathbf{Y}-\mathbf{H}\|_{\mathrm{F}}^2}{\|\mathbf{X}-\mathbf{H}\|_{\mathrm{F}}^2} d^2 \right\rceil$, then $-K_{\mathbf{H},\mathbf{Y}} \leq -\frac{\|\mathbf{Y}-\mathbf{H}\|_{\mathrm{F}}^2}{\|\mathbf{X}-\mathbf{H}\|_{\mathrm{F}}^2} d^2$, and we have

$$\begin{aligned}
\|\mathcal{C}_{\mathbf{H},\mathbf{Y}}(\mathbf{X}) - \mathbf{X}\|_{\mathrm{F}}^2 &= \|\mathbf{H} + \text{Top-}K_{\mathbf{H},\mathbf{Y}}(\mathbf{X} - \mathbf{H}) - \mathbf{X}\|_{\mathrm{F}}^2 \\
&\leq \left(1 - \frac{K_{\mathbf{H},\mathbf{Y}}}{d^2}\right) \|\mathbf{H} - \mathbf{X}\|_{\mathrm{F}}^2 \\
&\leq \left(1 - \frac{\|\mathbf{Y} - \mathbf{H}\|_{\mathrm{F}}^2}{\|\mathbf{X} - \mathbf{H}\|_{\mathrm{F}}^2}\right) \|\mathbf{H} - \mathbf{X}\|_{\mathrm{F}}^2 \\
&= \|\mathbf{H} - \mathbf{X}\|_{\mathrm{F}}^2 - \|\mathbf{Y} - \mathbf{H}\|_{\mathrm{F}}^2 \\
&\leq \frac{3}{2} \|\mathbf{H} - \mathbf{Y}\|_{\mathrm{F}}^2 + 3 \|\mathbf{Y} - \mathbf{X}\|_{\mathrm{F}}^2 - \|\mathbf{Y} - \mathbf{H}\|_{\mathrm{F}}^2 \\
&= \frac{1}{2} \|\mathbf{Y} - \mathbf{H}\|_{\mathrm{F}}^2 + 3 \|\mathbf{Y} - \mathbf{X}\|_{\mathrm{F}}^2,
\end{aligned}$$

where in the last inequality we use Young's inequality. Since we always have $\frac{d_0}{2d^2}$ (because $d_0 \leq d^2$), then $A = \frac{d_0}{2d^2}$. □

E.5 New 3PC: Rotation Compression

Qian et al. (2022) proposed a novel idea to change the basis in the space of matrices that allows to apply more aggresive compression mechanism. Following Section 2.3 from (Qian et al., 2022) one can show that for Generalized Linear Models local Hessians can be represented as $\nabla^2 f_i(x) = \mathbf{Q}_i \Lambda_i(x) \mathbf{Q}_i^\top$, where $\mathbf{Q}_i$ is properly designed basis matrix. This means that $\mathbf{Q}_i$ is orthogonal matrix. Their idea is based on the fact that $\Lambda_i(x)$ is potentially sparser matrix than $\nabla^2 f_i(x)$, and applying compression on $\Lambda_i(x)$ could require smaller compression level to obtain the same results than applying compression on dense standard representation $\nabla^2 f_i(x)$. We introduce the following compression based on this idea. Let $\mathcal{C}$ be an arbitrary contractive compressor with parameter α, and $\mathbf{Q}$ be an orthogonal matrix, then our new compressor is defined as follows

$$\mathcal{C}_{\mathbf{H},\mathbf{Y}}(\mathbf{X}) := \mathbf{H} + \mathbf{Q}\mathcal{C}\left(\mathbf{Q}^\top(\mathbf{X} - \mathbf{H})\mathbf{Q}\right)\mathbf{Q}^\top. \tag{20}$$

Now we prove that this compressor satisfy (3).

Lemma E.2. *The compressor $\mathcal{C}_{\mathbf{H},\mathbf{Q}}$ (20) based on a contractive compressor $\mathcal{C}$ with parameter $\alpha \in (0, 1]$ satisfy (3) with $A = \alpha/2$ and $B = (1 - \alpha)\left((2-\alpha)/\alpha\right)$.*

Proof. From the definition of contractive compressor

$$\mathbb{E}\left[\|\mathcal{C}(\mathbf{X}) - \mathbf{X}\|_{\mathrm{F}}^2\right] \leq (1 - \alpha) \|\mathbf{X}\|_{\mathrm{F}}^2.$$

□

Thus, we get

$$\begin{aligned}
\mathbb{E}\left[\|\mathcal{C}_{\mathbf{H},\mathbf{Y}}(\mathbf{X})-\mathbf{X}\|_{\mathrm{F}}^2\right] &= \mathbb{E}\left[\left\|\mathbf{Q}\mathcal{C}\left(\mathbf{Q}^\top(\mathbf{X}-\mathbf{H})\mathbf{Q}\right)\mathbf{Q}^\top-(\mathbf{X}-\mathbf{H})\right\|_{\mathrm{F}}^2\right]\\
&= \mathbb{E}\left[\left\|\mathbf{Q}\mathcal{C}\left(\mathbf{Q}^\top(\mathbf{X}-\mathbf{H})\mathbf{Q}\right)\mathbf{Q}^\top-\mathbf{Q}\mathbf{Q}^\top(\mathbf{X}-\mathbf{H})\mathbf{Q}\mathbf{Q}^\top\right\|_{\mathrm{F}}^2\right]\\
&= \mathbb{E}\left[\left\|\mathcal{C}\left(\mathbf{Q}^\top(\mathbf{X}-\mathbf{H})\mathbf{Q}\right)-\mathbf{Q}^\top(\mathbf{X}-\mathbf{H})\mathbf{Q}\right\|_{\mathrm{F}}^2\right]\\
&\le (1-\alpha)\left\|\mathbf{Q}^\top(\mathbf{X}-\mathbf{H})\mathbf{Q}\right\|_{\mathrm{F}}^2\\
&= (1-\alpha)\left\|\mathbf{X}-\mathbf{H}\right\|_{\mathrm{F}}^2\\
&\le (1-\alpha)(1+\beta)\left\|\mathbf{Y}-\mathbf{H}\right\|_{\mathrm{F}}^2+(1-\alpha)(1+\beta^{-1})\left\|\mathbf{Y}-\mathbf{X}\right\|_{\mathrm{F}}^2,
\end{aligned}$$

where we use the fact that an orthogonal matrix doesn't change a norm. Let $\beta=\frac{\alpha}{2(1-\alpha)}$, then

$$\mathbb{E}\left[\|\mathcal{C}_{\mathbf{H},\mathbf{Y}}(\mathbf{X})-\mathbf{X}\|_{\mathrm{F}}^2\right]\le\left(1-\frac{\alpha}{2}\right)\left\|\mathbf{Y}-\mathbf{H}\right\|_{\mathrm{F}}^2+(1-\alpha)\left(\frac{2-\alpha}{\alpha}\right)\left\|\mathbf{Y}-\mathbf{X}\right\|_{\mathrm{F}}^2. \tag{21}$$

F Deferred Proofs from Section 4 (Newton-3PC)

F.1 Auxiliary lemma

Denote by $\mathbb{E}_{k+1}[\cdot]$ the conditional expectation given $(k+1)^{th}$ iterate x^{k+1}. We first develop a lemma to handle the mismatch $\mathbb{E}_k\|\mathbf{H}_i^{k+1}-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2$ of the estimate $\mathbf{H}_i^{k+1}$ defined via 3PC compressor.

Lemma F.1. *Assume that* $\left\|x^{k+1}-x^*\right\|^2\le\frac{1}{2}\left\|x^k-x^*\right\|^2$ *for all* $k\ge 0$. *Then*

$$\mathbb{E}_{k+1}\left[\|\mathbf{H}_i^{k+1}-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\right]\le\left(1-\frac{A}{2}\right)\|\mathbf{H}_i^k-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2+\left(\frac{1}{A}+3B\right)L_{\mathrm{F}}^2\|x^k-x^*\|_{\mathrm{F}}^2$$

Proof. Using the defining inequality of 3PC compressor and the assumption of the error in terms of iterates, we expand the approximation error of the estimate $\mathbf{H}_i^{k+1}$ as follows:

$$\begin{aligned}
&\mathbb{E}_{k+1}\left[\|\mathbf{H}_i^{k+1}-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\right]\\
=\;&\mathbb{E}_{k+1}\left[\|\mathcal{C}_{\mathbf{H}_i^k,\nabla^2 f_i(x^k)}\left(\nabla^2 f_i(x^{k+1})\right)-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\right]\\
\le\;&(1+\beta)\mathbb{E}_{k+1}\left[\|\mathcal{C}_{\mathbf{H}_i^k,\nabla^2 f_i(x^k)}\left(\nabla^2 f_i(x^{k+1})\right)-\nabla^2 f_i(x^{k+1})\|_{\mathrm{F}}^2\right]+(1+1/\beta)\|\nabla^2 f_i(x^{k+1})-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\\
\le\;&(1+\beta)(1-A)\|\mathbf{H}_i^k-\nabla^2 f_i(x^k)\|_{\mathrm{F}}^2+B\|\nabla^2 f_i(x^{k+1})-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2+(1+1/\beta)\|\nabla^2 f_i(x^{k+1})-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\\
\le\;&(1+\beta)(1-A)\|\mathbf{H}_i^k-\nabla^2 f_i(x^k)\|_{\mathrm{F}}^2\\
&\quad+2B\|\nabla^2 f_i(x^k)-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2+(1+1/\beta+2B)\|\nabla^2 f_i(x^{k+1})-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\\
\le\;&(1+\beta)(1-A)\|\mathbf{H}_i^k-\nabla^2 f_i(x^k)\|_{\mathrm{F}}^2\\
&\quad+2BL_{\mathrm{F}}^2\|x^k-x^*\|_{\mathrm{F}}^2+(1+1/\beta+2B)L_{\mathrm{F}}^2\|x^{k+1}-x^*\|_{\mathrm{F}}^2\\
\le\;&(1+\beta)(1-A)\|\mathbf{H}_i^k-\nabla^2 f_i(x^k)\|_{\mathrm{F}}^2+\left(\frac{\beta+1}{2\beta}+3B\right)L_{\mathrm{F}}^2\|x^k-x^*\|_{\mathrm{F}}^2.
\end{aligned}$$

where we use Young's inequality for some $\beta>0$. By choosing $\beta=\frac{A}{2(1-A)}$ when $0<A<1$, we get

$$\mathbb{E}_{k+1}\left[\|\mathbf{H}_i^{k+1}-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\right]\le\left(1-\frac{A}{2}\right)\|\mathbf{H}_i^k-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2+\left(\frac{1}{A}+3B-\frac{1}{2}\right)L_{\mathrm{F}}^2\|x^k-x^*\|_{\mathrm{F}}^2$$

When $A=1$, we can choose $\beta=1$ and have

$$\mathbb{E}_{k+1}\left[\|\mathbf{H}_i^{k+1}-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\right]\le(3B+1)\,L_{\mathrm{F}}^2\|x^k-x^*\|_{\mathrm{F}}^2.$$

Thus, for all $0<A\le 1$ we get the desired bound. □

F.2 Proof of Theorem 4.2

The proof follows the same steps as for FedNL until the appearance of 3PC compressor. We derive recurrence relation for $\|x^k - x^*\|^2$ covering both options of updating the global model. If *Option 1.* is used in FedNL, then

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \left\|x^k - x^* - \left[\mathbf{H}_\mu^k\right]^{-1} \nabla f(x^k)\right\|^2 \\
&\leq \left\|\left[\mathbf{H}_\mu^k\right]^{-1}\right\|^2 \left\|\mathbf{H}_\mu^k(x^k - x^*) - \nabla f(x^k))\right\|^2 \\
&\leq \frac{2}{\mu^2}\left(\left\|\left(\mathbf{H}_\mu^k - \nabla^2 f(x^*)\right)(x^k - x^*)\right\|^2 + \left\|\nabla^2 f(x^*)(x^k - x^*) - \nabla f(x^k) + \nabla f(x^*)\right\|^2\right) \\
&= \frac{2}{\mu^2}\left(\left\|\left(\mathbf{H}_\mu^k - \nabla^2 f(x^*)\right)(x^k - x^*)\right\|^2 + \left\|\nabla f(x^k) - \nabla f(x^*) - \nabla^2 f(x^*)(x^k - x^*)\right\|^2\right) \\
&\leq \frac{2}{\mu^2}\left(\left\|\mathbf{H}_\mu^k - \nabla^2 f(x^*)\right\|^2 \|x^k - x^*\|^2 + \frac{L_*^2}{4}\|x^k - x^*\|^4\right) \\
&= \frac{2}{\mu^2}\|x^k - x^*\|^2\left(\left\|\mathbf{H}_\mu^k - \nabla^2 f(x^*)\right\|^2 + \frac{L_*^2}{4}\|x^k - x^*\|^2\right) \\
&\leq \frac{2}{\mu^2}\|x^k - x^*\|^2\left(\left\|\mathbf{H}^k - \nabla^2 f(x^*)\right\|^2 + \frac{L_*^2}{4}\|x^k - x^*\|^2\right) \\
&\leq \frac{2}{\mu^2}\|x^k - x^*\|^2\left(\left\|\mathbf{H}^k - \nabla^2 f(x^*)\right\|_{\mathrm{F}}^2 + \frac{L_*^2}{4}\|x^k - x^*\|^2\right),
\end{aligned}
$$

where we use $\mathbf{H}_\mu^k \succeq \mu\mathbf{I}$ in the second inequality, and $\nabla^2 f(x^*) \succeq \mu\mathbf{I}$ in the fourth inequality. From the convexity of $\|\cdot\|_{\mathrm{F}}^2$, we have

$$
\|\mathbf{H}^k - \nabla^2 f(x^*)\|_{\mathrm{F}}^2 = \left\|\frac{1}{n}\sum_{i=1}^n \left(\mathbf{H}_i^k - \nabla^2 f_i(x^*)\right)\right\|_{\mathrm{F}}^2 \leq \frac{1}{n}\sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 = \mathcal{H}^k.
$$

