A $\text{MenuGap}(X) = 1$ when $k = 1$

In this brief section we prove that when $k = 1$, for any sequence of $x_i \in \mathbb{R}^+_{\geq 0}$, $\text{MenuGap}(X) = 1$.

Claim 28. *When $k = 1$, for any $X = \{x_i\}_{i=1}^N$, $x_i \in \mathbb{R}^+_{\geq 0}$, $\text{MenuGap}(X) = 1$.*

Proof. Note that when $k = 1$, $||x_i||_1 = x_i$. Therefore, $\text{MenuGap}(X, Q) = \sum_i \min_{j<i}(q_i - q_j)$. We make the following observation which allows us to look at structured optimal solutions.

Observation 29. *Any optimal solution Q to $\text{MenuGap}(X)$ is monotone non-decreasing.*

Proof. For the sake of contradiction, suppose we are given an optimal solution Q that is not monotone non-decreasing. Let i be the smallest index for which $q_i < q_{i-1}$. Then $\text{gap}_i^{X,Q} = (q_i - q_{i-1})x_i < 0$. Consider instead a solution Q' where $q'_j = q_j$ for all $j \neq i$ and $q'_i = q_{i-1}$. Now, $\text{gap}_i^{X,Q'} = 0$. Since $Q_{\leq i-1} = Q'_{\leq i-1}$, $\text{gap}_j^{X,Q} = \text{gap}_j^{X,Q'}$ for all $j < i$. Since $q_{i-1} > q_i$, for any $j > i$ it holds that $(q_j - q_{i-1}) < (q_j - q_i)$. Therefore, q_i is not "setting the gap" for any point after it. Hence it also holds that $\text{gap}_j^{X,Q} = \text{gap}_j^{X,Q'}$ for all $j > i$. Putting everything together we get that $\text{MenuGap}(X, Q') - \text{MenuGap}(X, Q) = \text{gap}_i^{X,Q'} - \text{gap}_i^{X,Q} > 0$ contradicting the optimality of Q. ☐

With this observation in hand, since the q_i are monotone non-decreasing, without loss of generality it holds that $\text{gap}_i^{X,Q} = \min_{j<i} q_i - q_j = q_i - q_{i-1}$ ($q_{i-1} \geq q_j$ for all $j < i$). Therefore, we get $\text{MenuGap}(X, Q) = \sum_i q_i - q_{i-1} = q_N - q_0$. Since $q_0 = 0$ and $0 \leq q_N \leq 1$, we get that $\text{MenuGap}(X, Q) \leq 1$.

Finally, note that for any X, we can set $q_N = 1$ and $q_i = 0$ for all other i, proving that $\text{MenuGap}(X) \geq 1$.

☐

B Omitted Proofs

Proof of Lemma 8. We prove that for all X, C, $\text{AlignGap}(X, C) \leq \text{MenuGap}(X)$, which implies the lemma. For a given X, C, define:

- $\vec{q}_i := c_i \cdot \vec{x}_i$, if $\text{sgap}_i^{X,C} > 0$.
- $\vec{q}_i := \arg\max_{j<i}\{c_j \cdot \vec{x}_j\}$, if $\text{sgap}_i^{X,C} \leq 0$.

Observe first that each $\vec{q}_i \in [0, 1]^k$, as each $c_i\vec{x}_i \in [0, 1]^k$ (this follows because each component of $\vec{x}_i$ is at most $||\vec{x}_i||_\infty$, and each c_i is at most $1/||\vec{x}_i||_\infty$). Next, observe that if $\text{sgap}_i^{X,C} \leq 0$, then $\text{gap}_i^{X,Q} = 0$. This is by definition in bullet two above. Finally, observe that if $\text{sgap}_i^{X,C} > 0$, then $\text{gap}_i^{X,Q} \geq \text{sgap}_i^{X,C}$. This is because the set of $\{\vec{q}_j\}_{j<i}$ is a subset of $\{c_j\vec{x}_j\}_{j<i}$, and because $\vec{q}_i := c_i \cdot \vec{x}_i$ by bullet one. Therefore, $\text{gap}_i^{X,Q} \geq \max\{0, \text{sgap}_i^{X,C}\}$ for all i and the lemma follows. ☐

