

A Technical Appendices and Supplementary Material

A.1 Feature Regularity Conditions and General Log-concave Distributions

The literature on learning halfspaces with noise typically assumes that the feature vector \mathbf{x} has an isotropic log-concave distribution [Balcan and Long, 2013, Awasthi et al., 2017, Zhang et al., 2020, Shen, 2023]. The log-concave and isotropic log-concave distributions are defined as follows.

Definition 1 (Isotropic log-concave distribution [Lovász and Vempala, 2007]). *A random vector \mathbf{z} over \mathbb{R}^d with probability density function $\phi_{\mathbf{z}}(\cdot)$ follows a log-concave distribution if $\ln \phi_{\mathbf{z}}(\cdot)$ is concave. Moreover, it is isotropic if $\mathbb{E}[\mathbf{z}] = \mathbf{0}$ and $\mathbb{E}[\mathbf{z}\mathbf{z}^T] = I$.*

The following lemma summarizes some important properties of (isotropic) log-concave distributions that have been proved by literature.

Lemma 2. *Suppose $\mathbf{z} \in \mathbb{R}^d$ with probability density function $\phi_{\mathbf{z}}(\cdot)$ follows a log-concave distribution. Then, the following holds.*

- (a) (Klivans et al. [2009] Lemma 5.17) *For $d = 1$, assume that $\mathbb{E}[\mathbf{z}^2] = C^2$, then for every $t > 0$, $\mathbb{P}(|\mathbf{z}| > t) \leq e^{-Ct+1}$.*

Moreover, if \mathbf{z} is isotropic, then,

- (b) (Lovász and Vempala [2007] Lemma 5.2) $\phi_{\mathbf{z}}(\mathbf{z}) \geq \beta_1(d)$ for all $0 \leq \|\mathbf{z}\|_2 \leq 1/9$, where $\beta_1(d) = 2^{-8d}$.
- (c) (Lovász and Vempala [2007] Lemma 5.5) For $d = 1$, $\phi_{\mathbf{z}}(\mathbf{z}) \leq 1$.
- (d) (Klivans et al. [2009] Lemma 7) For $d \geq 2$, $\phi_{\mathbf{z}}(\mathbf{z}) \leq \beta_2(d)e^{-\beta_3(d)\|\mathbf{z}\|_2}$, where $\beta_2(d) = 2^{8d}d^{d/2}e$ and $\beta_3(d) = \frac{2^{-7d}}{2(d-1)(20(d-1))^{(d-1)/2}}$.

Following Lemma 2, one can show that any mean-zero isotropic log-concave distribution satisfies the regularity conditions in assumption 2. Importantly, in this part, we show that the regularity conditions can hold even for a mean-zero log-concave distribution that is not isotropic. In this case, eigenvalue bounds on the covariance matrix of the distribution determine the corresponding regularity parameters.

Lemma 3. *Let $\mathbf{x} \in \mathbb{R}^d$ ($d \geq 2$) have zero mean and a log-concave distribution. Suppose the eigenvalues of its covariance matrix $\Sigma = \mathbb{E}[\mathbf{x}\mathbf{x}^T]$ are all bounded within $[\underline{\lambda}, \bar{\lambda}]$ for some positive constants $\underline{\lambda}, \bar{\lambda}$, then the distribution of \mathbf{x} satisfies the regularity conditions in assumption 2, with parameters $L_1 = \frac{\beta_1(1)}{\sqrt{\bar{\lambda}}}$, $L_2 = \frac{\beta_1(2)}{\bar{\lambda}}$, $R = \frac{1}{9}\sqrt{\bar{\lambda}}$, $U_1 = \frac{1}{\sqrt{\bar{\lambda}}}$, $U_2 = \frac{\beta_2(2)}{\bar{\lambda}}$, $\delta = \frac{\beta_3(2)}{\sqrt{\bar{\lambda}}}$, $Q = \sqrt{\bar{\lambda}}$ for $\beta_1(1), \beta_1(2), \beta_2(2), \beta_3(2)$ given in Lemma 2.*

A.2 Numerical Experiments

In this subsection, we conduct numerical experiments to evaluate our proposed algorithm. To highlight the challenges posed by Massart Noise and strategic behavior, and to demonstrate the effectiveness of our algorithm, we compare its regret against two benchmarks: (1) the Strategic Perceptron algorithm from Ahmadi et al. [2021], designed for noiseless online strategic classification, and (2) the PAC learning algorithm for halfspaces with Massart Noise from Zhang et al. [2020], designed for non-strategic classification. Note that these benchmarks are both originally designed for full feedback settings, whereas our work focuses on partial feedback. We evaluate the performance of these benchmark algorithms both when they have access to full feedback (while our algorithm does not) and when they only use partial feedback as our algorithm.

We test the algorithms under two different settings, with key parameters outlined in Table 1. Each setting is replicated 30 times, and we report the average regret for each algorithm. Our analysis includes a performance comparison of the different algorithms and an investigation of how various problem parameters influence our proposed algorithm.

Benchmark against Strategic Perceptron by Ahmadi et al. [2021] To understand the impact of Massart Noise, we compare our algorithm with the Strategic Perceptron algorithm from Ahmadi et al.

Table 1: Numerical experiment settings.

Index	$\mathcal{D}_{\mathbf{x}}$	$\bar{\eta}$	$\eta(\mathbf{x})$	γ	\mathbf{w}^*
Setting 1	Standard Normal	0.1	$\eta(\mathbf{x}) = \bar{\eta}(1 - \exp(-\ \mathbf{x}\ _2))$	0.1	(1,0)
Setting 2	Unit Ball	0.1	$\eta(\mathbf{x}) = \bar{\eta}$	0.1	(1,0)

[2021]. This algorithm provably achieves only a finite number of mistakes under a noiseless model where the feature-label relationship is deterministic and the true and negative classes are strictly separated by a positive margin. Ahmadi et al. [2021] modify the classical Perceptron algorithm by setting a higher threshold for classifying an agent as positive and proxy surrogate features to estimate the agents' true features. Their proxy feature is defined as follows.

Definition 2 ($\tilde{\mathbf{x}}_t$, proxy feature, Ahmadi et al. [2021]). *For a given classifier $\tilde{h}(\cdot) = \text{sgn}(\langle \mathbf{w}, \cdot \rangle + m)$, an agent (\mathbf{x}, y) reports his feature as \mathbf{r} according to Lemma 1. Then the corresponding proxy feature $\tilde{\mathbf{x}}_t$ in Ahmadi et al. [2021] is defined as*

$$\tilde{\mathbf{x}}_t = \begin{cases} \mathbf{r}_t - \gamma \mathbf{w} & \langle \mathbf{w}, \mathbf{r}_t \rangle = \gamma \text{ and } y_t = -1; \\ \mathbf{r}_t & \text{otherwise.} \end{cases} \quad (2)$$

Algorithm 5: Original Strategic Perceptron with Full Feedback (Ahmadi et al. [2021])

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1 Accept the first agent without declaring any classifier
2 if  $y_1 = 1$  then
3    $\hat{\mathbf{w}}_2 \leftarrow \mathbf{r}_1$ 
4 else
5    $\hat{\mathbf{w}}_2 \leftarrow -\mathbf{r}_1$ 
6  $\mathbf{w}_2 \leftarrow \hat{\mathbf{w}}_2 / \|\hat{\mathbf{w}}_2\|_2$ 
7 for  $t = 2 \dots T$  do
8   Declare classifier  $\tilde{h}_t(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}_t, \mathbf{r} \rangle - \gamma)$ , receive agent response  $\mathbf{r}_t$ 
9   Classify the agent as  $\hat{y}_t = \tilde{h}_t(\mathbf{r}_t)$ 
10  if  $y_t \neq \hat{y}_t$  then
11     $\hat{\mathbf{w}}_{t+1} \leftarrow \mathbf{w}_t + y_t \tilde{\mathbf{x}}_t$ ,  $\mathbf{w}_{t+1} \leftarrow \hat{\mathbf{w}}_{t+1} / \|\hat{\mathbf{w}}_{t+1}\|_2$ 
12  else
13     $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t$ 

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Their original algorithm, designed for the full feedback setting, is presented in Algorithm 5. Algorithm 6 below directly adapts this algorithm to our partial feedback setting. Specifically, instead of using all data points that incur misclassifications for update, the refined algorithm uses only positively classified agents with true labels -1 to adjust the coefficient vector.

Algorithm 6: Strategic Perceptron with Partial Feedback

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1 Accept the first agent without declaring any classifier
2 if  $y_1 = 1$  then
3    $\hat{\mathbf{w}}_2 \leftarrow \mathbf{r}_1$ 
4 else
5    $\hat{\mathbf{w}}_2 \leftarrow -\mathbf{r}_1$ 
6  $\mathbf{w}_2 \leftarrow \hat{\mathbf{w}}_2 / \|\hat{\mathbf{w}}_2\|_2$ 
7 for  $t = 2 \dots T$  do
8   Declare classifier  $\tilde{h}_t(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}_t, \mathbf{r} \rangle - \gamma)$ , receive agent response  $\mathbf{r}_t$ 
9   if  $\langle \mathbf{w}_t, \mathbf{r}_t \rangle \geq \gamma$  then
10    Accept the agent and receive his true label  $y_t$ 
11  else
12    Reject the agent without getting his true label
13  if  $y_t = -1$  then
14     $\hat{\mathbf{w}}_{t+1} \leftarrow \mathbf{w}_t - \tilde{\mathbf{x}}_t$ ,  $\mathbf{w}_{t+1} \leftarrow \hat{\mathbf{w}}_{t+1} / \|\hat{\mathbf{w}}_{t+1}\|_2$ 
15  else
16     $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t$ 

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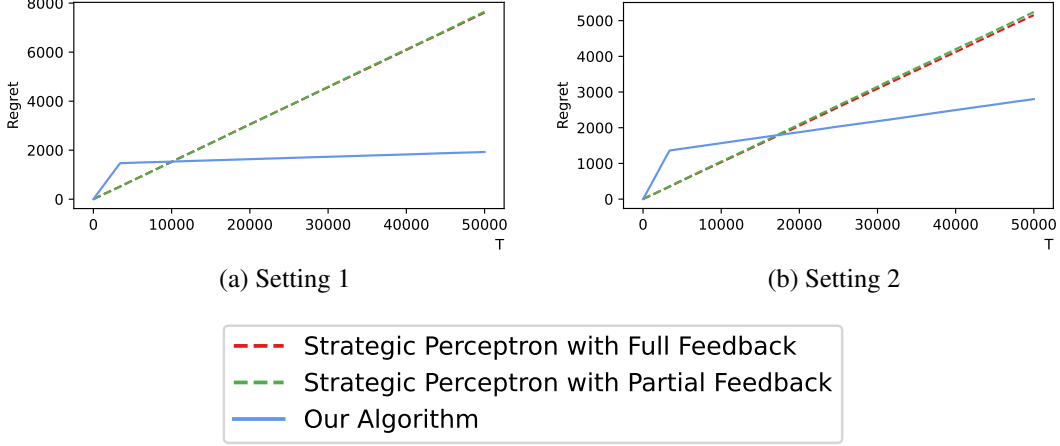


Figure 3: Average regrets of our algorithm and two Strategic Perceptron based-benchmark algorithms over different time horizons T under the two settings listed in Table 1. Results are based on 30 independent replications of the experiment.

Figure 3 illustrates the average regret of each algorithm over 50,000 cycles in both settings listed in Table 1. We observe that our algorithm’s regret grows sublinearly, while the two benchmarks’ regrets may accumulate linearly. The original Strategic Perceptron with full feedback (the dashed red line) demonstrates slightly better performance compared to its partially feedback-modified counterpart (the dashed green line). However, the improvement remains marginal, suggesting that our partial feedback setting is not the primary cause of the Strategic Perceptron’s failure. Intuitively, the ineffectiveness of the two benchmarks stems from their sensitivity to noise when updating with data from all admitted agents. When $\theta(\mathbf{w}^*, \mathbf{w}_t)$ is relatively small but $\langle \mathbf{w}_t, \mathbf{r}_t \rangle$ (equals $\langle \mathbf{w}_t, \mathbf{x}_t \rangle$ when $\langle \mathbf{w}_t, \mathbf{r}_t \rangle > \gamma$) is large, the algorithm is more likely to admit a ‘wrong’ agent ($y_t = -1$ but $\tilde{h}(\mathbf{r}_t) = 1$) due to noise rather than an inaccurate classifier. Since the perceptron algorithm updates are based on mistakes (Rosenblatt [1958]), the presence of noise increases the probability of misleading updates for the classifiers. In contrast, our algorithm explores a small band near the decision boundary, whose bandwidth decreases proportionally to $\theta(\mathbf{w}^*, \mathbf{w}_k)$, $k = 1, 2, \dots, K$ across batches. Within this band, wrong admissions are more likely due to suboptimal classifiers than noise, making the update more effective. This enables our algorithm to gradually converge to the optimal decision.

Benchmark against Non-Strategic Learning under Massart Noise by Zhang et al. [2020] Next, to highlight the impact of agents’ strategic behavior, we compare our algorithm against the algorithm proposed by Zhang et al. [2020] (see Algorithm 7), which is designed for adaptively learning halfspaces with Massart Noise in the *non-strategic* classification setting. Their algorithm also adopts a localization scheme that focuses on data within an increasingly narrow band near the classification boundary and uses online mirror descent in batches for classifier updates. However, they do not consider the impact of agents’ strategic behavior. We test the performance of their algorithm under strategic manipulation in both full feedback and partial feedback settings. The two settings differ in: 1) whether the principal can collect labels of those who are negatively classified (full feedback) or not (partial feedback) and 2) the algorithm chooses different bandwidths for updates, namely, $\{\mathbf{r} \mid -b_k \leq \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i} \rangle \leq b_k\}$ for the full feedback setting and $\{\mathbf{r} \mid 0 < \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i} \rangle \leq b_k\}$ for the partial feedback setting.

As shown in Figure 4, after a common pure exploration phase, the regret of the non-strategic learning algorithm in both full feedback and partial feedback settings grows linearly. This is because the non-strategic learning algorithm ignores agents’ strategic manipulation. Consequently, true negative agents may misreport their features to be positively labeled. In contrast, our algorithm accounts for agents’ strategic behavior and is able to efficiently learn the ground truth distribution.

