(Optimal) Online Bipartite Matching with Degree Information

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Abstract

We propose a model for online graph problems where algorithms are given access to an oracle that predicts (e.g., based on modeling assumptions or on past data) the degrees of nodes in the graph. Within this model, we study the classic problem of online bipartite matching, and a natural greedy matching algorithm called MinPredictedDegree, which uses predictions of the degrees of offline nodes. For the bipartite version of a stochastic graph model due to Chung, Lu, and Vu where the expected values of the offline degrees are known and used as predictions, we show that MinPredictedDegree stochastically dominates *any* other online algorithm, i.e., it is optimal for graphs drawn from this model. Since the "symmetric" version of the model, where all online nodes are identical, is a special case of the well-studied "known i.i.d. model", it follows that the competitive ratio of MinPredictedDegree on such inputs is at least 0.7299. For the special case of graphs with power law degree distributions, we show that MinPredictedDegree frequently produces matchings almost as large as the true maximum matching on such graphs. We complement these results with an extensive empirical evaluation showing that MinPredictedDegree compares favorably to state-of-the-art online algorithms for online matching.

1 Introduction

Online algorithms are algorithms that process their inputs "on the fly", making irrevocable decisions based only on the data seen so far. Since they do not make any assumptions about the future, they are versatile and work even for adversarial inputs. Unfortunately, by focusing on the worst case, their performance in "typical" cases can be sub-optimal. As a result there has been a large body of research studying various relaxations of the worst-case model, where some extra information about the inputs, or the distribution they are selected from, is available [1].

Motivated by the developments in machine learning, over the last few years, many papers have studied online algorithms with predictions [47]. Such algorithms are equipped with a predictor that, when invoked, provides an (imperfect) prediction of some features of the future part of the input, which is then used by the algorithm to improve its performance. The specific information provided by such predictors is problem-dependent. For graph problems studied in this paper, predictions could include: the list of edges incident to a given vertex [33], the weight of an edge adjacent to a given node in an optimal solution [4], or vertex weights that guide a proportional allocation scheme [36].

In this paper we focus on online graph problems, and propose a model where an algorithm is equipped with a "degree predictor", i.e., an oracle that, given any vertex, predicts the degree of that vertex in the full graph (containing yet-unseen edges). This predictor has multiple appealing features. First it is simple, natural, and easy to interpret. Second, it is useful: vertex degree information is employed in many heuristic and approximation algorithms for graph optimization, for problems such as maximum independent set [24] or maximum matching [53]. Third (as demonstrated in Section 7)

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such predictors can be easily obtained. Finally, degree prediction is closely related to the problem of estimating the frequencies of elements in a data set[1], and frequency predictors have been already shown to improve the performance of algorithms for multiple data analysis problems [25, 28, 19, 17].

The specific graph problem studied in this paper is *online bipartite matching*, where we are given a bipartite graph $G = (U \cup V, E)$, and the goal is to find a maximum matching in G. In the online setting, the set U is known beforehand, while the vertices in V arrive online one by one. When a new vertex v arrives, the edges in G adjacent to v are provided as well. Online maximum bipartite matching is a classic question studied in the online algorithms literature, with many applications [41]. It is known that a randomized online greedy algorithm, called Ranking, computes a matching of size at least $1 - 1/e \approx 0.632$ times the optimum [31], and that this bound is tight in the worst-case. A large body of work studied various relaxations of the problem, obtained by assuming that vertex arrivals are random [21] or that the graph itself is randomly generated from a given "known i.i.d. model" [20]. In this paper we extend the basic online model by assuming access to a predictor that, given any "offline" vertex $u \in U$, returns an estimate of its degree. (Note that the degree of any vertex in V is known immediately upon its arrival.)

Our results   We study the following simple greedy algorithm for bipartite matching: upon the arrival of a vertex v, if the set of neighbors $N(v)$ of v in G contains any yet-unmatched vertex, the algorithm selects $u \in N(v)$ of minimum predicted degree in G and adds the edge (u, v) to the matching. This algorithm, which we call *MinPredictedDegree (MPD)*, is essentially identical[2] to the algorithm proposed in [10] which in turn was inspired by the offline matching algorithm called MinGreedy [53]. The intuition is that vertices with higher degree will have more chances to be matched in the future.

Our main contributions are as follows. First, following in a long line of work on the average-case analysis of matching algorithms initiated by [30], we analyze MPD under a natural random bipartite graph model we refer to as *CLV-B*, a bipartite version of the Chung-Lu-Vu random graph model [12]. A CLV-B random graph is parameterized by $n = |U|$, $m = |V|$, and two weight vectors $\mathbf{p} = \{p_i\}_{i=1}^n \in [0, 1]^n$ and $\mathbf{q} = \{q_i\}_{i=1}^m \in [0, 1]^m$. For any $u_i \in U$ and $v_j \in V$, the edge $\{u_i, v_j\}$ appears in the graph with probability $p_i q_j$ and these events are mutually independent. This model corresponds to the setting where consumers pick their edges with probabilities proportional to the vector $\mathbf{p}$ which describes the relative distribution over the producers.

Many natural families of random graphs can be described in the CLV-B model. Of particular interest is the case when $q = (1, \ldots, 1)$, corresponding to the consumers picking their edges i.i.d.; we will refer to this case as the *symmetric CLV-B model*. The symmetric version can be viewed as a special case of the well-studied known i.i.d. model of [20]. If we further let $\mathbf{p} = (p, \ldots, p)$, then the CLV-B graph is an Erdős-Rényi random bipartite graph with edge probability p.

Theoretical Results   For the CLV-B model and the MPD algorithm which uses the expected degrees as predictions, we make the following theoretical contributions:

- We show that MPD stochastically dominates *any* other online algorithm, i.e., it is optimal for graphs drawn from the CLV-B model (Section 5). Specifically, we show that for any degree distribution, any algorithm A and any integer t, the probability that A produces matching of size at least t is upper bounded by the analogous probability for MPD. Since symmetric CLV-B is a special case of the known i.i.d. model, it follows that the competitive ratio of MPD for this model is at least equal to the best competitive ratio of any algorithm that works for the known i.i.d. model. By the result of [11], this ratio is at least 0.7299.

- We analyze MPD on symmetric CLV-B model with power law degree distribution (Section 6). Our theoretical predictions demonstrate that the competitive ratio achieved by our algorithm on such graphs is very high. In particular, for several different power law distributions, it exceeds 0.99.

- We also analyze MPD on Erdős-Rényi bipartite random graphs where all edges appear with the same probability (Appendix K). In particular, the competitive ratio of the algorithm on

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[1]The degree of a node is simply the number of times the node appears in the union of all edges.

[2]The main differences are syntactic: the algorithm of [10] computes the degrees based on the given "type graph" (see Section 2), while in this paper we allow arbitrary predictors.

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such graphs is at least 0.831. Since in this case all expected degrees are equal, the prediction oracle is of no help. Thus, we conjecture that this is the worst distribution for MPD among all distributions in the CLV-B model class.

- Finally, we observe that the competitive ratio of MPD is $1/2$ for *worst case* graphs, and that this bound is tight. In addition, we show that the worst-case competitive ratio of *any* algorithm with access to the offline degrees is at most $1 - 1/e$, implying that degree predictions do not help in the worst-case though they prove to be useful in the random model as well as in practice. See Appendix E for details.

Experiments We complement our theoretical studies with an extensive empirical evaluation of MPD for multiple random graph models and real graph benchmarks in Section 7. Our experiments show that, on most benchmarks, MPD has the best performance among about a dozen state-of-the-art online algorithms, even when compared to algorithms that use much more information about the input. These experimental results demonstrate that MPD performs well beyond the average-case instances we study theoretically.

Prediction Error For our theoretical results on the CLV-B graphs, MPD is given only the expected (as opposed to the actual) degrees. Although this models the uncertainty in the input, it is natural to ask how MPD performs when even the expected degrees are mispredicted. To this end, in Appendix D, we suppose that the offline nodes are prioritized in an arbitrary order π' which may be different from the order π obtained by sorting the nodes according to their expected degrees. Letting Δ be the minimum number of offline nodes that needs to be deleted such that π and π' induce the same order on the remaining nodes, we prove that using a noisy degree predictor which induces π' instead of π can shrink the size of the matching produced by MPD by at most Δ. We note that the number of mispredicted nodes is an upper bound on Δ, but in general Δ could be much smaller.

Importantly, we also note that the empirical performance of MPD shows its resilience to prediction error. Our experiments on real graphs use predictors which are noisy and which degrade over time but still find large matchings. Furthermore, on synthetic Zipfian data, we experiment with artifically adding noise and show a gradual degradation of MPD's performance as error increases.

2   Preliminaries

CLV-B model CLV-B is the bipartite version of the Chung-Lu-Vu model used in prior work [12, 42]. Given vectors $\mathbf{p} = \{p_i\}_{i=1}^n$ and $\mathbf{q} = \{q_j\}_{j=1}^m$, the edge $\{u_i, v_j\}$ appears in the graph independently with probability $p_i q_j$. From the vectors $\mathbf{p}$ and $\mathbf{q}$, we obtain the vector of offline expected degrees $\mathbf{d} = \{d_i\}_{i=1}^n = \{p_i \cdot \|\mathbf{q}\|_1\}_{i=1}^n$. For our theoretical results within this model, our algorithm MinPredictedDegree uses the degree predictor which returns the *expected* degree for each offline node: $\sigma(u_i) = d_i$ (see Appendix D for extension to noisy predictors). The particular case of symmetric CLV-B where $q = (1, \ldots, 1)$ corresponds to the case where consumers (online) pick their edges i.i.d. over producers (offline) and MPD has knowledge of the average preferences over producers.

Known i.i.d. model In the known i.i.d. model of [20], algorithms are given access to a *type graph* $G = (U \cup V, E)$ and a distribution $\mathcal{P} : V \to [0, 1]$. The nodes in V and their incident edges represent "types" of online nodes. An input instance $\hat{G} = (U \cup \hat{V}, \hat{E})$ is formed by picking m online nodes i.i.d. from V according to the probabilities described by $\mathcal{P}$. Note that the symmetric CLV-B model defined earlier is a special case of this model. In our experiments, the degree predictions are given by the expected degrees of the offline nodes.

3   Related Work

Online bipartite matching and its generalizations have been investigated extensively. The survey [41] and the recent paper [9] provide excellent overviews of this area. The state of the art competitive ratios are $1 - 1/e \approx 0.632$ in the worst case [31] and ≈ 0.7299 for the known i.i.d. model [11]. See [9] for an extensive empirical study of the existing algorithms. Other algorithms examined in the experimental section include [20, 5, 40, 26, 18, 10].

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Algorithm 1 MinPredictedDegree

Input: Offline nodes U and degree predictor $\sigma : U \rightarrow \mathbb{R}_{\geq 0}$
Output: Matching M
Initialize $M \leftarrow \emptyset$.
while online node $v \in V$ arrives **do**
   $N(v) \leftarrow$ unmatched neighbors of v
   if $|N(v)| > 0$ **then**
      $u^* \leftarrow \arg\min_{u \in N(v)} \sigma(u)$ (ties broken arbitrarily)
      $M \leftarrow M \cup \{(u^*, v)\}$
   end if
end while

More generally, there has been lots of interest in online algorithms with predictions over the last few years, for problems like caching [39, 50, 55, 29], ski-rental and its generalizations [49, 22, 2, 3], scheduling [45, 35] matching [33, 4, 36] and learning [14, 7]. Other areas impacted by learning-based algorithms include combinatorial optimization [13, 6, 15], similarity search [52, 56, 27, 54, 16], data structures [32, 44] and streaming/sampling algorithms [25, 28, 19]. See [47] for an excellent survey of this area.

4 Algorithm

Online Bipartite Matching The online bipartite matching problem is defined as follows. Given a bipartite graph $G = (U \cup V, E)$, we call U the "offline" side and V the "online" side of the bipartition. Let $n = |U|$ and $m = |V|$. The nodes in U are known beforehand and the nodes in V arrive one at a time, along with their incident edges. An online bipartite matching algorithm maintains a matching throughout the process, with the goal of maximizing the size of the matching. As each node $v \in V$ arrives, the algorithm can pick one of its neighboring edges to add to the matching.

MinPredictedDegree In addition to knowing the offline nodes U beforehand, MinPredictedDegree (MPD) is given a degree predictor $\sigma : U \rightarrow \mathbb{R}_{\geq 0}$. In practice, this predictor could be inferred from additional knowledge about the graph or from past data. When a node $v \in V$ arrives, MPD (see Algorithm 1) uses this predictor to greedily select the minimum predicted degree neighbor u^* of v that is not already covered in the matching and then adds the edge $\{u^*, v\}$ to the matching. If no such valid neighbor exists, MPD does nothing with v. Intuitively, low degree offline nodes should be matched as early as possible as they only appear a few times while we will have many chances to match high degree offline nodes.

The MPD algorithm has similar structure to the worst-case optimal Ranking algorithm [31] which assigns a random cost to each offline node and at each step greedily matches with the lowest cost offline neighbor. Specifically, if the degree predictor is random, MPD and Ranking are equivalent. As we show in the later sections, if the predictor is "good enough", MPD often performs much better than Ranking, both in theory and in practice.

5 Optimality of MPD on CLV-B graphs

In this section we show that the size of the matching found by the the MPD algorithm stochastically dominates the size of the matching found by any other algorithm. We start by providing some preliminaries for the analysis.

Preliminaries For $p \in [0,1]^n$ and $q \in [0,1]^m$, let $I_{p,q}$ denote an instance of a CLV-B graph with $n = |U|$ offline nodes, $m = |V|$ online nodes, and weight vectors p and q, such that the probability that an edge (u_i, v_j) exists is equal to $p_i q_j$ for any $i \in [n], j \in [m]$. Assume with no loss of generality that p is ordered, $p_1 \leq p_2 \leq \ldots \leq p_n$. Note that the expected degree of the offline node u_i is $p_i \|q\|_1$, i.e., it is proportional to the weight p_i.

In the online setting, the nodes of V arrive sequentially in the order $v_1, \ldots, v_m$ with the random neighborhood of $v_j \in V$ being revealed at the arrival of v_j. When v_j arrives, an online bipartite

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algorithm A can match v_j to any of its unmatched neighbors in U but cannot change its decision later. For any online bipartite matching algorithm A, let $A(I_{p,q})$ denote the size of the matching attained by A on the instance $I_{p,q}$. Let A_0 be the MinPredictedDegree algorithm which matches a node v_j with neighborhood S to an unmatched node $u_i \in S$ such that p_i minimal, i.e. to an available node in S with minimal expected degree (ties broken arbitrarily but consistently, e.g. by sorted order of the offline node id's). Let $p(A, S)$ be the resulting set of weights after algorithm A (potentially) chooses a neighbor in S to match with.

Consider two ordered weight vectors p, p' both of length n. We say that p' *dominates* p, equivalently $p \preceq p'$, if $p_i \leq p'_i$ for all $i \in [n]$. We are now ready to state our main result on the optimality of MPD.