Thus,

$$
\|x^{k+1} - x^*\|^2 \leq \frac{2}{\mu^2}\|x^k - x^*\|^2\mathcal{H}^k + \frac{L_*^2}{2\mu^2}\|x^k - x^*\|^4. \tag{22}
$$

If *Option 2.* is used in FedNL, then as $\mathbf{H}^k + l^k\mathbf{I} \succeq \nabla^2 f(x^k) \succeq \mu\mathbf{I}$ and $\nabla f(x^*) = 0$, we have

$$
\begin{aligned}
\|x^{k+1} - x^*\| &= \|x^k - x^* - [\mathbf{H}^k + l^k\mathbf{I}]^{-1}\nabla f(x^k)\| \\
&\leq \|[\mathbf{H}^k + l^k\mathbf{I}]^{-1}\| \cdot \|(\mathbf{H}^k + l^k\mathbf{I})(x^k - x^*) - \nabla f(x^k) + \nabla f(x^*)\| \\
&\leq \frac{1}{\mu}\|(\mathbf{H}^k + l^k\mathbf{I} - \nabla^2 f(x^*))(x^k - x^*)\| + \frac{1}{\mu}\|\nabla f(x^k) - \nabla f(x^*) - \nabla^2 f(x^*)(x^k - x^*)\| \\
&\leq \frac{1}{\mu}\|\mathbf{H}^k + l^k\mathbf{I} - \nabla^2 f(x^*)\|\|x^k - x^*\| + \frac{L_*}{2\mu}\|x^k - x^*\|^2 \\
&\leq \frac{1}{n\mu}\sum_{i=1}^n \|\mathbf{H}_i^k + l_i^k\mathbf{I} - \nabla^2 f_i(x^*)\|\|x^k - x^*\| + \frac{L_*}{2\mu}\|x^k - x^*\|^2 \\
&\leq \frac{1}{n\mu}\sum_{i=1}^n (\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\| + l_i^k)\|x^k - x^*\| + \frac{L_*}{2\mu}\|x^k - x^*\|^2.
\end{aligned}
$$

From the definition of l_i^k, we have

$$
l_i^k = \|\mathbf{H}_i^k - \nabla^2 f_i(x^k)\|_{\mathrm{F}} \leq \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}} + L_{\mathrm{F}}\|x^k - x^*\|.
$$

Thus,

$$\|x^{k+1}-x^*\| \leq \frac{2}{n\mu}\sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}\|x^k-x^*\| + \frac{L_* + 2L_{\mathrm{F}}}{2\mu}\|x^k-x^*\|^2.$$

From Young's inequality, we further have

$$\begin{aligned}
\|x^{k+1}-x^*\|^2 &\leq \frac{8}{\mu^2}\left(\frac{1}{n}\sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}\|x^k-x^*\|\right)^2 + \frac{(L_* + 2L_{\mathrm{F}})^2}{2\mu^2}\|x^k-x^*\|^4 \\
&\leq \frac{8}{\mu^2}\|x^k-x^*\|^2\left(\frac{1}{n}\sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\right) + \frac{(L_* + 2L_{\mathrm{F}})^2}{2\mu^2}\|x^k-x^*\|^4 \\
&= \frac{8}{\mu^2}\|x^k-x^*\|^2\mathcal{H}^k + \frac{(L_* + 2L_{\mathrm{F}})^2}{2\mu^2}\|x^k-x^*\|^4, \qquad (23)
\end{aligned}$$

where we use the convexity of $\|\cdot\|_{\mathrm{F}}^2$ in the second inequality.

Thus, from (22) and (23), we have the following unified bound for both *Option 1* and *Option 2*:

$$\|x^{k+1}-x^*\|^2 \leq \frac{C}{\mu^2}\|x^k-x^*\|^2\mathcal{H}^k + \frac{D}{2\mu^2}\|x^k-x^*\|^4. \qquad (24)$$

Assume $\|x^0-x^*\|^2 \leq \frac{\mu^2}{2D}$ and $\mathcal{H}^k \leq \frac{\mu^2}{4C}$ for all $k \geq 0$. Then we show that $\|x^k-x^*\|^2 \leq \frac{\mu^2}{2D}$ for all $k \geq 0$ by induction. Assume $\|x^k-x^*\|^2 \leq \frac{\mu^2}{2D}$ for all $k \leq K$. Then from (24), we have

$$\|x^{K+1}-x^*\|^2 \leq \frac{1}{4}\|x^K-x^*\|^2 + \frac{1}{4}\|x^K-x^*\|^2 \leq \frac{\mu^2}{2D}.$$

Thus we have $\|x^k-x^*\|^2 \leq \frac{\mu^2}{2D}$ and $\mathcal{H}^k \leq \frac{\mu^2}{4C}$ for $k \geq 0$. Using (24) again, we obtain

$$\|x^{k+1}-x^*\|^2 \leq \frac{1}{2}\|x^k-x^*\|^2. \qquad (25)$$

Assume $\|x^0-x^*\|^2 \leq \frac{\mu^2}{2D}$ and $\mathcal{H}^k \leq \frac{\mu^2}{4C}$ for all $k \geq 0$. Then we show that $\|x^k-x^*\|^2 \leq \frac{\mu^2}{2D}$ for all $k \geq 0$ by induction. Assume $\|x^k-x^*\|^2 \leq \frac{\mu^2}{2D}$ for all $k \leq K$. Then from (24), we have

$$\|x^{K+1}-x^*\|^2 \leq \frac{1}{4}\|x^K-x^*\|^2 + \frac{1}{4}\|x^K-x^*\|^2 \leq \frac{\mu^2}{2D}.$$

Thus we have $\|x^k-x^*\|^2 \leq \frac{\mu^2}{2D}$ and $\mathcal{H}^k \leq \frac{\mu^2}{4C}$ for $k \geq 0$. Using (24) again, we obtain

$$\|x^{k+1}-x^*\|^2 \leq \frac{1}{2}\|x^k-x^*\|^2. \qquad (26)$$

Thus, we derived the first rate of the theorem. Next, we invoke Lemma F.1 to have an upper bound for $\mathcal{H}^{k+1}$:

$$\mathbb{E}_k[\mathcal{H}^{k+1}] \leq \left(1-\frac{A}{2}\right)\mathcal{H}^k + \left(\frac{1}{A}+3B\right)L_{\mathrm{F}}^2\|x^k-x^*\|^2.$$

Using the above inequality and (26), for Lyapunov function Φ^k we deduce

$$\begin{aligned}
\mathbb{E}_k[\Phi^{k+1}] &\leq \left(1-\frac{A}{2}\right)\mathcal{H}^k + \left(\frac{1}{A}+3B\right)L_{\mathrm{F}}^2\|x^k-x^*\|^2 + 3\left(\frac{1}{A}+3B\right)L_{\mathrm{F}}^2\|x^k-x^*\|^2 \\
&= \left(1-\frac{A}{2}\right)\mathcal{H}^k + \left(1-\frac{1}{3}\right)6\left(\frac{1}{A}+3B\right)L_{\mathrm{F}}^2\|x^k-x^*\|^2 \\
&\leq \left(1-\min\left\{\frac{A}{2},\frac{1}{3}\right\}\right)\Phi^k.
\end{aligned}$$

Hence $\mathbb{E}_k[\Phi^k] \leq \left(1 - \min\left\{\frac{A}{2}, \frac{1}{3}\right\}\right)^k \Phi^0$. Clearly, we further have $\mathbb{E}[\mathcal{H}^k] \leq \left(1 - \min\left\{\frac{A}{2}, \frac{1}{3}\right\}\right)^k \Phi^0$ and $\mathbb{E}[\|x^k - x^*\|^2] \leq \frac{A}{6(1+3AB)L_{\mathrm{F}}^2}\left(1 - \min\left\{\frac{A}{2}, \frac{1}{3}\right\}\right)^k \Phi^0$ for $k \geq 0$. Assume $x^k \neq x^*$ for all k. Then from (24), we have

$$\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2} \leq \frac{C}{\mu^2}\mathcal{H}^k + \frac{D}{2\mu^2}\|x^k - x^*\|^2,$$

and by taking expectation, we have

$$\begin{aligned}\mathbb{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] &\leq \frac{C}{\mu^2}\mathbb{E}[\mathcal{H}^k] + \frac{D}{2\mu^2}\mathbb{E}[\|x^k - x^*\|^2] \\ &\leq \left(1 - \min\left\{\frac{A}{2}, \frac{1}{3}\right\}\right)^k \left(C + \frac{AD}{12(1+3AB)L_{\mathrm{F}}^2}\right)\frac{\Phi^0}{\mu^2},\end{aligned}$$

which concludes the proof.

F.3 Proof of Lemma 4.3

We prove this by induction. Assume $\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 \leq \frac{\mu^2}{4C}$ and $\|x^k - x^*\|^2 \leq e_1^2$ for $k \leq K$. Then we also have $\mathcal{H}^k \leq \frac{\mu^2}{4C}$ for $k \leq K$. From (24), we can get

$$\begin{aligned}\|x^{K+1} - x^*\|^2 &\leq \frac{C}{\mu^2}\|x^K - x^*\|^2\mathcal{H}^K + \frac{D}{2\mu^2}\|x^K - x^*\|^4 \\ &\leq \frac{1}{4}\|x^K - x^*\|^2 + \frac{1}{4}\|x^K - x^*\|^2 \\ &\leq \|x^K - x^*\|^2 \leq e_1^2.\end{aligned}$$

Using Lemma F.1 and the assumptions that we use non-random 3PC compressor, we have

$$\begin{aligned}\|\mathbf{H}_i^{K+1} - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 &\leq \left(1 - \frac{A}{2}\right)\|\mathbf{H}_i^K - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 + \frac{1+3AB}{A}L_{\mathrm{F}}^2\|x^K - x^*\|^2 \\ &\leq \left(1 - \frac{A}{2}\right)\frac{\mu^2}{4C} + \frac{1+3AB}{A}L_{\mathrm{F}}^2 \cdot \frac{A^2\mu^2}{8(1+3AB)CL_{\mathrm{F}}^2} \\ &= \frac{\mu^2}{4C}.\end{aligned}$$

F.4 Proof of Lemma 4.4

We prove this by induction. Assume $\|x^k - x^*\| \leq e_1$ and $\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 \leq \frac{\mu^2}{4C}$ for $k \leq K$. Then we also have $\mathcal{H}^k \leq \frac{\mu^2}{4C}$ for $k \leq K$. From (24), we can get

$$\begin{aligned}\|x^{K+1} - x^*\|^2 &\leq \frac{C}{\mu^2}\|x^K - x^*\|^2\mathcal{H}^K + \frac{D}{2\mu^2}\|x^K - x^*\|^4 \\ &\leq \frac{1}{4}\|x^K - x^*\|^2 + \frac{1}{4}\|x^K - x^*\|^2 \leq e_1^2.\end{aligned}$$

From the definition

$$\mathbf{H}_i^{k+1} = \begin{cases}\mathbf{H}_i^k + \mathcal{C}(\nabla^2 f_i(x^{k+1}) - \mathbf{H}_i^k) & \text{with probability } p, \\ \mathbf{H}_i^k & \text{with probability } 1 - p.\end{cases} \tag{27}$$

we have two cases for $\mathbf{H}_i^{k+1}$ we need to upper bound individually instead of in expectation. Note that the case $\mathbf{H}_i^{k+1} = \mathbf{H}_i^k$ is trivial as $\|\mathbf{H}_i^{k+1} - \nabla^2 f_i(x^*)\|_{\mathrm{F}} = \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}} \leq \frac{\mu}{2\sqrt{C}}$. For the other case when

$\mathbf{H}_i^{k+1} = \mathbf{H}_i^k + \mathcal{C}(\nabla^2 f_i(x^{k+1}) - \mathbf{H}_i^k)$, we have

$$
\begin{aligned}
& \|\mathbf{H}_i^{k+1} - \nabla^2 f_i(x^*)\|_{\mathrm{F}} \\
= \; & \|\mathbf{H}_i^k + \mathcal{C}(\nabla^2 f_i(x^{k+1}) - \mathbf{H}_i^k) - \nabla^2 f_i(x^*)\|_{\mathrm{F}} \\
\leq \; & \|\mathcal{C}(\nabla^2 f_i(x^{k+1}) - \mathbf{H}_i^k) - (\nabla^2 f_i(x^{k+1}) - \mathbf{H}_i^k)\|_{\mathrm{F}} + \|\nabla^2 f_i(x^{k+1}) - \nabla^2 f_i(x^*)\|_{\mathrm{F}} \\
\leq \; & \sqrt{1-\alpha}\|\nabla^2 f_i(x^{k+1}) - \mathbf{H}_i^k\|_{\mathrm{F}} + L_{\mathrm{F}}\|x^{k+1} - x^*\| \\
\leq \; & \sqrt{1-\alpha}\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}} + \sqrt{1-\alpha}\|\nabla^2 f_i(x^{k+1}) - \nabla^2 f_i(x^*)\|_{\mathrm{F}} + L_{\mathrm{F}}\|x^{k+1} - x^*\| \\
\leq \; & \sqrt{1-\alpha}\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}} + 2L_{\mathrm{F}}\|x^{k+1} - x^*\| \\
\leq \; & \sqrt{1-\alpha}\frac{\mu}{2\sqrt{C}} + 2L_{\mathrm{F}} \cdot \frac{(1-\sqrt{1-\alpha})\mu}{4\sqrt{C}L_{\mathrm{F}}} = \frac{\mu}{2\sqrt{C}},
\end{aligned}
$$

which completes our induction step and the proof.