Algorithm 7: Non-Strategic Learning under Massart Noise

Input: Feedback setting F , noise level bound $\bar{\eta}$, lengths $\{T_{\text{init}}\} \cup \{T_k\}_{k=0}^K$, bandwidths $\{b_k\}_{k=0}^K$, step sizes $\{\alpha_k\}_{k=0}^K$, feature dimension d

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1  $\bar{\mathbf{w}}_0 = \text{Non-Strategic-Initialization}(F, T_{\text{init}})$  // See Algorithm 8
2  $\mathbf{w}_1 = \text{Non-Strategic-Refinement}(F, \bar{\mathbf{w}}_0, \bar{\eta}, T_0, b_0, \alpha_0, d)$  // See Algorithm 9
3 for  $k \leftarrow 1$  to  $K$  do
4    $\mathbf{w}_{k+1} = \text{Non-Strategic-Batched-Enhancement}(F, \mathbf{w}_1, \bar{\eta}, k, T_k, b_k, \alpha_k, d)$  // See Algorithm 10
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Algorithm 8: Non-Strategic-Initialization

Input: Feedback setting F , iteration length T_{init}

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1 for  $i \leftarrow 1$  to  $T_{\text{init}}$  do
2   Uniformly draw  $\mathbf{w}_{\text{init},i} \in \mathbb{S}^d$ 
3   Declare  $\tilde{h}_{\text{init},i}(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}_{\text{init},i}, \mathbf{r} \rangle)$ , agent  $(\mathbf{x}_{\text{init},i}, y_{\text{init},i})$  arrives and reports  $\mathbf{r}_{\text{init},i}$ 
4   Make classification decision  $\tilde{h}_{\text{init},i}(\mathbf{r}_{\text{init},i})$ 
5   if  $F = \text{"full"}$  then
6     collect label  $y_{\text{init},i}$ 
7   if  $F = \text{"partial"}$  then
8     collect label  $y_{\text{init},i}$  only if  $\tilde{h}_{\text{init},i}(\mathbf{r}_{\text{init},i}) = 1$ 
9 if  $F = \text{"full"}$  then
10  return  $\bar{\mathbf{w}}_0 = \frac{1}{T_{\text{init}}} \sum_{i=1}^{T_{\text{init}}} \mathbf{r}_{\text{init},i} y_{\text{init},i}$ 
11 if  $F = \text{"partial"}$  then
12  return  $\bar{\mathbf{w}}_0 = \frac{1}{T_{\text{init}}} \sum_{i=1}^{T_{\text{init}}} \mathbf{r}_{\text{init},i} y_{\text{init},i} \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i} \rangle > 0)$ 
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Algorithm 9: Non-Strategic-Refinement

Input: Feedback setting F , Initial vector $\bar{\mathbf{w}}_0$, noise level $\bar{\eta}$, iteration length T_0 , bandwidth b_0 , step size α_0 , feature dimension d

Initialization: $\mathbf{w}_{0,1} = \bar{\mathbf{w}}_0 / \|\bar{\mathbf{w}}_0\|_2$

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1 for  $i \leftarrow 1$  to  $T_0$  do
2   Declare classifier  $\hat{y}_{0,i} = \tilde{h}_{0,i}(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}_{0,i}, \mathbf{r} \rangle)$ , agent  $(\mathbf{x}_{0,i}, y_{0,i})$  arrives and reports  $\mathbf{r}_{0,i}$ 
3   if  $F = \text{"full"}$  then
4     Make classification decision  $\hat{y}_{0,1} = \tilde{h}_{0,i}(\mathbf{r}_{0,i})$  and collect label  $y_{0,i}$ 
5     Compute gradient:
6        $\tilde{\mathbf{g}}_{0,i} = [-\bar{\eta} \mathbf{r}_{0,i} \mathbb{I}(y_{0,i} = \hat{y}_{0,1}) + (1 - \bar{\eta}) \mathbf{r}_{0,i} \mathbb{I}(y_{0,i} \neq \hat{y}_{0,1})] \mathbb{I}(-b_0 \leq \langle \mathbf{w}_{0,i}, \mathbf{r}_{0,i} \rangle \leq b_0)$ 
7   if  $F = \text{"partial"}$  then
8     Make classification decision  $\hat{y}_{0,1} = \tilde{h}_{0,i}(\mathbf{r}_{0,i})$  and collect label  $y_{0,i}$  only if  $\hat{y}_{0,i} = 1$ 
9     Compute gradient:
10     $\tilde{\mathbf{g}}_{0,i} = [-\bar{\eta} \mathbf{r}_{0,i} \mathbb{I}(y_{0,i} = \hat{y}_{0,1}) + (1 - \bar{\eta}) \mathbf{r}_{0,i} \mathbb{I}(y_{0,i} \neq \hat{y}_{0,1})] \mathbb{I}(0 < \langle \mathbf{w}_{0,i}, \mathbf{r}_{0,i} \rangle \leq b_0)$ 
11  Set constraint set:  $\mathcal{W}_0 = \{\mathbf{w} \mid \|\mathbf{w}\|_2 \leq 1, \langle \mathbf{w}, \bar{\mathbf{w}}_0 \rangle \geq c_1(1 - 2\bar{\eta})\}$ 
12  Update  $\mathbf{w}$ :  $\hat{\mathbf{w}}_{0,i+1} = \arg \min_{\mathbf{w} \in \mathcal{W}_0} \langle \tilde{\mathbf{g}}_{0,i}, \mathbf{w} \rangle + \frac{1}{\alpha_0} \frac{\|\mathbf{w} - \mathbf{w}_{0,i}\|_p^2}{2(p-1)}$ , where  $p = \frac{\ln(8d)}{\ln(8d)-1}$ 
13  Normalize:  $\mathbf{w}_{0,i+1} = \hat{\mathbf{w}}_{0,i+1} / \|\hat{\mathbf{w}}_{0,i+1}\|_2$ 
14 Compute mean vector:  $\bar{\mathbf{w}}_1 = \frac{1}{T_0} \sum_{i=1}^{T_0} \mathbf{w}_{0,i}$ 
15 return  $\mathbf{w}_1 = \bar{\mathbf{w}}_1 / \|\bar{\mathbf{w}}_1\|_2$ 
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Impact of Different Parameters We examine three groups of additional settings to analyze the impact of different parameters. For each group, we test both settings from Table 1. The average regret over 30 independent experiments for each group is depicted in fig. 5 up to 50,000 cycles.

We first examine how different maximum manipulation distances $\gamma = 0.1, 0.2, 0.5$ affect the regret of our algorithm. As depicted in Figure 5 (a1) and (a2), larger γ values result in higher regret. Intuitively, a larger γ permits more agents to manipulate their features, so the strategic manipulation problem becomes more severe. This causes all algorithms to have worse performance.

Next, we vary the feature space dimension, setting \mathbf{w}^* to be $(1, 0)$ ($d = 2$), $(1, 0, 0, 0)$ ($d = 4$), and $(1, 0, 0, 0, 0, 0)$ ($d = 6$), respectively. The average regret across different time horizons is shown in

Algorithm 10: Non-Strategic-Batched Enhancement

Input: Feedback setting F , initial vector \mathbf{w}_k , noise level $\bar{\eta}$, batch index k , iteration length T_k , bandwidth b_k , step size α_k , feature dimension d

Initialization: $\mathbf{w}_{k,1} = \mathbf{w}_k$

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1 for  $i \leftarrow 1$  to  $T_k$  do
2   Declare classifier  $\tilde{h}_{k,i}(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}_{k,i}, \mathbf{r} \rangle)$ , agent  $(\mathbf{x}_{k,i}, y_{k,i})$  arrives and reports  $\mathbf{r}_{k,i}$ 
3   if  $F = \text{"full"}$  then
4     Make classification decision  $\hat{y}_{k,i} = \tilde{h}_{k,i}(\mathbf{r}_{k,i})$  and collect label  $y_{k,i}$ 
5     Compute gradient:
6        $\tilde{\mathbf{g}}_{k,i} = [-\bar{\eta}\mathbf{r}_{k,i}\mathbb{I}(y_{0,i} = \hat{y}_{k,i}) + (1 - \bar{\eta})\mathbf{r}_{0,i}\mathbb{I}(y_{0,i} \neq \hat{y}_{k,i})]\mathbb{I}(-b_0 \leq \langle \mathbf{w}_{0,i}, \mathbf{r}_{0,i} \rangle \leq b_0)$ 
7   if  $F = \text{"partial"}$  then
8     Make classification decision  $\hat{y}_{k,i} = \tilde{h}_{k,i}(\mathbf{r}_{k,i})$  and collect label  $y_{k,i}$  only if  $\hat{y}_{k,i} = 1$ 
9     Compute gradient:
10       $\tilde{\mathbf{g}}_{k,i} = [-\bar{\eta}\mathbf{r}_{k,i}\mathbb{I}(y_{0,i} = \hat{y}_{k,i}) + (1 - \bar{\eta})\mathbf{r}_{0,i}\mathbb{I}(y_{0,i} \neq \hat{y}_{k,i})]\mathbb{I}(0 < \langle \mathbf{w}_{0,i}, \mathbf{r}_{0,i} \rangle \leq b_0)$ 
11   Update:  $\hat{\mathbf{w}}_{k,i+1} \leftarrow \arg \min_{\mathbf{w} \in \mathcal{W}_k} \langle \tilde{\mathbf{g}}_{k,i}, \mathbf{w} \rangle + \frac{1}{\alpha_k} \frac{\|\mathbf{w} - \mathbf{w}_{k,i}\|_2^2}{2(p-1)}$ , where  $p = \frac{\ln(8d)}{\ln(8d)-1}$ , the constraint set
12    $\mathcal{W}_k = \{\mathbf{w} \mid \|\mathbf{w}\|_2 \leq 1, \|\mathbf{w} - \mathbf{w}_k\|_2 \leq \theta_k, \text{ starting angle } \theta_k = \frac{\pi}{2^{k+1}}\}$ 
13   Normalize:  $\mathbf{w}_{k,i+1} = \hat{\mathbf{w}}_{k,i+1} / \|\hat{\mathbf{w}}_{k,i+1}\|_2$ 
14 Compute mean vector  $\hat{\mathbf{w}}_{k+1} = \frac{1}{T_k} \sum_{i=1}^{T_k} \mathbf{w}_{k,i}$ 
15 return  $\mathbf{w}_{k+1} = \hat{\mathbf{w}}_{k+1} / \|\hat{\mathbf{w}}_{k+1}\|_2$ 
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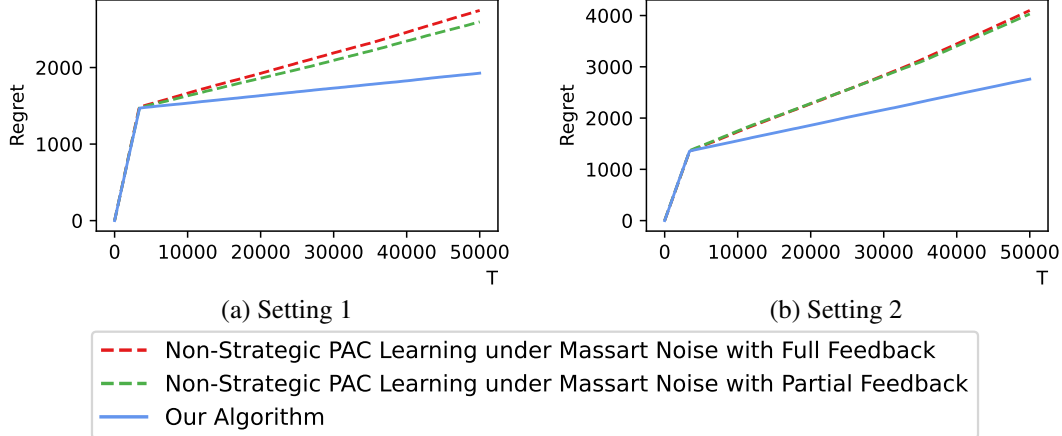


Figure 4: Average regrets of our algorithm and two Non-strategic learning based-benchmark algorithms over different time horizons T under the two settings listed in Table 1. Results are based on 30 independent replications of the experiment.

fig. 5 (b1) and (b2). As expected, the $d = 2$ setting yields the lowest regret, while $d = 6$ setting yields the highest, consistent with our regret bound.

We finally investigate the impact of the noise level $\bar{\eta}$ on our algorithm's convergence in fig. 5 (c1) and (c2), setting $\bar{\eta}$ to 0.1, 0.2 and 0.4. Surprisingly, the impact of the noise level manifests in opposite trends across the two settings. In setting 1, a higher noise level results in greater regret when T is large enough. Conversely, in setting 2, increased noise levels lead to a lower regret rate. This discrepancy might stem from the fact that as the noise level rises, the learning accuracy of the clairvoyant optimal classifier diminishes. Given that regret is defined as the difference in cumulative error between our algorithm's classifiers and the clairvoyant optimal ones, the noisier environment could potentially narrow this gap.

A.3 Technical Lemmas

In this subsection, we list some technical lemmas as instruments for our further proofs.

Properties of Regular Distributions

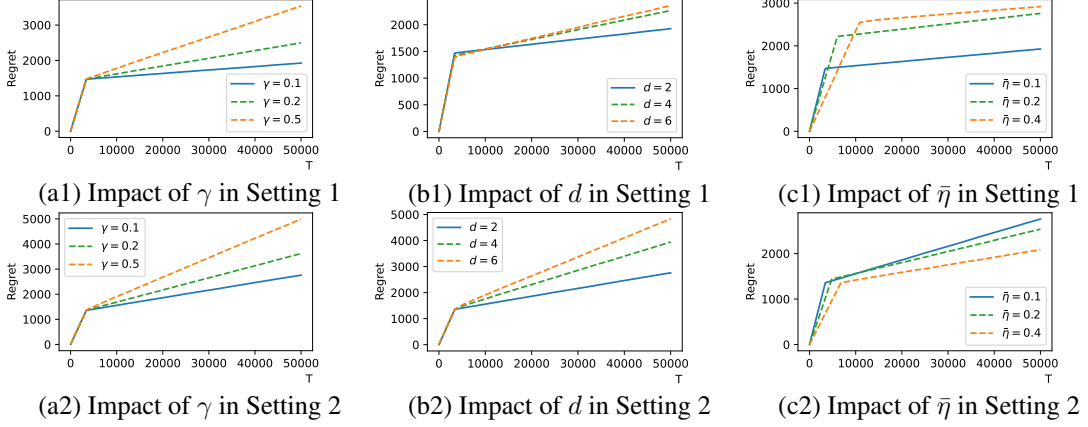


Figure 5: Average regrets of our algorithm over different time horizons T on various parameters. Results are based on 30 independent replications of the experiment.

Lemma 4. Suppose that the distribution of a random vector $\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}$ satisfies the regularity conditions outlined in Assumption 2, then, it has the following properties.

- (a) For $\forall \mathbf{w} \in \mathbb{S}^d$ and $\forall b > 0$, $L_1 \min\{R, b\} \leq \mathbb{P}(0 \leq \langle \mathbf{w}, \mathbf{x} \rangle \leq b) \leq U_1 b$.
- (b) There exist positive constants $c_8, c_9 > 0$, such that for any two unit vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{S}^d$, if $0 \leq \theta(\mathbf{v}_1, \mathbf{v}_2) \leq \frac{\pi}{2}$, then
$$c_8 \mathbb{P}(\text{sgn}(\langle \mathbf{v}_1, \mathbf{x} \rangle) \neq \text{sgn}(\langle \mathbf{v}_2, \mathbf{x} \rangle)) \leq \theta(\mathbf{v}_1, \mathbf{v}_2) \leq c_9 \mathbb{P}(\text{sgn}(\langle \mathbf{v}_1, \mathbf{x} \rangle) \neq \text{sgn}(\langle \mathbf{v}_2, \mathbf{x} \rangle)). \quad (3)$$

Proof. (a) Since $\langle \mathbf{w}, \mathbf{x} \rangle$ forms a projection of \mathbf{x} onto a certain 1-dimensional hyperplane, property (a) trivially holds by conditions 1 and 2 in Assumption 2.