Proof of Claim 13. Take M' to be exactly the same as M, except having removed all entries with price $< c$. For every value in the support of $\mathcal{D}$ with $p^M(\vec{v}) \geq c$ in M, we still have $p^{M'}(\vec{v}) \geq c$. This is simply $\vec{v}$'s favorite option in M is still available in M', and all options in M' were also available in M. For any value with $p^M(\vec{v}) < c$, we clearly have $p^{M'}(\vec{v}) \geq 0$. So for all $\vec{v}$, we have $p^{M'}(\vec{v}) \geq p^M(\vec{v}) - c$, and the claim follows by taking an expectation with respect to $\vec{v}$. ☐

Proof of Claim 15. Simply let M_1 denote the set of menu options from M whose price lies in $[c \cdot 2^i, c \cdot 2^{i+1})$ for an odd integer i, and M_2 denote the remaining menu options (which lie in $[c \cdot 2^i, c \cdot 2^{i+1})$ for an even power of i). It is easy to see that M_1 is oddly-priced and M_2 is evenly-priced. Then for all $\vec{v}$, we must have $p^{M_1}(\vec{v}) + p^{M_2}(\vec{v}) \geq p^M(\vec{v})$. This is because $\vec{v}$'s favorite menu option from M appears in one of the two menus, and is necessarily $\vec{v}$'s favorite option on that menu (and they pay non-zero from the other menu). Taking an expectation with respect to $\vec{v}$ yields that $\text{Rev}(\mathcal{D}, M_1) + \text{Rev}(\mathcal{D}, M_2) \geq \text{Rev}(\mathcal{D}, M)$, completing the proof. □

Proof of Claim 18. Recall that $(1+\varepsilon) \cdot ||\vec{v}||_1 \geq ||\vec{x}_i||_1$ for all $\vec{v} \in B_i$. Therefore, if we set a price of $\|\vec{x}_i\|_1/(1+\varepsilon)$ for the grand bundle, every $\vec{v} \in B_i$ would choose to purchase the grand bundle. This immediately implies the claim, as: $\text{BRev}(\mathcal{D}) \geq \frac{||\vec{x}_i||_1}{1+\varepsilon} \cdot \Pr_{\vec{v}\sim\mathcal{D}}\left[||\vec{v}||_1 \geq \frac{||\vec{x}_i||_1}{1+\varepsilon}\right] \geq \frac{||\vec{x}_i||_1}{1+\varepsilon} \cdot \Pr_{\vec{v}\sim\mathcal{D}}[\vec{v} \in B_i]$. □

Proof of Claim 19. Recall that $\text{gap}_i^{X,Q} := \min_{j<i}\{\vec{x}_i \cdot (\vec{q}_i - \vec{q}_j)\}$, and that $\vec{q}_i := \vec{q}^M(\vec{x}_i)$. For any fixed $j < i$, recall that because M was a truthful mechanism, we must have:

$$\begin{aligned}
&\vec{x}_i \cdot \vec{q}^M(\vec{x}_i) - p^M(\vec{x}_i) \geq \vec{x}_i \cdot \vec{q}^M(\vec{x}_j) - p^M(\vec{x}_j)\\
\Rightarrow &\vec{x}_i \cdot (\vec{q}_i - \vec{q}_j) \geq p^M(\vec{x}_i) - p^M(\vec{x}_j)\\
\Rightarrow &\vec{x}_i \cdot (\vec{q}_i - \vec{q}_j) \geq p^M(\vec{x}_i)/2.
\end{aligned}$$

The first line is simply restating incentive compatibility. The second line is basic algebra, and substituting $\vec{q}_i := \vec{q}^M(\vec{x}_i)$. The third line invokes the fact that $p^M(\vec{x}_i) \geq 2^{2(i-1)+a}$, while $p^M(\vec{x}_j) < 2^{2(j-1)+a+1} \leq 2^{2(i-1)+a-1}$. □

Proof of Observation 21. This follows immediately from weak Lagrangian duality. For a quick refresher on weak Lagrangian duality, observe that for any feasible solution to the LP defining $\text{AlignGap}'(X)$ we must have $\vec{x}_i \cdot (c_i\vec{x}_i - c_{i-1}\vec{x}_{i-1}) - \text{sgap}_i \geq 0$. Therefore, for any feasible solution to the original LP, that solution is also feasible for $\text{LagRel}_1(X)$, and the objective is only larger. Therefore, the optimal solution to $\text{LagRel}_1(X)$ must be at least as large as $\text{AlignGap}'(X)$. □

Proof of Observation 22. For all i, $\max\{0, \text{sgap}_i\} - \text{sgap}_i \leq 0$. When $\text{sgap}_i = 0$, the maximum is achieved (and $\text{sgap}_i := 0$ is feasible). Substituting $\max\{0, \text{sgap}_i\} - \text{sgap}_i = 0$ for all i concludes the proof. □