Theorem 5.1. *Let $p \in [0,1]^n$ and $q \in [0,1]^m$. Let A be any online algorithm and let $t \geq 0$. Then,*

$$\mathbb{P}(A(I_{p,q}) \geq t) \leq \mathbb{P}(A_0(I_{p,q}) \geq t).$$

To prove the theorem, we will need two technical Lemmas. Informally, Lemma 5.2 states that for any $S \neq \emptyset$, it is an advantage for A_0 if the neighborhood of the first arriving node is S rather than the empty set. Lemma 5.3 (the proof of which is the main technical challenge) states that if $p \preceq p'$, then $A_0(I_{p',q})$ stochastically dominates $A_0(I_{p,q})$. Intuitively, Theorem 5.1 then follows from Lemma 5.3 by inducting on the number of online nodes m. For any algorithm A and non-empty subset $S \subseteq [n]$, if A matches v_1 to a node in the neighborhood S, then $p(A, S) \preceq p(A_0, S)$, and we can apply Lemma 5.3 together with the induction hypothesis with $m - 1$ online nodes. We need Lemma 5.2 to handle the issue that A may not to match v_1 even in the case that S is non-empty. The proofs of the two lemmas and of Theorem 5.1 are postponed to Appendices A, B and C.

Lemma 5.2. *Let $p \in [0,1]^n$ and $q \in [0,1]^m$ be weight vectors. Let $p^* \in [0,1]^{n-1}$ be obtained from p by removing its i'th entry for some $i \in [n]$. For any $t \geq 0$,*

$$\mathbb{P}(A_0(I_{p,q}) \geq t) \leq \mathbb{P}(A_0(I_{p^*,q}) \geq t - 1).$$

Lemma 5.3. *Let $p, p' \in [0,1]^n$, be ordered weight vectors and $q \in [0,1]^m$. Suppose that $p \preceq p'$. For any $t \geq 0$,*

$$\mathbb{P}(A_0(I_{p,q}) \geq t) \leq \mathbb{P}(A_0(I_{p',q}) \geq t).$$

While optimally only holds when the predicted degrees are the expected degrees (or at least induce the same ordering over the offline nodes), the performance of MPD cannot be much worse if the predictions are slightly off. Formally, for an arbitrary degree predictor σ, let $p[\sigma]$ be the array of CLV-B offline weights ordered by σ and let $\mathrm{LIS}(p[\sigma])$ be the size of the longest increasing subsequence in this array. We show (via a more general result) in Appendix D that MPD will match at most $n - \mathrm{LIS}(p[\sigma])$ fewer nodes than when given the expected degrees as predictions.

6 Competitive ratio of MPD on symmetric CLV-B random graphs

Though we know that MPD is optimal within the CLV-B model, this result does not give explicit competitive ratios for MPD. In this section we analyze MPD under the symmetric CLV-B model, and derive a set of equations that give a lower bound on MPD's competitive ratio. To recap, the symmetric model is parameterized by $n = |U|$, $m = |V|$, and a vector $\mathbf{d} = \{d_i\}_{i=1}^n$ corresponding to the expected degrees of the offline nodes. Formally, for any $u_i \in U$ and $v_j \in V$, the edge $\{u_i, v_j\}$ appears in the graph with probability d_i/m.

As in the previous section, we analyze MPD when the degree predictions are given by the expected degrees $\mathbf{d}$. Our main results within this model are a set of equations that describe the size of the matching produced by MPD as well as the size of the maximum matching.

- Given a set of expected degrees $\mathbf{d}$, Equation 4 models the behavior of MPD on a symmetric CLV-B($\mathbf{d}$) graph. We extend these results to the asymptotic case in Appendix J, giving the expected matching size as $n, m \to \infty$ for a given distribution of expected degrees.

- Given a set of expected degrees $\mathbf{d}$,in Appendix I, we give an upper bound on the expected size of the maximum matching on a symmetric CLV-B($\mathbf{d}$) graph, and in Appendix J, we give the asymptotic equivalent. Empirically, we find this upper bound to be close to the maximum matching size when $\mathbf{d}$ follows a power law distribution.

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- Using these equations, we show that in expectation MPD returns matchings almost as large as the maximum when the expected degrees of the offline nodes follow a power law distribution (see Table 1 and Figure 6). For both MPD and the maximum matching, we show that the matching sizes are concentrated about their expectations (Appendix L and M), implying that on these graphs, MPD achieves a large competitive ratio.

6.1 Competitive ratios on power law graphs

Cutoff λ	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$
10	0.967	0.948	0.934	0.928
100	0.998	0.986	0.958	0.937
1000	1.000	0.995	0.966	0.940
10000	1.000	0.997	0.969	0.940
100000	1.000	0.998	0.970	0.940

Table 1: Lower bound on the competitive ratio of MPD on symmetric CLV-B graphs with offline expected degrees following a power law with exponential cutoff distribution as $n, m \to \infty$. The fraction of offline nodes with expected degree d is proportional to $d^{-\alpha}e^{-d/\lambda}$ for $d = \{1, 2, ...\}$.

In Table 1, we show the competitive ratio of MPD on symmetric CLV-B graphs with expected offline degrees following a power law with exponential cutoff distribution [9, 46] and with $n, m \to \infty$. For $d = \{1, 2, ...\}$, the fraction of offline nodes with expected degree d is proportional to $d^{-\alpha}e^{-d/\lambda}$ for exponent α and cutoff λ. Note that in the asymptotic case, as the sizes of MPD's matching and the maximum matching are concentrated about their expectations (Theorems L.1, M.1), the ratio of expectations is equivalent to the competitive ratio (expectation of ratio). When the exponent is small or the cutoff is large, MPD achieves a better competitive ratio, with the ratio exceeding 0.99 when both occur. When $\alpha = 2$, while MPD still achieves a competitive ratio of up to 0.94, the competitive ratio is not as affected by a larger cutoff as with smaller exponents (the power law factor is already significantly limiting the fraction of offline nodes with large expected degree). The analysis we develop is general and can be used to evaluate MPD on symmetric CLV-B graphs with different parameters than those we have considered.

6.2 Differential equation analysis of MPD

Let Y_d^t be the number of offline nodes with expected degree d who are unmatched by MPD after seeing the tth online node. Within this random graph model, $\{Y_d^t\}_{t=0}^m$ form a Markov chain with the following expected evolution:

$$\mathbb{E}[Y_d^{t+1} - Y_d^t] = -\left(1 - (1 - d/m)^{Y_d^t}\right) \prod_{d' < d} (1 - d'/m)^{Y_{d'}^t}. \tag{1}$$

The first term corresponds to the probability that at least one unmatched offline node with expected degree d is incident on the $(t + 1)$st online node while the second term corresponds to the probability that this online node has no neighboring unmatched offline nodes with smaller expected degree (which would be prioritized).

Let $k_d = -\log(1 - d/m)$. To simplify the analysis of MPD, it will be helpful to consider the random variables $Z_d^t = -k_d * Y_d^t$ where

$$\mathbb{E}[Z_d^{t+1} - Z_d^t] = k_d \left(1 - e^{Z_d^t}\right) \prod_{d' < d} e^{Z_{d'}^t}. \tag{2}$$

Following the work of Kurtz and many subsequent researchers [34, 57, 43, 38, 48], we show that the behavior of MPD as described by these Markov chains is well approximated by the trajectory of the following system of differential equations for all unique expected degrees d in $\mathbf{d}$:

$$\frac{dz_d(t)}{dt} = k_d \left(1 - e^{z_d(t)}\right) \prod_{d' < d} e^{z_{d'}(t)}. \tag{3}$$

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These functions $z_d(t)$ represent continuous-time approximations of the Markov chains with their derivatives corresponding to expected change from Equation 2. In Appendix G, we give the solution to these differential equations. Relying on past work [38], we give the following theorem (see Appendix H for proof).

Theorem 6.1. *Let G be a symmetric CLV-B random graph with unique expected offline degrees $\{\delta_i\}_{i=1}^{\ell}$. Let $f_d = \lambda_d \cdot n$ be the number of offline nodes with expected degree d. Then, the expected (over the randomness in G) size of the matching formed by MPD approaches*

$$\sum_{i=1}^{\ell} f_{\delta_i} + z_{\delta_i}(m)/k \qquad (4)$$

as $n = m$ approach infinity, where $z_{\delta_i}(t)$ for $i \in \{1, \ldots, \ell\}$ form the solution to the system of differential equations in Equation 3.

The solution to the system of differential equations gives us a closed form continuous-time approximation for expected performance of MPD in terms of $\mathbf{d}$. In particular, in the asymptotic case, the equations give the exact expected performance and in the non-asymptotic case give an approximation on the number of unmatched offline nodes (and thus the matching size).

7 Experiments

In this section, we evaluate the empirical performance of MPD on real and synthetic data. For each dataset, we report the empirical competitive ratio of MPD and a variety of baselines. In each case, the empirical competitive ratio is the average, over 100 trials, of the ratios of the sizes of the matchings outputted by a given algorithm and the sizes of the maximum matching. In addition to the average ratio, we report one standard deviation of the ratio across the trials.

Datasets We evaluate MPD on the following datasets.

- **Oregon:** 9 graphs[3] sampled over 3 months representing a communication network of internet routers from the Stanford SNAP Repository [37]. Each graph has $\sim 10k$ nodes on each side of the bipartition and $\sim 40k$ edges. For MPD, the offline degree predictor $\sigma : U \to \mathbb{R}$ is based on the first graph: if an offline node u (i.e. a specific router) appeared in the first graph, $\sigma(u)$ is the degree of u in that graph. If an offline node u did not appear in the first graph, $\sigma(u) = 1$. For each trial, the order of arrival of the online nodes is randomized.

- **CAIDA:** 122 graphs[3] sampled approximately weekly over 4 years representing a communication network of internet routers from the Stanford SNAP Repository [37]. Each graph has $\sim 20k$ nodes on each side of the bipartition and $\sim 100k$ edges. The degree predictor is the same as for the Oregon dataset (for each year, the first graph of the year is used to form the predictor). As seen in Figure 8 (see Appendix N), the degree distribution of the graphs for both the Oregon and Caida datasets are long-tailed and the error of the first graph predictor increases over time as the underlying graph evolves. For each trial, the order of arrival of the online nodes is randomized.

- **Symmetric CLV-B random graph:** We consider symmetric CLV-B model where the expected offline degrees are distributed according to Zipf's Law, a popular power law distribution where $d_i = C \cdot i^{-\alpha}$ [46]. In our experiments, we set size $n = m = 1000$, set $C = m/2$, and vary the exponent α.

- **Known i.i.d.:** Finally, we compare MPD to algorithms for the known i.i.d. model, copying the methodology of Borodin et al. [9] for synthetic power law graphs (Molloy Reed and Preferential Attachment) and real world graphs. In the Molloy Reed experiments, the type graph is sample from a family of random graphs with degrees distributed according to a power law with exponential cutoff. In the Preferential Attachment experiments, the type graph is formed by the preferential attachment model in which edges are added sequentially with edges between high degree nodes being more likely. The Real World graphs are comprised of a variety of graphs from the Network Repository [51]. See Appendix N for more results on Real World graphs.

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[3]The graphs in the Oregon and CAIDA datasets are made bipartite following the *bipartite double cover* or *duplicating method* used in prior work [9]. Given a graph $G = (V, E)$, the bipartite double cover of G is the graph $G' = (U' \cup V', E')$ where U' and V' are copies of V and there is an edge $\{u_i', v_j'\} \in E'$ if and only if $\{v_i, v_j\} \in E$.

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Baselines We compare our algorithm to a variety of baseline algorithms.

- **Ranking** In all experiments, we compare to the classic, worst-case optimal Ranking algorithm [31].
- **MinDegree** The MinDegree algorithm is a version of MPD with a perfect oracle, i.e. $\sigma(u)$ returns the true degree of u. In comparison with MPD, MinDegree shows the effect of prediction error on the performance of MPD.
- **Known i.i.d. baselines** For the experiments in the known i.i.d. case, we also compare to the baselines in the extensive empirical study of [9]–see their paper for detailed descriptions of all algorithms. The code is distributed under the GPL license. Notably, the algorithms Category-Advice and 3-Pass are *not* strictly online algorithms: they take multiple passes over the data, using some limited information from previous passes to make better decisions in the next pass. It should also be noted that BKPMinDegree is distinct from either the MPD or MinDegree algorithms we have described–it does not use the type graph but rather maintains and updates an estimate of the degree of the offline nodes throughout the runtime of the algorithm.

  Most known i.i.d. baselines are *not* greedy–they do not always match an online node even if it has unmatched neighbors. [9] evaluate greedy augmentations of these algorithms (denoted by Algorithm(g)) which match to an arbitrary unmatched neighbor in these cases and generally show them to outperform their non-greedy counterparts. We additionally evaluate MPD augmented versions of these algorithms (denoted by Algorithm(MPD)) which applies the MPD rule in these cases using the expected degrees as predictions.

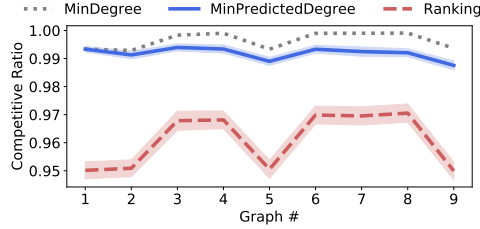



Figure 1: Comparison of empirical competitive ratios on the Oregon dataset. The first graph is used to form predictions.

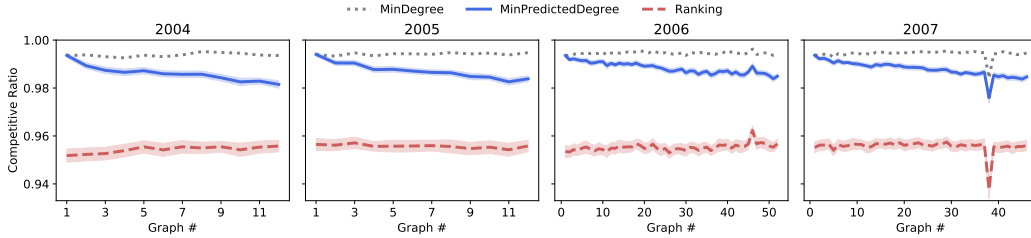



Figure 2: Comparison of empirical competitive ratios on the CAIDA dataset. For each subfigure, the first graph of the year is used to form predictions for the rest of the year.

Results Across the various datasets, MPD performs well compared to the baseline algorithms. For the Oregon, CAIDA, and symmetric CLV-B random graph datasets, MPD significantly outperforms Ranking, and for Oregon and CAIDA, the performance of the algorithm mildly declines as the degree predictions degrade. For the known i.i.d. datasets, MPD often outperforms all *online* baselines, despite making only limited use of the known i.i.d. model. Additionally, augmenting the known i.i.d. algorithms with the (MPD) rule often improves their performance over both the base and the greedy (g) versions of the algorithms.

- **Oregon and CAIDA (Figures 1, 2):** On the Oregon dataset, MPD achieves a competitive ratio of ~ 0.99 across the graphs compared with competitive ratios ranging from 0.95 to 0.97 for

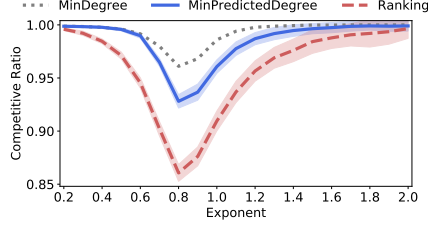
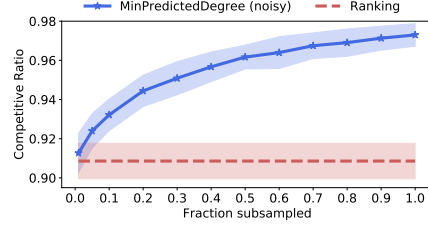
8

(a) Comparison across Zipf's Law exponents.