(b) Let $\mathbf{z} := (\langle \mathbf{v}_1, \mathbf{x} \rangle, \langle \mathbf{v}_2, \mathbf{x} \rangle)$, which is a projection of \mathbf{x} onto a 2-dimensional subspace V_2 spanned by \mathbf{v}_1 and \mathbf{v}_2 . Let $\phi_{V_2}(\cdot)$ and D_{V_2} denote its density and distribution, respectively. Let $G_{V_2} := \{\mathbf{z} \mid \text{sgn}(\mathbf{z}_1) \neq \text{sgn}(\mathbf{z}_2)\}$, then,

$$\begin{aligned} \mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}(\text{sgn}(\langle \mathbf{v}_1, \mathbf{x} \rangle) \neq \text{sgn}(\langle \mathbf{v}_2, \mathbf{x} \rangle)) &= \mathbb{P}_{\mathbf{z} \sim D_{V_2}}(\mathbf{z} \in G_{V_2}) \\ &= \int_{\mathbf{z} \in G_{V_2}} \phi_{V_2}(\mathbf{z}) d\mathbf{z} \\ &\geq \int_{\mathbf{z} \in G_{V_2} \cap \mathbb{B}^2(R)} L_2 d\mathbf{z} \\ &\geq L_2 R^2 \theta(\mathbf{v}_1, \mathbf{v}_2). \end{aligned}$$

Where the first inequality holds by condition 1 of Assumption 2 that $L_2 \leq \phi_{V_2}(\mathbf{z})$ for all $\|\mathbf{z}\|_2 \leq R$. The last inequality holds by an observation that $\int_{\mathbf{z} \in G_{V_2} \cap \mathbb{B}^2(R)} 1 d\mathbf{z} \geq R^2 \theta(\mathbf{v}_1, \mathbf{v}_2)$. Hence, we prove the first inequality of (3).

To prove the second inequality of (3), for $\forall \epsilon > 0$, we have

$$\begin{aligned} &\mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}(\text{sgn}(\langle \mathbf{v}_1, \mathbf{x} \rangle) \neq \text{sgn}(\langle \mathbf{v}_2, \mathbf{x} \rangle)) \\ &= \mathbb{P}_{\mathbf{z} \sim D_{V_2}}(\mathbf{z} \in G_{V_2}) \\ &\leq \mathbb{P}_{\mathbf{z} \sim D_{V_2}}(\mathbf{z} \in G_{V_2}, \|\mathbf{z}\|_2 \leq \epsilon) + \mathbb{P}_{\mathbf{z} \sim D_{V_2}}(\|\mathbf{z}\|_2 > \epsilon) \\ &\leq \int_{\mathbf{z} \in G_{V_2} \cap \mathbb{B}^2(\epsilon)} \phi_{V_2}(\mathbf{z}) + \mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}(|\langle \mathbf{v}_1, \mathbf{x} \rangle| > \epsilon) + \mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}(|\langle \mathbf{v}_2, \mathbf{x} \rangle| > \epsilon) \\ &= \int_{\mathbf{z} \in G_{V_2} \cap \mathbb{B}^2(\epsilon)} \phi_{V_2}(\mathbf{z}) + \mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}(|\langle \mathbf{v}_1, \mathbf{x} \rangle| > \epsilon) + \mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}(|\langle \mathbf{v}_2, \mathbf{x} \rangle| > \epsilon) \\ &\leq U_2 \theta(\mathbf{v}_1, \mathbf{v}_2) \epsilon^2 + 2 \exp(1 - Q\epsilon). \end{aligned}$$

Where the last inequality holds by the fact that $\phi_{V_2}(\mathbf{z}) \leq U_2 \exp(-\delta \|\mathbf{z}\|_2) \leq U_2$ according to Assumption 2 condition 2 and $\mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}(|\langle \mathbf{w}, \mathbf{x} \rangle| > \epsilon) \leq \exp(1 - Q\epsilon)$ for $\forall \mathbf{w} \in \mathbb{S}^d$ according to Assumption 2 Condition 3. Taking $\epsilon = \frac{1 - \ln(\theta(\mathbf{v}_1, \mathbf{v}_2))}{Q}$, then we have

$$\mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}(\text{sgn}(\langle \mathbf{v}_1, \mathbf{x} \rangle) \neq \text{sgn}(\langle \mathbf{v}_2, \mathbf{x} \rangle)) \leq \left(\frac{U_2}{Q^2} + 2 \right) \theta(\mathbf{v}_1, \mathbf{v}_2)$$

Thus, we complete the proof of the second inequality in (3). \square

Probability Tail bounds

Definition 3. ((σ, b) -subexponential, Wainwright [2019], Definition 2.7) A random variable X with mean $\mu = \mathbb{E}[X]$ is (σ, b) -subexponential, if for $\forall \lambda \in [-\frac{1}{b}, \frac{1}{b}]$,

$$\mathbb{E}[\exp(\lambda(X - \mu))] \leq \exp\left(\frac{\sigma^2 \lambda^2}{2}\right).$$

Lemma 5. ((σ, b) -subexponential tail bound, another form of Wainwright [2019], Proposition 2.9) Suppose X is a (σ, b) -subexponential random variable with mean $\mathbb{E}[X] = \mu$, then with probability at least $1 - \delta$,

$$X \leq \mu + \sqrt{2\sigma^2 \ln \frac{1}{\delta}} + 2b \ln \frac{1}{\delta},$$

also, with probability at least $1 - \delta$,

$$X \geq \mu - \sqrt{2\sigma^2 \ln \frac{1}{\delta}} - 2b \ln \frac{1}{\delta}.$$

Lemma 6. (A Bernstein-type bound for i.i.d. random variables, another form of Wainwright [2019], Equation (2.18)) Suppose $\{X\}_{i=1}^N$ is sequence of i.i.d. (σ, b) -subexponential random variables, then, with probability at least $1 - \delta$,

$$\sum_{i=1}^N X_i \leq \sum_{i=1}^N \mathbb{E}[X_i] + \sigma \sqrt{2N \ln \frac{1}{\delta}} + 2b \ln \frac{1}{\delta},$$

and, with probability at least $1 - \delta$,

$$\sum_{i=1}^N X_i \geq \sum_{i=1}^N \mathbb{E}[X_i] - \sigma \sqrt{2N \ln \frac{1}{\delta}} - 2b \ln \frac{1}{\delta}.$$

Lemma 7. (A Bernstein-type bound for a martingale difference sequence, another form of Wainwright [2019], Theorem 2.19) Suppose $\{X\}_{i=1}^N$ is a sequence of conditionally (σ, b) -subexponential random variables adapted from filtration $\{\mathcal{F}_i\}_{i=1}^N$, i.e.,

$$\mathbb{E}[\exp(\lambda(X_i - \mathbb{E}[X_i | \mathcal{F}_{i-1}])) | \mathcal{F}_{i-1}] \leq \exp\left(\frac{\sigma^2 \lambda^2}{2}\right), \quad \forall \lambda \in \left[-\frac{1}{b}, \frac{1}{b}\right].$$

Then, with probability at least $1 - \delta$,

$$\sum_{i=1}^N X_i \leq \sum_{i=1}^N \mathbb{E}[X_i | \mathcal{F}_{i-1}] + \sigma \sqrt{2N \ln \frac{1}{\delta}} + 2b \ln \frac{1}{\delta},$$

and, with probability at least $1 - \delta$,

$$\sum_{i=1}^N X_i \geq \sum_{i=1}^N \mathbb{E}[X_i | \mathcal{F}_{i-1}] - \sigma \sqrt{2N \ln \frac{1}{\delta}} - 2b \ln \frac{1}{\delta}.$$

Lemma 8. (Azuma-Hoeffding's Inequality, another form of Wainwright [2019], Corollary 2.20) Suppose $\{X\}_{i=1}^N$ is a sequence adapted from filtration $\{\mathcal{F}_i\}_{i=1}^N$ such that $X_i \in [a, b]$, Then, with probability at least $1 - \delta$,

$$\sum_{i=1}^N X_i \leq \sum_{i=1}^N \mathbb{E}[X_i | \mathcal{F}_{i-1}] + (b - a) \sqrt{\frac{1}{2} N \ln \frac{1}{\delta}},$$

and, with probability at least $1 - \delta$,

$$\sum_{i=1}^N X_i \geq \sum_{i=1}^N \mathbb{E}[X_i | \mathcal{F}_{i-1}] - (b - a) \sqrt{\frac{1}{2} N \ln \frac{1}{\delta}}.$$

We show in the following lemma how to determine the parameters (σ, b) prescribed in Definition 3 by a given probability tail bound.

Lemma 9. Suppose a random variable satisfies $\mathbb{P}(|X| \geq a) \leq C \exp(-\frac{a}{\nu})$ for given $C, \nu > 0$, then X is $(6\nu\sqrt{1+2C}, 6\nu)$ -subexponential. Also, if Y is a random variable that satisfy $|Y| \leq M$, then, XY is $(6M\nu\sqrt{1+2C}, 6M\nu)$ -subexponential.

Proof. First, consider $|X|$'s moment generating function $\mathbb{E}[e^{\lambda|X|}]$, for $\forall \lambda > 0$, we have:

$$\begin{aligned} \mathbb{E}[e^{\lambda|X|}] &= \int_0^{+\infty} \mathbb{P}\left(e^{\lambda|X|} \geq u\right) du \\ &\leq 1 + \int_1^{+\infty} \mathbb{P}\left(|X| \geq \frac{\ln u}{\lambda}\right) du \\ &\leq 1 + C \int_1^{+\infty} u^{-\frac{1}{\lambda\nu}} du. \end{aligned}$$

From the above inequality, we get that $\mathbb{E}[e^{\lambda|X|}] \leq 1 + \frac{C\lambda\nu}{1-\lambda\nu} < \infty$ if $0 < \lambda < \frac{1}{\nu}$. Set $\lambda = \frac{2}{3\nu}$, as $\mathbb{E}[e^{\frac{2}{3\nu}|X|}] = \sum_{i=0}^{\infty} \frac{\mathbb{E}[|X|^i]}{(\frac{3}{2}\nu)^i i!}$, we have:

$$\frac{\mathbb{E}[|X|^i]}{(\frac{3}{2}\nu)^i i!} \leq \mathbb{E}\left[e^{\frac{2}{3\nu}|X|}\right] \leq 1 + 2C. \quad (4)$$

Now we introduce a new random variable X' that is an independent copy of X , then we can bound $\mathbb{E}[\exp(\lambda(X - \mathbb{E}[X]))]$ by Jensen's inequality, $\mathbb{E}[\exp(\lambda(X - \mathbb{E}[X]))] \leq \mathbb{E}[\exp(\lambda(X - X'))]$. Therefore, we only need to bound $\mathbb{E}[\exp(\lambda(X - X'))]$. For $\forall \lambda \in [-\frac{1}{6\nu}, \frac{1}{6\nu}]$,

$$\begin{aligned} \mathbb{E}[\exp(\lambda(X - X'))] &= \sum_{i=0}^{\infty} \frac{\mathbb{E}[(X - X')^i \lambda^i]}{i!} = \sum_{i=0}^{\infty} \frac{\mathbb{E}[(X - X')^{2i} \lambda^{2i}]}{(2i)!} \\ &\leq 1 + \sum_{i=1}^{\infty} \frac{\mathbb{E}[|X|^{2i}] 2^{2i} \lambda^{2i}}{(2i)!} \leq 1 + (1 + 2C) \sum_{i=1}^{\infty} \left(\frac{3}{2}\nu\right)^{2i} 2^{2i} \lambda^{2i} \\ &= 1 + (1 + 2C) \sum_{i=1}^{\infty} (3\nu\lambda)^{2i} \leq 1 + 2(1 + 2C)(3\nu\lambda)^2 \\ &\leq \exp(2(1 + 2C)(3\nu\lambda)^2) = \exp\left(\frac{(6\nu\sqrt{1+2C})^2 \lambda^2}{2}\right), \end{aligned}$$

where the first equality holds by Taylor expansion, the second equality holds since $\mathbb{E}[(X - X')^i] = 0$ for all i 's that are odd. The first inequality holds by the fact that $|x - x'|^i \leq 2^{i-1}(|x|^i + |x'|^i)$ for all $i \geq 1$, and that X and X' have the same distribution. The second inequality holds by (4). The third inequality holds since $\sum_{i=1}^{\infty} (3\nu\lambda)^{2i} = \frac{(3\nu\lambda)^2}{1-(3\nu\lambda)^2} \leq 2 \times (3\nu\lambda)^2$ when $\lambda \in [-\frac{1}{6\nu}, \frac{1}{6\nu}]$. The last inequality holds by the fact that $1 + x \leq e^x$ for all $x \in \mathbb{R}$.

Thus, by definition of (σ, b) -subexponential, we conclude that X is $(6\nu\sqrt{1+2C}, 6\nu)$ -subexponential. Now we prove the subexponential property of XY . Since $|Y| \leq M$ and $\mathbb{P}(|X| \geq a) \leq C \exp(-\frac{a}{\nu})$,

$$\mathbb{P}(|XY| \geq a) \leq \mathbb{P}\left(|X| \geq \frac{a}{M}\right) \leq C \exp\left(-\frac{a}{M\nu}\right).$$

Replacing ν by $M\nu$, we conclude that XY is $(6M\nu\sqrt{1+2C}, 6M\nu)$ -subexponential. \square

Lemma 9 directly accommodates to our regularity assumption and leads to the following corollary.

Corollary 2. (Subexponential property of regular distributions) Suppose \mathbf{x} is a random variable that satisfies Assumption 2, then, for $\forall \mathbf{w} \in \mathbb{B}^d$, $\langle \mathbf{w}, \mathbf{x} \rangle$ is $(\frac{16}{Q}, \frac{6}{Q})$ -subexponential.

Proof. By Assumption 2 Condition 3, $\mathbb{P}(|\langle \mathbf{w}, \mathbf{x} \rangle| \geq t) \leq \exp(1 - Qt)$ for $\forall t > 0$. Then by Lemma 9, set $C = e$, $\nu = \frac{1}{Q}$, then, $6\nu\sqrt{1+2C} = \frac{6\sqrt{1+2e}}{Q} < \frac{16}{Q}$, $6\nu = \frac{6}{Q}$, we conclude that \mathbf{x} is $(\frac{16}{Q}, \frac{6}{Q})$ -subexponential. \square

The relationship between $\mathbb{E}[\langle \mathbf{w}^*, -\mathbf{g}_{k,i} \rangle]$ and $\theta(\mathbf{w}_{k,i}, \mathbf{w}^*)$ Recall that in the non-strategic setting, we shall adjust the coefficient by solving a sequence of adaptively constructed online regret minimization problems $\min_{\mathbf{w} \in \mathcal{W}_k} \sum_{i=1}^{T_k} \langle \mathbf{w}, \mathbf{g}_{k,i} \rangle - \sum_{i=1}^{T_k} \langle \mathbf{w}^*, \mathbf{g}_{k,i} \rangle$ with $\mathbf{g}_{k,i} = [-\bar{\eta}\mathbf{x}_{k,i}\mathbb{I}(y_{k,i} = 1) + (1 - \bar{\eta})\mathbf{x}_{k,i}\mathbb{I}(y_{k,i} = -1)]\mathbb{I}(\mathbf{x}_{k,i} \in D_{k,i})$ via mirror descent over $k = 0 \dots K$ batches, using local data within increasingly narrow bands $D_{k,i} = \{\mathbf{x} : 0 < \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq b_k\}$. The key ingredient underlying this guarantee is that the gradients $\mathbf{g}_{k,i}$ are well constructed so that $|\langle \mathbf{w}_{k,i}, \mathbf{g}_{k,i} \rangle|$ is small and meanwhile $\mathbb{E}[\langle \mathbf{w}^*, -\mathbf{g}_{k,i} \rangle]$ upper bounds $\theta(\mathbf{w}^*, \mathbf{w}_{k,i})$. Here, we show the relationship between $\mathbb{E}[\langle \mathbf{w}^*, -\mathbf{g}_{k,i} \rangle]$ and $\theta(\mathbf{w}_{k,i}, \mathbf{w}^*)$ for $k = 0, 1, \dots, K$, which is critical in the guarantees of Algorithm 3 and Algorithm 4.