G Deferred Proofs from Section 5 (Newton-3PC-BC)

G.1 Proof of Theorem 5.1

First we have

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|z^k - x^* - [\mathbf{H}^k]_\mu^{-1} g^k\|^2 \\
&= \left\|[\mathbf{H}^k]_\mu^{-1}\left([\mathbf{H}^k]_\mu(z^k - x^*) - (g^k - \nabla f(x^*))\right)\right\|^2 \\
&\leq \frac{1}{\mu^2}\left\|[\mathbf{H}^k]_\mu(z^k - x^*) - (g^k - \nabla f(x^*))\right\|^2, \qquad (28)
\end{aligned}
$$

where we use $\nabla f(x^*) = 0$ in the second equality, and $\|[\mathbf{H}^k]_\mu^{-1}\| \leq \frac{1}{\mu}$ in the last inequality.

If $\xi^k = 1$, then

$$
\begin{aligned}
& \left\|[\mathbf{H}^k]_\mu(z^k - x^*) - (g^k - \nabla f(x^*))\right\|^2 \\
= \; & \left\|\nabla f(z^k) - \nabla f(x^*) - \nabla^2 f(x^*)(z^k - x^*) + (\nabla^2 f(x^*) - [\mathbf{H}^k]_\mu)(z^k - x^*)\right\|^2 \\
\leq \; & 2\left\|\nabla f(z^k) - \nabla f(x^*) - \nabla^2 f(x^*)(z^k - x^*)\right\|^2 + 2\left\|(\nabla^2 f(x^*) - [\mathbf{H}^k]_\mu)(z^k - x^*)\right\|^2 \\
\leq \; & \frac{L_*^2}{2}\|z^k - x^*\|^4 + 2\|[\mathbf{H}^k]_\mu - \nabla^2 f(x^*)\|^2 \cdot \|z^k - x^*\|^2 \\
\leq \; & \frac{L_*^2}{2}\|z^k - x^*\|^4 + 2\|\mathbf{H}^k - \nabla^2 f(x^*)\|_{\mathrm{F}}^2\|z^k - x^*\|^2 \\
= \; & \frac{L_*^2}{2}\|z^k - x^*\|^4 + 2\left\|\frac{1}{n}\mathbf{H}_i^k - \frac{1}{n}\nabla^2 f_i(x^*)\right\|_{\mathrm{F}}^2\|z^k - x^*\|^2 \\
\leq \; & \frac{L_*^2}{2}\|z^k - x^*\|^4 + \frac{2}{n}\sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\|z^k - x^*\|^2, \qquad (29)
\end{aligned}
$$

where in the second inequality, we use the Lipschitz continuity of the Hessian of f, and in the last inequality, we use the convexity of $\|\cdot\|_{\mathrm{F}}^2$.

If $\xi^k = 0$, then

$$\begin{aligned}
&\left\|[\mathbf{H}^k]_\mu(z^k - x^*) - (g^k - \nabla f(x^*))\right\|^2 \\
&= \left\|[\mathbf{H}^k]_\mu(z^k - w^k) + \nabla f(w^k) - \nabla f(x^*) - [\mathbf{H}^k]_\mu(z^k - x^*)\right\|^2 \\
&= \left\|[\mathbf{H}^k]_\mu(x^* - w^k) + \nabla f(w^k) - \nabla f(x^*)\right\|^2 \\
&= \left\|\nabla f(w^k) - \nabla f(x^*) - \nabla^2 f(x^*)(w^k - x^*) + (\nabla^2 f(x^*) - [\mathbf{H}^k]_\mu)(w^k - x^*)\right\|^2 \\
&\le \frac{L_*^2}{2}\|w^k - x^*\|^4 + 2\|\mathbf{H}^k - \nabla^2 f(x^*)\|_{\mathrm{F}}^2\|w^k - x^*\|^2 \\
&\le \frac{L_*^2}{2}\|w^k - x^*\|^4 + \frac{2}{n}\sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\|w^k - x^*\|^2. \qquad (30)
\end{aligned}$$

For $k \ge 1$, from the above three inequalities, we can obtain

$$\begin{aligned}
\mathbb{E}_k\|x^{k+1} - x^*\|^2 &\le \frac{L_*^2 p}{2\mu^2}\|z^k - x^*\|^4 + \frac{2p}{n\mu^2}\sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\|z^k - x^*\|^2 \\
&\quad + \frac{L_*^2(1-p)}{2\mu^2}\|w^k - x^*\|^4 + \frac{2(1-p)}{n\mu^2}\sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2\|w^k - x^*\|^2 \\
&= \frac{p}{2\mu^2}\left(L_*^2\|z^k - x^*\|^2 + 4\mathcal{H}^k\right)\|z^k - x^*\|^2 \\
&\quad + \frac{(1-p)}{2\mu^2}\left(L_*^2\|w^k - x^*\|^2 + 4\mathcal{H}^k\right)\|w^k - x^*\|^2, \qquad (31)
\end{aligned}$$

where we denote $\mathcal{H}^k := \frac{1}{n}\sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2$.

For $k = 0$, since $z^0 = w^0$, it is easy to verify that the above equality also holds.

From the update rule of z^k, we have

$$\begin{aligned}
\mathbb{E}_k\|z^{k+1} - x^*\|^2 &\le (1+\alpha)\mathbb{E}_k\|z^{k+1} - x^{k+1}\|^2 + \left(1 + \frac{1}{\alpha}\right)\mathbb{E}_k\|x^{k+1} - x^*\|^2 \\
&\le (1+\alpha)(1 - A_M)\|z^k - x^k\|^2 + (1+\alpha)B_M\mathbb{E}_k\|x^{k+1} - x^k\|^2 + \left(1 + \frac{1}{\alpha}\right)\mathbb{E}_k\|x^{k+1} - x^*\|^2 \\
&\le (1+\alpha)(1 - A_M)(1+\beta)\|z^k - x^*\|^2 + (1+\alpha)(1 - A_M)\left(1 + \frac{1}{\beta}\right)\|x^k - x^*\|^2 \\
&\quad + 2(1+\alpha)B_M\|x^k - x^*\|^2 + \left(2(1+\alpha)B_M + 1 + \frac{1}{\alpha}\right)\mathbb{E}_k\|x^{k+1} - x^*\|^2,
\end{aligned}$$

for any $\alpha > 0$, $\beta > 0$. By choosing $\alpha = \frac{A_M}{4}$ and $\beta = \frac{A_M}{4(1 - \frac{3A_M}{4})}$, we arrive at

$$\begin{aligned}
\mathbb{E}_k\|z^{k+1} - x^*\|^2 &\le \left(1 - \frac{A_M}{2}\right)\|z^k - x^*\|^2 + \left(\frac{4}{A_M} - 3 + \frac{5B_M}{2}\right)\|x^k - x^*\|^2 + \left(\frac{4}{A_M} + 1 + \frac{5B_M}{2}\right)\mathbb{E}_k\|x^{k+1} - x^*\|^2 \\
&\le \left(1 - \frac{A_M}{2}\right)\|z^k - x^*\|^2 + C_M\|x^k - x^*\|^2 + C_M\mathbb{E}_k\|x^{k+1} - x^*\|^2, \qquad (32)
\end{aligned}$$

where we denote $C_M := \frac{4}{A_M} + 1 + \frac{5B_M}{2}$. Then we have

$$\begin{aligned}\mathbb{E}_k[\|z^{k+1} - x^*\|^2 + 2C_M\|x^{k+1} - x^*\|^2] &\leq \left(1 - \frac{A_M}{2}\right)\|z^k - x^*\|^2 + C_M\|x^k - x^*\|^2 + 3C_M\mathbb{E}_k\|x^{k+1} - x^*\|^2 \\ &\overset{(31)}{\leq} \left(1 - \frac{A_M}{2}\right)\|z^k - x^*\|^2 + \frac{3C_M p}{2\mu^2}\left(L_*^2\|z^k - x^*\|^2 + 4\mathcal{H}^k\right)\|z^k - x^*\|^2 \\ &\quad + \frac{3C_M(1-p)}{2\mu^2}\left(L_*^2\|w^k - x^*\|^2 + 4\mathcal{H}^k\right)\|w^k - x^*\|^2 + C_M\|x^k - x^*\|^2.\end{aligned}$$

Assume $\|z^k - x^*\|^2 \leq \frac{A_M\mu^2}{24C_M L_*^2}$ and $\mathcal{H}^k \leq \frac{A_M\mu^2}{96C_M}$ for $k \geq 0$. Then from the update rule of w^k, we also have $\|w^k - x^*\|^2 \leq \frac{A_M\mu^2}{24C_M L_*^2}$ for $k \geq 0$. Therefore, we have

$$\mathbb{E}_k[\|z^{k+1} - x^*\|^2 + 2C_M\|x^{k+1} - x^*\|^2] \leq \left(1 - \frac{A_M}{2} + \frac{A_M p}{8}\right)\|z^k - x^*\|^2 + \frac{A_M(1-p)}{8}\|w^k - x^*\|^2 + C_M\|x^k - x^*\|^2. \tag{33}$$

From the update rule of w^k, we have

$$\mathbb{E}_k\|w^{k+1} - x^*\|^2 = p\|z^{k+1} - x^*\|^2 + (1-p)\|w^k - x^*\|^2. \tag{34}$$

Define $\Phi_1^k := \|z^k - x^*\|^2 + C_M\|x^k - x^*\|^2 + \frac{A_M(1-p)}{4p}\|w^k - x^*\|^2$. Then we have

$$\begin{aligned}\mathbb{E}_k[\Phi_1^{k+1}] &= \mathbb{E}_k[\|z^{k+1} - x^*\|^2 + 2C_M\|x^{k+1} - x^*\|^2] + \frac{A_M(1-p)}{4p}\mathbb{E}_k\|w^{k+1} - x^*\|^2 \\ &\overset{(34)}{\leq} \left(1 + \frac{A_M(1-p)}{4}\right)\mathbb{E}_k[\|z^{k+1} - x^*\|^2 + 2C_M\|x^{k+1} - x^*\|^2] + \frac{A_M(1-p)^2}{4p}\|w^k - x^*\|^2 \\ &\overset{(33)}{\leq} \left(1 + \frac{A_M(1-p)}{4}\right)\left(1 - \frac{A_M}{2} + \frac{A_M p}{8}\right)\|z^k - x^*\|^2 + \left(1 + \frac{A_M(1-p)}{4}\right)C_M\|x^k - x^*\|^2 \\ &\quad + \left(\left(1 + \frac{A_M(1-p)}{4}\right)\frac{A_M(1-p)}{8} + \frac{A_M(1-p)^2}{4p}\right)\|w^k - x^*\|^2 \\ &\leq \left(1 - \frac{A_M}{4}\right)\|z^k - x^*\|^2 + \left(1 - \frac{3}{8}\right)2C_M\|x^k - x^*\|^2 + \frac{A_M(1-p)}{4p}\left(1 - \frac{3p}{8}\right)\|w^k - x^*\|^2 \\ &\leq \left(1 - \frac{\min\{2A_M, 3p\}}{8}\right)\Phi_1^k.\end{aligned}$$

By applying the tower property, we have

$$\mathbb{E}[\Phi_1^{k+1}] \leq \left(1 - \frac{\min\{2A_{\mathrm{M}}, 3p\}}{8}\right)\mathbb{E}[\Phi_1^k].$$

Unrolling the recursion, we can get the result.

G.2 Proof of Lemma 5.2

We prove the results by mathematical induction. Assume the results hold for $k \leq K$. From the update rule of w^k, we know $\|w^k - x^*\|^2 \leq \min\{\frac{A_M\mu^2}{24C_M L_*^2}, \frac{A_W A_M\mu^2}{384C_M C_W L_{\mathrm{F}}^2}\}$ for $k \leq K$. If $\xi^K = 1$, from (28) and (29), we have

$$\begin{aligned}\|x^{K+1} - x^*\|^2 &\leq \frac{1}{\mu^2}\left(\frac{L_*^2}{2}\|z^K - x^*\|^2 + 2\mathcal{H}^K\right)\|z^K - x^*\|^2 \\ &\leq \frac{A_M}{24C_M}\|z^K - x^*\|^2.\end{aligned} \tag{35}$$

If $\xi^K = 0$, from $\|w^K - x^*\|^2 \leq \min\{\frac{A_M\mu^2}{24C_M L_*^2}, \frac{A_W A_M \mu^2}{384 C_M C_W L_{\mathrm{F}}^2}\}$ and (30), we can obtain the above inequality similarly. From the upper bound of $\|z^K - x^*\|^2$, we further have $\|x^{K+1} - x^*\| \leq \frac{11A_M}{24C_M} \min\{\frac{A_M\mu^2}{24C_M^2 L_*^2}, \frac{A_W A_M \mu^2}{384 C_M C_W L_{\mathrm{F}}^2}\}$. Then from (32) and the fact that $\mathcal{C}^M_{z^k, x^k}(x^{k+1})$ is deterministic, we have

$$\begin{aligned}
\|z^{K+1} - x^*\|^2 &\leq \left(1 - \frac{A_M}{2}\right)\|z^K - x^*\|^2 + C_M\|x^K - x^*\|^2 + C_M\|x^{K+1} - x^*\|^2 \\
&\leq \left(1 - \frac{A_M}{2} + \frac{A_M}{24}\right)\|z^K - x^*\|^2 + C_M \cdot \frac{11A_M}{24C_M} \min\{\frac{A_M\mu^2}{24C_M^2 L_*^2}, \frac{A_W A_M \mu^2}{384 C_M C_W L_{\mathrm{F}}^2}\} \\
&\leq \min\{\frac{A_M\mu^2}{24C_M^2 L_*^2}, \frac{A_W A_M \mu^2}{384 C_M C_W L_{\mathrm{F}}^2}\}.
\end{aligned}$$