(b) Analysis of predictor noise (exponent $\alpha = 1$).

Figure 3: Comparison of empirical competitive ratios on symmetric CLV-B random graphs with offline expected degrees following Zipf's Law with exponent α. In (a), we vary α and MPD uses the expected degree as its predictor. In (b), the degree predictor is the offline degree in a random subgraph using a (varying) fraction of the online nodes.

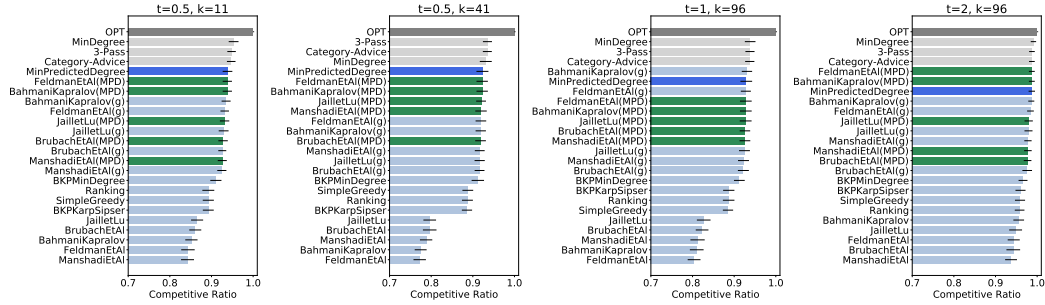



Figure 4: Comparison of empirical competitive ratios for Known i.i.d. Molloy-Reed graphs. Algorithms depicted in gray are *not* online algorithms (they use extra information or multiple passes). Algorithms in green are augmented with MPD.

Ranking. Compared with MinDegree, which uses knowledge of the true offline degrees, MPD's performance slowly degrades over time as the graphs become less similar to Graph #1 (see Figure 8 in Appendix N for quantitative details).

Similarly, on the CAIDA dataset, MPD does significantly better than Ranking, achieving competitive ratios almost always greater than 0.98 compared to ratios around 0.95, respectively. As the performance of the degree predictor degrades over time, the performance of MPD gradually declines (though it still significantly outperforms Ranking for both datasets).

- **Symmetric CLV-B random graph (Figure 3):** For symmetric CLV-B random graphs with offline expected degrees following Zipf's Law, MPD outperforms Ranking across a spectrum of exponents α ranging from 0.2 to 2. For exponents less than 0.5 and greater than 1.5, MPD achieves a competitive ratio close to 1 (greater than 0.995). All of the online algorithms have worse competitive ratios when the exponent is closer to one with MPD achieving a ratio of ~ 0.93 and Ranking achieving a ratio of ~ 0.86 when $\alpha = 0.8$. Though MPD does worst at $\alpha = 0.8$, it also achieves its greatest improvement over Ranking at this setting.

  In Figure 3b, we analyze the performance of MPD with a noisy degree predictor on Zipf's Law symmetric CLV-B random graphs with exponent 1. To introduce noise, the degree predictor $\sigma(u)$ is given by the number of neighbors u has with a random subset of the online nodes V. As we decrease the fraction of V we subsample, thus increasing the variance of the predictor, the performance of MPD steadily declines. Even when the degree predictor only uses 10% or even 1% (the leftmost point on the graph) of the online nodes, it still outperforms Ranking.

- **Known i.i.d. (Figures 4, 5):** Across all of the experiments in the known i.i.d. model, MPD is among the top online algorithms, and is often the best performing online algorithm (note the algorithms in gray are *not* strictly online algorithms). Most of the algorithms (e.g. BahamiKapralov and ManshadiEtAl) rely heavily on the type graph, including precomputing an optimal matching on the type graph. By contrast, MPD only uses first-order information: it only looks at degrees

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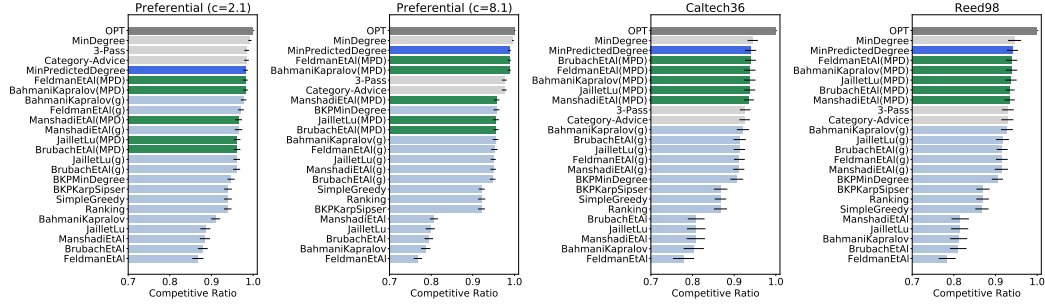
Figure 5: Comparison of empirical competitive ratios on Known i.i.d. Preferential Attachment graphs and Real World graphs. Algorithms depicted in gray are *not* online algorithms (they use extra information or multiple passes). Algorithms in green are augmented with MPD. See Appendix N for more Real World results.

and does not rely on any information about specific edges. Even so, in most cases, it outperforms all of the other online algorithms. Additionally, the (MPD) augmented versions of the known i.i.d. algorithms always beat the base algorithms and often beat the greedy (g) versions, indicating the potential of predicted degrees to be integrated with other algorithms. Note that while the standard deviations are quite wide (the known i.i.d. model is inherently stochastic), as the results are summarized over 100 trials, relatively small differences in the *average* performance of these algorithms are statistically significant as the standard error is small.

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References

[1] *Special Semester on Algorithms and Uncertainty*, 2016. `https://simons.berkeley.edu/programs/uncertainty2016`.

[2] Keerti Anand, Rong Ge, and Debmalya Panigrahi. Customizing ml predictions for online algorithms. In *International Conference on Machine Learning*, pages 303–313. PMLR, 2020.

[3] Spyros Angelopoulos, Christoph Dürr, Shendan Jin, Shahin Kamali, and Marc Renault. Online computation with untrusted advice. In *11th Innovations in Theoretical Computer Science Conference (ITCS 2020)*, 2020.

[4] Antonios Antoniadis, Themis Gouleakis, Pieter Kleer, and Pavel Kolev. Secretary and online matching problems with machine learned advice. In *34th Conference on Neural Information Processing Systems*, 2020.

[5] Bahman Bahmani and Michael Kapralov. Improved bounds for online stochastic matching. In *European Symposium on Algorithms*, pages 170–181. Springer, 2010.

[6] Maria-Florina Balcan, Travis Dick, Tuomas Sandholm, and Ellen Vitercik. Learning to branch. In *International Conference on Machine Learning*, 2018.

[7] Aditya Bhaskara, Ashok Cutkosky, Ravi Kumar, and Manish Purohit. Online learning with imperfect hints. In *International Conference on Machine Learning*, pages 822–831. PMLR, 2020.

[8] Benjamin Birnbaum and Claire Mathieu. On-line bipartite matching made simple. *SIGACT News*, 39(1):80–87, March 2008.

10

[9] Allan Borodin, Christodoulos Karavasilis, and Denis Pankratov. An experimental study of algorithms for online bipartite matching. *Journal of Experimental Algorithmics (JEA)*, 25:1–37, 2020.

[10] Allan Borodin, Denis Pankratov, and Amirali Salehi-Abari. On conceptually simple algorithms for variants of online bipartite matching. *Theory of Computing Systems*, 63(8):1781–1818, 2019.

[11] Brian Brubach, Karthik Abinav Sankararaman, Aravind Srinivasan, and Pan Xu. New algorithms, better bounds, and a novel model for online stochastic matching. In *24th Annual European Symposium on Algorithms (ESA 2016)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2016.

[12] Fan Chung, Linyuan Lu, and Van Vu. The spectra of random graphs with given expected degrees. *Internet Mathematics*, 1(3):257–275, 2004.

[13] Hanjun Dai, Elias Khalil, Yuyu Zhang, Bistra Dilkina, and Le Song. Learning combinatorial optimization algorithms over graphs. In *Advances in Neural Information Processing Systems*, pages 6351–6361, 2017.

[14] Ofer Dekel, Arthur Flajolet, Nika Haghtalab, and Patrick Jaillet. Online learning with a hint. In *NIPS*, pages 5299–5308, 2017.

[15] Michael Dinitz, Sungjin Im, Thomas Lavastida, Benjamin Moseley, and Sergei Vassilvitskii. Faster matchings via learned duals. *Advances in Neural Information Processing Systems*, 34, 2021.

[16] Yihe Dong, Piotr Indyk, Ilya P Razenshteyn, and Tal Wagner. Learning space partitions for nearest neighbor search. *ICLR*, 2020.

[17] Elbert Du, Franklyn Wang, and Michael Mitzenmacher. Putting the "learning" into learning-augmented algorithms for frequency estimation. In *International Conference on Machine Learning*, 2021.

[18] Christoph Dürr, Christian Konrad, and Marc Renault. On the power of advice and randomization for online bipartite matching. In *The 24th European Symposium on Algorithms (ESA)*, 2016.

[19] Talya Eden, Piotr Indyk, Shyam Narayanan, Ronitt Rubinfeld, Sandeep Silwal, and Tal Wagner. Learning-based support estimation in sublinear time. In *International Conference on Learning Representations*, 2021.

[20] Jon Feldman, Aranyak Mehta, Vahab Mirrokni, and Shan Muthukrishnan. Online stochastic matching: Beating 1-1/e. In *2009 50th Annual IEEE Symposium on Foundations of Computer Science*, pages 117–126. IEEE, 2009.

[21] Gagan Goel and Aranyak Mehta. Online budgeted matching in random input models with applications to adwords. In *SODA*, volume 8, pages 982–991, 2008.

[22] Sreenivas Gollapudi and Debmalya Panigrahi. Online algorithms for rent-or-buy with expert advice. In *International Conference on Machine Learning*, pages 2319–2327. PMLR, 2019.

[23] Philip Hall. On representatives of subsets. *Journal of The London Mathematical Society-second Series*, pages 26–30, 1935.

[24] Magnús M Halldórsson and Jaikumar Radhakrishnan. Greed is good: Approximating independent sets in sparse and bounded-degree graphs. *Algorithmica*, 18(1):145–163, 1997.

[25] Chen-Yu Hsu, Piotr Indyk, Dina Katabi, and Ali Vakilian. Learning-based frequency estimation algorithms. In *International Conference on Learning Representations*, 2019.

[26] Patrick Jaillet and Xin Lu. Online stochastic matching: New algorithms with better bounds. *Mathematics of Operations Research*, 39(3):624–646, 2014.

[27] Herve Jegou, Matthijs Douze, and Cordelia Schmid. Product quantization for nearest neighbor search. *IEEE transactions on pattern analysis and machine intelligence*, 33(1):117–128, 2011.

11

[28] Tanqiu Jiang, Yi Li, Honghao Lin, Yisong Ruan, and David P Woodruff. Learning-augmented data stream algorithms. *ICLR*, 2020.

[29] Zhihao Jiang, Debmalya Panigrahi, and Kevin Sun. Online algorithms for weighted paging with predictions. In *47th International Colloquium on Automata, Languages, and Programming (ICALP 2020)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.

[30] R. Karp and M. Sipser. Maximum matchings in sparse random graphs. In *FOCS 1981*, 1981.

[31] Richard M Karp, Umesh V Vazirani, and Vijay V Vazirani. An optimal algorithm for on-line bipartite matching. In *Proceedings of the twenty-second annual ACM symposium on Theory of computing*, pages 352–358, 1990.

[32] Tim Kraska, Alex Beutel, Ed H Chi, Jeffrey Dean, and Neoklis Polyzotis. The case for learned index structures. In *Proceedings of the 2018 International Conference on Management of Data (SIGMOD)*, pages 489–504. ACM, 2018.

[33] Ravi Kumar, Manish Purohit, Aaron Schild, Zoya Svitkina, and Erik Vee. Semi-online bipartite matching. *10th Innovations in Theoretical Computer Science*, 2019.

[34] Thomas Kurtz. *8. Approximations of Density Dependent Jump Markov Processes*, pages 51–61. 1981.

[35] Silvio Lattanzi, Thomas Lavastida, Benjamin Moseley, and Sergei Vassilvitskii. Online scheduling via learned weights. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1859–1877. SIAM, 2020.

[36] Thomas Lavastida, Benjamin Moseley, R Ravi, and Chenyang Xu. Learnable and instance-robust predictions for online matching, flows and load balancing. *arXiv preprint arXiv:2011.11743*, 2020.

[37] Jure Leskovec and Andrej Krevl. SNAP Datasets: Stanford large network dataset collection. http://snap.stanford.edu/data, June 2014.

[38] Michael Luby, Michael Mitzenmacher, M Amin Shokrollahi, and Daniel Spielman. Efficient erasure correcting codes. *IEEE Transactions on Information Theory*, 47(2):569–584, 2001.

[39] Thodoris Lykouris and Sergei Vassilvtiskii. Competitive caching with machine learned advice. In *International Conference on Machine Learning*, pages 3296–3305. PMLR, 2018.

[40] Vahideh H Manshadi, Shayan Oveis Gharan, and Amin Saberi. Online stochastic matching: Online actions based on offline statistics. *Mathematics of Operations Research*, 37(4):559–573, 2012.

[41] Aranyak Mehta. Online matching and ad allocation. *Foundations and Trends® in Theoretical Computer Science*, 8(4):265–368, 2013.

[42] Raghu Meka, Prateek Jain, and Inderjit Dhillon. Matrix completion from power-law distributed samples. In *Advances in Neural Information Processing Systems*, volume 22, pages 1258–1266. Curran Associates, Inc., 2009.

[43] Michael Mitzenmacher. Studying balanced allocations with differential equations. *Combinatorics, Probability, and Computing*, 8:473–482, 1997.

[44] Michael Mitzenmacher. A model for learned bloom filters and optimizing by sandwiching. In *Advances in Neural Information Processing Systems*, 2018.

[45] Michael Mitzenmacher. Scheduling with predictions and the price of misprediction. In *ITCS*, 2020.

[46] Michael Mitzenmacher and Eli Upfal. *Probability and Computing*. 01 2005.

[47] Michael Mitzenmacher and Sergei Vassilvitskii. Algorithms with predictions. *arXiv preprint arXiv:2006.09123*, 2020.

12

[48] Nathan Noiry, Vianney Perchet, and Flore Sentenac. Online matching in sparse random graphs: Non-asymptotic performances of greedy algorithm. In *35th Conference on Neural Information Processing Systems*, 2021.

[49] Manish Purohit, Zoya Svitkina, and Ravi Kumar. Improving online algorithms via ml predictions. In *Advances in Neural Information Processing Systems*, pages 9661–9670, 2018.

[50] Dhruv Rohatgi. Near-optimal bounds for online caching with machine learned advice. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1834–1845. SIAM, 2020.

[51] Ryan A. Rossi and Nesreen K. Ahmed. The network data repository with interactive graph analytics and visualization. In *AAAI*, 2015.

[52] Ruslan Salakhutdinov and Geoffrey Hinton. Semantic hashing. *International Journal of Approximate Reasoning*, 50(7):969–978, 2009.

[53] Gottfried Tinhofer. A probabilistic analysis of some greedy cardinality matching algorithms. *Annals of Operations Research*, 1(3):239–254, 1984.