Fix batch k and iteration i , to connect $\mathbb{E}[\langle \mathbf{w}^*, -\mathbf{g}_{k,i} \rangle]$ and $\theta(\mathbf{w}_{k,i}, \mathbf{w}^*)$ we introduce a new variable $f_{k,i}(\mathbf{w}_{k,i})$ in (5). Later, we will show how $\mathbb{E}[\langle \mathbf{w}^*, -\mathbf{g}_{k,i} \rangle]$ upper bounds $f_{k,i}(\mathbf{w}_{k,i})$ and how $f_{k,i}(\mathbf{w}_{k,i})$ approximates $\theta(\mathbf{w}_{k,i}, \mathbf{w}^*)$.

$$f_{k,i}(\mathbf{w}_{k,i}) := \mathbb{E}[|\langle \mathbf{w}^*, \mathbf{x} \rangle| \mathbb{I}(\langle \mathbf{w}^*, \mathbf{x} \rangle < 0) \mid \mathbf{x} \in D_{k,i}]. \quad (5)$$

Now, we show that $\mathbb{E}[\langle \mathbf{w}^*, -\mathbf{g}_{k,i} \rangle]$ can upper bound $f_{k,i}(\mathbf{w}_{k,i})$ by the following lemma.

Lemma 10. Given a unit vector $\mathbf{w}_{k,i} \in \mathbb{S}^d$ and an agent with true feature-label pair $(\mathbf{x}_{k,i}, y_{k,i})$. For $\mathbf{g}_{k,i} = [-\bar{\eta}\mathbf{x}_{k,i}\mathbb{I}(y_{k,i} = 1) + (1 - \bar{\eta})\mathbf{x}_{k,i}\mathbb{I}(y_{k,i} = -1)]\mathbb{I}(\mathbf{x}_{k,i} \in D_{k,i})$ and $f_{k,i}(\mathbf{w}_{k,i})$ defined in (5). The following holds.

$$\mathbb{E}[\langle \mathbf{w}^*, -\mathbf{g}_{k,i} \rangle] \geq (1 - 2\bar{\eta})f_{k,i}(\mathbf{w}_{k,i})\mathbb{P}(\mathbf{x} \in D_{k,i}).$$

Proof. First, for convenience, we rewrite $\mathbf{g}_{k,i}$ as the following.

$$\begin{aligned} \mathbf{g}_{k,i} &= [-\bar{\eta}\mathbf{x}_{k,i}\mathbb{I}(y_{k,i} = 1) + (1 - \bar{\eta})\mathbf{x}_{k,i}\mathbb{I}(y_{k,i} = -1)]\mathbb{I}(\mathbf{x}_{k,i} \in D_{k,i}) \\ &= \left(-\frac{1}{2}y_{k,i} + \left(\frac{1}{2} - \bar{\eta}\right)\right)\mathbf{x}_{k,i}\mathbb{I}(\mathbf{x}_{k,i} \in D_{k,i}). \end{aligned}$$

Then, we have

$$\begin{aligned}
\mathbb{E}[\langle \mathbf{w}^*, -\mathbf{g}_{k,i} \rangle] &= \mathbb{E} \left[\left\langle \mathbf{w}^*, \left(\frac{1}{2} y_{k,i} - \left(\frac{1}{2} - \bar{\eta} \right) \right) \mathbf{x}_{k,i} \right\rangle \mathbb{I}(\mathbf{x}_{k,i} \in D_{k,i}) \right] \\
&= \mathbb{E} \left[\left\langle \mathbf{w}^*, \left(\frac{1}{2} y_{k,i} - \left(\frac{1}{2} - \bar{\eta} \right) \right) \mathbf{x}_{k,i} \right\rangle \middle| \mathbf{x}_{k,i} \in D_{k,i} \right] \mathbb{P}(\mathbf{x}_{k,i} \in D_{k,i}) + 0 \\
&= \frac{1}{2} \mathbb{E}[\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle \mathbb{E}[y_{k,i} \mid \mathbf{x}_{k,i} \mid \mathbf{x}_{k,i} \in D_{k,i}] \mathbb{P}(\mathbf{x} \in D_{k,i}) \\
&\quad - \left(\frac{1}{2} - \bar{\eta} \right) \mathbb{E}[\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle \mid \mathbf{x}_{k,i} \in D_{k,i}] \mathbb{P}(\mathbf{x} \in D_{k,i})] \\
&\geq \left(\frac{1}{2} - \bar{\eta} \right) \mathbb{E}[\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle \mid \mathbf{x}_{k,i} \in D_{k,i}] \mathbb{P}(\mathbf{x} \in D_{k,i}) \\
&\quad - \left(\frac{1}{2} - \bar{\eta} \right) \mathbb{E}[\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle \mid \mathbf{x}_{k,i} \in D_{k,i}] \mathbb{P}(\mathbf{x} \in D_{k,i}) \\
&= (1 - 2\bar{\eta}) \mathbb{E}[\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle \mid \mathbb{I}(\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle < 0) \mid \mathbf{x}_{k,i} \in D_{k,i}] \mathbb{P}(\mathbf{x} \in D_{k,i}) \\
&= (1 - 2\bar{\eta}) f_{k,i}(\mathbf{w}_{k,i}) \mathbb{P}(\mathbf{x} \in D_{k,i}),
\end{aligned}$$

where the second and third equality hold by the law of iterated expectations, and the inequality holds as the following.

$$\begin{aligned}
\mathbb{E}[y_{k,i} \mid \mathbf{x}_{k,i}] &= (1 - \eta(\mathbf{x}_{k,i})) \text{sgn}(\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle) - \eta(\mathbf{x}_{k,i}) \text{sgn}(\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle) \\
&= (1 - 2\eta(\mathbf{x}_{k,i})) \text{sgn}(\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle) \\
&\geq (1 - 2\bar{\eta}) \text{sgn}(\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle).
\end{aligned}$$

□

Next, in the following lemma, we show that $f_{k,i}(\mathbf{w}_{k,i})$ measures the closeness of \mathbf{w}^* and $\mathbf{w}_{k,i}$.

Lemma 11. *For fixed batch k and iteration i , if $\theta(\mathbf{w}^*, \mathbf{w}_{k,i}) = \phi$, then the following holds.*

1. When $0 < b_k \leq \frac{R}{4}$ and $\phi \in [\frac{4b_k}{R}, \frac{\pi}{2}]$, we have

$$f_{k,i}(\mathbf{w}_{k,i}) \geq \frac{L_2}{32U_1} R^2 \phi.$$

2. When $0 < b_k \leq \frac{R}{4}$ and $\phi \in [\frac{\pi}{2}, \pi - \frac{4b_k}{R}]$, we have

$$f_{k,i}(\mathbf{w}_{k,i}) \geq \frac{L_2}{32U_1} R^2 (\pi - \phi).$$

Proof. We prove the two cases respectively. For Case 1, define the region $G_1 := \{\mathbf{x} \mid 0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq b_k, -\frac{1}{2}R \sin \phi \leq \langle \mathbf{w}^*, \mathbf{x} \rangle \leq -\frac{1}{4}R \sin \phi\}$, see Figure 6 as an illustration. We have the following

$$\begin{aligned}
&\mathbb{E}[\langle \mathbf{w}^*, \mathbf{x} \rangle \mid \mathbb{I}(\langle \mathbf{w}^*, \mathbf{x} \rangle < 0) \mathbb{I}(0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq b_k)] \\
&\geq \mathbb{E}[\langle \mathbf{w}^*, \mathbf{x} \rangle \mid \mathbb{I}(\mathbf{x} \in G_1)] \\
&\geq \frac{1}{4} R \sin \phi \mathbb{E}[\mathbb{I}(\mathbf{x} \in G_1)] \\
&\geq \frac{1}{8} R \phi \mathbb{P}(\mathbf{x} \in G_1) \\
&\geq \frac{1}{32} L_2 R^2 \phi b_k.
\end{aligned}$$

Where the first inequality holds since $G_1 \subseteq \{\mathbf{x} \mid 0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq b, \langle \mathbf{w}^*, \mathbf{x} \rangle < 0\}$. The third inequality holds by the fact that $\sin \phi \geq \frac{\phi}{2}$ for $0 \leq \phi \leq \frac{\pi}{2}$. And the last inequality holds by the claim that $\mathbb{P}(\mathbf{x} \in G_1) \geq \frac{1}{4} L_2 R b_k$ which we will show later.

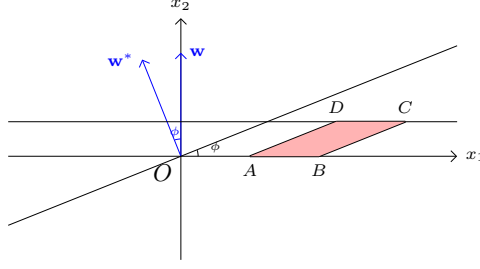


Figure 6: Illustration of region G_1 (the red region) in Case 1. Which satisfies $G_1 = \{\mathbf{x} \mid 0 \leq \langle \mathbf{w}, \mathbf{x} \rangle \leq b_k, -\frac{1}{2}R \sin \phi \leq \langle \mathbf{w}^*, \mathbf{x} \rangle \leq -\frac{1}{4}R \sin \phi\}$.

Hence, we can establish the lower bound of $f_{k,i}(\mathbf{w}_{k,i})$ by:

$$\begin{aligned}
 f_{k,i}(\mathbf{w}_{k,i}) &= \mathbb{E}[\langle \mathbf{w}^*, \mathbf{x} \rangle \mid \mathbb{I}(\langle \mathbf{w}^*, \mathbf{x} \rangle < 0) \mid 0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq b_k] \\
 &= \frac{\mathbb{E}[\langle \mathbf{w}^*, \mathbf{x} \rangle \mid \mathbb{I}(\langle \mathbf{w}^*, \mathbf{x} \rangle < 0) \mid 0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq b_k]}{\mathbb{P}(0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq b_k)} \\
 &\geq \frac{\frac{1}{32}L_2R^2\phi b_k}{\mathbb{P}(0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq b_k)} \\
 &\geq \frac{L_2}{32U_1}R^2\phi.
 \end{aligned}$$

Where the last inequality holds by Lemma 4, property (a).

Now we show the claim that $\mathbb{P}(\mathbf{x} \in G_1) \geq \frac{1}{4}L_2Rb_k$. For a given vector \mathbf{x} , we first project \mathbf{x} down to the subspace $V_2 \subseteq \mathbb{R}^2$ spanned by \mathbf{w}^* and $\mathbf{w}_{k,i}$ and denote the projected value $\mathbf{z} := (\langle \mathbf{w}^*, \mathbf{x} \rangle, \langle \mathbf{w}, \mathbf{x} \rangle)$.

Without loss of generality, let $\mathbf{w} = (0, 1)$ and $\mathbf{w}^* = (\sin \phi, \cos \phi)$. As illustrated in Figure 6, the parallelogram ABCD denotes the region G_1 , where $A = (\frac{1}{4}R, 0)$, $B = (\frac{1}{2}R, 0)$, $C = (\frac{1}{2}R + \frac{b_k}{\tan \phi}, b_k)$, $D = (\frac{1}{4}R + \frac{b_k}{\tan \phi}, b_k)$. Since C is the farthest point to the origin with respect to the Euclidean Norm, and $\left\| \left(\frac{1}{2}R + \frac{b_k}{\tan \phi}, b_k \right) \right\|_2 \leq \left\| \left(\frac{1}{2}R + \frac{b_k}{\tan \phi}, b_k \right) \right\|_1 = \frac{1}{2}R + \frac{b_k}{\tan \phi} + b_k \leq \frac{1}{2}R + \frac{b_k}{\phi} + b_k \leq R$, then for all $\mathbf{z} \in \{\mathbf{z} = (z_1, z_2) \mid -\frac{1}{2}R \sin \phi \leq z_1 \leq -\frac{1}{4}R \sin \phi, 0 \leq z_2 \leq b_k\}$, we have $\|\mathbf{z}\|_2 \leq R$. Also, the area of parallelogram ABCD is $b_k \cdot \frac{1}{4}R = \frac{1}{4}Rb_k$. In addition, by Assumption 2, condition 1, the density $\phi_{V_2}(\mathbf{z})$ of projected value \mathbf{z} satisfies $\phi_{V_2}(\mathbf{z}) \geq L_2$ for all $\mathbf{z} \in V_2 \cap \mathbb{B}^2(R)$. Hence, we can lower bound $\mathbb{P}(\mathbf{x} \in G_1)$ by

$$\mathbb{P}(\mathbf{x} \in G_1) \geq L_2 \cdot \frac{1}{4}Rb_k = \frac{1}{4}L_2Rb_k.$$

The proof of Case 2 is similar to that of Case 1. Define the region $G_2 = \{\mathbf{x} \mid 0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq b_k, -\frac{1}{2}R \sin(\pi - \phi) \leq \langle \mathbf{w}^*, \mathbf{x} \rangle \leq -\frac{1}{4}R \sin(\pi - \phi)\}$ (see Figure 7). We replace ϕ in the proof of Case 1 by $\pi - \phi$, by choosing $A = (-\frac{1}{4}R, 0)$, $B = (-\frac{1}{2}R, 0)$, $C = (-\frac{1}{2}R + \frac{b_k}{\tan(\pi - \phi)}, b_k)$, $D = (-\frac{1}{4}R + \frac{b_k}{\tan(\pi - \phi)}, b_k)$ and then we complete the proof. \square

Lemma 11 directly leads to the following corollary.

Corollary 3. If $\theta(\mathbf{w}^*, \mathbf{w}_{k,i}) \leq \frac{\pi}{2}$, and $f_{k,i}(\mathbf{w}_{k,i}) \leq \frac{L_2}{160U_1}R^2\theta_k$, then $\theta(\mathbf{w}^*, \mathbf{w}_{k,i}) \leq \frac{\theta_k}{5}$.

Proof. We conduct a case analysis.

1. If $\theta(\mathbf{w}^*, \mathbf{w}_{k,i}) < \frac{4b_k}{R}$, then by our setting of b_k in Algorithm 3 and Algorithm 4, $\theta(\mathbf{w}^*, \mathbf{w}_{k,i}) < \frac{4b_k}{R} \leq \frac{\theta_k}{5}$;

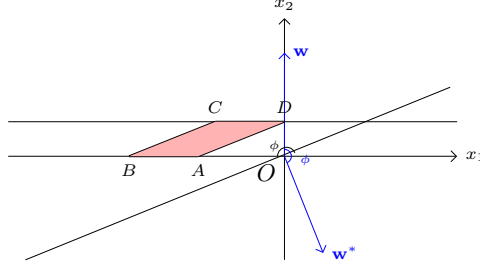


Figure 7: Illustration of region G_2 (the red region) in Case 2. Which satisfies $G_2 = \{\mathbf{x} \mid 0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq b_k, -\frac{1}{2}R \sin(\pi - \phi) \leq \langle \mathbf{w}^*, \mathbf{x} \rangle \leq -\frac{1}{4}R \sin(\pi - \phi)\}$.