For $\|\mathbf{H}_i^{k+1} - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2$, we have

$$\begin{aligned}
\mathbb{E}_k\|\mathbf{H}_i^{k+1} - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 &\leq (1+\alpha)\mathbb{E}_k\|\mathbf{H}_i^k - \nabla^2 f_i(z^{k+1})\|_{\mathrm{F}}^2 + \left(1 + \frac{1}{\alpha}\right)\mathbb{E}_k\|\nabla^2 f_i(z^{k+1}) - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 \\
&\leq (1+\alpha)(1 - A_W)\|\mathbf{H}_i^k - \nabla^2 f_i(z^k)\|_{\mathrm{F}}^2 + (1+\alpha)B_W\mathbb{E}_k\|\nabla^2 f_i(z^k) - \nabla^2 f_i(z^{k+1})\|_{\mathrm{F}}^2 \\
&\quad + \left(1 + \frac{1}{\alpha}\right)\mathbb{E}_k\|\nabla^2 f_i(z^{k+1}) - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 \\
&\leq (1+\alpha)(1 - A_W)\|\mathbf{H}_i^k - \nabla^2 f_i(z^k)\|_{\mathrm{F}}^2 + (1+\alpha)B_W L_{\mathrm{F}}^2\mathbb{E}_k\|z^k - z^{k+1}\|^2 \\
&\quad + \left(1 + \frac{1}{\alpha}\right)L_{\mathrm{F}}^2\mathbb{E}_k\|z^{k+1} - x^*\|^2 \\
&\leq (1+\alpha)(1 - A_W)(1+\beta)\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 + (1+\alpha)(1 - A_W)\left(1 + \frac{1}{\beta}\right)L_{\mathrm{F}}^2\|z^k - x^*\|^2 \\
&\quad + 2(1+\alpha)B_W L_{\mathrm{F}}^2\|z^k - x^*\|^2 + \left(2(1+\alpha)B_W + 1 + \frac{1}{\alpha}\right)L_{\mathrm{F}}^2\|z^{k+1} - x^*\|^2,
\end{aligned}$$

for any $\alpha > 0$, $\beta > 0$. By choosing $\alpha = \frac{A_W}{4}$ and $\beta = \frac{A_W}{4(1 - \frac{3A_W}{4})}$, we arrive at

$$\mathbb{E}_k\|\mathbf{H}_i^{k+1} - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 \leq \left(1 - \frac{A_W}{2}\right)\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 + C_W L_{\mathrm{F}}^2\|z^k - x^*\|^2 + C_W L_{\mathrm{F}}^2\mathbb{E}_k\|z^{k+1} - x^*\|^2, \quad (36)$$

where we denote $C_W := \frac{4}{A_W} + 1 + \frac{5B_W}{2}$. Since $\mathcal{C}^W_{\mathbf{H}_i^k, \nabla^2 f_i(z^k)}(z^{k+1})$ is disterministic, from (36), we have

$$\begin{aligned}
\mathcal{H}^{K+1} &\leq \left(1 - \frac{A_W}{2}\right)\mathcal{H}^K + C_W L_{\mathrm{F}}^2\|z^K - x^*\|^2 + C_W L_{\mathrm{F}}^2\|z^{K+1} - x^*\|^2 \\
&\leq \left(1 - \frac{A_W}{2}\right)\frac{A_M\mu^2}{96C_M} + 2C_W L_{\mathrm{F}}^2 \cdot \frac{A_W A_M \mu^2}{384 C_M C_W L_{\mathrm{F}}^2} \\
&\leq \frac{A_M\mu^2}{96C_M}.
\end{aligned}$$

G.3 Proof of Lemma 5.3

We prove the results by mathematical induction. From the assumption on $\mathbf{H}_i^k$, we have

$$\begin{aligned}
\mathcal{H}^k &= \frac{1}{n}\sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|^2 \\
&\leq \frac{1}{n}\sum_{i=1}^n d^2 \max_{jl}\{|(\mathbf{H}_i^k)_{jl} - (\nabla^2 f(x^*))_{jl}|^2\} \\
&\leq d^2 L_\infty^2 \max_{0 \leq t \leq k}\|z^t - x^*\|^2. \quad (37)
\end{aligned}$$

Then from $\|x^0 - x^*\|^2 \leq \tilde{c}_1$, we have $\mathcal{H}^0 \leq \min\{\frac{A_M\mu^2}{96C_M}, \frac{\mu^2}{4d}\}$. Assume the results hold for all $k \leq K$. If $\xi^K = 1$, from (35), we have

$$\begin{aligned}\|x^{K+1} - x^*\|^2 &\leq \frac{1}{\mu^2}\left(\frac{L_*^2}{2}\|z^K - x^*\|^2 + 2\mathcal{H}^K\right)\|z^K - x^*\|^2 \\ &\leq \frac{1}{d}\|z^K - x^*\|^2 \\ &\leq \tilde{c}_1.\end{aligned}$$

If $\xi^K = 0$, from $\|w^K - x^*\|^2 \leq d\tilde{c}_1$ and (30), we can obtain the above inequality similarly. From the assumption on z^k, we have

$$\begin{aligned}\|z^{K+1} - x^*\|^2 &\leq d\max_j |z_j^{K+1} - x_j^*|^2 \\ &\leq d\max_{0\leq t\leq K+1}\|x^t - x^*\|^2 \\ &\leq d\tilde{c}_1.\end{aligned}$$

At last, using (37), we can get $\mathcal{H}^{K+1} \leq \min\{\frac{A_M\mu^2}{96C_M}, \frac{\mu^2}{4d}\}$, which completes the proof.

H Extension to Bidirectional Compression and Partial Participation

In this section, we unify the bidirectional compression and partial participation in Algorithm 3. The algorithm can also be regarded as an extension of BL2 in (Qian et al., 2022) by the three point compressor. Here the symmetrization operator $[\cdot]_s$ is defined as

$$[\mathbf{A}]_s := \frac{\mathbf{A} + \mathbf{A}^\top}{2}$$

for any $\mathbf{A} \in \mathbb{R}^{d\times d}$. The update of the global model at k-th iteration is

$$x^{k+1} = \left([\mathbf{H}^k]_s + l^k\mathbf{I}\right)^{-1} g^k,$$

where $\mathbf{H}^k$, l^k, and g^k are the average of $\mathbf{H}_i^k$, l_i^k, and g_i^k respectively. This update is based on the following step in Stochastic Newton method (Kovalev et al., 2019)

$$x^{k+1} = \left[\frac{1}{n}\sum_{i=1}^n \nabla^2 f_i(w_i^k)\right]^{-1}\left[\frac{1}{n}\sum_{i=1}^n\left(\nabla^2 f_i(w_i^k)w_i^k - \nabla f_i(w_i^k)\right)\right].$$

We use $[\mathbf{H}_i^k]_s + l_i^k\mathbf{I}$ to estimate $\nabla^2 f_i(w_i^k)$, and g_i^k to estimate $\nabla^2 f_i(w_i^k)w_i^k - \nabla f_i(w_i^k)$, where $l_i^k = \|[\mathbf{H}_i^k]_s - \nabla^2 f_i(z_i^k)\|_{\mathrm{F}}$ is adopted to guarantee the positive definiteness of $[\mathbf{H}^k]_s + l^k\mathbf{I}$. Hence, like BL2 in (Qian et al., 2022), we maintain the key relation

$$g_i^k = ([\mathbf{H}_i^k]_s + l_i^k\mathbf{I})w_i^k - \nabla f_i(w_i^k). \tag{38}$$

Since each node has a local model w_i^k, we introduce z_i^k to apply the bidirectional compression with the three point compressor and $\mathbf{H}_i^k$ is expected to learn $h^i(\nabla^2 f_i(z_i^k))$ iteratively. For the update of g_i^k on the server when $\xi_i^k = 0$, from (38), it is natural to let

$$g_i^{k+1} - g_i^k = ([\mathbf{H}_i^{k+1}]_s - [\mathbf{H}_i^k]_s + l_i^{k+1}\mathbf{I} - l_i^k\mathbf{I})w_i^{k+1},$$

since we have $w_i^{k+1} = w_i^k$ when $\xi_i^k = 0$. The convergence results of Newton-3PC-BC-PP are stated in the following two theorems.

Algorithm 3 Newton-3PC-BC-PP (Newton's method with 3PC, BC and Partial Participation)

Parameters: Worker's ($\mathcal{C}^W$) and Master's ($\mathcal{C}^M$) 3PC; probability $p \in (0,1]$; $0 < \tau \leq n$
Initialization: $w_i^0 = z_i^0 = x^0 \in \mathbb{R}^d$; $\mathbf{H}_i^0 \in \mathbb{R}^{d\times d}$; $l_i^0 = \|[\mathbf{H}_i^0]_s - \nabla^2 f_i(w_i^0)\|_{\mathrm{F}}$; $g_i^0 = ([\mathbf{H}_i^0]_s + l_i^0\mathbf{I})w_i^0 - \nabla f_i(w_i^0)$; Moreover: $\mathbf{H}^0 = \frac{1}{n}\sum_{i=1}^n \mathbf{H}_i^0$; $l^0 = \frac{1}{n}\sum_{i=1}^n l_i^0$; $g^0 = \frac{1}{n}\sum_{i=1}^n g_i^0$
on server
  $x^{k+1} = \left([\mathbf{H}^k]_s + l^k\mathbf{I}\right)^{-1} g^k$,
  choose a subset $S^k \subseteq [n]$ such that $\mathbb{P}[i \in S^k] = \tau/n$ for all $i \in [n]$
  $z_i^{k+1} = \mathcal{C}^M_{z_i^k, x^k}(x^{k+1})$ for $i \in S^k$
  $z_i^{k+1} = z_i^k, \quad w_i^{k+1} = w_i^k$ for $i \notin S^k$
  Send $\mathcal{C}^M_{z_i^k, x^k}(x^{k+1})$ to the selected devices $i \in S^k$
for each device $i = 1, \ldots, n$ in parallel **do**
  for participating devices $i \in S^k$ **do**
  $z_i^{k+1} = \mathcal{C}^M_{z_i^k, x^k}(x^{k+1})$
  $\mathbf{H}_i^{k+1} = \mathcal{C}^W_{\mathbf{H}_i^k, \nabla^2 f_i(z_i^k)}(\nabla^2 f_i(z_i^{k+1}))$
  $l_i^{k+1} = \|[\mathbf{H}_i^{k+1}]_s - \nabla^2 f_i(z_i^{k+1})\|_{\mathrm{F}}$
  Sample $\xi_i^{k+1} \sim$ Bernoulli(p)
  if $\xi_i^{k+1} = 1$
    $w_i^{k+1} = z_i^{k+1}$, $g_i^{k+1} = ([\mathbf{H}_i^{k+1}]_s + l_i^{k+1}\mathbf{I})w_i^{k+1} - \nabla f_i(w_i^{k+1})$, send $g_i^{k+1} - g_i^k$ to server
  if $\xi_i^{k+1} = 0$
    $w_i^{k+1} = w_i^k$, $g_i^{k+1} = ([\mathbf{H}_i^{k+1}]_s + l_i^{k+1}\mathbf{I})w_i^{k+1} - \nabla f_i(w_i^{k+1})$
  Send $\mathbf{H}_i^{k+1}$, $l_i^{k+1} - l_i^k$, and ξ_i^{k+1} to the server
  for non-participating devices $i \notin S^k$ **do**
  $z_i^{k+1} = z_i^k$, $w_i^{k+1} = w_i^k$, $\mathbf{H}_i^{k+1} = \mathbf{H}_i^k$, $l_i^{k+1} = l_i^k$, $g_i^{k+1} = g_i^k$
end for
on server
  if $\xi_i^{k+1} = 1$
    $w_i^{k+1} = z_i^{k+1}$, receive $g_i^{k+1} - g_i^k$
  if $\xi_i^{k+1} = 0$
    $w_i^{k+1} = w_i^k$, $g_i^{k+1} - g_i^k = \left[\mathbf{H}_i^{k+1} - \mathbf{H}_i^k\right]_s w_i^{k+1} + (l_i^{k+1} - l_i^k)w_i^{k+1}$
  $g^{k+1} = g^k + \frac{1}{n}\sum_{i\in S^k}\left(g_i^{k+1} - g_i^k\right)$
  $\mathbf{H}^{k+1} = \frac{1}{n}\sum_{i=1}^n \mathbf{H}_i^{k+1}$
  $l^{k+1} = l^k + \frac{1}{n}\sum_{i\in S^k}\left(l_i^{k+1} - l_i^k\right)$

For $k \geq 0$, define Lyapunov function

$$\Phi_3^k := \mathcal{Z}^k + \frac{2\tau C_M}{n}\|x^k - x^*\|^2 + \frac{A_M}{4p}\mathcal{W}^k,$$

where $\tau \in [n]$ is the number of devices participating in each round.

Theorem H.1. *Let Assumption 4.1. Assume* $\|z_i^k - x^*\|^2 \leq \frac{A_M\mu^2}{36(H^2+4L_{\mathrm{F}}^2)C_M}$ *and* $\mathcal{H}^k \leq \frac{A_M\mu^2}{576C_M}$ *for all* $i \in [n]$ *and* $k \geq 0$. *Then we have*

$$\mathbb{E}[\Phi_3^k] \leq \left(1 - \frac{\tau\min\{2A_M, 3p\}}{8n}\right)^k \Phi_3^0,$$

for $k \geq 0$.