[54] Jun Wang, Wei Liu, Sanjiv Kumar, and Shih-Fu Chang. Learning to hash for indexing big data - a survey. *Proceedings of the IEEE*, 104(1):34–57, 2016.

[55] Alexander Wei. Better and simpler learning-augmented online caching. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2020)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.

[56] Yair Weiss, Antonio Torralba, and Rob Fergus. Spectral hashing. In *Advances in neural information processing systems*, pages 1753–1760, 2009.

[57] Nicholas C. Wormald. Differential equations for random processes and random graphs. *Annals of Applied Probability*, 5(4):1217–1235, 1995.

Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See, for example, Appendix E.
   (c) Did you discuss any potential negative societal impacts of your work? [No] We did not include a discussion of this in the text of the paper as our paper is focused on fundamental research. That being said, as online bipartite matching has a wide variety of applications, developing and better understanding algorithms for this problem may have impact on these diverse applications. In particular, metrics other than the competitive ratio of algorithms (such as certain notions of fairness) are important to consider in applications involving individuals and are being studied in other works.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes]

3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] See supplementary material.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes]
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes]

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(d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [No] The experiments are relatively lightweight and the focus of this paper is not on computational resources (rather, we focus on the quality of the algorithms' output). That being said, MPD is a very simple and computationally efficient algorithm (assuming accessing the degree predictor is not too expensive). The experiments were all run on a 2018 MacBook Pro.

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...

   (a) If your work uses existing assets, did you cite the creators? [Yes]

   (b) Did you mention the license of the assets? [Yes]

   (c) Did you include any new assets either in the supplemental material or as a URL? [Yes] See code in supplement.

   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A] All data was gathered from open sources intended for research use.

   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A] None of our data falls within this category.

5. If you used crowdsourcing or conducted research with human subjects...

   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

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A The Proof of Lemma 5.2

In this appendix, we provide the proof of Lemma 5.2.

Proof of Lemma 5.2. The result follows by coupling the instances $I_{p,q}$ and $I_{p^*,q}$. Let $U = (u_1, \ldots, u_n)$, $V = (v_1, \ldots, v_m)$, and $U^* = (u_1, \ldots u_{i-1}, u_{i+1}, \ldots, u_n)$. For any instance of a bipartite graph I on (U, V) we can generate an instance I^* on (U^*, V) as follows: For each $j \in [m]$, if $N(v_j)$ is the neigborhood of v_j in I, then the neighborhood of v_j in I^* is $N(v_j) \cap U^*$. It is readily checked that if I is distributed as $I_{p,q}$, then I^* is distributed as $I_{p^*,q}$. To prove the result, it therefore suffices to show that for any instance I, if $A_0(I) \geq t$, then $A_0(I^*) \geq t - 1$, or equivalently, that $A_0(I^*) \geq A_0(I) - 1$. We prove this by induction on m. The case $m = 0$ is trivial, so assume that $m > 0$ and inductively that the result holds for smaller values of m. Let $M_j \subseteq U$ denote the set of matched nodes by MPD on I after the arrival of $v_1, \ldots, v_j$ and let similarly $M_j^* \subseteq U^*$ denote the set of matched nodes by MPD on I^* after the arrival of $v_1, \ldots, v_j$.

We will proceed by cases. First, consider the case in which MPD does not match any vertex v_j to u_i during its run on I. Then MPD's behavior on I and I^* will be identical as other than u_i and its incident edges, the two graphs are the same. In particular, $A_0(I) = |M_m| = |M_m^*| = A_0(I^*)$.

Consider next the case in which MPD does match u_i to some vertex v_j during its run on I. Up until the arrival of v_j, MPD has behaved similarly on I and I^*, and in particular, $M_{j-1} = M_{j-1}^*$. Now if the neighborhood of v_j in U^* does not contain an unmatched node, then $M_j = M_j^* \cup \{u_i\}$, and from this point on, MPD will behave similarly on the two instances I and I^*. In particular, $A_0(I) = |M_m(I)| = |M_m^*(I) \cup \{u_i\}| = A_0(I^*) + 1$, as desired. If on the other hand, MPD on I^* matches v_j to some node $u_{i'} \neq u_i$ in U^*, then $|M_j| = |M_j^*|$. Moreover, by the induction hypothesis, $|M_m \setminus M_j| \leq |M_m^* \setminus M_j^*| + 1$. It follows that $A_0(I) = |M_j| + |M_m \setminus M_j| \leq |M_j^*| + |M_m^* \setminus M_j^*| + 1 = A_0(I^*) + 1$, as desired. $\square$

B The Proof of Lemma 5.3

In this appendix, we provide the proof of Lemma 5.3. We start with some preliminaries. For some subset of offline nodes $S \subseteq [n]$, let $\mathbb{P}_p(N_S)$ be the probability that the first online node's neighborhood is exactly the set S in $I_{p,q}$. Note that $\mathbb{P}_p(N_S)$ depends only on p and q_1 and equals $\prod_{i \in S}(p_i q_1) \prod_{i \in [n] \setminus S}(1 - p_i q_1)$. Let $p(A, S)$ be the ordered vector of weights of the unmatched offline nodes remaining after running A on the first online node if this node has neighborhood S. If $A = A_0$, then $p(A, S)$ is obtained by removing the entry p_i of p corresponding to the $u_i \in S$ with minimal p_i (if $S = \emptyset$, then $p(A, S) = p$), but generally, A could behave differently even choosing not to match v_1 to any node in $S \neq \emptyset$.

Proof of Lemma 5.3. We will prove the lemma by induction on the number of online nodes, m. The base case $m = 0$ is trivial. Indeed, in this case, $A_0(I_{p,q}) = A_0(I_{p',q}) = 0$ with probability 1, so the probabilities of attaining matching size at least t are equal for the two weight vectors p and p'.

Now, we will consider the inductive case. Consider any $m > 0$ and assume the statement holds for $m - 1$ online nodes. Define the $(m - 1)$-dimensional vector $q^* = (q_2, \ldots, q_m)$. Then,

$$\mathbb{P}_p(A_0(I_{p,q}) \geq t) = \mathbb{P}_p(N_\emptyset) \, \mathbb{P}(A_0(I_{p,q^*}) \geq t) + \sum_{S \subseteq [n], S \neq \emptyset} \mathbb{P}_p(N_S) \, \mathbb{P}(A_0(I_{p(A_0,S),q^*}) \geq t - 1), \quad (5)$$

and a similar identity holds with p replaced by p'.

Denote by $r = q_1 p = (q_1 p_1, \ldots, q_1 p_n)$, and $r' = q_1 p' = (q_1 p'_1, \ldots, q_1 p'_n)$, so that r_i and r'_i are the probabilities that v_1 has an edge to u_i and u'_i in respectively $I_{p,q}$ and $I_{p',q}$. As $p \preceq p'$, also $r \preceq r'$. In particular, we can write $1 - r'_i = (1 - s_i)(1 - r_i)$ for some $s_i \in [0, 1]$. For $i \in [n]$, we let X_i and Y_i be independent Bernoulli variables with $\mathbb{P}[X_i = 1] = r_i$ and $\mathbb{P}[Y_i = 1] = s_i$. Let furthermore Z_i be the Bernoulli variable which is 1 if either $X_i = 1$ or $Y_i = 1$, and zero otherwise. Then $\mathbb{P}[Z_i = 1] = r'_i$. We now let $N = \{i \in [n] \mid X_i = 1\}$ and $N' = \{i \in [n] \mid Z_i = 1\}$, noting that $N \subseteq N'$. For any $T \subseteq [n]$ we can then write

$$\mathbb{P}_{p'}(N_T) = \mathbb{P}[N' = T] = \sum_{S \subseteq T} \mathbb{P}[N = S] \, \mathbb{P}[N' = T \mid N = S] = \sum_{S \subseteq T} \mathbb{P}_p(N_S) \Delta(S, T),$$

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where we have put $\mathbb{P}[N' = T \mid N = S] = \Delta(S,T)$. We note for later use that for any $S \subseteq [n]$, it holds that $\sum_{T \supseteq S} \Delta(S,T) = 1$. Indeed, conditioned on $N = S$, it holds that $S \subseteq N'$ with probability 1. Combining this with (5), we can write the probability of MPD exceeding size t on the graphs parameterized by p' as

$$\mathbb{P}(A_0(I_{p',q}) \geq t) = \mathbb{P}_{p'}(N_\emptyset)\,\mathbb{P}(A_0(I_{p',q^*}) \geq t) + \sum_{T \subseteq [n], T \neq \emptyset} \mathbb{P}_{p'}(N_T)\,\mathbb{P}(A_0(I_{p'(A_0,T),q^*}) \geq t-1)$$

$$= \mathbb{P}_{p'}(N_\emptyset)\,\mathbb{P}(A_0(I_{p',q^*}) \geq t) +$$

$$\sum_{T \subseteq [n], T \neq \emptyset} \sum_{S \subseteq T} \mathbb{P}_p(N_S)\Delta(S,T)\,\mathbb{P}(A_0(I_{p'(A_0,T),q^*}) \geq t-1)$$

$$= \mathbb{P}_p(N_\emptyset)\Delta(\emptyset,\emptyset)\,\mathbb{P}(A_0(I_{p',q^*}) \geq t) +$$

$$\mathbb{P}_p(N_\emptyset) \sum_{T \subseteq [n], T \neq \emptyset} \Delta(\emptyset,T)\,\mathbb{P}(A_0(I_{p'(A_0,T),q^*}) \geq t-1) +$$

$$\sum_{S \subseteq [n], S \neq \emptyset} \mathbb{P}_p(N_S) \sum_{T \supseteq S} \Delta(S,T)\,\mathbb{P}(A_0(I_{p'(A_0,T),q^*}) \geq t-1), \tag{6}$$

where the final steps follows by interchanging summations in the second term, and splitting into the cases $S = \emptyset$ and $S \neq \emptyset$.

Note that if S is non-empty and $S \subseteq T$, then by the MPD rule, $p'(A_0,S) \preceq p'(A_0,T)$ as the minimum degree within the set of neighbors cannot be larger in T than in S. Moreover, it is readily checked that the assumption $p \preceq p'$ implies that $p(A_0,S) \preceq p'(A_0,S)$ (after an appropriate permutation). Using the induction hypothesis, we get that for $S \neq \emptyset$ and $T \supseteq S$,

$$\mathbb{P}(A_0(I_{p'(A_0,T),q^*}) \geq t-1) \geq \mathbb{P}(A_0(I_{p(A_0,S),q^*}) \geq t-1), \tag{7}$$

and that

$$\mathbb{P}(A_0(I_{p',q^*}) \geq t) \geq \mathbb{P}(A_0(I_{p,q^*}) \geq t). \tag{8}$$

Moreover, an application of Lemma 5.2 and the induction hypothesis gives that

$$\mathbb{P}(A_0(I_{p'(A_0,T),q^*}) \geq t-1) \geq \mathbb{P}(A_0(I_{p',q^*}) \geq t) \geq \mathbb{P}(A_0(I_{p,q^*}) \geq t). \tag{9}$$

Plugging the bounds of (7),(8), and (9) into (6), it follows that

$$\mathbb{P}(A_0(I_{p',q}) \geq t) \geq \mathbb{P}_p(N_\emptyset)\Delta(\emptyset,\emptyset)\,\mathbb{P}(A_0(I_{p,q^*}) \geq t) +$$

$$\mathbb{P}_p(N_\emptyset) \sum_{T \subseteq [n], T \neq \emptyset} \Delta(\emptyset,T)\,\mathbb{P}(A_0(I_{p,q^*}) \geq t) +$$

$$\sum_{S \subseteq [n], S \neq \emptyset} \mathbb{P}_p(N_S) \sum_{T \supseteq S} \Delta(S,T)\,\mathbb{P}(A_0(I_{p(A_0,S),q^*}) \geq t-1).$$

Now combining the first two terms above and using that $\sum_{T \supseteq S} \Delta(S,T) = 1$ for any $S \subseteq [n]$, we obtain that

$$\mathbb{P}(A_0(I_{p',q}) \geq t) \geq \mathbb{P}_p(N_\emptyset)\,\mathbb{P}(A_0(I_{p,q^*}) \geq t) + \sum_{S \subseteq [n], S \neq \emptyset} \mathbb{P}_p(N_S)\,\mathbb{P}(A_0(I_{p(A_0,S),q^*}) \geq t-1)$$

$$= \mathbb{P}(A_0(I_{p,q}) \geq t),$$

where the final equality follows from (5). This is the desired result. $\qquad\square$

C The Proof of Theorem 5.1

In this appendix, we provide the proof of Theorem 5.1. As in Appendix B, for subsets of offline nodes $S \subseteq [n]$, we let $\mathbb{P}_p(N_S)$ denote the probability that the first online node's neighborhood is exactly the set S in $I_{p,q}$.

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Proof of Theorem 5.1. We will prove the theorem by induction over m. For the base case, $m = 0$, the inequality holds with equality as both algorithms yield empty matchings.

For the inductive case, let $m > 0$ and assume the inequality holds with $m - 1$ online nodes. Let $q^* = (q_2, \ldots, q_m)$. Let $R_A \subseteq \mathcal{P}([n])$ be set of subsets of $[n]$ s.t. for any $S \in R_A$, $p(A, S) = p$, i.e. algorithm A matches no edges if the set of neighbors of the current online node is S. Consider the probability of algorithm A attaining a matching with size at least t. By the induction hypothesis,

$$\mathbb{P}(A(I_{p,q}) \geq t) = \sum_{S \in R_A} \mathbb{P}_p(N_S) \, \mathbb{P}(A(I_{p,q^*}) \geq t) + \sum_{S \subseteq [n], S \notin R_A} \mathbb{P}_p(N_S) \, \mathbb{P}(A(I_{p(A,S),q^*}) \geq t - 1)$$

$$\leq \sum_{S \in R_A} \mathbb{P}_p(N_S) \, \mathbb{P}(A_0(I_{p,q^*}) \geq t) + \sum_{S \subseteq [n], S \notin R_A} \mathbb{P}_p(N_S) \, \mathbb{P}(A_0(I_{p(A,S),q^*}) \geq t - 1).$$

Now if $S \in R_A$, then $p(A, S) \preceq p(A_0, S)$. Thus, by applying Lemma 5.2 to the first term and Lemma 5.3 to the second term above,

$$\mathbb{P}(A(I_{p,q}) \geq t) \leq \mathbb{P}_p(N_\emptyset) \, \mathbb{P}(A_0(I_{p,q^*}) \geq t) + \sum_{S \in R_A \setminus \{\emptyset\}} \mathbb{P}_p(N_S) \, \mathbb{P}(A_0(I_{p(A_0,S),q^*}) \geq t - 1) +$$

$$\sum_{S \subseteq [n], S \notin R_A} \mathbb{P}_p(N_S) \, \mathbb{P}(A_0(I_{p(A_0,S),q^*}) \geq t - 1)$$

$$= \mathbb{P}_p(N_\emptyset) \, \mathbb{P}(A_0(I_{p,q^*}) \geq t) + \sum_{S \subseteq [n], S \neq \{\emptyset\}} \mathbb{P}_p(N_S) \, \mathbb{P}(A_0(I_{p(A_0,S),q^*}) \geq t - 1)$$

$$= \mathbb{P}(A_0(I_{p,q^*} \geq t),$$

completing the proof. $\qquad\square$

D MPD with Noisy Predictions

In Section 5, we show that given expected degrees as predictions, MPD is the optimal algorithm on CLV-B graphs. Here, we extend that analysis to show that even if the predictor is noisy, MPD can still return a large matching.