2. If $\frac{4b_k}{R} \leq \theta(\mathbf{w}^*, \mathbf{w}_{k,i}) \leq \frac{\pi}{2}$, then by Lemma 11, we have $f_{k,i}(\mathbf{w}_{k,i}) \geq \frac{L_2}{32U_1} R^2 \theta(\mathbf{w}^*, \mathbf{w}_{k,i})$, combining it with the condition that $f_{k,i}(\mathbf{w}_{k,i}) \leq \frac{L_2}{160U_1} R^2 \theta_k$, we have $\theta(\mathbf{w}^*, \mathbf{w}_{k,i}) \leq \frac{\theta_k}{5}$. \square

Other lemmas We also outline some other lemmas that are used in the subsequent proof.

Lemma 12. (Triangular inequality of angles) Suppose vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^d$ satisfy $0 \leq \theta(\mathbf{x}, \mathbf{y}) \leq \frac{\pi}{2}$ and $0 \leq \theta(\mathbf{x}, \mathbf{z}) \leq \frac{\pi}{2}$, then,

$$|\theta(\mathbf{x}, \mathbf{y}) - \theta(\mathbf{x}, \mathbf{z})| \leq \theta(\mathbf{y}, \mathbf{z}) \leq \theta(\mathbf{x}, \mathbf{y}) + \theta(\mathbf{x}, \mathbf{z}).$$

Proof. Without loss of generality, assume that $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = \|\mathbf{z}\|_2 = 1$. First, we decompose vectors \mathbf{y} and \mathbf{z} into components along the direction of \mathbf{x} and perpendicular to \mathbf{x} , respectively, as:

$$\begin{aligned} \mathbf{y} &= \cos(\theta(\mathbf{x}, \mathbf{y}))\mathbf{x} + \sin(\theta(\mathbf{x}, \mathbf{y}))\mathbf{y}_{\perp\mathbf{x}}, \\ \mathbf{z} &= \cos(\theta(\mathbf{x}, \mathbf{z}))\mathbf{x} + \sin(\theta(\mathbf{x}, \mathbf{z}))\mathbf{z}_{\perp\mathbf{x}}, \end{aligned}$$

where $\mathbf{y}_{\perp\mathbf{x}}$ and $\mathbf{z}_{\perp\mathbf{x}}$ are unit vectors that are perpendicular to \mathbf{x} . Hence, we have

$$\cos(\theta(\mathbf{y}, \mathbf{z})) = \langle \mathbf{y}, \mathbf{z} \rangle = \cos(\theta(\mathbf{x}, \mathbf{y}))\cos(\theta(\mathbf{x}, \mathbf{z})) + \sin(\theta(\mathbf{x}, \mathbf{y}))\sin(\theta(\mathbf{x}, \mathbf{z}))\langle \mathbf{y}_{\perp\mathbf{x}}, \mathbf{z}_{\perp\mathbf{x}} \rangle,$$

where the second equality holds since $\langle \mathbf{x}, \mathbf{y}_{\perp\mathbf{x}} \rangle = \langle \mathbf{x}, \mathbf{z}_{\perp\mathbf{x}} \rangle = 0$. Since $\mathbf{y}_{\perp\mathbf{x}}$ and $\mathbf{z}_{\perp\mathbf{x}}$ are both unit vectors, by Cauchy-Schwarz inequality, $-1 \leq \langle \mathbf{y}_{\perp\mathbf{x}}, \mathbf{z}_{\perp\mathbf{x}} \rangle \leq 1$. Also, since $0 \leq \theta(\mathbf{x}, \mathbf{y}) \leq \frac{\pi}{2}$ and $0 \leq \theta(\mathbf{x}, \mathbf{z}) \leq \frac{\pi}{2}$, we have $\sin(\theta(\mathbf{x}, \mathbf{y}))\sin(\theta(\mathbf{x}, \mathbf{z})) \geq 0$. Putting all together, we have:

$$\begin{aligned} \cos(\theta(\mathbf{y}, \mathbf{z})) &\leq \cos(\theta(\mathbf{x}, \mathbf{y}))\cos(\theta(\mathbf{x}, \mathbf{z})) + \sin(\theta(\mathbf{x}, \mathbf{y}))\sin(\theta(\mathbf{x}, \mathbf{z})) \\ &= \cos(\theta(\mathbf{x}, \mathbf{y}) - \theta(\mathbf{x}, \mathbf{z})), \end{aligned} \tag{6}$$

and

$$\begin{aligned} \cos(\theta(\mathbf{y}, \mathbf{z})) &\geq \cos(\theta(\mathbf{x}, \mathbf{y}))\cos(\theta(\mathbf{x}, \mathbf{z})) - \sin(\theta(\mathbf{x}, \mathbf{y}))\sin(\theta(\mathbf{x}, \mathbf{z})) \\ &= \cos(\theta(\mathbf{x}, \mathbf{y}) + \theta(\mathbf{x}, \mathbf{z})). \end{aligned} \tag{7}$$

Since $\cos(x)$ is decreasing in $x \in [0, \pi]$, and $\cos(x) = \cos(-x)$ for all $x \in \mathbb{R}$, we get $\theta(\mathbf{y}, \mathbf{z}) \geq |\theta(\mathbf{x}, \mathbf{y}) - \theta(\mathbf{x}, \mathbf{z})|$ from (6), and $\theta(\mathbf{y}, \mathbf{z}) \leq \theta(\mathbf{x}, \mathbf{y}) + \theta(\mathbf{x}, \mathbf{z})$ from (7). \square

Lemma 13. Given a random vector $\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}$ that satisfy Assumption 2, then with probability at least $1 - \delta$,

$$\|\mathbf{x}\|_{\infty} \leq \frac{1}{Q} \left(1 + \ln \left(\frac{d}{\delta} \right) \right)$$

Proof. We bound $\|\mathbf{x}\|_{\infty}$ element-wisely. Given $\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}$ and $j \in [d]$, let x_j be the j -th coordinate of \mathbf{x} . Let $\mathbf{e}^{[j]} \in \mathbb{R}^d$ denote the unit vector whose j 'th coordinate is 1 while other coordinate is 0, then, by Assumption 2, condition 3, for $\forall a > 0$

$$\mathbb{P}(|x_j| \geq a) = \mathbb{P}\left(\left|\langle \mathbf{e}^{[j]}, \mathbf{x} \rangle\right| \geq a\right) \leq \exp(1 - Qa).$$

Taking union bound over all coordinates, we have

$$\mathbb{P}(\|\mathbf{x}\|_{\infty} \geq a) \leq d \exp(1 - Qa).$$

Taking $a = \frac{1}{Q} \left(1 + \ln \left(\frac{d}{\delta} \right) \right)$ in the above inequality and hence we complete the proof. \square

A.4 Proofs for Section 2

In this subsection, we outline some important results to describe the relationships between classifier $h(\cdot)$ for unmanipulated features \mathbf{x} and classifier \tilde{h} for reported features $\mathbf{r} = \mathbf{r}^*(\mathbf{x}, \tilde{h})$, which is critical for the subsequent algorithm design.

Proof of Proposition 1

Proposition 1. *For any $(\mathbf{w}, m) \in \mathbb{S}^d \times \mathbb{R}$, the output of $\tilde{h}(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}, \mathbf{r} \rangle + m - \gamma)$ for $\mathbf{r} = \mathbf{r}^*(\mathbf{x}, \tilde{h})$ is identical to the output of $h(\mathbf{x}) = \text{sgn}(\langle \mathbf{w}, \mathbf{x} \rangle + m)$ for any $\mathbf{x} \in \mathbb{R}^d$.*

Proof. For fixed $\mathbf{w} \in \mathbb{S}^d$ and $m \in \mathbb{R}$, we categorize the agent population into three classes according to their true features \mathbf{x} : $\{\mathbf{x} \mid \langle \mathbf{w}, \mathbf{x} \rangle + m < 0\}$, $\{\mathbf{x} \mid 0 \leq \langle \mathbf{w}, \mathbf{x} \rangle + m < \gamma\}$ and $\{\mathbf{x} \mid \langle \mathbf{w}, \mathbf{x} \rangle + m \geq \gamma\}$. Then, we discuss their classification output by $h(\mathbf{x})$ and $\tilde{h}(\mathbf{r})$ with respect to $\mathbf{r} = \mathbf{r}^*(\mathbf{x}, \tilde{h})$, respectively.

1) When $\langle \mathbf{w}, \mathbf{x} \rangle + m < 0$, $h(\mathbf{x}) = -1$. At the same time, $\langle \mathbf{w}, \mathbf{x} \rangle + m - \gamma < -\gamma$, by Lemma 1, agent in this region will report his feature truthfully, i.e., $\mathbf{r} = \mathbf{x}$. Thus, $\tilde{h}(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}, \mathbf{r} \rangle + m - \gamma) = \text{sgn}(\langle \mathbf{w}, \mathbf{x} \rangle + m - \gamma) = -1$. Hence, we have $h(\mathbf{x}) = \tilde{h}(\mathbf{r}) = -1$ for $\forall \mathbf{x} \in \{\mathbf{x} \mid \langle \mathbf{w}, \mathbf{x} \rangle + m < 0\}$.

2) When $0 \leq \langle \mathbf{w}, \mathbf{x} \rangle + m < \gamma$, $h(\mathbf{x}) = 1$. At the same time, $-\gamma \leq \langle \mathbf{w}, \mathbf{x} \rangle + m - \gamma < 0$, by Lemma 1, agent in this region will manipulate his feature as $\mathbf{r} = \mathbf{x} + (\gamma - m - \langle \mathbf{w}, \mathbf{x} \rangle)\mathbf{w}$. Thus, $\tilde{h}(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}, \mathbf{r} \rangle + m - \gamma) = \text{sgn}(\langle \mathbf{w}, \mathbf{x} + (\gamma - m - \langle \mathbf{w}, \mathbf{x} \rangle)\mathbf{w} \rangle + m - \gamma) = 1$. Hence, we have $h(\mathbf{x}) = \tilde{h}(\mathbf{r}) = 1$ for $\forall \mathbf{x} \in \{\mathbf{x} \mid 0 \leq \langle \mathbf{w}, \mathbf{x} \rangle + m < \gamma\}$.

3) When $\langle \mathbf{w}, \mathbf{x} \rangle + m \geq \gamma$, $h(\mathbf{x}) = 1$. At the same time, $\langle \mathbf{w}, \mathbf{x} \rangle + m - \gamma \geq 0$, by Lemma 1, agent in this region will report his feature truthfully, i.e., $\mathbf{r} = \mathbf{x}$. Thus, $\tilde{h}(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}, \mathbf{r} \rangle + m - \gamma) = \text{sgn}(\langle \mathbf{w}, \mathbf{x} \rangle + m - \gamma) = 1$. Hence, we have $h(\mathbf{x}) = \tilde{h}(\mathbf{r}) = 1$ for $\forall \mathbf{x} \in \{\mathbf{x} \mid \langle \mathbf{w}, \mathbf{x} \rangle + m \geq \gamma\}$. \square

Inferring agents' true features from their reported features Proposition 1 directly leads to the following corollary, enabling us to infer an agent's true features \mathbf{x} given a classification rule $\tilde{h}(\cdot)$ and his corresponding reported features \mathbf{r} .

Corollary 4. *For given announced classifier $\tilde{h}(\cdot) = \text{sgn}(\langle \mathbf{w}, \cdot \rangle + m)$ and agent response \mathbf{r} , then, his true features \mathbf{x} satisfy the following.*

1. if $\langle \mathbf{w}, \mathbf{r} \rangle + m \neq 0$, then $\mathbf{x} = \mathbf{r}$;
2. if $\langle \mathbf{w}, \mathbf{r} \rangle + m = 0$, then $-\gamma \leq \langle \mathbf{w}, \mathbf{x} \rangle + m \leq 0$.

A.5 Proofs for Section 3

In this subsection, we show the theoretic guarantees of Algorithm 2, Algorithm 3 and Algorithm 4, respectively.

Theoretical Guarantees of Algorithm 2 A key observation in the non-strategic and noiseless classification scenario is that $y \langle \mathbf{w}^*, \mathbf{x} \rangle > 0$ holds for all (\mathbf{x}, y) . Consequently, $\mathbf{x}y$ always forms an acute angle with the optimal normal vector \mathbf{w}^* . Considering agents' strategic responses and the bandit feedback setting, we introduce Lemma 14 to show that, under our construction in Algorithm 2, $\bar{\mathbf{w}}_0$ is an unbiased estimator of $\mathbb{E}[\mathbf{x}y]$.

Lemma 14. *In Algorithm 2, we have*

$$\mathbb{E}[\bar{\mathbf{w}}_0] = \mathbb{E}[\mathbf{x}y].$$

Proof. In Algorithm 2, at iteration i , the principal declares classifier $\tilde{h}_{\text{init},i}^{(1)}(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}_{\text{init},i}, \mathbf{r} \rangle)$ and $\tilde{h}_{\text{init},i}^{(2)}(\mathbf{r}) = \text{sgn}(\langle -\mathbf{w}_{\text{init},i}, \mathbf{r} \rangle)$, and receives response $\mathbf{r}_{\text{init},i}^{(1)}$ and $\mathbf{r}_{\text{init},i}^{(2)}$ respectively. By Corollary 4, we get that $\mathbf{r}_{\text{init},i}^{(1)} \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(1)} \rangle > 0) = \mathbf{x}_{\text{init},i}^{(1)} \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{x}_{\text{init},i}^{(1)} \rangle > 0)$ and

$\mathbf{r}_{\text{init},i}^{(2)} \mathbb{I}(\langle -\mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(2)} \rangle > 0) = \mathbf{x}_{\text{init},i}^{(2)} \mathbb{I}(\langle -\mathbf{w}_{\text{init},i}, \mathbf{x}_{\text{init},i}^{(2)} \rangle > 0)$. Also, $(\mathbf{x}_{\text{init},i}^{(1)}, y_{\text{init},i}^{(1)})$ and $(\mathbf{x}_{\text{init},i}^{(2)}, y_{\text{init},i}^{(2)})$ are i.i.d. drawn from \mathcal{D} , hence,

$$\begin{aligned} & \mathbb{E} \left[\mathbf{r}_{\text{init},i}^{(1)} y_{\text{init},i}^{(1)} \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(1)} \rangle > 0) + \mathbf{r}_{\text{init},i}^{(2)} y_{\text{init},i}^{(2)} \mathbb{I}(\langle -\mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(2)} \rangle > 0) \right] \\ &= \mathbb{E} \left[\mathbf{x}_{\text{init},i}^{(1)} y_{\text{init},i}^{(1)} \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{x}_{\text{init},i}^{(1)} \rangle > 0) + \mathbf{x}_{\text{init},i}^{(2)} y_{\text{init},i}^{(2)} \mathbb{I}(\langle -\mathbf{w}_{\text{init},i}, \mathbf{x}_{\text{init},i}^{(2)} \rangle > 0) \right] \\ &= \mathbb{E} [\mathbf{x} y \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{x} \rangle > 0)] + \mathbb{E} [\mathbf{x} y \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{x} \rangle < 0)] \\ &= \mathbb{E} [\mathbf{x} y] . \end{aligned}$$

Thus, we have

$$\begin{aligned} \mathbb{E} [\bar{\mathbf{w}}_0] &= \frac{1}{T_{\text{init}}} \sum_{i=1}^{T_{\text{init}}} \mathbb{E} \left[\mathbf{r}_{\text{init},i}^{(1)} y_{\text{init},i}^{(1)} \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(1)} \rangle > 0) + \mathbf{r}_{\text{init},i}^{(2)} y_{\text{init},i}^{(2)} \mathbb{I}(\langle -\mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(2)} \rangle > 0) \right] \\ &= \mathbb{E} [\mathbf{x} y] \end{aligned}$$

□

Now we show that $\bar{\mathbf{w}}_0$ constructed by Algorithm 2 has a positive inner product with the optimal coefficient \mathbf{w}^* with high probability.