Proof. First, similar to (30) in (Qian et al., 2022), we can get

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &\le \frac{3L_*^2}{4\mu^2}(\mathcal{W}^k)^2+\frac{12\mathcal{W}^k}{n\mu^2}\sum_{i=1}^n\|\mathbf{H}_i^k-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2+\frac{3L_{\mathrm{F}}^2}{\mu^2}\mathcal{Z}^k\mathcal{W}^k\\ &= \frac{3L_*^2}{4\mu^2}(\mathcal{W}^k)^2+\frac{12\mathcal{W}^k}{\mu^2}\mathcal{H}^k+\frac{3L_{\mathrm{F}}^2}{\mu^2}\mathcal{Z}^k\mathcal{W}^k,\end{aligned}\tag{39}$$

where $\mathcal{W}^k=\frac{1}{n}\sum_{i=1}^n\|w_i^k-x^*\|^2$ and $\mathcal{Z}^k=\frac{1}{n}\sum_{i=1}^n\|z_i^k-x^*\|^2$. For $i\in S^k$, we have $z_i^{k+1}=\mathcal{C}^M_{z_i^k,x^k}(x^{k+1})$. Then, similar to (32), we have

$$\mathbb{E}_k\|z_i^{k+1}-x^*\|^2\le\left(1-\frac{A_M}{2}\right)\|z_i^k-x^*\|^2+C_M\|x^k-x^*\|^2+C_M\|x^{k+1}-x^*\|^2.$$

Noticing that $\mathbb{P}[i\in S^k]=\tau/n$ and $z_i^{k+1}=z_i^k$ for $i\notin S^k$, we further have

$$\begin{aligned}\mathbb{E}_k\|z_i^{k+1}-x^*\|^2 &= \frac{\tau}{n}\mathbb{E}_k[\|z_i^{k+1}-x^*\|^2\mid i\in S^k]+\left(1-\frac{\tau}{n}\right)\mathbb{E}_k[\|z_i^{k+1}-x^*\|^2\mid i\notin S^k]\\ &\le \frac{\tau}{n}\left(1-\frac{A_M}{2}\right)\|z_i^k-x^*\|^2+\frac{\tau C_M}{n}\|x^k-x^*\|^2+\frac{\tau C_M}{n}\|x^{k+1}-x^*\|^2+\left(1-\frac{\tau}{n}\right)\|z_i^k-x^*\|^2\\ &= \left(1-\frac{\tau A_{\mathrm{M}}}{2n}\right)\|z_i^k-x^*\|^2+\frac{\tau C_M}{n}\|x^k-x^*\|^2+\frac{\tau C_M}{n}\|x^{k+1}-x^*\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\mathbb{E}_k[\mathcal{Z}^{k+1}] &= \frac{1}{n}\sum_{i=1}^n\mathbb{E}_k\|z_i^{k+1}-x^*\|^2\\ &\le \frac{1}{n}\sum_{i=1}^n\left(1-\frac{\tau A_{\mathrm{M}}}{2n}\right)\|z_i^k-x^*\|^2+\frac{\tau C_M}{n}\|x^k-x^*\|^2+\frac{\tau C_M}{n}\|x^{k+1}-x^*\|^2\\ &= \left(1-\frac{\tau A_{\mathrm{M}}}{2n}\right)\mathcal{Z}^k+\frac{\tau C_M}{n}\|x^k-x^*\|^2+\frac{\tau C_M}{n}\|x^{k+1}-x^*\|^2.\end{aligned}\tag{40}$$

Combining (39) and (40), we have

$$\begin{aligned}&\mathbb{E}_k[\mathcal{Z}^{k+1}+\frac{2\tau C_M}{n}\|x^{k+1}-x^*\|^2]\\ &\le\left(1-\frac{\tau A_{\mathrm{M}}}{2n}\right)\mathcal{Z}^k+\frac{\tau C_M}{n}\|x^k-x^*\|^2+\frac{3\tau C_M}{n}\|x^{k+1}-x^*\|^2\\ &\le\left(1-\frac{\tau A_{\mathrm{M}}}{2n}\right)\mathcal{Z}^k+\frac{\tau C_M}{n}\|x^k-x^*\|^2+\frac{3\tau C_M}{n}\left(\frac{3L_*^2}{4\mu^2}\mathcal{W}^k+\frac{12\mathcal{H}^k}{\mu^2}+\frac{3L_{\mathrm{F}}^2\mathcal{Z}^k}{\mu^2}\right)\mathcal{W}^k.\end{aligned}$$

Assume $\|z_i^k-x^*\|^2\le\frac{A_M\mu^2}{36(L_*^2+4L_{\mathrm{F}}^2)C_M}$ and $\mathcal{H}^k\le\frac{A_M\mu^2}{576C_M}$ for all $i\in[n]$ and $k\ge0$. Then we have

$$\frac{3L_*^2}{4\mu^2}\mathcal{W}^k+\frac{12\mathcal{H}^k}{\mu^2}+\frac{3L_{\mathrm{F}}^2\mathcal{Z}^k}{\mu^2}\le\frac{A_M}{24C_M},$$

which indicates that

$$\mathbb{E}_k[\mathcal{Z}^{k+1}+\frac{2\tau C_M}{n}\|x^{k+1}-x^*\|^2]\le\left(1-\frac{\tau A_{\mathrm{M}}}{2n}\right)\mathcal{Z}^k+\frac{\tau C_M}{n}\|x^k-x^*\|^2+\frac{\tau A_M}{8n}\mathcal{W}^k.\tag{41}$$

For $\mathcal{W}^k$, similar to (32) in (Qian et al., 2022), we have

$$\mathbb{E}_k[\mathcal{W}^{k+1}]=\left(1-\frac{\tau p}{n}\right)\mathcal{W}^k+\frac{\tau p}{n}\mathbb{E}[\mathcal{Z}^{k+1}].$$

Then from the above two inequalities we have

$$\begin{aligned}
&\mathbb{E}_k[\Phi_3^{k+1}] \\
&\leq \left(1+\frac{\tau A_M}{4n}\right)\mathbb{E}_k[\mathcal{Z}^{k+1}+\frac{2\tau C_M}{n}\|x^{k+1}-x^*\|^2]+\frac{A_M}{4p}\left(1-\frac{\tau p}{n}\right)\mathcal{W}^k \\
&\overset{(41)}{\leq} \left(1-\frac{\tau A_M}{4n}\right)\mathcal{Z}^k+\left(1+\frac{\tau A_M}{4n}\right)\frac{\tau C_M}{n}\|x^k-x^*\|^2+\frac{A_M}{4p}\left(1-\frac{\tau p}{n}+\frac{\tau p}{2n}\left(1+\frac{\tau A_M}{4n}\right)\right)\mathcal{W}^k \\
&\leq \left(1-\frac{\tau\min\{2A_M,3p\}}{8n}\right)\Phi_3^k.
\end{aligned}$$

By applying the tower property, we have

$$\mathbb{E}[\Phi_3^{k+1}] \leq \left(1-\frac{\tau\min\{2A_M,3p\}}{8n}\right)\mathbb{E}[\Phi_3^k].$$

Unrolling the recursion, we can obtain the result.

□

Define $\Phi_4^k = \mathcal{H}^k + \frac{16C_W L_{\mathrm{F}}^2}{A_M}\|x^k-x^*\|^2$ for $k\geq 0$, where $C_W := \frac{4}{A}+1+\frac{5B}{2}$.

Theorem H.2. *Let Assumption 4.1 holds, $\xi^k \equiv 1$, $S^k \equiv [n]$, and $\mathcal{C}^M_{z_i^k,x^k}(x^{k+1}) \equiv x^{k+1}$ for all $i\in[n]$ and $k\geq 0$. Assume $\|z_i^k-x^*\|^2 \leq \frac{A_M\mu^2}{36(L_*^2+4L_{\mathrm{F}}^2)C_M}$ and $\mathcal{H}^k \leq \frac{A_M\mu^2}{576C_M}$ for all $i\in[n]$ and $k\geq 0$. Then we have*

$$\mathbb{E}[\Phi_4^k] \leq \theta_2^k\Phi_4^0,$$

$$\mathbb{E}\left[\frac{\|x^{k+1}-x^*\|^2}{\|x^k-x^*\|^2}\right] \leq \theta_2^k\left(\frac{3(L_*^2+4L_{\mathrm{F}}^2)A_{\mathrm{M}}}{64C_W L_{\mathrm{F}}^2\mu^2}+\frac{12}{\mu^2}\right)\Phi_4^0.$$

for $k\geq 0$, where $\theta_2 := \left(1-\frac{\min\{2A_W,A_M\}}{4}\right)$.

Proof. Since $\xi^k\equiv 1$, $S^k\equiv[n]$, and $\mathcal{C}^M_{z_i^k,x^k}(x^{k+1})\equiv x^{k+1}$ for all $i\in[n]$ and $k\geq 0$, we have $z_i^k \equiv w_i^k \equiv x^k$ for all $i\in[n]$ and $k\geq 0$. Then from (41), we have

$$\mathbb{E}_k\|x^{k+1}-x^*\|^2 \leq \left(1-\frac{3A_M}{8}\right)\|x^k-x^*\|^2. \tag{42}$$

For $\|\mathbf{H}_i^{k+1}-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2$, similar to (36), we have

$$\mathbb{E}_k\|\mathbf{H}_i^{k+1}-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 \leq \left(1-\frac{A_W}{2}\right)\|\mathbf{H}_i^k-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 + C_W L_{\mathrm{F}}^2\|z_i^k-x^*\|^2 + C_W L_{\mathrm{F}}^2\mathbb{E}_k\|z_i^{k+1}-x^*\|^2.$$

Considering $z_i^k \equiv x^k$, we further have

$$\begin{aligned}
\mathbb{E}_k\|\mathbf{H}_i^{k+1}-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 &\leq \left(1-\frac{A_W}{2}\right)\|\mathbf{H}_i^k-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 + C_W L_{\mathrm{F}}^2\|x^k-x^*\|^2 + C_W L_{\mathrm{F}}^2\mathbb{E}_k\|x^{k+1}-x^*\|^2 \\
&\overset{(42)}{\leq} \left(1-\frac{A_W}{2}\right)\|\mathbf{H}_i^k-\nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 + 2C_W L_{\mathrm{F}}^2\|x^k-x^*\|^2,
\end{aligned}$$

which implies that

$$\mathbb{E}_k[\mathcal{H}^{k+1}] \leq \left(1-\frac{A_W}{2}\right)\mathcal{H}^k + 2C_W L_{\mathrm{F}}^2\|x^k-x^*\|^2. \tag{43}$$

Thus, we have

$$\begin{aligned}\mathbb{E}_k[\Phi_4^{k+1}] &= \mathbb{E}_k[\mathcal{H}^{k+1}] + \frac{16C_W L_{\mathrm{F}}^2}{A_M}\mathbb{E}_k\|x^{k+1}-x^*\|^2 \\ &\le \left(1-\frac{A_W}{2}\right)\mathcal{H}^k + 2C_W L_{\mathrm{F}}^2\|x^k-x^*\|^2 + \frac{16C_W L_{\mathrm{F}}^2}{A_M}\mathbb{E}_k\|x^{k+1}-x^*\|^2 \\ &\overset{(42)}{\le} \left(1-\frac{\min\{2A_W, A_M\}}{4}\right)\Phi_4^k.\end{aligned}$$

By applying the tower property, we have $\mathbb{E}[\Phi_4^{k+1}] \le \theta_1 \mathbb{E}[\Phi_4^k]$. Unrolling the recursion, we have $\mathbb{E}[\Phi_4^k] \le \theta_2^k \Phi_4^0$. Then we further have $\mathbb{E}[\mathcal{H}^k] \le \theta_2^k \Phi_4^0$ and $\mathbb{E}\|x^k - x^*\|^2 \le \frac{A_{\mathrm{M}}}{16C_W L_{\mathrm{F}}^2}\theta_2^k \Phi_4^0$.

From (39), we can get

$$\|x^{k+1}-x^*\|^2 \le \frac{1}{\mu^2}\left(\frac{3(L_*^2+4L_{\mathrm{F}}^2)}{4}\|x^k-x^*\|^2 + 12\mathcal{H}^k\right)\|x^k-x^*\|^2.$$

Assume $x^k \neq x^*$ for all $k \ge 0$. Then we have

$$\frac{\|x^{k+1}-x^*\|^2}{\|x^k-x^*\|^2} \le \frac{1}{\mu^2}\left(\frac{3(L_*^2+4L_{\mathrm{F}}^2)}{4}\|x^k-x^*\|^2 + 12\mathcal{H}^k\right),$$

and by taking expectation, we arrive at

$$\begin{aligned}\mathbb{E}\left[\frac{\|x^{k+1}-x^*\|^2}{\|x^k-x^*\|^2}\right] &\le \frac{3(L_*^2+4L_{\mathrm{F}}^2)}{4\mu^2}\mathbb{E}\|x^k-x^*\|^2 + \frac{12}{\mu^2}\mathbb{E}[\mathcal{H}^k] \\ &\le \theta_2^k\left(\frac{3(L_*^2+4L_{\mathrm{F}}^2)A_{\mathrm{M}}}{64C_W L_{\mathrm{F}}^2\mu^2} + \frac{12}{\mu^2}\right)\Phi_4^0.\end{aligned}$$

□

Next, we explore under what conditions we can guarantee the boundedness of $\|z_i^k - x^*\|^2$ and $\mathcal{H}^k$.