Consider two degree predictors σ and σ' (or more generally, two orderings of the offline nodes). Let $\Delta(\sigma, \sigma')$ denote the minimum number of offline nodes which must be removed such that σ and σ' induce the same ordering over the remaining nodes.

Theorem D.1. *For any graph $G = (U \cup V, E)$, the matching returned by MPD with degree predictor σ has at most $\Delta(\sigma, \sigma')$ more edges than the matching returned by MPD with degree predictor σ'.*

On CLV-B graphs, let $p[\sigma]$ be the array of offline weights ordered by σ. So, if σ^* returns the expected degrees, $p[\sigma^*]$ is in sorted order. Let $\mathrm{LIS}(p)$ denote the size of the longest increasing subsequence of p. Note that $\Delta(\sigma^*, \sigma) = n - \mathrm{LIS}(p)$. Then, as a corollary to Theorem D.1, we can bound the performance of a noisy degree predictor compared to the performance of MPD with expected degrees.

Corollary D.2. *On CLV-B graphs, the expected size of the matching returned by MPD with degree predictor σ has at most $n - LIS(p[\sigma])$ fewer nodes than the expected size of the matching returned by MPD given the expected degrees.*

As MPD with expected degrees is the optimal online algorithm for CLV-B graphs, this implies that as long as $n - LIS(p[\sigma])$ is small, MPD is still near-optimal. Note that $n - LIS(p[\sigma])$ is equal to the minimum number of offline nodes that would need to be removed s.t. the offline weights are in sorted order. This quantity is clearly upper bounded by the number of mispredicted nodes as removing each mispredicted node will leave a remaining sequence which is sorted.

In order to prove the theorem, we will use the following lemmas.

Lemma D.3 (Corollary of Lemma 2 of [8]). *Consider any graph $G = (U \cup V, E)$, offline node $u \in U$, and degree predictor σ. MPD with degree predictor σ when run on $G' = (U \setminus \{u\} \cup V, E)$ will produce a matching with at most one fewer edge than when run on G.*

This follows by [8] as these two matchings will differ by at most one alternating path, which can reduce the matching size by at most one.

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Lemma D.4. *Consider any graph $G = (U \cup V, E)$, offline node $u \in U$, and degree predictor σ. MPD with degree predictor σ when run on $G' = (U \setminus \{u\} \cup V, E)$ will produce a matching no bigger than when run on G.*

Proof. Let $N_G(v)$ be the neighborhood of a node $v \in V$ in the graph G, only including *unmatched* offline nodes at the point that v arrives. We will prove via induction that for any online node v, $N_{G'}(v) \subseteq N_G(v)$. For the base case, consider the first online node v_1. In this case, if $u \notin N_G(v_1)$, $N_{G'}(v_1) = N_G(v_1)$. Otherwise $N_{G'}(v_1) = N_G(v_1) \setminus \{u\}$.

Now, consider the inductive case for some online node v_i for $i > 1$. Assume that for all $j < i$, $N_{G'}(v_j) \subseteq N_G(v_j)$. Assume, for the sake of contradiction, that there exists some u' s.t. $u' \in N_{G'}(v_i)$ but $u' \notin N_G(v_i)$. This can only happen if at some point previously, u' was matched by MPD running on G but was never matched by MPD running on G'. Let v' be the online node that matched with u' when run on G. Note that as $u' \in N_{G'}(v_i)$, it must be the case that $u' \in N_{G'}(v')$ as MPD running on G' has not yet matched u' and u' was in $N_G(v')$. Let u^* be the node that v' matched with in G'. As MPD had the choice to match v' with u' or u^* on G', it must be the case that $\sigma(u^*) < \sigma(u')$ (breaking ties arbitrarily but consistently). On the other hand, by the inductive hypothesis, $u', u^* \in N_G(v')$, so MPD run on G chose to match v' with u' over u^*, so $\sigma(u') < \sigma(u^*)$, and we have reached a contradiction.

To complete the proof, note that for MPD (or any greedy algorithm), the size of the matching is exactly m minus the number of times an online node has no available neighbors. The inductive statement implies that for all online nodes v, $|N_{G'}(v)| \leq |N_G(v)|$. Therefore, the number of times an online node will have no available neighbors is at least as large for MPD when run on G' as on G. $\qquad \square$

Proof of Theorem D.1. Let $S \subset U$ be the $\Delta(\sigma, \sigma')$ online nodes which, if removed, would leave the remaining offline weights in sorted order. Let $A_\sigma(G)$ refer to size of matching returned by MPD with degree predictor σ on graph G. Let $G_{-S} = ((U \setminus S) \cup V, E)$.

By repeated invocation of Lemma D.4,

$$A_{\sigma'}(G) \geq A_{\sigma'}(G_{-S})$$

as removing an offline node never increases the size of the matching returned by MPD. By repeated invocation of Lemma D.3,

$$A_\sigma(G) \leq A_\sigma(G_{-S}) + \Delta(\sigma, \sigma')$$

as removing an offline node can decrease the size of the matching returned by MPD by at most 1.

To combine these bounds, note that $A_\sigma(G_{-S}) = A_{\sigma'}(G_{-S})$ as σ and σ' induce the same ordering after removing the offline nodes in S. Therefore,

$$A_{\sigma'}(G) \geq A_\sigma(G) - \Delta(\sigma, \sigma'),$$

completing the proof. $\qquad \square$

E Worst-Case Bound and Failure Modes

In the worst-case, MinPredictedDegree achieves a competitive ratio of $1/2$. When σ gives arbitrary predictions, MinPredictedDegree is equivalent to the simple greedy algorithm for online bipartite matching which achieves a competitive ratio of $1/2$ [31, 41]. Even if σ is the perfect degree predictor where $\sigma(u) = deg(u)$ for all $u \in U$, MinPredictedDegree is still $(1/2)$-competitive in the worst-case.

As MinPredictedDegree forms a *maximal* matching, the matching it returns is always at least half the size of the maximum matching. For the matching upper bound on the competitive ratio in the perfect predictor case, consider the graph $G = (U \cup V, E)$ where $n = m = 6$ with the following adjacency

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list:
$$\{u_1 : v_1, v_2, v_3\},$$
$$\{u_2 : v_1, v_2, v_3\},$$
$$\{u_3 : v_1, v_2, v_3\},$$
$$\{u_4 : v_1, v_4\},$$
$$\{u_5 : v_2, v_5\},$$
$$\{u_6 : v_3, v_6\}.$$

The first half of the offline and online nodes form a complete bipartite subgraph while the second half of the offline nodes each connect to one online node in the first half and one online node in the second half. MinPredictedDegree with a perfect degree predictor will return the matching $M = \{\{u_4, v_1\}, \{u_5, v_2\}, \{u_6, v_3\}\}$, matching the first half of the online nodes with their lower degree neighbors in the second half of the offline nodes, therefore leaving the first half of the offline nodes unmatched.

More generally, MinPredictedDegree can perform poorly if the degree predictor is arbitrary or if high degree nodes have poor edge expansion compared to low degree nodes (making it disadvantageous to always prioritize low degree nodes as seen in the example above). However, often these adversarial structures do not appear in practice and matching low degree nodes first leads to better results. In fact, the hard instance given in the seminal paper by Karp, Vazirani, and Vazirani [31] that introduced the online bipartite matching problem and the Ranking algorithm relies on the fact that algorithms with no extra information on the graph (e.g. no degree predictions) must often mistakenly match high degree left nodes rather than low degree left nodes when given a choice.

Better worst-case algorithms with degree predictions? As demonstrated in [31], no algorithm for the online matching problem has a competitive ratio better than $1 - 1/e$. It is however a natural question whether an algorithm with access to the offline degrees can obtain a better worst-case competitive ratio. We answer this question in the negative by modifying the example in [31] to show that no algorithm with knowledge of the true offline degrees has a competitive ratio of more than $1 - 1/e$. The hard example in [31] is an n by n bipartite graph with offline nodes $U = \{u_1, \ldots, u_n\}$ and online nodes $V = \{v_1, \ldots, v_n\}$. Node v_i has an edge to each node in $\{u_j : i \leq j \leq n\}$. The maximum matching clearly has size n but it is shown in [31] that if the online nodes arrive in a random order, then no algorithm matches more than $(1 - 1/e + o(1))n$ nodes in expectation. Note that given the true offline degrees as predictions, MPD will actually achieve the maximum matching for this example.

We augment this result in a black box manner to construct a bad example even when the offline degrees are known to the algorithm. Assume with no loss of generality that n is a square. Let $U_1, \ldots, U_{\sqrt{n}}$ and $V_1, \ldots, V_{\sqrt{n}}$ be disjoint vertex sets each of size $\sqrt{n}$. Let the offline nodes be $U = \bigcup_{i \leq \sqrt{n}} U_i$ and the online nodes be $V = \bigcup_{i \leq \sqrt{n}} V_i$. For each $1 \leq i \leq \sqrt{n} - 1$, we let (U_i, V_i) form an instance of the hard graph from [31] (as described above) with $\sqrt{n}$ online/offline nodes. Moreover, $(U_{\sqrt{n}}, V_{\sqrt{n}})$ form a complete bipartite graph. Finally, for each offline node $u \in U$, we add $\sqrt{n} - deg(u)$ edges from u to $V_{\sqrt{n}}$. Then all of the offline degrees are the same (they are all $\sqrt{n}$), so the degree oracle is of no use. If we sequentially for $1 \leq i \leq \sqrt{n} - 1$ let the nodes of V_i arrive in a random order (and finally the nodes from $V_{\sqrt{n}}$ in any order), it follows from the result in [31] that the expected size of the produced matching is at most

$$(\sqrt{n} - 1)(1 - 1/e + o(1))\sqrt{n} + \sqrt{n} = n(1 - 1/e + o(1))$$

As the maximum matching has size n, the upper bound on the competitive ratio follows.

F Additional Competitive Ratio Results

Using the equations derived in Section 6, in Figure 6 we plot the competitive ratio for symmetric CLV-B graphs with expected offline degrees following Zipf's Law (at $n = m = 1000$, this setup corresponds to the experiment shown in Figure 3). Across all choices of the exponent other than $\alpha = 1$, MPD's performance relative to the size of the maximum matching increases as the size of the graph grows. Further, for larger graphs, in many settings MPD achieves a ratio close to 1 and even for smaller graph achieves ratios above 0.9 for almost all settings.

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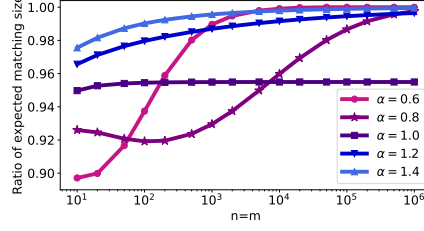
Figure 6: Ratio of MPD's expected matching size and the expected maximum matching size on symmetric CLV-B random graphs with offline expected degrees following Zipf's Law with exponent α. The ith offline node has expected degree $d_i = Ci^{-\alpha}$ with $C = m/2$.

Notably, at $\alpha = 1$, the ratio plateaus around 0.955, and at $\alpha = 0.8$, the ratio decreases from $n = 10$ to $n = 100$ before rising again as n increases. At $\alpha = 1$ across all values of n as well as at $\alpha = 0.8$ with $n = 100$, a large fraction of the offline nodes have expected degree close to one. Many of these nodes will have actual degree 1 and many will have actual degree ≥ 2. MPD has no way of distinguishing between these two types of nodes as it only uses expected degrees and will mistakenly not match some offline nodes that only appear once. While there is always a discrepancy between actual and expected degrees, the issue of prioritizing a node with actual degree ≥ 2 over a node with actual degree 1 is most detrimental, leading to worse performance when there are many offline nodes with expected degree close to one.

G Solution to System of Differential Equations in Equation 3

Recall the system of differential equations from Equation 3:

$$\frac{dz_d(t)}{dt} = k_d \left(1 - e^{z_d(t)}\right) \prod_{d' < d} e^{z_{d'}(t)} \tag{10}$$

for all unique expected degrees d in $\mathbf{d}$.

The solution to the system of differential equations is given by the following equations. Let $\{\delta_i\}_{i=1}^{\ell}$ be the ordered set of unique expected degrees and let f_d be the number of offline nodes with expected degree d. We will define the auxiliary functions $\alpha_{\delta_i}(t)$ for $i \in \{2, \ldots, \ell\}$ and variables C_{δ_i} for $i = \{1, \ldots, \ell\}$ as follows:

$$\alpha_{\delta_2}(t) = C_{\delta_1} + e^{k_{\delta_1} t}$$
$$\alpha_{\delta_i}(t) = (\alpha_{\delta_{i-1}}(t))^{k_{\delta_{i-1}}/k_{\delta_{i-2}}} + C_{\delta_{i-1}} \text{ (for } i \geq 3)$$

where

$$C_{\delta_1} = e^{k_{\delta_1} f_{\delta_1}} - 1$$
$$C_{\delta_i} = (\alpha_{\delta_i}(0))^{k_{\delta_i}/k_{\delta_{i-1}}} (e^{k_{\delta_i} f_{\delta_i}} - 1) \text{ (for } i \geq 2).$$

Then,

$$z_{\delta_1}(t) = -\log(C_{\delta_1} e^{-k_{\delta_1} t} + 1)$$
$$z_{\delta_i}(t) = -\log(C_{\delta_i} (\alpha_{\delta_i}(t))^{-k_{\delta_i}/k_{\delta_{i-1}}} + 1) \text{ (for } i \geq 2). \tag{11}$$

In the rest of this section, we show that Equation 11 give the correct solutions to the differential equations in Equation 3.