Proposition 2. *For some constants $c_0, c_1 > 0$, when Algorithm 2 runs for $T_{\text{init}} = c_0 \ln T / (1 - 2\bar{\eta})^2$ iterations, its output $\bar{\mathbf{w}}_0$ satisfies $\langle \mathbf{w}^*, \bar{\mathbf{w}}_0 \rangle > c_1 (1 - 2\bar{\eta}) > 0$ and $\theta(\mathbf{w}^*, \bar{\mathbf{w}}_0) \leq \frac{\pi}{2}$ with probability at least $1 - 2/T^2$.*

Proof. First, considering the non-strategic classification problem, we establish a lower bound of $\mathbb{E}[\langle \mathbf{w}^*, \mathbf{x} \rangle y]$ as the following.

$$\begin{aligned} \mathbb{E}[\langle \mathbf{w}^*, \mathbf{x} \rangle y] &= \mathbb{E} [\mathbb{E}[y | \mathbf{x}] \langle \mathbf{w}^*, \mathbf{x} \rangle] \\ &= \mathbb{E} [(1 - \eta(\mathbf{x})) \text{sgn}(\langle \mathbf{w}^*, \mathbf{x} \rangle) - \eta(\mathbf{x}) \text{sgn}(\langle \mathbf{w}^*, \mathbf{x} \rangle)] \langle \mathbf{w}^*, \mathbf{x} \rangle] \\ &= \mathbb{E} [(1 - 2\eta(\mathbf{x})) | \langle \mathbf{w}^*, \mathbf{x} \rangle |] \\ &\geq (1 - 2\bar{\eta}) \mathbb{E} [| \langle \mathbf{w}^*, \mathbf{x} \rangle |] \\ &\geq (1 - 2\bar{\eta}) L_1 R^2 , \end{aligned} \tag{8}$$

where the first equality holds by the law of iterated expectations. The second equality holds by the definition of Massart Noise. The third equality holds since $\text{sgn}(\langle \mathbf{w}^*, \mathbf{x} \rangle \langle \mathbf{w}^*, \mathbf{x} \rangle) = | \langle \mathbf{w}^*, \mathbf{x} \rangle |$. The first inequality holds because $\eta(\mathbf{x}) \leq \bar{\eta}$, $\forall \mathbf{x} \in \mathbb{R}^d$. Now we prove $\mathbb{E} [| \langle \mathbf{w}^*, \mathbf{x} \rangle |] \geq L_1 R^2$ for the last inequality as the following: let $\mathbf{x}_{V_1} := \langle \mathbf{w}^*, \mathbf{x} \rangle$ denote a 1-dimensional projection of \mathbf{x} . Then, by Assumption 2, condition 1, $\phi_{V_1}(\mathbf{x}_{V_1}) \geq L_1$ for all $-R \leq \mathbf{x}_{V_1} \leq R$. Hence, we have

$$\mathbb{E} [| \langle \mathbf{w}^*, \mathbf{x} \rangle |] \geq 2 \int_0^R \mathbf{x}_{V_1} \phi_{V_1}(\mathbf{x}_{V_1}) d\mathbf{x}_{V_1} \geq L_1 \times 2 \int_0^R x dx = L_1 R^2 .$$

Second, considering agents' strategic response, we find an unbiased estimator of $\mathbb{E}[\langle \mathbf{w}^*, \mathbf{x} \rangle y]$ through samples. By Lemma 14, $\bar{\mathbf{w}}_0 = \frac{1}{T_{\text{init}}} \sum_{i=1}^{T_{\text{init}}} \mathbf{r}_{\text{init},i}^{(1)} y_{\text{init},i}^{(1)} \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(1)} \rangle > 0) + \mathbf{r}_{\text{init},i}^{(2)} y_{\text{init},i}^{(2)} \mathbb{I}(\langle -\mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(2)} \rangle > 0)$ is an unbiased estimator of $\mathbb{E}[\mathbf{x} y]$. Therefore, $\mathbb{E}[\langle \mathbf{w}^*, \bar{\mathbf{w}}_0 \rangle] = \mathbb{E}[\langle \mathbf{w}^*, \mathbf{x} \rangle y] \geq (1 - 2\bar{\eta}) L_1 R^2$.

Finally, using the results of concentration inequalities, we establish the high probability bound of $\langle \mathbf{w}^*, \bar{\mathbf{w}}_0 \rangle$. By Corollary 1, $\langle \mathbf{w}^*, \mathbf{r}_{\text{init},i}^{(1)} \rangle \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(1)} \rangle > 0) = \langle \mathbf{w}^*, \mathbf{x}_{\text{init},i}^{(1)} \rangle \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{x}_{\text{init},i}^{(1)} \rangle > 0)$ and $\langle \mathbf{w}^*, \mathbf{r}_{\text{init},i}^{(2)} \rangle \mathbb{I}(\langle -\mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(2)} \rangle > 0) = \langle \mathbf{w}^*, \mathbf{x}_{\text{init},i}^{(2)} \rangle \mathbb{I}(\langle -\mathbf{w}_{\text{init},i}, \mathbf{x}_{\text{init},i}^{(2)} \rangle > 0)$. Then, by Assumption 2 condition 3 and Lemma 9, we get that both $\langle \mathbf{w}^*, \mathbf{r}_{\text{init},i}^{(1)} \rangle y_{\text{init},i}^{(1)} \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(1)} \rangle > 0)$ and $\langle \mathbf{w}^*, \mathbf{r}_{\text{init},i}^{(2)} \rangle y_{\text{init},i}^{(2)} \mathbb{I}(\langle -\mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(2)} \rangle > 0)$

are $\left(\frac{16}{Q}, \frac{6}{Q}\right)$ -subexponential. Thus, by Lemma 6, we have that with probability at least $1 - \frac{1}{T^2}$,

$$\begin{aligned} & \frac{1}{T_{\text{init}}} \sum_{i=1}^{T_{\text{init}}} \left\langle \mathbf{w}^*, \mathbf{r}_{\text{init},i}^{(1)} \right\rangle y_{\text{init},i}^{(1)} \mathbb{I} \left(\left\langle \mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(1)} \right\rangle > 0 \right) \\ & \geq \frac{1}{T_{\text{init}}} \sum_{i=1}^{T_{\text{init}}} \mathbb{E} [\langle \mathbf{w}^*, \mathbf{x} \rangle y \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{x} \rangle > 0)] - \frac{32}{Q} \sqrt{\frac{\ln T}{T_{\text{init}}}} - \frac{24}{Q} \frac{\ln T}{T_{\text{init}}}, \end{aligned} \quad (9)$$

and with probability $1 - \frac{1}{T^2}$,

$$\begin{aligned} & \frac{1}{T_{\text{init}}} \sum_{i=1}^{T_{\text{init}}} \left\langle \mathbf{w}^*, \mathbf{r}_{\text{init},i}^{(2)} \right\rangle y_{\text{init},i}^{(2)} \mathbb{I} \left(\left\langle -\mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(2)} \right\rangle > 0 \right) \\ & \geq \frac{1}{T_{\text{init}}} \sum_{i=1}^{T_{\text{init}}} \mathbb{E} [\langle \mathbf{w}^*, \mathbf{x} \rangle y \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{x} \rangle < 0)] - \frac{32}{Q} \sqrt{\frac{\ln T}{T_{\text{init}}}} - \frac{24}{Q} \frac{\ln T}{T_{\text{init}}}. \end{aligned} \quad (10)$$

Taking the union bound for (9) and (10), then, with probability at least $1 - \frac{2}{T^2}$,

$$\begin{aligned} \langle \mathbf{w}^*, \bar{\mathbf{w}}_0 \rangle &= \frac{1}{T_{\text{init}}} \sum_{i=1}^{T_{\text{init}}} \left\langle \mathbf{w}^*, \mathbf{r}_{\text{init},i}^{(1)} \right\rangle y_{\text{init},i}^{(1)} \mathbb{I} \left(\left\langle \mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(1)} \right\rangle > 0 \right) \\ & \quad + \frac{1}{T_{\text{init}}} \sum_{i=1}^{T_{\text{init}}} \left\langle \mathbf{w}^*, \mathbf{r}_{\text{init},i}^{(2)} \right\rangle y_{\text{init},i}^{(2)} \mathbb{I} \left(\left\langle -\mathbf{w}_{\text{init},i}, \mathbf{r}_{\text{init},i}^{(2)} \right\rangle > 0 \right) \\ & \geq \frac{1}{T_{\text{init}}} \sum_{i=1}^{T_{\text{init}}} \mathbb{E} [\langle \mathbf{w}^*, \mathbf{x} \rangle y \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{x} \rangle > 0)] - \frac{32}{Q} \sqrt{\frac{\ln T}{T_{\text{init}}}} - \frac{24}{Q} \frac{\ln T}{T_{\text{init}}} \\ & \quad + \frac{1}{T_{\text{init}}} \sum_{i=1}^{T_{\text{init}}} \mathbb{E} [\langle \mathbf{w}^*, \mathbf{x} \rangle y \mathbb{I}(\langle \mathbf{w}_{\text{init},i}, \mathbf{x} \rangle < 0)] - \frac{32}{Q} \sqrt{\frac{\ln T}{T_{\text{init}}}} - \frac{24}{Q} \frac{\ln T}{T_{\text{init}}} \\ & = \mathbb{E} [\langle \mathbf{w}^*, \mathbf{x} \rangle y] - \frac{64}{Q} \sqrt{\frac{\ln T}{T_{\text{init}}}} - \frac{48}{Q} \frac{\ln T}{T_{\text{init}}} \\ & \geq (1 - 2\bar{\eta}) L_1 R^2 - \frac{64}{Q} \sqrt{\frac{\ln T}{T_{\text{init}}}} - \frac{48}{Q} \frac{\ln T}{T_{\text{init}}} \\ & \geq \frac{1}{2} (1 - 2\bar{\eta}) L_1 R^2. \end{aligned}$$

The first inequality holds by (9) and (10). The second inequality holds by Lemma 14. The last inequality holds by setting $T_{\text{init}} = \left\lceil \frac{20736 \ln T}{(1-2\bar{\eta})^2 L_1^2 Q^2 R^4} \right\rceil$, then $\frac{64}{Q} \sqrt{\frac{\ln T}{T_{\text{init}}}} \leq \frac{4}{9} (1-2\bar{\eta}) L_1 R^2$ and $\frac{48}{Q} \frac{\ln T}{T_{\text{init}}} \ll \frac{1}{18} (1-2\bar{\eta}) L_1 R^2$ for large enough T . Thus, we complete the proof of Proposition 2. \square

Theoretical Guarantees of Algorithm 3 Recall that in the i 'th iteration of Algorithm 3, we declare classifier $\tilde{h}_{0,i}(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}_{0,i}, \mathbf{r} \rangle)$ and construct the gradient as $\tilde{\mathbf{g}}_{0,i} = [-\bar{\eta} \mathbf{r}_{0,i} \mathbb{I}(y_{0,i} = 1) + (1 - \bar{\eta}) \mathbf{r}_{0,i} \mathbb{I}(y_{0,i} = -1)] \mathbb{I}(0 < \langle \mathbf{w}_{0,i}, \mathbf{r}_{0,i} \rangle \leq b_0)$. In Lemma 15, we establish the high probability upperbound of $\sum_{i=1}^{T_0} \mathbb{E}[\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle] | \mathcal{F}_{0,i-1}]$ by $\sum_{i=1}^{T_0} \langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle$.

Lemma 15. *In Algorithm 3, with probability at least $1 - 1/T^2$, we have*

$$\begin{aligned} & \sum_{i=1}^{T_0} \mathbb{E}[\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle | \mathcal{F}_{0,i-1}] \leq \sum_{i=1}^{T_0} \langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle \\ & \quad + \frac{24}{\delta} \sqrt{1 + 2\rho b_0} \sqrt{T_0 \ln T} + \frac{48}{\delta} \ln T, \end{aligned}$$

where $\rho = \max \left\{ U_1 \exp(\delta), \frac{U_2 \exp(\delta)}{\delta} \right\}$.

Proof. Since $\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle = \langle \mathbf{w}^*, \mathbf{x}_{0,i} \rangle \mathbb{I}(0 < \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0) \left(\frac{1}{2} y_{0,i} - \left(\frac{1}{2} - \bar{\eta} \right) \right)$, we first establish the probability tail bound of $\langle \mathbf{w}^*, \mathbf{x}_{0,i} \rangle \mathbb{I}(0 < \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0)$.

We partition $\mathbf{x}_{0,i}$ into two orthonormal vectors, for notational convenience, we omit the subindex 0, i of $\mathbf{x}_{0,i}$ and let $\mathbf{x}_{\parallel \mathbf{w}}$ denote the ingredient of $\mathbf{x}_{0,i}$ that is parallel to $\mathbf{w}_{0,i}$, *i.e.*, $\mathbf{x}_{\parallel \mathbf{w}} = \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \mathbf{w}_{0,i}$ and $\mathbf{x}_{\perp \mathbf{w}}$ denote the ingredient of $\mathbf{x}_{0,i}$ that is vertical to $\mathbf{w}_{0,i}$, *i.e.*, $\mathbf{x}_{\perp \mathbf{w}} = \mathbf{x}_{0,i} - \mathbf{x}_{\parallel \mathbf{w}}$. Then,

$$\begin{aligned} & \langle \mathbf{w}^*, \mathbf{x}_{0,i} \rangle \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0) \\ &= \underbrace{\langle \mathbf{w}^*, \mathbf{x}_{\parallel \mathbf{w}} \rangle \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0)}_{(a)} + \underbrace{\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}} \rangle \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0)}_{(b)}. \end{aligned} \quad (11)$$

Thus, we have to bound part (a) and (b) in (11), respectively. First, we bound part (a) as

$$\begin{aligned} \langle \mathbf{w}^*, \mathbf{x}_{\parallel \mathbf{w}} \rangle \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0) &= \langle \mathbf{w}^*, \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \mathbf{w}_{0,i} \rangle \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0) \\ &\leq b_0 \langle \mathbf{w}^*, \mathbf{w}_{0,i} \rangle \\ &\leq b_0. \end{aligned} \quad (12)$$

Where the first equality holds since $\mathbf{x}_{\parallel \mathbf{w}} = \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \mathbf{w}_{0,i}$. The last inequality holds by Cauchy-Schwarz Inequality. Next, we bound part (b) in (11). For $\|\mathbf{w}_{0,i} - \mathbf{w}^*\|_2 \leq r_0$ and $a > b_0$, we have

$$\begin{aligned} & \mathbb{P}(|\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}} \rangle| \geq a - b_0) \\ &= \mathbb{P}(|\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}} \rangle| \geq a - b_0, 0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0) \\ &= \mathbb{P}(|\langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{0,i} \rangle| \geq a - b_0, 0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0), \end{aligned}$$

where we prove the second equality as the following

$$\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}} \rangle = \langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{\perp \mathbf{w}} \rangle + \langle \mathbf{w}_{\parallel \mathbf{w}}^*, \mathbf{x}_{\perp \mathbf{w}} \rangle,$$

$$\langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{0,i} \rangle = \langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{\perp \mathbf{w}} \rangle + \langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{\parallel \mathbf{w}} \rangle.$$

since $\langle \mathbf{w}_{\parallel \mathbf{w}}^*, \mathbf{x}_{\perp \mathbf{w}} \rangle = \langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{\parallel \mathbf{w}} \rangle = 0$, we have $\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}} \rangle = \langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{0,i} \rangle$.