Theorem H.3. *Let Assumption 4.1 holds.*
(i) Let $\mathcal{C}^M$ and $\mathcal{C}^W$ be deterministic. Assume $\|x^0-x^*\|^2 \le \frac{11A_M}{24C_M}\min\{\frac{A_M\mu^2}{36(L_*^2+4L_{\mathrm{F}}^2)C_M}, \frac{A_W A_M \mu^2}{2304 C_M C_W L_{\mathrm{F}}^2}\}$ and $\mathcal{H}^0 \le \frac{A_M\mu^2}{576C_M}$. Then we have $\|x^k-x^*\| \le \frac{11A_M}{24C_M}\min\{\frac{A_M\mu^2}{36(L_*^2+4L_{\mathrm{F}}^2)C_M}, \frac{A_W A_M \mu^2}{2304 C_M C_W L_{\mathrm{F}}^2}\}$, $\|z_i^k - x^*\|^2 \le \min\{\frac{A_M\mu^2}{36(L_*^2+4L_{\mathrm{F}}^2)C_M}, \frac{A_W A_M \mu^2}{2304 C_M C_W L_{\mathrm{F}}^2}\}$ and $\mathcal{H}^k \le \frac{A_M\mu^2}{576C_M}$ for all $i \in [n]$ and $k \ge 0$.

(ii) Assume $(z_i^k)_j$ is a convex combination of $\{(x^t)_j\}_{t=0}^k$, and $(\mathbf{H}_i^k)_{jl}$ is a convex combination of $\{(\nabla^2 f_i(z_i^k))_{jl}\}_{t=0}^k$ for all $i \in [n]$, $j,l \in [d]$, and $k \ge 0$. If $\|x^0-x^*\|^2 \le \tilde{c}_2 := \min\{\frac{2\mu^2}{3d^2(L_*^2+4L_{\mathrm{F}}^2)}, \frac{A_M\mu^2}{36dC_M(L_*^2+4L_{\mathrm{F}}^2)}, \frac{A_M\mu^2}{576d^3C_M L_\infty^2}, \frac{\mu^2}{24d^4L_\infty^2}\}$, then $\|z_i^k - x^*\|^2 \le d\tilde{c}_2$ and $\mathcal{H}^k \le \min\{\frac{A_M\mu^2}{576C_M}, \frac{\mu^2}{24d}\}$ for all $i \in [n]$ and $k \ge 0$.

Proof. The proof is similar to that of Lemmas 5.2 and 5.3. Hence we omit it.

□

I Globalization Through Cubic Regularization and Line Search Procedure

So far, we have discussed only the local convergence of our methods. To prove global rates, one must incorporate additional regularization mechanisms. Otherwise, global convergence cannot be guaranteed. Due

to the smooth transition from contractive compressors to general 3PC mechanism, we can easily adapt two globalization strategies of FedNL (equivalent to Newton-EF21) to our Newton-3PC algorithm.

The two globalization strategies are *cubic regularization* and *line search procedure*. We only present the extension with cubic regularization Newton-3PC-CR (Algorithm 4) analogous to FedNL-CR (Safaryan et al., 2022). Similarly, line search procedure can be combined as it was done in FedNL-LS (Safaryan et al., 2022).

Algorithm 4 Newton-3PC-CR (Newton's method with 3PC and Cubic Regularization)

Input: $x^0 \in \mathbb{R}^d$, $\mathbf{H}_1^0, \ldots, \mathbf{H}_n^0 \in \mathbb{R}^{d\times d}$, $\mathbf{H}^0 := \frac{1}{n}\sum_{i=1}^n \mathbf{H}_i^0$, $l^0 = \frac{1}{n}\sum_{i=1}^n \|\mathbf{H}_i^0 - \nabla^2 f_i(x^0)\|_{\mathrm{F}}$
on master
  $h^k = \arg\min_{h\in\mathbb{R}^d} T_k(h)$, where $T_k(h) := \left\langle \nabla f(x^k), h\right\rangle + \frac{1}{2}\left\langle (\mathbf{H}^k + l^k\mathbf{I})h, h\right\rangle + \frac{L_*}{6}\|h\|^3$
  Update global model to $x^{k+1} = x^k + h^k$ and send to the nodes
for each device $i = 1, \ldots, n$ in parallel **do**
  Get x^{k+1} and compute local gradient $\nabla f_i(x^{k+1})$ and local Hessian $\nabla^2 f_i(x^{k+1})$
  Take $\nabla^2 f_i(x^k)$ from memory and update $\mathbf{H}_i^{k+1} = \mathcal{C}_{\mathbf{H}_i^k, \nabla^2 f_i(x^k)}(\nabla^2 f_i(x^{k+1}))$
  Send $\nabla f_i(x^{k+1})$, $\mathbf{H}_i^{k+1}$ and $l_i^{k+1} := \|\mathbf{H}_i^{k+1} - \nabla^2 f_i(x^{k+1})\|_{\mathrm{F}}$ to the server
end for
on server
  Aggregate $\nabla f(x^{k+1}) = \frac{1}{n}\sum_{i=1}^n \nabla f_i(x^{k+1}), \mathbf{H}^{k+1} = \frac{1}{n}\sum_{i=1}^n \mathbf{H}_i^{k+1}, l^{k+1} = \frac{1}{n}\sum_{i=1}^n l_i^{k+1}$

We omit theoretical analysis of these extension as they can be obtained directly from FedNL approach with minor adaptations. In particular, one can get global linear rate for Newton-3PC-CR, global $\mathcal{O}(\frac{1}{k})$ rate for general convex case and the same fast local rates (9) and (11) of Newton-3PC.

J Additional Experiments and Extended Numerical Analysis

In this section we provide extended variety of experiments to analyze the empirical performance of Newton-3PC. We study the efficiency of Newton-3PC in different settings changing 3PC compressor and comparing with other second-order state-of-the-art algorithms. Tests were carried out on logistic regression problem with L2 regularization

$$\min_{x\in\mathbb{R}^d}\left\{\frac{1}{n}\sum_{i=1}^n f_i(x) + \frac{\lambda}{2}\|x\|^2\right\}, \quad f_i(x) = \frac{1}{m}\sum_{j=1}^m \log\left(1+\exp(-b_{ij}a_{ij}^\top x)\right), \tag{44}$$

where $\{a_{ij}, b_{ij}\}_{j\in[m]}$ are data points at the i-th device. On top of that, we also consider L2 regularized Softmax problem of the form

$$\min_{x\in\mathbb{R}^d}\left\{\frac{1}{n}\sum_{i=1}^n f_i(x) + \frac{\lambda}{2}\|x\|^2\right\}, \quad f_i(x) = \sigma\log\left(\sum_{j=1}^m \exp\left(\frac{a_{ij}^\top x - b_{ij}}{\sigma}\right)\right), \tag{45}$$

where $\sigma > 0$ is a smoothing parameter. One can show that this function has both Lipschitz continuous gradient and Lipschitz continuous Hessian. Let $\tilde{a}_{ij}$ be initial data points, and $\tilde{f}_i$ be defined as

$$\tilde{f}_i(x) = \sigma\log\left(\sum_{j=1}^m \exp\left(\frac{\tilde{a}_{ij}^\top x - b_{ij}}{\sigma}\right)\right).$$

Then data shift is performed as follows

$$a_{ij} = \tilde{a}_{ij} - \tilde{f}_i(0), j \in [m], i \in [n].$$

After such shift we may claim that 0 is the optimum since $\nabla f(0) = 0$. Note that this problem does not belong to the class of *generalized linear models.*

J.1 Datasets split

We use standard datasets from LibSVM library (Chang & Lin, 2011). We shuffle and split each dataset into n equal parts representing a local data of i-th client. Exact names of datasets and values of n are shown in Table 2.

Table 2: Datasets used in the experiments with the number of worker nodes n used in each case.

Data set	**# workers** n	**total # of data points** $(= nm)$	**# features** d
a1a	16	1600	123
a9a	80	32560	123
w2a	50	3450	300
w8a	142	49700	300
phishing	100	11000	68

J.2 Choice of parameters

We follow the authors' choice of DINGO (Crane & Roosta, 2019) in choosing hyperparameters: $\theta = 10^{-4}, \phi = 10^{-6}, \rho = 10^{-4}$. Besides, DINGO uses a backtracking line search that selects the largest stepsize from $\{1, 2^{-1}, \ldots, 2^{-10}\}$. The initialization of $\mathbf{H}_i^0$ for Newton-3PC, FedNL (Safaryan et al., 2022) and its extensions, NL1 (Islamov et al., 2021) is $\nabla^2 f_i(x^0)$ if it is not specified directly. For Fib-IOS (Fabbro et al., 2022) we set $d_k^i = 1$. Local Hessians are computed following the partial sums of Fibonacci number and the parameter $\rho = \lambda_{q_{j+1}}$. This is stated in the description of the method. The parameters of backtracking line search for Fib-IOS are $\alpha = 0.5$ and $\beta = 0.9$.

We conduct experiments for two values of regularization parameter $\lambda \in \{10^{-3}, 10^{-4}\}$. In the figures we plot the relation of the optimality gap $f(x^k) - f(x^*)$ and the number of communicated bits per node. In the heatmaps numbers represent the communication complexity per client of Newton-3PC for some specific choice of 3PC compression mechanism (see the description in corresponding section). The optimal value $f(x^*)$ is chosen as the function value at the 20-th iterate of standard Newton's method.

In our experiments we use various compressors for the methods. Examples of classic compression mechanisms include Top-K and Rank-R. The parameters of these compressors are parsed in details in Section A.3 of Safaryan et al. (2022); we refer a reader to this paper for disaggregated description of aforementioned compression mechanisms. Besides, we use various 3PC compressors introduced in (Richtárik et al., 2022).

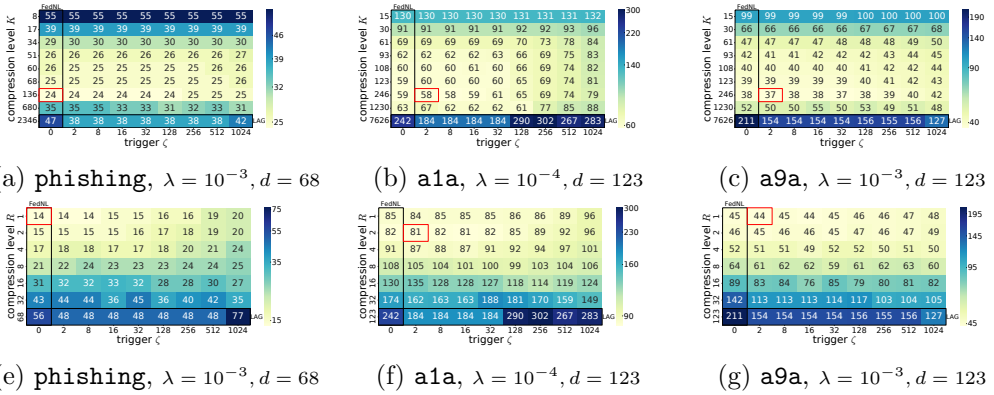
J.3 Behavior of Newton-CLAG based on Top-K and Rank-R compressors

Next, we study how the performance of Newton-CLAG changes when we vary parameters of biased compressor CLAG compression mechanism is based on. In particular, we test Newton-CLAG combined with Top-K and Rank-R compressors modifying compression level (parameters K and R respectively) and trigger parameter ζ. We present the results as heatmaps in Figure 3 indicating the communication complexity in Mbytes for particular choice of a pair of parameters ($(K,\ \zeta)$ or $(R,\ \zeta)$ for CLAG based on Top-K and Rank-R respectively) .

First, we can highlight that in special cases Newton-CLAG reduces to FedNL ($\zeta = 0$, left column) and Newton-LAG (compression is identity, bottom row). Second, we observe slight improvement from using the lazy aggregation.

J.4 Efficiency of Newton-3PCv2 under different compression levels

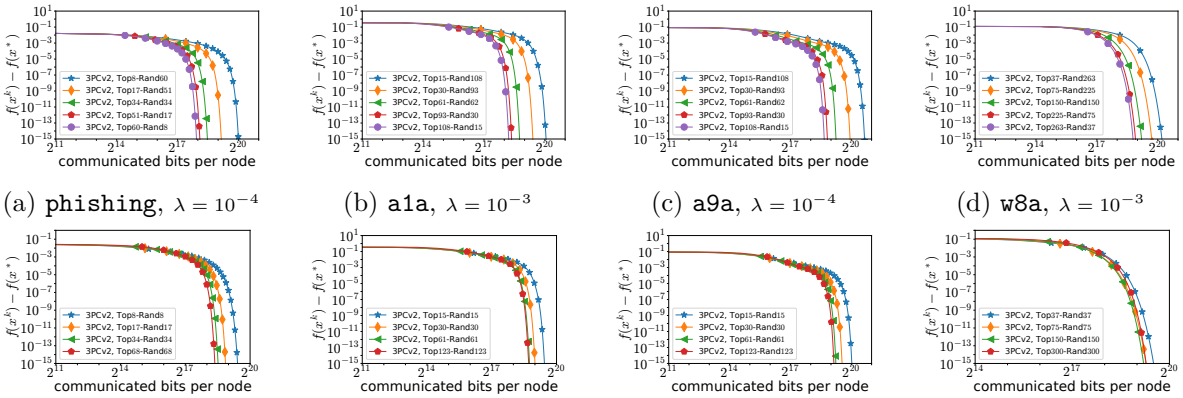
On the following step we study how Newton-3PCv2 behaves when the parameters of compressors 3PCv2 is based on are changing. In particular, in the first set of experiments we test the performance of Newton-3PCv2 assembled from Top-K_1 and Rand-K_2 compressors where $K_1 + K_2 = d$. Such constraint is forced to make the cost of one iteration to be $\mathcal{O}(d)$. In the second set of experiments we choose $K_1 = K_2 = K$ and vary K. The results are presented in Figure 4.