Proof of Proposition 27. To ease notation throughout the proof, we'll use the notation $\text{gap}_{\ell,j}^{X,Q} := \text{gap}_i^{X,Q}$, where $\vec{x}_i := \vec{x}_{\ell,j}$ ($\vec{x}_i$ is the j^{th} point on layer ℓ). We will also use the notation $(\ell', j') < (\ell, j)$ if $\ell' < \ell$, or $\ell' = \ell$ and $j' < j$ (that is, if the j'^{th} point in the ℓ'^{th} layer comes before the j^{th} point in the ℓ^{th} layer). To understand $\text{gap}_{\ell,j}^{X,Q}$, we need to understand which point "sets the gap" for $\vec{x}_{\ell,j}$, that is, which $(\ell', j') := \arg\min_{(\ell',j')<(\ell,j)}\{(\vec{q}_{\ell,j} - \vec{q}_{\ell',j'}) \cdot \vec{x}_i\}$.

We first analyze which point sets the gap for $\vec{x}_{\ell,j}$ (for even ℓ; for odd ℓ the gap is zero and we don't care which point sets it), and observe that it must either be $\vec{q}_{\ell,j-1}$ or $\vec{q}_{\ell-2,n_{\ell-2}-1}$ (that is, it must be the previous point in the same layer, or the final point in the previous even layer).

Claim 30. *For all j, and all even ℓ,* $\text{gap}_{\ell,j}^{X,Q} = \vec{x}_{\ell,j} \cdot \vec{q}_{\ell,j} - \max\{\vec{x}_{\ell,j} \cdot \vec{q}_{\ell-2,n_{\ell-2}-1}, \vec{x}_{\ell,j} \cdot \vec{q}_{\ell,j-1}\}$.[11]

Proof of Claim 30. First, note that $\text{gap}_{\ell,j}^{X,Q} := \min_{(\ell',j')<(\ell,j)}\{\vec{x}_{\ell,j} \cdot (\vec{q}_{\ell,j} - \vec{q}_{\ell',j'})\} = \vec{x}_{\ell,j} \cdot \vec{q}_{\ell,j} - \max_{(\ell',j')<(\ell,j)}\{\vec{x}_{\ell,j} \cdot \vec{q}_{\ell',j'}\}$. To conclude the proof, simply observe that the first component of $\vec{q}_{\ell',j'}$ is monotone increasing in ℓ' (for fixed j'), and the second component is monotone increasing in j' (for fixed ℓ'). Moreover, the second component of $\vec{q}_{\ell',n_{\ell'}-1}$ is 1, and this is the maximum possible. Also, both components of $\vec{x}_{\ell,j}$ are non-negative, and therefore we conclude that $\vec{x}_{\ell,j} \cdot \vec{q}_{\ell-2,n_{\ell-2}-1} \geq \vec{x}_{\ell,j} \cdot \vec{q}_{\ell',j'}$ whenever $(\ell', j') \leq (\ell-2, n_{\ell-2}-1)$ (in fact, this extends even to $(\ell', j') \leq (\ell-1, n_{\ell-1}-1)$ as no new $\vec{q}$ are introduced in layer $\ell - 1$). Also, $\vec{x}_{\ell,j} \cdot \vec{q}_{\ell,j-1} \geq \vec{x}_{\ell,j} \cdot \vec{q}_{\ell,j'}$ whenever $j' \leq j - 1$. □

Now that we know that the gap is set either by the last point in the previous layer, or the previous point in the current layer, we can nail down $\text{gap}_{\ell,j}^{X,Q}$ exactly.

Lemma 31. *For all even $\ell > 2$, and all $j \in [0, n_\ell - 1]$:* $\text{gap}_{\ell,j}^{X,Q} \geq \delta_\ell \frac{\sin(\theta_\ell)}{\sin((j+1)\theta_\ell)}$.

[11] For simplicity of notation, define $\vec{q}_{0,j} = \vec{0} = \vec{q}_{\ell,-1}$ for all ℓ, j.