Denote $X := \left\langle \frac{\mathbf{w}_{\perp \mathbf{w}}^*}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}, \mathbf{x}_{0,i} \right\rangle$ and $Y := \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle$. Then, (X, Y) forms a projection of $\mathbf{x}_{0,i}$ onto a 2-dimensional subspace V_2 spanned by $\frac{\mathbf{w}_{\perp \mathbf{w}}^*}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}$ and $\mathbf{w}_{0,i}$. Let ϕ_{V_2} denote the density of (X, Y) . By Assumption 2 condition 2, we have $\phi_{V_2}(X, Y) \leq U_2 \exp(-\delta \|(X, Y)\|_2) = U_2 \exp(-\delta \sqrt{X^2 + Y^2})$. Hence, we can bound the above probability by

$$\begin{aligned} & \mathbb{P}(|\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}} \rangle| \geq a - b_0) \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0) \\ &= \mathbb{P}\left(\left|\left\langle \frac{\mathbf{w}_{\perp \mathbf{w}}^*}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}, \mathbf{x}_{0,i} \right\rangle\right| \geq \frac{a - b_0}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}, 0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0\right) \\ &= \int_{\frac{a-b_0}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}}^{+\infty} \int_0^{b_0} \phi_{V_2}(X, Y) dX dY \\ &\leq U_2 \int_{\frac{a-b_0}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}}^{+\infty} \int_0^{b_0} \exp(-\delta \sqrt{X^2 + Y^2}) dX dY \\ &\leq U_2 b_0 \int_{\frac{a-b_0}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}}^{+\infty} \exp(-\delta X) dX \\ &= \frac{U_2}{\delta} b_0 \exp\left(-\delta \frac{a - b_0}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}\right) \\ &\leq \frac{U_2 \exp(\delta)}{\delta} b_0 \exp\left(-\delta \frac{a}{r_0}\right). \end{aligned} \quad (13)$$

The last inequality holds by the fact that $\|\mathbf{w}^* - \mathbf{w}_{0,i}\|_2 \leq r_0$, which implies $\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2 \leq r_0$, and that $b_0 < r_0 = 2$, which implies $\exp\left(\delta \frac{b_0}{r_0}\right) < \exp(\delta)$. Combing (12) and (13), for $a > b_0$,

$$\begin{aligned} \mathbb{P}(\langle \mathbf{w}^*, \mathbf{x}_{0,i} \rangle \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0) > a) &\leq \mathbb{P}(|\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}} \rangle \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0)| \geq a - b_0) \\ &\leq \frac{U_2 \exp(\delta)}{\delta} b_0 \exp\left(-\delta \frac{a}{r_0}\right). \end{aligned} \quad (14)$$

For $0 < a \leq b_0$,

$$\begin{aligned} \mathbb{P}(\langle \mathbf{w}^*, \mathbf{x}_{0,i} \rangle \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0) > a) &\leq \mathbb{P}(|\langle \mathbf{w}^*, \mathbf{x}_{0,i} \rangle \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0)| > 0) \\ &\leq \mathbb{P}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0) \\ &\leq U_1 b_0 \\ &\leq U_1 b_0 \exp\left(\delta \frac{b_0}{r_0}\right) \exp\left(-\delta \frac{a}{r_0}\right) \\ &\leq U_1 \exp(\delta) b_0 \exp\left(-\delta \frac{a}{r_0}\right). \end{aligned} \quad (15)$$

The third inequality hold by Lemma 4 property (a). The fourth inequality holds by $0 < a < b_0$. The last inequality holds since by our construction, $b_0 < r_0 = 2$.

Let $\rho = \max \left\{ U_1 \exp(\delta), \frac{U_2 \exp(\delta)}{\delta} \right\}$, by (14) and (15), we conclude that for $\forall a > 0$,

$$\mathbb{P}(\mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x} \rangle \leq b_0) \langle \mathbf{w}^*, \mathbf{x} \rangle > a) \leq \rho b_0 \exp\left(-\delta \frac{a}{r_0}\right) \quad (16)$$

Thus,

$$\begin{aligned} &\mathbb{P}(|\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle| \geq a) \\ &= \mathbb{P}\left(\left|\langle \mathbf{w}^*, \mathbf{x}_{0,i} \rangle \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0) \left(\frac{1}{2}y_{0,i} - \left(\frac{1}{2} - \bar{\eta}\right)\right)\right| \geq a\right) \\ &\leq \mathbb{P}(|\langle \mathbf{w}^*, \mathbf{x}_{0,i} \rangle \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0)| \geq a) \\ &\leq \rho b_0 \exp\left(-\delta \frac{a}{2}\right). \end{aligned}$$

Where the first inequality holds since $|\frac{1}{2}y_{0,i} - (\frac{1}{2} - \bar{\eta})| \leq 1$. In the last inequality, since $\|\mathbf{w}_{0,i} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}_{0,i}\|_2 + \|\mathbf{w}^*\|_2 = 2$, we take $r_0 = 2$ in (16) and get the upper bound.

By Lemma 9, $\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle$ is $(\frac{12}{\delta}\sqrt{1+2\rho b_0}, \frac{12}{\delta})$ -subexponential. By Lemma 7, we can get that with probability at least $(1 - 1/T^2)$,

$$\begin{aligned} &\sum_{i=1}^{T_0} \mathbb{E}[\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle | \mathcal{F}_{0,i-1}] \leq \sum_{i=1}^{T_0} \langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle \\ &+ \frac{24}{\delta} \sqrt{1+2\rho b_0} \sqrt{T_0 \ln T} + \frac{48}{\delta} \ln T. \end{aligned}$$

□

The following lemma establishes a high-probability upper bound for the average of $f_{0,i}(\mathbf{w}_{0,i})$ over T_0 iterations.

Lemma 16. *In Algorithm 3, if $\langle \mathbf{w}^*, \bar{\mathbf{w}}_0 \rangle \geq \frac{1}{2}(1 - 2\eta)L_1 R^2$, then there exist some constants $c_2, c_3, c_4 > 0$, when setting bandwidth $b_0 = c_2(1 - 2\eta)^2$, iteration number $T_0 = c_3 \frac{1}{(1-2\eta)^8} d \ln d (\ln T)^2$, step size $\alpha_0 = c_4 \frac{\sqrt{d \ln d}}{\sqrt{T_0 \ln T}}$, then with probability at least $1 - 3/T^2$, we have*

$$\frac{1}{T_0} \sum_{i=1}^{T_0} f_{0,i}(\mathbf{w}_{0,i}) \leq \min\{1, L_1 R^2\} \frac{\pi(1 - 2\eta)L_2 R^2}{2880U_1}.$$

Proof. Recall that in the non-strategic setting, $\mathbf{g}_{k,i} = [-\bar{\eta}\mathbf{x}_{0,i}\mathbb{I}(y_{0,i} = 1) + (1 - \bar{\eta})\mathbf{x}_{0,i}\mathbb{I}(y_{0,i} = -1)]\mathbb{I}(\mathbf{x}_{0,i} \in D_{0,i})$, where $D_{0,i} = \{\mathbf{x} | 0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x} \rangle \leq b_0\}$ is just the localization

region. Since the announced classifier is $\tilde{h}_{0,i}(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}_{0,i}, \mathbf{r} \rangle)$, by the construction of $\tilde{\mathbf{g}}_{0,i}$ in Algorithm 3 and Corollary 4, we have

$$\begin{aligned}\tilde{\mathbf{g}}_{0,i} &= [-\bar{\eta}\mathbf{r}_{0,i}\mathbb{I}(y_{0,i} = 1) + (1 - \bar{\eta})\mathbf{r}_{0,i}\mathbb{I}(y_{0,i} = -1)]\mathbb{I}(0 < \langle \mathbf{w}_{0,i}, \mathbf{r}_{0,i} \rangle \leq b_0) \\ &= [-\bar{\eta}\mathbf{x}_{0,i}\mathbb{I}(y_{0,i} = 1) + (1 - \bar{\eta})\mathbf{x}_{0,i}\mathbb{I}(y_{0,i} = -1)]\mathbb{I}(\mathbf{x}_{0,i} \in D_{0,i}) \\ &= \mathbf{g}_{0,i}.\end{aligned}$$

Then, by Lemma 10, we have

$$\mathbb{E}[\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle] = \mathbb{E}[\langle \mathbf{w}^*, -\mathbf{g}_{0,i} \rangle] \geq (1 - 2\bar{\eta})f_{0,i}(\mathbf{w}_{0,i})\mathbb{P}(\mathbf{x} \in D_{0,i}). \quad (17)$$

We proceed to establish the high probability bound of $\sum_{i=1}^{T_0} \mathbb{E}[\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle]$. By Lemma 15, with probability at least $1 - 1/T^2$, we have

$$\begin{aligned}\sum_{i=1}^{T_0} \mathbb{E}[\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle | \mathcal{F}_{0,i-1}] &\leq \sum_{i=1}^{T_0} \langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle \\ &+ \frac{24}{\delta} \sqrt{1 + 2\rho b_0} \sqrt{T_0 \ln T} + \frac{48}{\delta} \ln T,\end{aligned} \quad (18)$$

where $\rho = \max \left\{ U_1 \exp(\delta), \frac{U_2 \exp(\delta)}{\delta} \right\}$.

Next, we move on to upper bound $\sum_{i=1}^{T_0} \langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle$ through a nonstandard regret analysis of online mirror decent.

Let $B(\mathbf{v}_1, \mathbf{v}_2) := \frac{1}{2(p-1)} \|\mathbf{v}_1 - \mathbf{v}_2\|_p^2$ denote the Bregman divergence w.r.t. $\frac{1}{2(p-1)} \|\cdot\|_p^2$, where $p = \frac{\ln(8d)}{\ln(8d)-1}$. In each iteration i , the regularizer, $B(\cdot, \mathbf{w}_{0,i-1})$ is 1-strongly convex with respect to $\|\cdot\|_p$ [see Shalev-Shwartz [2007]]. From the analysis of online mirror descent [see Orabona [2023], Lemma 6.9], with step size α_0 , we have

$$\langle \alpha_0 \tilde{\mathbf{g}}_{0,i}, \mathbf{w}_{0,i} - \mathbf{w}^* \rangle \leq B(\mathbf{w}^*, \mathbf{w}_{0,i}) - B(\mathbf{w}^*, \mathbf{w}_{0,i+1}) + \frac{\alpha_0^2}{2} \|\tilde{\mathbf{g}}_{0,i}\|_q^2,$$

where $q = \ln(8d) > 2$. Summing the above equality over $i \in [T_0]$, we get

$$\sum_{i=1}^{T_0} \langle \alpha_0 \tilde{\mathbf{g}}_{0,i}, \mathbf{w}_{0,i} - \mathbf{w}^* \rangle \leq B(\mathbf{w}^*, \mathbf{w}_{0,1}) - B(\mathbf{w}^*, \mathbf{w}_{0,T_0+1}) + \frac{\alpha_0^2}{2} \sum_{i=1}^{T_0} \|\tilde{\mathbf{g}}_{0,i}\|_q^2.$$

Dividing both sides by α_0 , and moving $\sum_{i=1}^{T_0} \langle \mathbf{w}_{0,i}, \tilde{\mathbf{g}}_{0,i} \rangle$ to RHS, we get

$$\begin{aligned}\sum_{i=1}^{T_0} \langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle &\leq \frac{1}{\alpha_0} [B(\mathbf{w}^*, \mathbf{w}_{0,1}) - B(\mathbf{w}^*, \mathbf{w}_{0,T_0+1})] + \sum_{i=1}^{T_0} \langle \mathbf{w}_{0,i}, -\tilde{\mathbf{g}}_{0,i} \rangle + \frac{\alpha_0}{2} \sum_{i=1}^{T_0} \|\tilde{\mathbf{g}}_{0,i}\|_q^2 \\ &\leq \frac{1}{\alpha_0} B(\mathbf{w}^*, \mathbf{w}_{0,1}) + \sum_{i=1}^{T_0} \langle \mathbf{w}_{0,i}, -\tilde{\mathbf{g}}_{0,i} \rangle + \frac{\alpha_0}{2} \sum_{i=1}^{T_0} \|\tilde{\mathbf{g}}_{0,i}\|_q^2.\end{aligned} \quad (19)$$

Now we need to bound the three terms in RHS of (19) respectively.

First, we bound $B(\mathbf{w}^*, \mathbf{w}_{0,1}) = B(\mathbf{w}^*, \mathbf{w}_0)$

$$B(\mathbf{w}^*, \mathbf{w}_{0,1}) = \frac{\|\mathbf{w}^* - \mathbf{w}_0\|_p^2}{2(p-1)} \stackrel{(a)}{\leq} \frac{\|\mathbf{w}^* - \mathbf{w}_0\|_1^2}{2(p-1)} \stackrel{(b)}{\leq} \frac{d\|\mathbf{w}^* - \mathbf{w}_0\|_2^2}{2(p-1)} \stackrel{(c)}{\leq} 2d \ln(8d). \quad (20)$$

Where inequality (a) holds by the fact that $\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_1$ for all $p > 1$ and $\mathbf{x} \in \mathbb{R}^d$. Inequality (b) holds since $\|\mathbf{x}\|_1 \leq \sqrt{d}\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^d$. Inequality (c) holds since $\|\mathbf{w}^* - \mathbf{w}_0\|_2 \leq \|\mathbf{w}^*\|_2 + \|\mathbf{w}_0\|_2 = 2$ and $\frac{1}{p-1} \leq \ln(8d) - 1 < \ln(8d)$.

Next, we bound $\sum_{i=1}^{T_0} \langle -\mathbf{w}_{0,i}, \tilde{\mathbf{g}}_{0,i} \rangle$.