(a) `phishing`, $\lambda = 10^{-3}, d = 68$ (b) `a1a`, $\lambda = 10^{-4}, d = 123$ (c) `a9a`, $\lambda = 10^{-3}, d = 123$

(e) `phishing`, $\lambda = 10^{-3}, d = 68$ (f) `a1a`, $\lambda = 10^{-4}, d = 123$ (g) `a9a`, $\lambda = 10^{-3}, d = 123$

Figure 3: First row: The performance of Newton-CLAG based on Top-K varying values of (ζ, K) in terms of communication complexity (in Mbytes). **Second row:** The performance of Newton-CLAG based on Rank-R varying values of (ζ, R) in terms of communication complexity (in Mbytes).



(a) `phishing`, $\lambda = 10^{-4}$ (b) `a1a`, $\lambda = 10^{-3}$ (c) `a9a`, $\lambda = 10^{-4}$ (d) `w8a`, $\lambda = 10^{-3}$

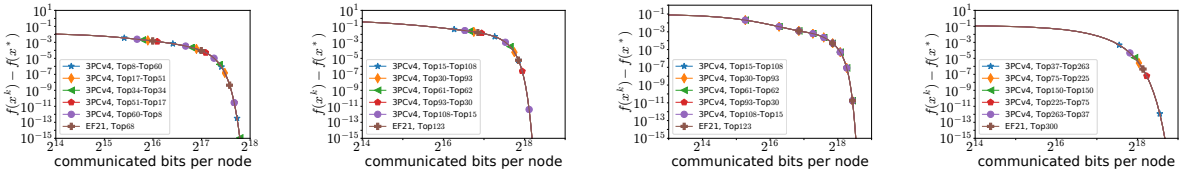
(e) `phishing`, $\lambda = 10^{-4}$ (f) `a1a`, $\lambda = 10^{-3}$ (g) `a9a`, $\lambda = 10^{-4}$ (h) `w8a`, $\lambda = 10^{-3}$

Figure 4: First row: The performance of Newton-3PCv2 where 3PCv2 compression mechanism is based on Top-K_1 and Rand-K_2 compressors with $K_1 + K_2 = d$ in terms of communication complexity. **Second row:** The performance of Newton-3PCv2 where 3PCv2 compression mechanism is based on Top-K_1 and Rand-K_2 compressors with $K_1 = K_2 \in \{d/8, d/4, d/2, d\}$ in terms of communication complexity.

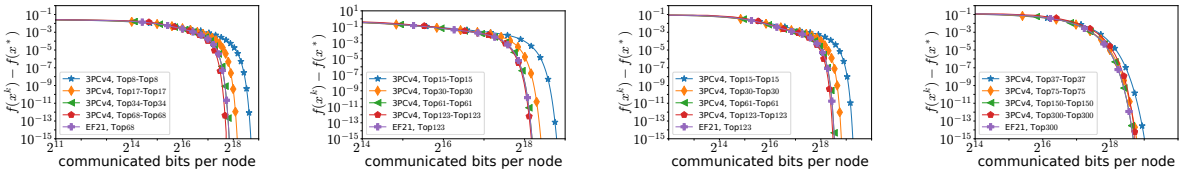
For the first set of experiments, one can notice that randomness hurts the convergence since the larger the value of K_2, the worse the convergence in terms of communication complexity. In all cases a weaker level of randomness is preferable. For the second set of experiments, we observe that the larger K, the better communication complexity of Newton-3PCv2 except the case of `w8a` where the results for $K = 150$ are slightly better than those for $K = 300$.

J.5 Behavior of Newton-3PCv4 under different compression levels

Now we test the behavior of Newton-3PCv4 where 3PCv4 is based on a pair (Top-K_1, Top-K_2) of compressors. Again, we have to sets of experiments: in the first one we examine the performance of Newton-3PCv4 when $K_1 + K_2 = d$; in the second one we check the efficiency of Newton-3PCv4 when $K_1 = K_2 = K$ varying K. In both cases we provide the behavior of Newton-EF21 (equivalent to FedNL) for comparison. All results are presented in Figure 5.

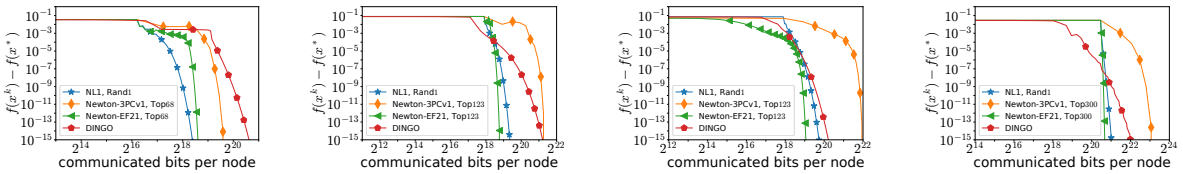


(a) `phishing`, $\lambda = 10^{-4}$ (b) `a1a`, $\lambda = 10^{-3}$ (c) `a9a`, $\lambda = 10^{-4}$ (d) `w8a`, $\lambda = 10^{-3}$



(e) `phishing`, $\lambda = 10^{-4}$ (f) `a1a`, $\lambda = 10^{-3}$ (g) `a9a`, $\lambda = 10^{-4}$ (h) `w8a`, $\lambda = 10^{-3}$

Figure 5: First row: The performance of Newton-3PCv4 where 3PCv4 compression mechanism is based on Top-K_1 and Top-K_2 compressors with $K_1 + K_2 = d$ in terms of communication complexity. **Second row:** The performance of Newton-3PCv4 where 3PCv4 compression mechanism is based on Top-K_1 and Top-K_2 compressors with $K_1 = K_2 \in \{d/8, d/4, d/2, d\}$ in terms of communication complexity. Performance of Newton-EF21 with Top-d is given for comparison.



(a) `phishing`, $\lambda = 10^{-4}$ (b) `a1a`, $\lambda = 10^{-3}$ (c) `a9a`, $\lambda = 10^{-4}$ (d) `w8a`, $\lambda = 10^{-3}$

Figure 6: The performance of Newton-3PCv1 with 3PCv1 based on Top-d, Newton-EF21 (equivalent to FedNL) with Top-d, NL1 with Rand-1, and DINGO in terms of communication complexity.

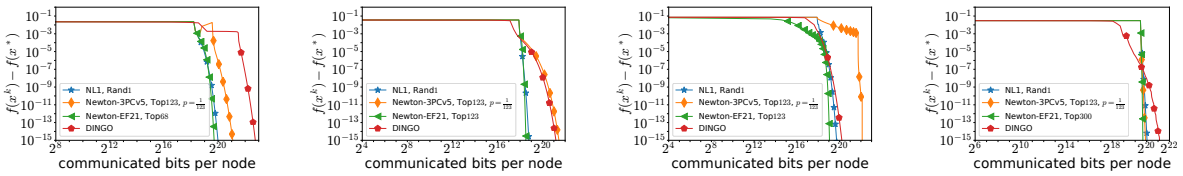
As we can see, in the first set of experiments it does not matter how we distribute d between K_1 and K_2 since it does not affect the performance. Regarding the second set of experiments, we can say that in some cases less aggressive compression ($K_1 = K_2 = d$) could be better than Newton-EF21.

J.6 Study of Newton-3PCv1

Next, we investigate the performance of Newton-3PCv1 where 3PC compression mechanism is based on Top-K. We compare its performance with Newton-EF21 (equivalent to FedNL) with Top-d, NL1 with Rand-1, and DINGO. We observe in Figure 6 that Newton-3PCv1 is not efficient method since it fails in all cases.

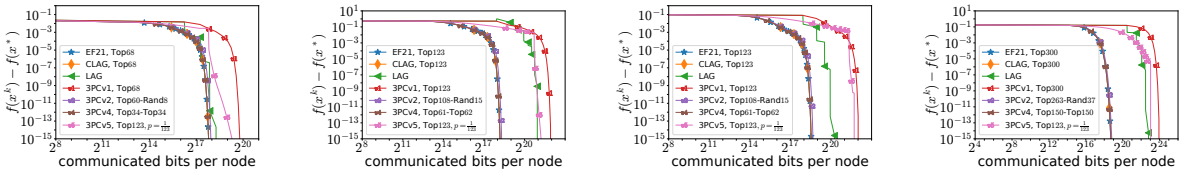
J.7 Performance of Newton-3PCv5

In this section we investigate the performance of Newton-3PCv5 where 3PC compression mechanism is based on Top-K. We compare its performance with Newton-EF21 (equivalent to FedNL) with Top-d, NL1 with Rand-1, and DINGO. According to the plots presented in Figure 7, we conclude that Newton-3PCv5 is not as effective as NL1 and Newton-EF21, but it is comparable with DINGO. The reason why Newton-3PCv5 is not efficient in terms of communication complexity is that we still need to send true Hessians with some nonzero probability which hurts the communication complexity of this method.



(a) `phishing`, $\lambda = 10^{-4}$ (b) `a1a`, $\lambda = 10^{-3}$ (c) `a9a`, $\lambda = 10^{-4}$ (d) `w8a`, $\lambda = 10^{-3}$

Figure 7: The performance of Newton-3PCv5 with 3PCv5 based on Top-d, Newton-EF21 (equivalent to FedNL) with Top-d, NL1 with Rand-1, and DINGO in terms of communication complexity.



(a) `phishing`, $\lambda = 10^{-4}$ (b) `a1a`, $\lambda = 10^{-3}$ (c) `a9a`, $\lambda = 10^{-4}$ (d) `w8a`, $\lambda = 10^{-3}$

Figure 8: The performance of Newton-3PC with different choice of 3PC compression mechanism in terms of communication complexity.

J.8 Newton-3PC with different choice of 3PC compression mechanism

Now we investigate how the choice of 3PC compressor influences the communication complexity of Newton-3PC. We test the performance of Newton-3PC with EF21, CLAG, LAG, 3PCv1 (based on Top-K), 3PCv2 (based on Top-K_1 and Rand-K_2), 3PCv4 (based on Top-K_1 and Top-K_2), and 3PCv5 (based on Top-K). We choose $p = 1/d$ for Newton-3PCv5 in order to make the communication cost of one iteration to be $\mathcal{O}(d)$. The choice of K, K_1, and K_2 is justified by the same logic.

We clearly see that Newton-3PC combined with EF21 (Newton-3PC with this 3PC compressor reduces to FedNL), CLAG, 3PCv2, 3PCv4 demonstrates almost identical results in terms of communication complexity. Newton-LAG performs worse than previous methods except the case of `phishing` dataset. Surprisingly, Newton-3PCv1, where only true Hessian differences is compressed, demonstrates the worst performance among all 3PC compression mechanisms. This probably caused by the fact that communication cost of one iteration of Newton-3PCv1 is significantly larger than those of other Newton-3PC methods.

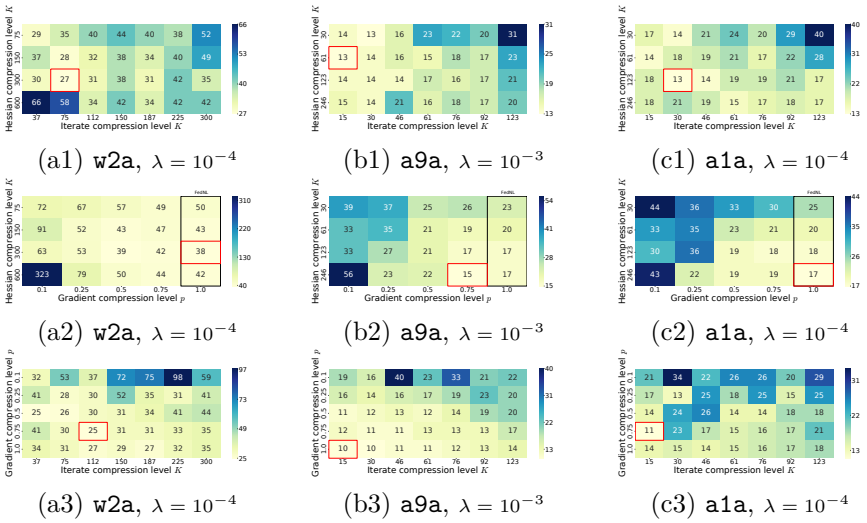
J.9 Analysis of Bidirectional Newton-3PC

J.9.1 EF21 compression mechanism

In this section we analyze how each type of compression (Hessians, iterates, and gradients) affects the performance of Newton-3PC. In particular, we choose Newton-EF21 (equivalent to FedNL) and change parameters of each compression mechanism. For Hessians and iterates we use Top-K_1 and Top-K_2 compressors respectively. In Figure 9 we present the results when we vary the parameter K_1, K_2 of Top-K compressor and probability p of Bernoulli Aggregation. The results are presented as heatmaps indicating the number of Mbytes transmitted in uplink and downlink directions by each client.

In the first row in Figure 9 we test different combinations of compression parameters for Hessians and iterates keeping the probability p of BAG for gradients to be equal 0.5. In the second row we analyze various combinations of pairs of parameters (K, p) for Hessians and gradients when the compression on iterates is not applied. Finally, the third row corresponds to the case when Hessians compression is fixed (we use Top-d), and we vary pairs of parameters (K, p) for iterates and gradients compression.

According to the results in the heatmaps, we can conclude that Newton-EF21 benefits from the iterates compression. Indeed, in both cases (when we vary compression level applied on Hessians or gradients) the best result is given in the case when we do apply the compression on iterates. This is not the case for gradients (see second row) since the best results are given for high probability p; usually for $p = 1$ and rarely for $p = 0.75$. Nevertheless, we clearly see that bidirectional compression is indeed useful in almost all cases.