Lemma G.1. *For $i \in \{2, 3, \ldots, \ell\}$,*

$$\frac{\frac{d}{dt}(\alpha_{\delta_i}(t))}{k_{\delta_{i-1}} \alpha_{\delta_i}(t)} = \prod_{j=1}^{i-1} e^{z_{\delta_j}(t)}. \tag{12}$$

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Proof. We will prove the lemma by induction. Consider the base case of $i = 2$:

$$\frac{\frac{d}{dt}\left(\alpha_{\delta_2}(t)\right)}{k_{\delta_1}\alpha_{\delta_2}(t)} = \frac{k_{\delta_1}e^{k_{\delta_1}t}}{k_{\delta_1}\left(C_{\delta_1} + e^{k_{\delta_1}t}\right)}$$

$$= \frac{1}{C_{\delta_1}e^{-k_{\delta_1}t} + 1}$$

$$= e^{z_{\delta_1}(t)}.$$

Now consider the inductive case of $i > 2$ under the assumption that $\frac{\frac{d}{dt}\left(\alpha_{\delta_{i-1}}\right)}{k_{\delta_{i-2}}\alpha_{\delta_{i-1}}} = \prod_{j=1}^{i-2} e^{z_{\delta_j}}$:

$$\frac{\frac{d}{dt}\left(\alpha_{\delta_i}(t)\right)}{k_{\delta_{i-1}}\alpha_{\delta_i}(t)} = \frac{\frac{k_{\delta_{i-1}}}{k_{\delta_{i-2}}}\alpha_{\delta_{i-1}}^{(k_{\delta_{i-1}}/k_{\delta_{i-2}})-1}\frac{d}{dt}\left(\alpha_{\delta_{i-1}}\right)}{k_{\delta_{i-1}}\left(\alpha_{\delta_{i-1}}^{k_{\delta_{i-1}}/k_{\delta_{i-2}}} + C_{\delta_{i-1}}\right)}$$

$$= \frac{\alpha_{\delta_{i-1}}^{k_{\delta_{i-1}}/k_{\delta_{i-2}}}}{\alpha_{\delta_{i-1}}^{k_{\delta_{i-1}}/k_{\delta_{i-2}}} + C_{\delta_{i-1}}} \cdot \frac{\frac{d}{dt}\left(\alpha_{\delta_{i-1}}\right)}{k_{\delta_{i-2}}\alpha_{\delta_{i-1}}}$$

$$= \frac{1}{C_{\delta_{i-1}}\alpha_{\delta_{i-1}}^{-k_{\delta_{i-1}}/k_{\delta_{i-2}}} + 1} \cdot \prod_{j=1}^{i-2} e^{z_{\delta_j}(t)}$$

$$= \prod_{j=1}^{i-1} e^{z_{\delta_j}(t)}.$$

This completes the proof. $\qquad\square$

Lemma G.2. *The expressions for $z_{\delta_i}(t)$ in Equation 11 give a solution to system of differential equations in Equation 3 with initial conditions $z_{\delta_i}(0) = -k_{\delta_1}f_{\delta_i}$.*

Proof. We will split the proof into two cases for δ_1 and for δ_i with $i \geq 2$. Starting with $i = 1$, recall that

$$z_{\delta_1}(t) = -\log(C_{\delta_1}e^{-k_{\delta_1}t} + 1).$$

First, we will show that this function has the correct derivative.

$$\frac{dz_{\delta_1}(t)}{dt} = -\frac{1}{C_{\delta_1}e^{-k_{\delta_1}t} + 1}(C_{\delta_1})(-k_{\delta_1})e^{-k_{\delta_1}t}$$

$$= k_{\delta_1}\frac{C_{\delta_1}e^{-k_{\delta_1}t}}{C_{\delta_1}e^{-k_{\delta_1}t} + 1}$$

$$= k_{\delta_1}\left(1 - e^{z_{\delta_1}(t)}\right)$$

It remains to be shown that $z_{\delta_1}(0) = -k_{\delta_1}f_{\delta_1}$:

$$z_{\delta_1}(0) = -\log(C_{\delta_1} + 1)$$

$$= -\log(e^{k_{\delta_1}f_{\delta_1}})$$

$$= -k_{\delta_1}f_{\delta_1}.$$

Now, consider the case where $i \geq 2$. Then,

$$\frac{dz_{\delta_i}(t)}{dt} = \frac{d}{dt}\left(-\log(C_{\delta_i}(\alpha_{\delta_i}(t))^{-k_{\delta_i}/k_{\delta_{i-1}}} + 1)\right)$$

$$= -\frac{C_{\delta_i}\frac{-k_{\delta_i}}{k_{\delta_{i-1}}}(\alpha_{\delta_i}(t))^{-k_{\delta_i}/k_{\delta_{i-1}}-1}}{C_{\delta_i}(\alpha_{\delta_i}(t))^{-k_{\delta_i}/k_{\delta_{i-1}}} + 1}\frac{d}{dt}\left(\alpha_{\delta_i}(t)\right)$$

$$= k_{\delta_i}\frac{C_{\delta_i}(\alpha_{\delta_i}(t))^{-k_{\delta_i}/k_{\delta_{i-1}}}}{C_{\delta_i}(\alpha_{\delta_i}(t))^{-k_{\delta_i}/k_{\delta_{i-1}}} + 1}\frac{\frac{d}{dt}\left(\alpha_{\delta_i}(t)\right)}{k_{\delta_{i-1}}\alpha_{\delta_i}(t)}$$

$$= k_{\delta_i}\left(1 - e^{z_{\delta_i}(t)}\right)\prod_{j=1}^{i-1} e^{z_{\delta_j}(t)}.$$

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The last step makes use of Lemma G.1. Finally, we must show that $z_{\delta_i}(0) = -k_{\delta_i} f_{\delta_i}$:

$$z_{\delta_i}(0) = -\log(C_{\delta_i}\alpha_{\delta_i}(0)^{-k_{\delta_i}/k_{\delta_{i-1}}} + 1)$$
$$= -\log(e^{k_{\delta_i} f_{\delta_i}} - 1 + 1)$$
$$= -k_{\delta_i} f_{\delta_i}.$$

Thus, the given solution to the system of differential equations is correct. □

H   Proof of Theorem 6.1

We give the proof of Theorem 6.1 which states that the solution to the differential equations models the size of the matching returned by MinPredictedDegree.

Proof of Theorem 6.1. The proof of Theorem 6.1 follows a direct application of Theorem 1 in Luby et al. [38]. We must show three conditions are satisfied: (i) that $|Z_d^{t+1} - Z_d^t|$ is bounded, (ii) that $\frac{dz_d(t)}{dt} = \mathbb{E}[Z_d^{t+1} - Z_d^t | H_t]$ (where H_t is the history up to time t, and (iii) that $\frac{dz_d(t)}{dt}$ satisfies a Lipschitz condition when $z_d(t) \le 0$ (recall that Z_d^t is always nonpositive as Y_d^t is always nonnegative).

If these conditions hold, then the solution to the system of differential equations gives the asymptotic expected behavior of the variables Z_d^t and thus the asymptotic expected behavior of the variables Y_d^t, which govern the size of the matching returned by MinPredictedDegree.

It remains to show that these three conditions are met. Condition (i) is satisfied as the number of nodes of a given expected degree can change by at most one per timestep, so $|Z_d^{t+1} - Z_d^t| \le k_d$. Condition (ii) is satisfied by construction in Equation 3. Finally, Condition (iii) is satisfied as $\frac{dz_d(t)}{dt}$ is comprised of a product of several terms resembling $C_1 \cdot e^{-C_2 x}$ for nonnegative constants C_1, C_2 and with x nonnegative. Therefore, $\frac{dz_d(t)}{dt}$ has constant bounded first derivatives when $z_d(t) \le 0$. □

I   Upper Bound on Expected Maximum Matching Size

I.1   Overview

To analyze the maximum matching size within this model, we rely on a upper bound based on the matching version of Hall's marriage theorem [23]. We first state the classic theorem.

Theorem I.1 (Hall's Theorem). *Let $G = (U \cup V, E)$ be a bipartite graph. For any subset of nodes S, let $N(S)$ be the set of neighbors of the nodes in X. Then, G has a perfect matching if and only if for all $S \subset U$ and $T \subset V$, $|S| \le |N(S)|$ and $|T| \le |N(T)|$.*

Intuitively, if there is any subset S with few neighbors, then only $|N(S)|$ of the members of S can possibly be matched. Let $\mu(G)$ correspond to the size of the maximum matching in G. For any bipartite graph $G = (U \cup V, E)$ and $S \subset U$,

$$\mu(G) \le n - (|S| - |N(S)|) \tag{13}$$

as out of all of the nodes in S, only $|N(S)|$ can be matched. Therefore, if we can calculate the expected size of $|S|$ and $|N(S)|$ for some subset of a random symmetric CLV-B graph G, we immediately get an upper bound on the expected size of the maximum matching in G.

Our upper bound involves constructing a specific subset S^* of the offline nodes that makes use of our focus on power law graphs to provide a useful bound that is easy to evaluate. We empirically test how good of a bound Equation 13 gives using S^* and find that for symmetric CLV-B random graphs with offline degrees following a power law distribution, the upper bound on $\mu(G)$ given by $n - (|S^*| - |N(S^*)|)$ is close to maximum matching size (often achieving the same value and in all trials was less than 2% greater than the true value). In addition to providing a good bound, we show that we can evaluate $\mathbb{E}[|S|]$ and $\mathbb{E}[|N(S)|]$ in Equations 14, 15, 16. By linearity of expectation, this directly gives us an upper bound on the expected size of the maximum matching.

In Appendix M, we show that on symmetric CLV-B random graphs with power law distributed degrees, our bound on the maximum matching size is concentrated about its expectation. Combined with Theorem L.1, this implies that our analytic results on the ratio of the expected sizes in Section 6 are closely related to the competitive ratio.

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I.2 Construction

Let S^* be the subset of U constructed as follows. Let U_1 be the set of degree 1 nodes in U (here degree 1 referring to the actual degree of the node rather than the expected degree in the CLV-B model). Then, S^* is the maximal set of nodes in U s.t. $N(S^*) \subseteq N(U_1)$. In other words, S^* is the maximal subset of nodes in U whose neighbors completely overlap with the neighbors of the degree 1 nodes of U.

As $n - (|S^*| - |N(S^*)|)$ gives an upper bound on the maximum matching size, $\mathbb{E}[n - (|S^*| - |N(S^*)|)]$ gives an upper bound on the expected maximum matching size. The expected sizes of S^* and $N(S^*)$ in a CLV-B random graph with expected degrees $\mathbf{d}$ are given by the following equations. The expected size of $N(S^*)$ is simply the sum over all online nodes $v \in V$ of the probability that v has at least one degree 1 neighbor. The expected size of S^* is broken down as the sum over all offline nodes $u \in U$ of the probability that u has actual degree Δ and then the probability that all Δ of u's neighbors are members of $N(S^*)$. Let S_Δ^* be the subset of nodes in S^* whose actual (as opposed to expected) degree are Δ and let β_i^Δ for $\Delta \in \{0, ..., m\}$ and $i \in \{1, ..., n\}$ be defined as

$$\beta_i^0 = \beta_i^1 = 1$$

$$\beta_i^\Delta = 1 + \sum_{r=1}^{\Delta} (-1)^r \binom{\Delta}{r} \prod_{i' \neq i} \left[1 - r \left(\frac{d_{i'}}{m} \right) \left(1 - \frac{d_{i'}}{m} \right)^{m-1} \right]$$

(for $\Delta \geq 2$).

β_i^Δ represents the probability of $u_i \in S$ conditioned on u_i having actual degree Δ. Then,

$$\mathbb{E}[|N(S^*)|] = m \left(1 - \prod_{i=1}^{n} \left[1 - \frac{d_i}{m} \left(1 - \frac{d_i}{m} \right)^{m-1} \right] \right) \tag{14}$$

and

$$\mathbb{E}[|S^*|] = \sum_{\Delta=0}^{m} \mathbb{E}[|S_\Delta^*|] \tag{15}$$

where

$$\mathbb{E}[|S_\Delta^*|] = \sum_{i=1}^{n} \binom{m}{\Delta} \left(\frac{d_i}{m} \right)^\Delta \left(1 - \frac{d_i}{m} \right)^{m-\Delta} \beta_i^\Delta. \tag{16}$$

In the rest of this section, we show that the equations for the expected size of $|S^*|$ and $|N(S^*)|$ are correct by showing their derivations.

First, consider $\mathbb{E}[|N(S^*)|]$. Recall that $N(S^*)$ is the set of online nodes that have a neighbor with actual degree 1. Therefore, the expected size of $N(S^*)$ is m minus the expected number of online nodes that have no degree one neighbors. For any online node $v \in V$, the probability that v has no degree 1 neighbors is

$$\prod_{u \in U} \mathbb{P}(u \text{ is not a deg 1 nbr of } v)$$

$$= \prod_{u \in U} [1 - \mathbb{P}(u \text{ nbr of } v) \, \mathbb{P}(u \text{ has no other nbrs})]$$

$$= \prod_{u \in U} \left[1 - \frac{d_u}{m} \left(1 - \frac{d_u}{m} \right)^{m-1} \right].$$

By linearity of expectation,

$$\mathbb{E}[|N(S)|] = m \left(1 - \prod_{u \in U} \left[1 - \frac{d_u}{m} \left(1 - \frac{d_u}{m} \right)^{m-1} \right] \right).$$

Now, we will deal with $\mathbb{E}[|S_\Delta^*|]$. Recall that S_Δ^* is the set of offline nodes with actual degree Δ with all of their online neighbors having at least one offline neighbor with actual degree 1. For a given

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offline node $u \in U$ with expected degree d_u, the probability of u being in S^*_Δ is the product of the probability of u having actual degree Δ and the conditional probability of all of u's neighbors having a degree 1 neighbor given u having actual degree Δ. We will call the first event $A_{u,\Delta}$ and the second, conditional event $B_{u,\Delta}|A_{u,\Delta}$. The probability of $A_{u,\Delta}$ occurring corresponds to a Binomial random variable with size parameter m and probability parameter d_u/m taking on value Δ:

$$\mathbb{P}(A_{u,\Delta}) = \binom{m}{\Delta} \left(\frac{d_u}{m}\right)^\Delta \left(1 - \frac{d_u}{m}\right)^{m-\Delta}.$$

The probability of $B_{u,\Delta}|A_{u,\Delta}$ equals 1 if $\Delta = 0$ or $\Delta = 1$ as either u has no neighbors or u is itself a degree 1 neighbor of its neighbors, respectively. If $\Delta \geq 2$, then $\mathbb{P}(B_{u,\Delta}|A_{u,\Delta})$ is equal to the complement of the event that at least one of u's neighbors has no degree 1 neighbor. Let $C_{u,\Delta,r}|A_{u,\Delta}$ be the event that any subset of r of u's Δ neighbors have no degree 1 neighbor given $A_{u,\Delta}$. $\mathbb{P}(C_{u,\Delta,r}|A_{u,\Delta})$ can be expressed as

$$\binom{\Delta}{r} \prod_{u' \neq u} \left[1 - r\left(\frac{d_{u'}}{m}\right)\left(1 - \frac{d_{u'}}{m}\right)^{m-1}\right]$$

where the term within the product represents the probability of no offline nodes (excluding u) being degree one neighbors of a specific set of r online nodes (similarly to when expressing $\mathbb{E}[|N(S^*)|]$ above). By the inclusion-exclusion rule,

$$\mathbb{P}(B_{u,\Delta}|A_{u,\Delta}) = 1 + \sum_{r=1}^{\Delta} (-1)^r \, \mathbb{P}(C_{u,\Delta,r}|A_{u,\Delta}),$$

thus completing the derivation of Equations 14, 15, and 16.

J Analysis on CLV-B random graphs in asymptotic case

In this section, we give slight modifications of the Equation 11 and Equations 14, 15, 16 in the case where $n, m \to \infty$ to allow us to evaluate the equations to produce the results in Table 1. The model will change slightly when considering the asymptotic case: we will describe the set of offline expected degrees $\mathbf{d}$ by a set of unique degrees $\{\delta_i\}_{i=1}^\ell$ and corresponding fractions $\{\lambda_i\}_{i=1}^\ell$ where a λ_i fraction of the offline nodes have expected degree δ_i.

Importantly for the asymptotic results in Table 1, while there are offline nodes with expected degree approaching infinity, a finite number of unique expected degrees account for all but an exponentially small fraction of the offline nodes, allowing us to evaluate the equations up to negligible error. In the following calculations, we will consider both ℓ as well as all δ_i for $i = \{1, ..., \ell\}$ to be finite.