Proof of Lemma 31. To prove the lemma, we simply compute the inner product of $\vec{x}_{\ell,j}$ with the three relevant vectors $\vec{q}_{\ell,j}, \vec{q}_{\ell-2,n_{\ell-2}-1}, \vec{q}_{\ell,j-1}$. To this end, recall that:

$$\begin{aligned}\vec{q}_{\ell,j} &= (z_\ell, 1-\delta_\ell \cot((j+1)\theta_\ell))\,,\\ \vec{q}_{\ell,j-1} &= (z_\ell, 1-\delta_\ell \cot(j\theta_\ell))\,,\\ \vec{q}_{\ell-2,n_{\ell-2}-1} &= (z_{\ell-2}, 1)\,.\end{aligned}$$

Therefore, observe that

$$\begin{aligned}\vec{x}_{\ell,j}\cdot(\vec{q}_{\ell,j}-\vec{q}_{\ell,j-1}) &= \sin(j\theta_\ell)\cdot\delta_\ell\cdot(\cot(j\theta_\ell)-\cot((j+1)\theta_\ell))\\ &= \sin(j\theta_\ell)\cdot\delta_\ell\cdot\left(\frac{\cos(j\theta_\ell)}{\sin(j\theta_\ell)}-\frac{\cos((j+1)\theta_\ell)}{\sin((j+1)\theta_\ell)}\right)\\ &= \delta_\ell\cdot\frac{\cos(j\theta_\ell)\sin((j+1)\theta_\ell)-\sin(j\theta_\ell)\cos((j+1)\theta_\ell)}{\sin((j+1)\theta_\ell)}\\ &= \delta_\ell\cdot\frac{\sin(\theta_\ell)}{\sin((j+1)\theta_\ell)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\vec{x}_{\ell,j}\cdot(\vec{q}_{\ell,j}-\vec{q}_{\ell-2,n_{\ell-2}-1}) &= (\delta_\ell+\delta_{\ell-1})\cdot\cos(j\theta_\ell)-\delta_\ell\cot((j+1)\theta_\ell)\cdot\sin(j\theta_\ell)\\ &\ge \delta_\ell\cdot\cos(j\theta_\ell)-\delta_\ell\cot((j+1)\theta_\ell)\cdot\sin(j\theta_\ell)\\ &= \frac{\delta_\ell}{\sin((j+1)\theta_\ell)}\left(\sin((j+1)\theta_\ell)\cos(j\theta_\ell)-\sin(j\theta_\ell)\cos((j+1)\theta_\ell)\right)\\ &= \delta_\ell\frac{\sin(\theta_\ell)}{\sin((j+1)\theta_\ell)}.\end{aligned}$$

This means that no matter which point sets the gap (or if one of the points does not exist), the gap is at least $\delta_\ell\frac{\sin(\theta_\ell)}{\sin((j+1)\theta_\ell)}$. ☐

Finally, we need to sum over each even layer.

Corollary 32. *For any even $\ell > 2$, $\sum_{j=0}^{n_\ell-1}\mathrm{gap}_{\ell,j}^{X,Q} \ge \delta_\ell\cdot\ln(n_\ell)/2$.*

Proof of Corollary 32. Consider the following sequence of calculations:

$$\begin{aligned}\sum_{j=0}^{n_\ell-1}\mathrm{gap}_{\ell,j}^{X,Q} &\ge \sum_{j=0}^{n_\ell-1}\delta_\ell\frac{\sin(\theta_\ell)}{\sin((j+1)\theta_\ell)}\\ &\ge \delta_\ell\cdot(\theta_\ell-\theta_\ell^3/6)\cdot\sum_{j=0}^{n_\ell-1}\frac{1}{(j+1)\theta_\ell}\\ &\ge \delta_\ell\cdot(1-\theta_\ell^2/6)\cdot\ln(n_\ell)\\ &\ge \delta_\ell\cdot\ln(n_\ell)/2\end{aligned}$$

Above, the first line follows from Lemma 31. The second line uses the fact that $\theta_\ell-\theta_\ell^3/6 \le \sin(\theta_\ell) \le \theta_\ell$, because $\theta_\ell \in [0,\pi/2]$. The third line follows as the n^{th} harmonic sum is at least $\ln(n)$. The final line follows as $\theta_\ell^2/6 = \pi^2/(24(n_\ell-1)^2) \le 1/2$. ☐

And finally, we can wrap up the proof of the proposition. Here, we just need to recall that $\delta_\ell := \frac{1}{\alpha n_\ell} = \frac{1}{\alpha\ell\ln^2(\ell)}$. Therefore, we conclude that:

$$\sum_{\ell\text{ even}}\sum_{j=0}^{n_\ell-1}\mathrm{gap}_{\ell,j}^{X,Q} \ge \sum_{\ell\text{ even}}\delta_\ell\cdot\ln(n_\ell)/2 = \sum_{\ell\text{ even}}\frac{1}{2\alpha\ell\ln(\ell)} = \infty. \qquad ☐$$