(a1) `w2a`, $\lambda = 10^{-4}$ (b1) `a9a`, $\lambda = 10^{-3}$ (c1) `a1a`, $\lambda = 10^{-4}$

(a2) `w2a`, $\lambda = 10^{-4}$ (b2) `a9a`, $\lambda = 10^{-3}$ (c2) `a1a`, $\lambda = 10^{-4}$

(a3) `w2a`, $\lambda = 10^{-4}$ (b3) `a9a`, $\lambda = 10^{-3}$ (c3) `a1a`, $\lambda = 10^{-4}$

Figure 9: First row: The performance of Newton-3PC-BC in terms of communication complexity for different values of (K_1, K_2) of Top-K_1 and Top-K_2 compressors applied on Hessians and iterates respectively while probability $p = 0.75$ of BAG applied on gradients is fixed. **Second row:** The performance of Newton-EF21 in terms of communication complexity for different values of (K_1, p) of Top-K_1 compressor applied on Hessians and probability p of BAG applied on gradients while $K_2 = d$ parameter of Top-K_2 applied on iterates is fixed. **Third row:** The performance of Newton-EF21 in terms of communication complexity for different values of (K_2, p) of Top-K_2 compressor applied on iterates and probability p of BAG applied on gradients while $K_1 = d$ parameter of Top-K_1 applied on Hessians is fixed.

J.9.2 3PCv4 compression mechanism

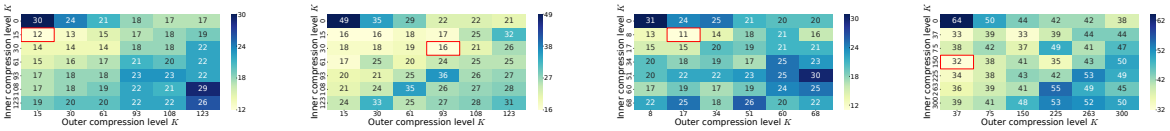
In our next set of experiments we fix EF21 compression mechanism based on Top-d compressor applied on Hessians and probability $p = 0.75$ of Bernoulli aggregation applied on gradients. Now we use 3PCv4 update rule on iterates based on outer and inner compressors (Top-K_1, Top-K_2) varying the values of pairs (K_1, K_2). We report the results as heatmaps in Figure 10.

We observe that in all cases it is better to apply relatively smaller outer and inner compression levels as this leads to better performance in terms of communication complexity. Note that the first row in heatmaps corresponds to Newton-3PC-BC when we apply just EF21 update rule on iterates. As a consequence, Newton-3PC-BC reduces to FedNL-BC method (Safaryan et al., 2022). We obtain that 3PCv4 compression mechanism applied on iterates in this setting is more communication efficient than EF21. This implies the fact that Newton-3PC-BC could be more efficient than FedNL-BC in terms of communication complexity.

J.10 BL1 (Qian et al., 2022) with 3PC compressor

As it was stated in Section 4.1 Newton-3PC covers methods introduced in (Qian et al., 2022) as a special case. Indeed, in order to run, for example, BL1 method we need to use rotation compression operator 20. The role of orthogonal matrix in the definition plays the basis matrix.

In this section we test the performance of BL1 in terms of communication complexity with different 3PC compressors: EF21, CBAG, CLAG. For CBAG update rule the probability $p = 0.5$, and for CLAG the

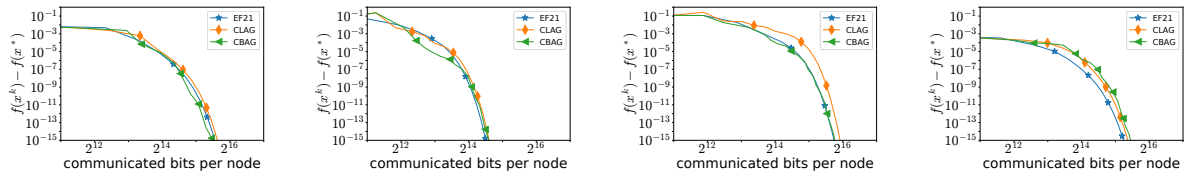


(a) a9a, $\lambda = 10^{-3}$ (b) a1a, $\lambda = 10^{-4}$ (c) phishing, $\lambda = 10^{-3}$ (d) a1a, $\lambda = 10^{-4}$

Figure 10: The performance of Newton-3PC-BC with EF21 update rule based on Top-d compressor applied on Hessians, BAG update rule with probability $p = 0.75$ applied on gradients, and 3PCv4 update rule based on (Top-K_1, Top-K_2) compressors applied on iterates for different values of pairs (K_1, K_2).

trigger $\zeta = 2$. All aforementioned 3PC compression operators are based on Top-τ compressor where τ is the dimension of local data (see Section 2.3 of (Qian et al., 2022) for detailed description).

Observing the results in Figure 11, we can notice that there is no improvement of one update rule over another. However, in EF21 is slightly better than other 3PC compressors in a half of the cases, and CBAG insignificantly outperform in other cases. This means that even if the performance of BL1 with EF21 and CBAG are almost identical, CBAG is still preferable since it is computationally less expensive since we do not need to compute local Hessians and their representations in new basis.



(a) a9a, $\lambda = 10^{-3}$ (b) phishing, $\lambda = 10^{-4}$ (c) a1a, $\lambda = 10^{-3}$ (d) w2a, $\lambda = 10^{-4}$

Figure 11: The performance of BL1 with EF21, CBAG and CLAG 3PC compression mechanisms in terms of communication complexity.

J.11 Analysis of Newton-3PC-BC-PP

J.11.1 3PC's parameters fine-tuning for Newton-3PC-BC-PP

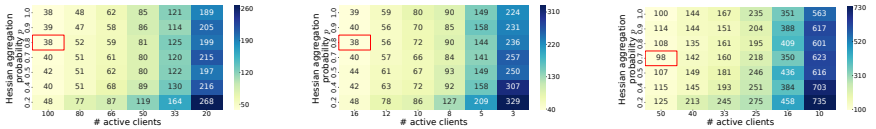
On the following step we study how the choice of parameters of 3PC compression mechanism and the number of active clients influence the performance of Newton-3PC-BC-PP.

In the first series of experiments we test Newton-3PC-BC-PP with CBAG compression combined with Top-$2d$ compressor and probability p applied on Hessians; EF21 with Top-$2d/3$ compressor applied on iterates; BAG update rule with probability $p = 0.75$ applied on gradients. We vary aggregation probability p of Hessians and the number of active clients τ. Looking at the numerical results in Figure 12 (first row), we may claim that the more clients are involved in the optimization process in each communication round, the faster the convergence since the best results in each case always belongs the first column. However, we do observe that lazy aggregation rule with probability $p < 1$ is still beneficial.

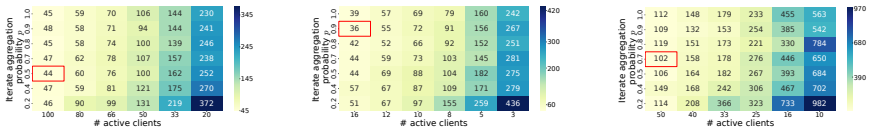
In the second row of Figure 12 we investigate Newton-3PC-BC-PP with CBAG compression based on Top-d and probability $p = 0.75$ applied on Hessians; 3PCv5 update rule combined with Top-$2d/3$ and probability p applied on iterates; BAG lazy aggregation rule with probability $p - 0.75$ applied gradients. In this case we modify iterate aggregation probability p and the number of clients participating in the training. We observe that again the fastest convergence is demonstrated when all clients are active, but aggregation parameter p of iterates smaller than 1.

Finally, we study the effect of BAG update rule on the communication complexity of Newton-3PC-BC-PP. As in previous cases, Newton-3PC-BC-PP is more efficient when all clients participate in the training process.

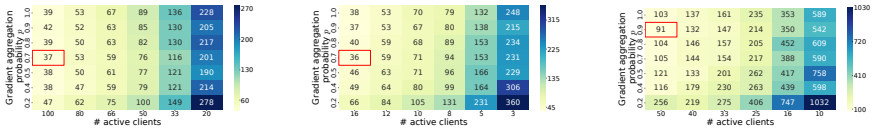
Nevertheless, lazy aggregation rule of BAG still brings the benefit to communication complexity of the method.



(a1) `phishing`, $\lambda = 10^{-4}$ (b1) `a1a`, $\lambda = 10^{-3}$ (c1) `w2a`, $\lambda = 10^{-4}$



(a2) `phishing`, $\lambda = 10^{-4}$ (b2) `a1a`, $\lambda = 10^{-3}$ (c2) `w2a`, $\lambda = 10^{-4}$

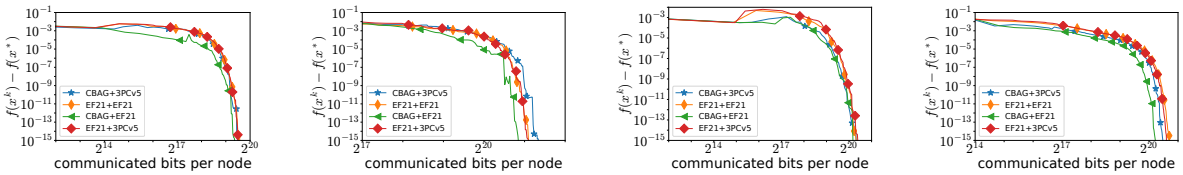


(a3) `phishing`, $\lambda = 10^{-4}$ (b3) `a1a`, $\lambda = 10^{-3}$ (c3) `w2a`, $\lambda = 10^{-4}$

Figure 12: The performance of Newton-3PC-BC-PP with various update strategies in terms of communication complexity (in Mbytes).

J.11.2 Comparison of different 3PC update rules

Now we test different combinations of 3PC compression mechanisms applied on Hessians and iterates. First, we fix probability parameter of BAG update rule applied on gradients to $p = 0.7$. The number of active clients in all cases $\tau = n/2$. We analyze various combinations of 3PC compressors: CBAG (Top-d and $p = 0.7$) and 3PCv5 (Top-$d/2$ and $p = 0.7$); EF21 (Top-d) and EF21 (Top-$d/2$); CBAG (Top-d and $p = 0.7$) and EF21 (Top-$d/2$); EF21 (Top-d) and 3PCv5 (Top-$d/2$ and $p = 0.7$) applied on Hessians and iterates respectively. Numerical results might be found in Figure 13. We can see that in all cases Newton-3PC-BC-PP performs the best with a combination of 3PC compressors that differ from EF21+EF21. This allows to conclude that EF21 update rule is not always the most effective since other 3PC compression mechanisms lead to better performance in terms of communication complexity. Nonetheless one can notice that it is useless to apply CBAG or LAG compression mechanisms on iterates. Indeed, in the case when we skip communication the iterates remain intact, and the next step is equivalent to previous one. Thus, there is no need to carry out the step again.



(a1) `a1a`, $\lambda = 10^{-3}$ (b1) `w2a`, $\lambda = 10^{-4}$ (c1) `a9a`, $\lambda = 10^{-3}$ (c1) `w8a`, $\lambda = 10^{-4}$

Figure 13: The performance of Newton-3PC-BC-PP with different combinations of 3PC compressors applied on Hessians and iterates respectively.

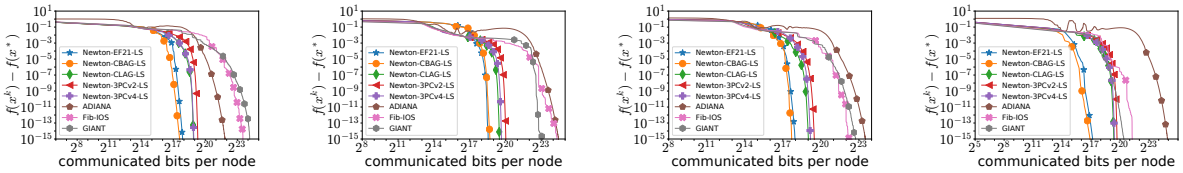
J.12 Global convergence of Newton-3PC

Now we investigate the performance of globally convergent Newton-3PC-LS — an extension of Newton-3PC — based on the line search as it performs significantly better than Newton-3PC-CR based on cubic

regularization. The experiments are done on synthetically generated datasets with heterogeneity control. A detailed description of how the datasets are created is given in section B.12 of (Safaryan et al., 2022). Roughly speaking, the generation function has 2 parameters α and β that control the heterogeneity of local data. We denote datasets created in a such way with parameters α and β as `Synt(`α`, `β`)`. All datasets are generated with dimension $d = 100$, split between $n = 20$ clients each of which has $m = 1000$ local data points. In all cases the regularization parameter is chosen $\lambda = 10^{-4}$.

We compare 5 versions of Newton-3PC-LS combined with EF21 (based on Rank-1 compressor), CBAG (based on Rank-1 compressor with probability 0.8), CLAG (based on Rank-1 compressor and communication trigger $\zeta = 2$), 3PCv2 (based on Top-$3d/4$ and Rand-$d/4$ compressors), and 3PCv4 (based on Top-$d/2$ and Top-$d/2$ compressors). In this series of experiments the initialization of $\mathbf{H}_i^0$ is equal to zero matrix. The comparison is performed against ADIANA (Li et al., 2020b) with random dithering ($s = \sqrt{d}$), Fib-IOS (Fabbro et al., 2022), and GIANT (Wang et al., 2018).

The numerical results are shown in Figure 14. According to them, we observe that Newton-3PC-LS is more resistant to heterogeneity than other methods since they outperform others *by several orders in magnitude*. Besides, we see that Newton-CBAG-LS and Newton-EF21-LS are the most efficient among all Newton-3PC-LS methods; in some cases, the difference is considerable.



(a) `Synt(0.5,0.5)` (b) `Synt(1,1)` (c) `Synt(1.5,1.5)` (d) `Synt(2,2)`

Figure 14: The performance of Newton-3PC-LS with different combinations of 3PC compressors applied on Hessians against ADIANA, Fib-IOS, and GIANT.