J.1 Asymptotic analysis of MinPredictedDegree

To start, we will replace f_d with $m \cdot \lambda_d$ and we will replace t with $\tau = t/m$. Recall $k_d = -\log(1 - d/m)$. The Taylor expansion of $\log(1 - x)$ at $x = 0$ is $-\sum_{n=1}^\infty \frac{x^n}{n}$. Within Equation 11, k_d appears in terms $k_d/k_{d'}$ and $k_d \cdot f_d = k_d \cdot m \cdot \lambda_d$. In the asymptotic case, we will use the following substitutions for those terms:

$$\lim_{m \to \infty} k_d/k_{d'} = d/d'.$$

and

$$\lim_{m \to \infty} k_d \cdot m \cdot \lambda_d = -d \cdot \lambda_d.$$

In both cases, we use the fact that as $m \to \infty$, the first term in the Taylor series (d/m) dominates.

Using these substitutions, we can rewrite the equations for MinPredictedDegree as follows.

$$\alpha_{\delta_2}(\tau) = C_{\delta_1} + e^{\delta_1 \tau}$$
$$\alpha_{\delta_i}(\tau) = (\alpha_{\delta_{i-1}}(\tau))^{\delta_{i-1}/\delta_{i-2}} + C_{\delta_{i-1}} \text{ (for } i \geq 3)$$

where

$$C_{\delta_1} = e^{\delta_1 \lambda_{\delta_1}} - 1$$
$$C_{\delta_i} = (\alpha_{\delta_i}(0))^{\delta_i/\delta_{i-1}}(e^{\delta_i \lambda_{\delta_i}} - 1) \text{ (for } i \geq 2).$$

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Then,

$$z_{\delta_1}(\tau) = -\log(C_{\delta_1} e^{-\delta_1 \tau} + 1)$$
$$z_{\delta_i}(\tau) = -\log(C_{\delta_i}(\alpha_{\delta_i}(\tau))^{-\delta_i/\delta_{i-1}} + 1) \text{ (for } i \geq 2). \tag{17}$$

Recall that the expected number of offline nodes with expected degree d is given by $-z_d(\tau)/k_d$ evaluated when $t = m$. Then, in the asymptotic case, the expected fraction of offline nodes matched is

$$\sum_{i=1}^{\ell} \lambda_i + z_{\delta_i}(1)/\delta_i \tag{18}$$

where $z_{\delta_i}(\tau)$ are given by Equation 17.

J.2 Asymptotic analysis of maximum matching

For the equations for the upper bound on the expected maximum matching size, the key fact we will use is $\lim_{x \to 0}(1 + x) = e^x$. Therefore, we can replace all terms of $(1 - \frac{d}{m})^{m-C}$ with e^{-d}. In addition, we can replace all terms of $\binom{m}{C}\left(\frac{d}{m}\right)^C$ with $\frac{d^C}{C!}$. These substitutions give the following equations.

$$\beta^0 = \beta^1 = 1$$

$$\beta^\Delta = 1 - \sum_{r=1}^{\Delta}(-1)^r \binom{\Delta}{r} \prod_{i=1}^{\ell} \left[1 - r\left(\frac{\delta_i}{m}\right)e^{-\delta_i}\right]^{m*\lambda_i}$$

$$= 1 - \sum_{r=1}^{\Delta}(-1)^r \binom{\Delta}{r} \prod_{i=1}^{\ell} e^{-r\delta_i \lambda_i e^{-\delta_i}} (\text{ for } \Delta \geq 2).$$

Note that the β^Δ terms are no longer indexed by i as conditioning on the actual degree of a single node makes no difference on the probability in the asymptotic case. Then,

$$\mathbb{E}\left[\frac{|N(S^*)|}{m}\right] = \left(1 - \prod_{i=1}^{\ell} e^{-\delta_i \lambda_i e^{-\delta_i}}\right) \tag{19}$$

and

$$\mathbb{E}\left[\frac{|S^*|}{m}\right] \geq \sum_{\Delta=0}^{C} \mathbb{E}\left[\frac{|S^*_\Delta|}{m}\right] \tag{20}$$

where

$$\mathbb{E}\left[\frac{|S^*_\Delta|}{m}\right] = \sum_{i=1}^{\ell} \lambda_i \frac{(\delta_i)^\Delta}{\Delta!} e^{-\delta_i} \beta_i^\Delta. \tag{21}$$

K Analysis of MPD for Erdős-Rényi Random Bipartite Graphs

We here analyze the performance of MPD on CLV-B instances, $I_{\mathbf{p},\mathbf{1}}$, where $\mathbf{p} = (p, \ldots, p) \in [0,1]^n$ and $\mathbf{1} = (1, \ldots, 1) \in [0,1]^m$. Such an instance is an Erdős-Rényi bipartite random graph where all edges appear with the same probability p. In particular, the expected degrees of the offline vertices are the same and equal to $d := mp$. Note that in this case, MPD is equivalent to any other greedy algorithm. Letting $c = m/n$, we show that for a wide range of the parameters m, n, and p, the expected fraction of matched offline vertices is $1 + c - \frac{c \ln(e^d + e^{d/c} - 1)}{d}$ up to a small additive error.

Combining this bound, with the asymptotic upper bound on the maximum matching of Appendix J, we obtain that for these graphs, the asymptotic competitive ratio of MPD is at least 0.831 which is significantly better than the 0.7299 bound from [11]. We conjecture that Erdős-Rényi random bipartite graphs are in fact worst-case instances for MPD, in the sense that a lower bound on the competitive ratio of MPD for Erdős-Rényi random graphs, also holds for general CLV-B random graphs.

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Theorem K.1. *Let $p \in [0, 1]$ and $n, m \in \mathbb{N}$. Assume that[4] $n^{1-o(1)} \leq m \leq n^{1+o(1)}$, $p = o(\log n)/m$, and $p \geq 1/n^{1+o(1)}$. Let $\boldsymbol{p} = (p, \ldots, p) \in [0, 1]^n$, and $\boldsymbol{q} = (1, \ldots, 1) \in [0, 1]^m$. Let M be the size of the matching output by any greedy algorithm $\mathcal{A}$ on input $I_{\boldsymbol{p}, \boldsymbol{q}}$. Let $c = m/n$. Then*

$$\frac{\mathbb{E}[M]}{n} = 1 + c - \frac{c \ln(e^d + e^{d/c} - 1)}{d} \pm n^{-1/2+o(1)}.$$

Moreover, $|M - \mathbb{E}[M]| = O(\sqrt{m \log n})$ with high probability in n.

Proof. Let $\mathcal{F}_i$ denote the σ-algebra generated by the neighborhoods of the first i arriving online nodes. Defining $X_i = \mathbb{E}[M \mid \mathcal{F}_i]$, we have that $(X_i)_{i=0}^m$ is a martingale with $X_0 = \mathbb{E}[M]$ and $X_n = M$. It is easy to check that $|X_i - X_{i-1}| \leq 1$ for $1 \leq i \leq m$, so it follows from Azuma's inequality that for any $t > 0$,

$$\mathbb{P}[|M - \mathbb{E}[M]| \geq t] \leq \exp\left(\frac{-t^2}{2m}\right)$$

In particular, $|M - \mathbb{E}[M]| = O(\sqrt{m \log n})$ with high probability, say at least $1 - n^{-10}$ as claimed in the theorem. Moreover, the assumption $pm = o(\log n)$ implies that $\mathbb{E}[M] \leq n(1 - n^{-o(1)})$. This is clear when $m \leq n/2$. On the other hand when $m > n/2$, the probability that any given offline node is never picked by an online node is $(1-p)^m = e^{-o(\log n)} = n^{-o(1)}$, using the assumption $pm = o(\log n)$.

For $0 \leq j \leq n - 1$, we let T_j denote the number of online vertices v such that when v arrives, the matching found by $\mathcal{A}$ so far has size j. Then T_j is geometrically distributed with parameter $1 - (1-p)^{n-j}$, and so, $\mathbb{E}[T_j] = \frac{1}{1-(1-p)^{n-j}}$. Moreover, $T_j = O(\frac{\log n}{1-(1-p)^{n-j}})$ with high probability.

Define $n^* = \lfloor \mathbb{E}[M] \rfloor$. Let $L_1 = n(1 - n^{-o(1)})$ be such that $\max(M, \mathbb{E}[M]) \leq L_1$ with probability at least $1 - n^{-10}$ and let A_1 be the event that $\max(M, \mathbb{E}[M]) > L_1$. Note that when $j \leq L_1$, then

$$\mathbb{E}[T_j] \leq \frac{1}{1 - (1-p)^{n^{1-o(1)}}} \leq \frac{1}{1 - \exp(-pn^{1-o(1)})} = \frac{1}{1 - \exp(-n^{-o(1)})} = n^{o(1)}.$$

We can therefore pick $L_2 = n^{o(1)}$ such that $\max(T_1, \ldots, T_{L_1}) \leq L_2$ with probability at least $1 - n^{-10}$ and we let A_2 be the event that $\max(T_1, \ldots, T_{L_1}) > L_2$. Finally, let $L_3 = O(\sqrt{m \log n})$ be such that $|M - \mathbb{E}[M]| \leq L_3$ with probability at least $1 - n^{-10}$ and let A_3 be the event that $|M - \mathbb{E}[M]| > L_3$

If neither A_1, A_2, or A_3 occur, which happens with probability at least $1 - 3n^{-10}$, then

$$\left| m - \sum_{i \leq n^*} T_i \right| \leq \left| m - \sum_{i \leq M} T_i \right| + L_2 L_3 \leq L_2 + L_2 L_3 = n^{1/2+o(1)}.$$

From this, it particularly follows that

$$\left| m - \mathbb{E}\sum_{i \leq n^*} T_i \right| \leq n^{1/2+o(1)}. \tag{22}$$

Let n_0 be minimal such that $\sum_{j < n_0} \mathbb{E}[T_j] \geq m$. Since $T_i \geq 1$ for every i, it follows from (22) that $|n_0 - n^*| \leq n^{1/2+o(1)}$, and in particular that $|n_0 - \mathbb{E}[M]| \leq n^{1/2+o(1)}$. To finish the proof, it therefore suffices to show that n_0/n satisfies the bound in the theorem.

For this, we first note that

$$m \leq \sum_{j < n_0} \mathbb{E}[T_j] = \sum_{j < n_0} \frac{1}{1 - (1-p)^{n-j}} \leq \sum_{j < n_0} \frac{1}{1 - e^{-p(n-j)}} \leq \int_0^{n_0} \frac{1}{1 - e^{-p(n-x)}} \, dx.$$

$$= n_0 - \frac{1}{p}\left(\ln(1 - e^{p(n_0-n)}) - \ln(1 - e^{-pn})\right)$$

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[4]Here, $o(1) \to 0$ as $n \to \infty$

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The right hand side is an increasing function of n_0 vanishing at 0 and turning to infinity as $n_0 \to n$. Solving for n_0 then gives that

$$n_0 \geq n + m - \frac{\ln(e^{pm} + e^{pn} - 1)}{p},$$

so that

$$n_0/n \geq 1 + m/n - \frac{\ln(e^{pm} + e^{pn} - 1)}{pn} = 1 + c - \frac{c\ln(e^d + e^{d/c} - 1)}{d}$$

which yields the lower bound in the proof of the theorem upon dividing by n.

For the upper bound, we use the inequality $(1 + \frac{x}{n})^n \geq e^x(1 - \frac{x^2}{n})$ holding for $n \geq 1$ and $|x| \leq n$, from which it follows that

$$\mathbb{E}[T_j] \geq \frac{1}{1 - e^{-p(n-j)}(1 - (n-j)p^2)} \geq \frac{1}{1 - e^{-p(n-j)}a},$$

where we have put $a = 1 - np^2$. Note that by the definition of n_0, $\sum_{j < n_0 - 1} \mathbb{E}[T_j] < m$, and so it follows similarly to above that

$$m_0 := m + \mathbb{E}[T_{n_0-1} + T_{n_0}] \geq \sum_{j \leq n_0} \mathbb{E}[T_j] \geq \int_0^{n_0} \frac{1}{1 - e^{-p(n-x)}a}\, dx$$

$$\geq n_0 - \frac{1}{p}\left(\ln(1 - ae^{p(n_0-n)}) - \ln(1 - ae^{-pn})\right).$$

The right hand side is again an increasing function of n_0, so solving for n_0 gives that

$$n_0 \leq m_0 + n - \frac{\ln(e^{pn} + ae^{pm} - a)}{p} = m_0 + n - \frac{\ln(e^{pn} + e^{pm} - 1 - p^2n(e^{pm} - 1))}{p}$$

$$= m_0 + n - \frac{\ln(e^{pn} + e^{pm} - 1)}{p} + O(pne^{pm}) = m + n - \frac{\ln(e^{pn} + e^{pm} - 1)}{p} + 2n^{o(1)}.$$

The desired result follows after dividing by n. $\qquad\square$

Competitive ratio of MPD with uniform expected degree sequence. We now demonstrate how to combine the bound of Theorem K.1 with the upper bounds on the maximum matching of Appendix J, to obtain the better bound of 0.831 on the asymptotic competitive ratio of MPD for Erdős-Rényi bipartite random graph. We start with the following lemma that allows us to focus on the case where both $c = m/n$ and $d = pm$ are constants, $d, c = \Theta(1)$.

Lemma K.2. *There exists an $\varepsilon > 0$, so that if $(c, d) \in [\varepsilon, 1/\varepsilon]^2$, and n is sufficiently large, then the competitive ratio of MPD on instance $I_{p,q}$ is at least 0.99. Here, $p = (p, \ldots, p) \in [0, 1]^n$, and $q = (1, \ldots, 1) \in [0, 1]^m$.*

Proof (sketch). Let S_1 denote the set of non-isolated offline nodes, S_2 the set of non-isolated online nodes, M^* the maximum matching, and M the matching found by MPD. First of all, it is easy to check that there exists $\varepsilon_1 > 0$, so that if if $c < \varepsilon_1$ and n is large enough, then both $|1 - \frac{\mathbb{E}[|M^*|]}{\mathbb{E}[|S_2|]}| < \frac{1}{1000}$ and $|1 - \frac{\mathbb{E}[|M|]}{\mathbb{E}[|S_2|]}| < \frac{1}{1000}$. The first bound follows from observing that for c small, nearly every non-isolated online node can be matched: Even if, we just consider a single random edge leaving each of the non-isolated online nodes, the number of pair of such edges that are both incident to the same offline node is $O(|S_2|^2/n) = O(c|S_2|)$, and thus the matching has size at least $|S_2|(1 - O(c))$. The second bound follows by observing that for each arriving non-isolated online node, MPD will match it with probability at least $1 - m/n = 1 - c$.

Similarly to above, there exists $\varepsilon_2 > 0$, such that if $c > 1/\varepsilon_2$, and n is large enough, then $|1 - \frac{\mathbb{E}[|M^*|]}{\mathbb{E}[|S_1|]}| < \frac{1}{1000}$. Moreover, we can choose ε_2 such that if $c > 1/\varepsilon_2$, and n is large enough, it also holds that $|1 - \frac{\mathbb{E}[|M|]}{\mathbb{E}[|S_1|]}| < \frac{1}{1000}$. Indeed, if $d \leq 10$, and ε_2 is small enough, then an $1 - 10/\varepsilon_2$ fraction of the nodes in S_1, will have a degree one neighbor, and therefore be matched, and the case $d \geq 10$ reduces to the case $d = 10$.

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Combining the above bounds it follows from some calculations that if we choose $\varepsilon_3 = min(\varepsilon_1, \varepsilon_2)$, then if $c \notin [\varepsilon_3, 1/\varepsilon_3]$, and n is large enough, the competitive ratio of MPD is at least 0.99.