C Proof of Corollary 11

We prove Corollary 11 by making use of Theorem 2 combined with the sequence X from Section 5. The only task is to confirm that $\text{ARev}(D) < \infty$ for the resulting $\mathcal{D}$, which essentially requires that we execute and analyze the construction fully. Let us quickly review the [HN19] construction, given as input a sequence X:

- Let B be a very large constant, to be defined later.
- Let $\vec{v}_i := B^{2^i} \cdot \vec{x}_i/||\vec{x}_i||_1$ (for all i).
- Let $\mathcal{D}$ sample $\vec{v}_i$ with probability $1/B^{2^i}$ (for all i).
- Let $\mathcal{D}$ sample $\vec{0}$ with probability $1 - \sum_{i\geq 1} 1/B^{2^i}$.

[HN19] establishes that the above construction yields Theorem 2 (for sufficiently large B, as a function of ε). To complete the proof of Corollary 11, we just need to relate $\text{ARev}(D)$ for this construction to $\text{AlignGap}(X)$.

Proposition 33. *The construction above yields a $\mathcal{D}$ satisfying* $\text{ARev}(\mathcal{D}) \leq \text{AlignGap}(X) + 1/B$.

Proof. Consider any mechanism M. We show that $\text{AlignGap}(X) \geq \text{ARev}(\mathcal{D}, M) - 1/B$. To see this, consider the following choice of C:

- If $\vec{v}_i$ is parallel to $\vec{q}^M(\vec{v}_i)$, set $c_i := ||\vec{q}^M(\vec{v}_i)||_2/||\vec{x}_i||_2$.
- If $\vec{v}_i$ is not parallel to $\vec{q}^M(\vec{v}_i)$, set $c_i := 0$.

We now need to lower bound $\text{sgap}_i^{X,C}$, when i satisfies the first bullet. Observe that, because M is truthful, we must have, for all $j < i$:

$$\begin{aligned}
\vec{v}_i \cdot \vec{q}^M(\vec{v}_i) - p^M(\vec{v}_i) &\geq \vec{v}_i \cdot \vec{q}^M(\vec{v}_j) - p^M(\vec{v}_j) \\
&\Rightarrow p^M(\vec{v}_i) \leq p^M(\vec{v}_j) + B^{2^i}\vec{x}_i \cdot (c_i\vec{x}_i - c_j\vec{x}_j)/||\vec{x}||_1 \\
&\Rightarrow p^M(\vec{v}_i) \leq 2B^{2^{i-1}} + B^{2^i}\text{sgap}_i^{X,C}/||\vec{x}_i||_1
\end{aligned}$$

Above, the first line follows from incentive compatibility. The second line follows as $\vec{q}^M(\vec{v}_i) = c_i\vec{x}_i$ for all i in the first bullet, and either $\vec{q}^M(\vec{v}_j) = c_j\vec{x}_j$, or $c_j = 0$. The final line follows by taking $j := \arg\min_{j<i}\{\vec{v}_i \cdot (c_i\vec{v}_i - c_j\vec{v}_j)\}$, and by observing that $\vec{v}_j$ cannot possibly pay more than their value for the grand bundle.

We can then conclude that:

$$\begin{aligned}
\text{ARev}(\mathcal{D}, M) &\leq \sum_i (2B^{2^{i-1}} + B^{2^i}\text{sgap}_i^{X,C}/||\vec{x}_i||_1)/B^{2^i} \\
&\leq \sum_i 2/B^{2^{i-1}} + \text{AlignGap}(X) \\
&\leq \text{AlignGap}(X) + 1/B.
\end{aligned}$$

□

Because we can take B as large as we like, we can construct a $\mathcal{D}$ such that $\text{ARev}(\mathcal{D})$ is arbitrarily close to $\text{AlignGap}(X)$, while also maintaining that $\text{Rev}(\mathcal{D})$ is arbitrarily close to $\text{MenuGap}(X)$. Because Theorem 9 provides a construction X such that $\text{MenuGap}(X)/\text{AlignGap}(X) = \infty$, the [HN19] construction, with sufficiently large B, yields a $\mathcal{D}$ with $\text{Rev}(\mathcal{D})/\text{ARev}(\mathcal{D}) = \infty$, completing the proof of Corollary 11.