We next assume that $c \in [\varepsilon_3, 1/\varepsilon_3]$. Using this assumption on c, we can then choose $\varepsilon_4 > 0$ such that if $d < \varepsilon_4$, then 99% of the edges (u, v) of the instance, will satisfy that the vertices u and v have degree 1. As MPD includes all those edges, and the total number of edges of the graph is an upper bound on $|M^*|$, it follows that in this case, the competitive ratio is at least 0.99. Finally, it is easy to check by splitting into the cases $m \leq n$ and $m > n$ that we can choose $\varepsilon_5 > 0$, such that if $d \geq 1/\varepsilon_5$, then $|1 - \frac{\mathbb{E}[|M|]}{\min(n,m)}| < \frac{1}{1000}$. As $|M^*| \leq \min(m, n)$, this gives that the competitive ratio in this case is at least 0.99. Setting $\varepsilon = \min(\varepsilon_3, \varepsilon_4, \varepsilon_5)$, we obtain the desired result. $\qquad\square$

Next, we show how to use the techniques of Appendix J to obtain an upper bound on the maximum matching size in the case of Erdős-Rényi random bipartite graphs in the case that $d, c = \Theta(1)$.

Lemma K.3. *Let c, d be given with $c, d = \Theta(1)$. Define*

$$A(c, d) = 1 - e^{-cde^{-d}}, \quad and \quad B(c, d) = e^{-d}(e^{-dA(c,d)} + d(1 - A(c, d)))$$

For instances $I_{p,q}$, where $p = (p, \ldots, p) \in [0, 1]^n$, and $q = (1, \ldots, 1) \in [0, 1]^m$, the expected fraction of matched offline vertices is at least $(1 \pm o(1))C(c, d)$ as $n \to \infty$, where $C(c, d) = 1 + cA(c, d) - B(c, d)$.

Proof. Let $N(S)$ denote the set of nodes of V with at least one degree one neighbor in U, and let S denote the nodes of U whose neighborhood is fully contained in $N(S)$. The suggestive notation is justified as $N(S)$ is indeed the neighborhood of S. As we saw in the Appendix J, for $v \in V$,

$$\mathbb{P}[v \in N(S)] = 1 - \left(1 - \frac{d}{n}\left(1 - \frac{d}{n}\right)^{m-1}\right)^n = (1 \pm o(1))\left(1 - e^{-cde^{-d}}\right) = (1 \pm o(1))A(c, d),$$

where we have used the approximation e^x for $1 + x$ which is sharp enough to get the bound above, since $d = \Theta(1)$ and $m = \Theta(n)$ with the assumptions on c and d. Next for a fixed node $u \in U$, we let A_k denote the event that u has degree k. Then for any Δ,

$$\mathbb{P}[u \in S] \geq \sum_{k=0}^{\Delta} \mathbb{P}[u \in S \cap A_k].$$

Assuming that $\Delta = O(1)$ and $2 \leq k \leq \Delta$, we can approximate

$$\mathbb{P}[u \in S \cap A_k] = (1 \pm o(1))\binom{m}{k}\left(\frac{d}{m}\right)^k\left(1 - \frac{d}{m}\right)^{m-k}A(c, d)^k = (1 \pm o(1))e^{-d}\frac{(A(c, d)d)^k}{k!}.$$

Moreover, $\mathbb{P}[u \in S \cap A_0] = \mathbb{P}[A_0]$ and $\mathbb{P}[u \in S \cap A_1] = \mathbb{P}[A_1]$, so we can bound

$$\mathbb{P}[u \in S \cap A_0] = (1 \pm o(1))e^{-d} \quad and \quad \mathbb{P}[u \in S \cap A_1] = (1 \pm o(1))de^{-d}.$$

Now for any constant Δ, we can upper bound the expected fraction of matched offline vertices by

$$1 - \mathbb{P}[u \in S] + c\mathbb{P}[v \in N(s)] \leq 1 - \left(\sum_{k=0}^{\Delta}\mathbb{P}[u \in S \cap A_k]\right) + c\mathbb{P}[v \in N(s)]$$

$$= 1 - (1 \pm o(1))e^{-d}\left(1 + d + \sum_{k=2}^{\Delta}\frac{(A(c, d)d)^k}{k!}\right) + (1 \pm o(1))cA(c, d)$$

Since this bound holds for any constant Δ, and the series converges, it also holds in the limit. Thus, we can conclude that as $n \to \infty$, the expected fraction of matched offline nodes is at most

$$(1 \pm o(1))\left(1 - e^{-d}(e^{A(c,d)d} + d(1 - A(c, d))) + cA(c, d)\right) = (1 \pm o(1))(1 - B(c, d) + cA(c, d)) = (1 \pm o(1))C(c, d),$$

as desired. $\qquad\square$

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Remark When bounding the size of the maximum matching, with Lemma K.3, we can switch the roles of m and n. Letting $d' = pn$ and $c' = n/m$, it holds that $c' = 1/c$ and $d' = d/c$, and we can use Lemma K.3, to conclude that the expected fraction of matched *online* nodes is at most $(1 \pm o(1))C(1/c, d/c)$ as $n \to \infty$. In particular, we can upper bound the expected fraction of *offline* nodes in a maximum matching by $(1 \pm o(1))D(c, d)$, where

$$D(c, d) = \min(C(c, d), cC(1/c, d/c))$$

We next show that for Erdős-Rényi bipartite random graph, the asymptotic competitive ratio of MPD is at least 0.831.

Lemma K.4. *Let $I_{\boldsymbol{p},\boldsymbol{q}}$, where $\boldsymbol{p} = (p, \dots, p) \in [0, 1]^n$, and $\boldsymbol{q} = (1, \dots, 1) \in [0, 1]^m$ be an Erdős-Rényi bipartite random graph. The asymptotic competitive ratio of MPD on $I_{\boldsymbol{p},\boldsymbol{q}}$ as $n \to \infty$ is at least 0.831.*

Proof. By Lemma K.2, we can assume that $(c, d) \in [\varepsilon, 1/\varepsilon]^2$ for some small enough constant, as otherwise, the competitive ratio of MPD is at least 0.99. Then we are in position to apply Theorem K.1 and Lemma K.3 (in fact the remark following the lemma). Letting $E(c, d) = 1 + c - \frac{c \ln(e^d + e^{d/c} - 1)}{d}$, we conclude that the asymptotic competitive ratio of MPD is at most $\frac{E(c,d)}{D(c,d)}$. Using some calculus, it can be checked that in any region $[\varepsilon, 1/\varepsilon]^2$, where ε is sufficiently large, $\frac{E(c,d)}{D(c,d)}$ has a unique minimum attained at (c_0, d_0), where $c_0 = 1$ and $d_0 \approx 2.7997$. Moreover, $\frac{E(c_0,d_0)}{D(c_0,d_0)} \approx 0.83105 \geq 0.831$. $\qquad\square$

L Concentration of MinPredictedDegree

In this section, we prove that MinPredictedDegree's performance on CLV-B random graphs is concentrated about its expectation.

Theorem L.1. *Let G be a symmetric CLV-B random graph with expected offline degrees $\mathbf{d}$ and let X be the random variable corresponding to the size of the matching returned by MPD. Then,*

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq 2\sqrt{m} \log m) \leq \frac{2}{m}. \tag{23}$$

Proof. Let H_j represent the state of MinPredictedDegree after it has processed the jth online node, and let $Y_j = \mathbb{E}[X|H_j]$ be the expectation of the size of the returned matching conditioned on the history of the algorithm up to time j. Then $\{Y_j\}_{j=0}^m$ form a Doob martingale. We will proceed by bounding $|Y_j - Y_{j-1}|$.

If an offline node i was matched with online node j, then the conditional expectation of the final matching size increases by $1 - \mathbb{P}(i \text{ matched}|H_{j-1})$. For each unmatched offline node that is not matched with online node j, the conditional expectation decreases by the sum over all such nodes i' of $\mathbb{P}(i' \text{ matched with } j|H_{j-1})$. As both the increment and decrement are bounded in magnitude by 1, the martingale has bounded differences $|Y_j - Y_{j-1}| \leq 1$.

Applying the standard Azuma's inequality bounds, we get the concentration result:

$$\mathbb{P}(|Y_m - Y_0| \geq 2\sqrt{m} \log m) \leq \frac{2}{m}.$$

As $Y_0 = \mathbb{E}[X]$ and $Y_m = X$, this completes the proof. $\qquad\square$

M Concentration of the Upper Bound on Maximum Matching Size

In this section, we show that our upper bound on the size of a maximum matching in CLV-B random graphs with power law distributed degrees is concentrated about its expectation.

Theorem M.1. *Let G be a CLV-B random graph with $n = m$ and with expected offline degrees following a power law distribution with exponent $\alpha > 3$, and let X be the random variable corresponding to the difference $|S^*| - |N(S^*)|$ where S^* and $N(S^*)$ are the subsets of the nodes in G described in Section I. Then, there exists some constant C s.t.*

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq C\sqrt{n} \log n) \leq \frac{1}{n}. \tag{24}$$

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Proof of Theorem M.1. Let H_t represent the t offline nodes with the smallest degrees (ties broken arbitrarily) as well as their incident edges and let $Y_t = \mathbb{E}[(|S^*| - |N(S^*)|) \,|H_t]$ be the expected difference in the sizes of S^* and $N(S^*)$ given knowledge of H_t. Note that here we are using true degrees and *not* expected degrees. Then, $\{Y_t\}_{t=0}^n$ form a Doob martingale.

Let u_t be the offline node with the tth smallest degree and let $deg(u_t)$ be the its degree. We will proceed by cases to show that the martingale has bounded differences.

1. Assume $deg(u_t) > 1$. Then, from H_{t-1} we know $N(S^*)$ as all degree one nodes have already been seen. Therefore, the contribution of u_t to the difference $(|S^*| - |N(S^*)|)$ is independent of any subsequent offline nodes. Specifically, if $u_t \in S^*$, $Y_t - Y_{t-1} = 1 - \mathbb{P}(u_t \in S^*|H_{t-1})$, and if $u_t \notin S^*$, $Y_t - Y_{t-1} = -\mathbb{P}(u_t \in S^*|H_{t-1})$ where

$$\mathbb{P}(u_t \in S^*|H_{t-1}) = \mathop{\mathbb{E}}_{deg(u_t)|H_{t-1}} \left[\prod_{k=0}^{deg(u_t)-1} \left(\frac{N(S^*) - k}{m} \right) \right]. \tag{25}$$

   In any case, $|Y_t - Y_{t-1}| \le 1$.

2. Assume $deg(u_t) \le 1$. In this case, we have to deal with the fact that u_t can affect $N(S^*)$ as well as S^*. Part of the difference $Y_t - Y_{t-1}$ is due to the inclusion of u_t in S^* and the subsequent possibility that u_t contributes a node to $N(S^*)$ if $deg(u_t) = 1$ and its neighbor is not already in $N(S^*)$. This part of the difference is bounded in magnitude by one as these events change the difference $S^* - N(S^*)$ by at most one.

   The other part of the difference $Y_t - Y_{t-1}$ is due to whether u_t increments the size of $N(S^*)$ via its neighbor, affecting the probabilities $\mathbb{P}(u_{t'} \in S^*)$ for t' where $deg(u_{t'}) > 1$ as in Equation 25. As the expected size of $N(S^*)$ can change by at most one, the change in $\mathbb{P}(u_{t'} \in S^*)$ for each t' where $deg(u_{t'}) > 1$ is at most

$$\prod_{k=0}^{deg(u_{t'})-1} \left(\frac{t' - k}{m} \right) - \prod_{k=0}^{deg(u_{t'})-1} \left(\frac{t' - k - 1}{m} \right). \tag{26}$$

   The factors of t' in the numerators come from the fact that $N(S^*) \le t'$ if $deg(u_{t'}) > 1$. As both parts of the difference contain many of the same terms, we can simplify Expression 26 as

$$\frac{deg(u_{t'})}{m} \prod_{k=1}^{deg(u_{t'})-1} \left(\frac{t' - k}{m} \right) \le \frac{deg(u_{t'})}{m}. \tag{27}$$

   As we assume that the expected degrees are distributed according to a power law distribution with exponent $\alpha > 3$, the expectation and variance of the degree of a given node u will be constant. Thus, with high probability, the sum over all offline degrees $\sum_u deg(u) = O(m)$. The contribution to $Y_t - Y_{t-1}$ by Expression 27 is thus bounded by $\sum_u \frac{deg(u)}{m} = O(1)$. Overall, with high probability, $|Y_t - Y_{t-1}| = O(1)$.

As in both cases, the martingale has constant bounded differences (with high probability), Azuma's inequality directly gives us the theorem. $\square$

N   Additional Experiments

Figure 7 shows additional experiments on Real World graphs from the known i.i.d. model (based on the methodology of [9]). Overall, the results are very similar to those in Section 7, MinPredictedDegree does very well compared to the other online baselines (depicted in light blue) despite making relatively little use of the type graph information. Additionally, augmenting the known i.i.d. baselines with MinPredictedDegree (e.g. using the MinPredictedDegree rule when the base algorithm does not match the current online node even though it has unmatched neighbors) often improves the performance over the baseline algorithm and the greedy augmentation.

Figure 8 shows the degree distribution of the Oregon and Caida 2004 datasets as well as the ℓ_2 prediction error (square root of the sum of the squared error of the degree prediction for each offline node in the current graph) over time of using the first days degrees as a prediction for future degrees. As the prediction quality degrades, the performance of MinPredictedDegree slowly declines.

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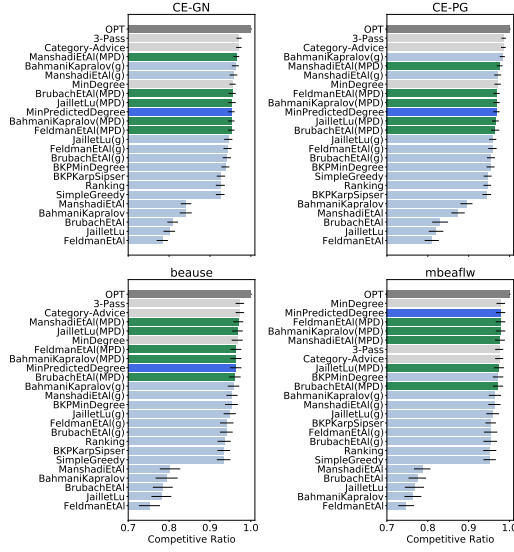
Figure 7: Additional comparison of empirical competitive ratios on Real World graphs. Algorithms depicted in gray are *not* online algorithms (they use extra information or multiple passes). Algorithms in green are augmented with MPD.
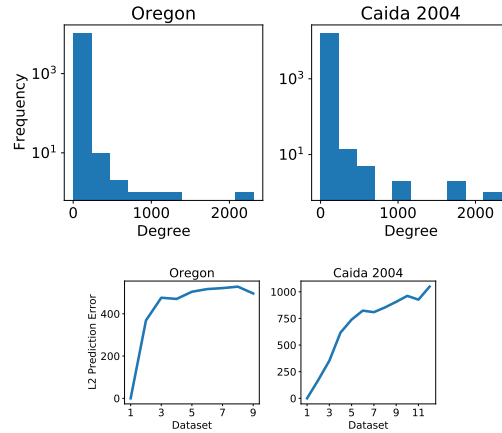
Figure 8: Degree distribution (left) and ℓ_2 prediction error over time (right) for the Oregon and Caida 2004 datasets.

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