Since $\langle -\mathbf{w}_{0,i}, \tilde{\mathbf{g}}_{0,i} \rangle = \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \left(\frac{1}{2} y_{0,i} - \left(\frac{1}{2} - \bar{\eta} \right) \right) \mathbb{I}(0 < \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0)$, then

$|\langle -\mathbf{w}_{0,i}, \tilde{\mathbf{g}}_{0,i} \rangle| \leq b_0$ and $\mathbb{E}[\langle -\mathbf{w}_{0,i}, \tilde{\mathbf{g}}_{0,i} \rangle] \leq b_0 \mathbb{P}(\mathbf{x} \in D_{0,i})$, by Lemma 8, with probability at least $1 - 1/T^2$, we have

$$\begin{aligned} \sum_{i=1}^{T_0} \langle -\mathbf{w}_{0,i}, \tilde{\mathbf{g}}_{0,i} \rangle &\leq \sum_{i=1}^{T_0} \mathbb{E}[\langle -\mathbf{w}_{0,i}, \tilde{\mathbf{g}}_{0,i} \rangle \mid \mathcal{F}_{0,i-1}] + b_0 \sqrt{T_0 \ln T} \\ &\leq b_0 \sum_{i=1}^{T_0} \Pr(\mathbf{x} \in D_{0,i}) + b_0 \sqrt{T_0 \ln T}. \end{aligned} \quad (21)$$

Finally, we bound $\sum_{i=1}^{T_0} \|\tilde{\mathbf{g}}_{0,i}\|_q^2$. Since $\|\mathbf{g}_{0,i}\|_q \leq 2\|\mathbf{g}_{0,i}\|_\infty$, we only need to upper bound $\sum_{i=1}^{T_0} \|\mathbf{g}_{0,i}\|_\infty^2$, which satisfies

$$\|\mathbf{g}_{0,i}\|_\infty = \left\| \mathbb{I}(0 \leq \langle \mathbf{w}_{0,i}, \mathbf{x}_{0,i} \rangle \leq b_0) \left(-\frac{1}{2} y_{0,i} + \left(\frac{1}{2} - \eta \right) \right) \mathbf{x}_{0,i} \right\|_\infty \leq \|\mathbf{x}_{0,i}\|_\infty.$$

By Lemma 13, we have with probability at least $1 - 1/T_0 T^2$,

$$\|\mathbf{x}_{0,i}\|_\infty \leq \frac{1}{Q} (1 + \ln(dT_0 T^2)) \leq \frac{3}{Q} \ln T.$$

Thus, taking the union bound over $i \in [T_0]$, we have with probability at least $1 - 1/T^2$,

$$\sum_{i=1}^{T_0} \|\mathbf{g}_{0,i}\|_\infty^2 \leq T_0 \times \left(\frac{3}{Q} \ln T \right)^2 = \frac{9}{Q^2} T_0 (\ln T)^2.$$

Hence, with probability at least $1 - 1/T^2$,

$$\sum_{i=1}^{T_0} \|\tilde{\mathbf{g}}_{0,i}\|_q^2 \leq 4 \sum_{i=1}^{T_0} \|\mathbf{g}_{0,i}\|_\infty^2 \leq \frac{36}{Q^2} T_0 (\ln T)^2 \quad (22)$$

Combing (18), (20), (21) and (22) together, and take the union bound, we get with probability at least $1 - 3/T^2$,

$$\begin{aligned} &(1 - 2\eta) \sum_{i=1}^{T_0} f_{0,i}(\mathbf{w}_{0,i}) \mathbb{P}(\mathbf{x} \in D_{0,i}) \\ &\leq \sum_{i=1}^{T_0} \mathbb{E}[\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle] \\ &\leq \sum_{i=1}^{T_0} \langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{0,i} \rangle + \frac{24}{\delta} \sqrt{1 + 2\rho b_0} \sqrt{T_0 \ln T} + \frac{48}{\delta} \ln T \\ &\leq \frac{2d \ln(8d)}{\alpha_0} + b_0 \sum_{i=1}^{T_0} \Pr(\mathbf{x} \in D_{0,i}) + b_0 \sqrt{T_0 \ln T} + \frac{18\alpha_0}{Q^2} T_0 (\ln T)^2 \\ &\quad + \frac{24}{\delta} \sqrt{1 + 2\rho b_0} \sqrt{T_0 \ln T} + \frac{48}{\delta} \ln T, \end{aligned} \quad (23)$$

where $\rho = \max \left\{ U_1 \exp(\delta), \frac{U_2 \exp(\delta)}{\delta} \right\}$. By Lemma 4, property (a), and the fact that $b_0 < R$, we have $L_1 b_0 \leq \mathbb{P}(\mathbf{x} \in D_{0,i}) \leq U_1 b_0$, dividing both sides of (23) by $(1 - 2\bar{\eta}) L_1 b_0 T_0$, we get

$$\begin{aligned} &\frac{1}{T_0} \sum_{i=1}^{T_0} f_{0,i}(\mathbf{w}_{0,i}) \\ &\leq \frac{1}{\alpha_0} \frac{2d \ln(8d)}{(1 - 2\bar{\eta}) L_1 b_0 T_0} + \frac{U_1}{(1 - 2\bar{\eta}) L_1} b_0 + \frac{1}{(1 - 2\bar{\eta}) L_1} \sqrt{\frac{\ln T}{T_0}} + \frac{18\alpha_0}{(1 - 2\bar{\eta}) Q^2 L_1 b_0} (\ln T)^2 \\ &\quad + \frac{24}{(1 - 2\bar{\eta}) \delta L_1 b_0} \sqrt{1 + 2\rho b_0} \sqrt{\frac{\ln T}{T_0}} + \frac{48}{(1 - 2\bar{\eta}) \delta L_1 b_0} \frac{\ln T}{T_0}. \end{aligned}$$

By our setting, $b_0 = \min\{1, L_1 R^2\} \frac{(1-2\bar{\eta})^2 L_1 L_2 R^2}{2880 U_1^2} = c_2(1-2\bar{\eta})^2$, $T_0 = \frac{576(1+2\rho b_0)d \ln(8d)(\ln T)^2}{U_1^2 \delta^2 b_0^4} = c_3 \frac{1}{(1-2\bar{\eta})^8} d \ln d (\ln T)^2$, $\alpha_0 = \frac{Q\sqrt{d \ln(8d)}}{3\sqrt{T_0} \ln T} = c_4 \frac{\sqrt{d \ln d}}{\sqrt{T_0} \ln T}$, then we finish the proof. \square

The following lemma established by Zhang et al. [2020] indicates that by the construction of the constraint set \mathcal{W}_0 , any two vectors in \mathcal{W}_0 form an angle that is no bigger than $\pi - \frac{1}{2}(1-2\bar{\eta})L_1 R^2$.

Lemma 17. (Zhang et al. [2020], Lemma 19) For any two vectors $\mathbf{u}, \mathbf{v} \in \mathcal{W}_0 = \{\mathbf{w} \mid \|\mathbf{w}\|_2 \leq 1, \langle \mathbf{w}, \bar{\mathbf{w}}_0 \rangle \geq \frac{1}{2}(1-2\bar{\eta})L_1 R^2\}$, we have $\theta(\mathbf{u}, \mathbf{v}) \leq \pi - \frac{1}{2}(1-2\bar{\eta})L_1 R^2$.

We use the following corollary to show that, in the i 'th iteration, a small value of $f_{0,i}(\mathbf{w}_{0,i})$ indicates that $\mathbf{w}_{0,i}$ and \mathbf{w}^* are close.

Corollary 5. If $\mathbf{w}^* \in \mathcal{W}_0$ and $f_{0,i}(\mathbf{w}_{0,i}) < \min\{1, L_1 R^2\} \frac{\pi(1-2\bar{\eta})L_2 R^2}{320 U_1}$, then $\theta(\mathbf{w}^*, \mathbf{w}_{0,i}) \leq \frac{\pi}{10}$.

Proof. We first exclude the case that $\theta(\mathbf{w}^*, \mathbf{w}_{0,i}) > \frac{\pi}{2}$, which we prove by contradiction. Suppose $\theta(\mathbf{w}^*, \mathbf{w}_{0,i}) > \frac{\pi}{2}$. By Lemma 17 and our choice of b_0 , $\theta(\mathbf{w}^*, \mathbf{w}_{0,i}) \leq \pi - \frac{1}{2}(1-2\bar{\eta})L_1 R^2 < \pi - b_0$. From Lemma 11, we get $f_{k,i}(\mathbf{w}_{k,i}) \geq \frac{L_2}{32 U_1} R^2 (\pi - \theta(\mathbf{w}^*, \mathbf{w}_{k,i}))$. Together with the condition that $f_{0,i}(\mathbf{w}_{0,i}) < \frac{\pi(1-2\bar{\eta})L_1 L_2 R^4}{320 U_1}$, we have $\theta(\mathbf{w}^*, \mathbf{w}_{0,i}) > \pi - \frac{\pi}{10}(1-2\bar{\eta})L_1 R^2 > \pi - \frac{1}{2}(1-2\bar{\eta})L_1 R^2$, which is a contradiction. Thus, we conclude that $\theta(\mathbf{w}^*, \mathbf{w}_{0,i}) \leq \frac{\pi}{2}$.

Next, since $f_{0,i}(\mathbf{w}_{0,i}) < \frac{\pi(1-2\bar{\eta})L_2 R^2}{320 U_1} \leq \frac{L_2 R^2}{160 U_1} \cdot \frac{\pi}{2}$, by Corollary 3, setting $\theta_0 = \frac{\pi}{2}$, then $\theta(\mathbf{w}^*, \mathbf{w}_{0,i}) \leq \frac{\theta_0}{5} = \frac{\pi}{10}$. \square

Putting all pieces together, now we are able to show the main theoretical guarantee of Algorithm 3.

Proposition 3. For the constant c_1 in Proposition 2 and some constants $c_2, c_3, c_4 > 0$, when the initial vector $\bar{\mathbf{w}}_0$ satisfies $\langle \mathbf{w}^*, \bar{\mathbf{w}}_0 \rangle \geq c_1(1-2\bar{\eta})$ and Algorithm 3 runs with bandwidth $b_0 = c_2(1-2\bar{\eta})^2$ for $T_0 = c_3 d \ln d (\ln T)^2 / (1-2\bar{\eta})^8$ iterations with step size $\alpha_0 = c_4 \sqrt{d \ln(d)} / (\sqrt{T_0} \ln T)$, then its output \mathbf{w}_1 satisfies $\theta(\mathbf{w}^*, \mathbf{w}_1) \leq \pi/4$ with probability at least $1 - 3/T^2$.

Proof. In Algorithm 3, the constraint set we choose for gradient update is $\mathcal{W}_0 = \{\mathbf{w} \mid \|\mathbf{w}\|_2 \leq 1, \langle \mathbf{w}, \bar{\mathbf{w}}_0 \rangle \geq c_1(1-2\bar{\eta})\}$, where $c_1 = \frac{1}{2}L_1 R^2$, since $\langle \mathbf{w}^*, \bar{\mathbf{w}}_0 \rangle \geq c_1(1-2\bar{\eta})$, we can conclude that $\mathbf{w}^* \in \mathcal{W}_0$.

Next, Lemma 16 shows that with probability at least $1 - 3/T^2$,

$$\frac{1}{T_0} \sum_{i=1}^{T_0} f_{0,i}(\mathbf{w}_{0,i}) \leq \min\{1, L_1 R^2\} \frac{\pi(1-2\bar{\eta})L_2 R^2}{2880 U_1}. \quad (24)$$

Let A_0 denote the set $\{i \mid f_{0,i}(\mathbf{w}_{0,i}) > \min\{1, L_1 R^2\} \frac{\pi(1-2\bar{\eta})L_2 R^2}{320 U_1}\}$. Combing (24), we have:

$$\min\{1, L_1 R^2\} \frac{\pi(1-2\bar{\eta})L_2 R^2}{2880 U_1} \geq \frac{1}{T_0} \sum_{i=1}^{T_0} f_{0,i}(\mathbf{w}_{0,i}) \geq \frac{|A_0|}{T_0} \min\{1, L_1 R^2\} \frac{\pi(1-2\bar{\eta})L_2 R^2}{320 U_1}.$$

Solve the above inequality and we get $\frac{|A_0|}{T_0} \leq \frac{1}{9}$.

From Corollary 3, set $\theta_0 = \frac{\pi}{2}$, we know that when $i \in \bar{A}_0$, $\theta(\mathbf{w}^*, \mathbf{w}_{0,i}) \leq \frac{\pi}{10}$. Thus,

$$\begin{aligned} \frac{1}{T_0} \sum_{i=1}^{T_0} \cos(\theta(\mathbf{w}^*, \mathbf{w}_{0,i})) &\geq \frac{|\bar{A}_0|}{T_0} \cos\left(\frac{\pi}{10}\right) - \frac{|A_0|}{T_0} \\ &\geq \left(1 - \frac{|A_0|}{T_0}\right) \left(1 - \frac{1}{2} \left(\frac{\pi}{10}\right)^2\right) - \frac{|A_0|}{T_0} \\ &\geq \left(1 - \frac{1}{9}\right) \left(1 - \frac{1}{2} \left(\frac{\pi}{10}\right)^2\right) - \frac{1}{9} \\ &\geq \cos\left(\frac{\pi}{4}\right), \end{aligned}$$

where the first inequality holds since $\cos x$ is decreasing in $x \in [0, \pi]$ and $\cos x \geq -1$. The second inequality holds since $\cos x \geq 1 - \frac{1}{2}x^2$ for all $x \in [0, \pi]$. The last inequality holds since $(1 - \frac{1}{9}) \left(1 - \frac{1}{2} \left(\frac{\pi}{10}\right)^2\right) - \frac{1}{9} \approx 0.73 > 0.71 \approx \cos\left(\frac{\pi}{4}\right)$. By the concavity of $\cos(\theta(\mathbf{w}^*, \cdot))$, using Jensen's inequality, we conclude that when the above inequality holds, we have

$$\cos\left(\theta\left(\mathbf{w}^*, \frac{1}{T_0} \sum_{i=1}^{T_0} \mathbf{w}_{0,i}\right)\right) \geq \frac{1}{T_0} \sum_{i=1}^{T_0} \cos(\theta(\mathbf{w}^*, \mathbf{w}_{0,i})) \geq \cos\left(\frac{\pi}{4}\right).$$

Thus, we can get that with probability at least $1 - 3/T^2$, Algorithm 3 returns a vector \mathbf{w}_1 such that $\theta(\mathbf{w}^*, \mathbf{w}_1) \leq \frac{\pi}{4}$. \square

Theoretical Guarantees of Algorithm 4 The following lemma shows that if, in batch k , iteration i , we can identify agents whose true features lie in the localization region $D_{k,i}$, and we use proxy features to construct a proxy gradient $\tilde{\mathbf{g}}_{k,i}$, then $\mathbb{E}[\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{k,i} \rangle]$ upper bounds $\mathbb{E}[\langle \mathbf{w}^*, -\mathbf{g}_{k,i} \rangle]$ (and hence upper bounds $\theta(\mathbf{w}^*, \mathbf{w}_{k,i})$).

Lemma 18. *Given a classification rule $\tilde{h}(\mathbf{r}) = \text{sgn}(\langle \mathbf{w}_{k,i}, \mathbf{r} \rangle + m_{k,i})$ with fixed $\mathbf{w}_{k,i} \in \mathbb{S}^d$ and arbitrary $m_{k,i} < 0$, an agent $(\mathbf{x}_{k,i}, y_{k,i})$ reports his feature as $\mathbf{r}_{k,i}$ according to Lemma 1. Construct proxy data as*

$$\begin{aligned} \tilde{\mathbf{x}}_{k,i}^+ &:= (\mathbf{r}_{k,i} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i} \rangle) \mathbf{w}_{k,i}) \mathbb{I}(y_{k,i} = 1, \mathbf{x}_{k,i} \in D_{k,i}), \\ \tilde{\mathbf{x}}_{k,i}^- &:= (\mathbf{r}_{k,i} - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i} \rangle \mathbf{w}_{k,i}) \mathbb{I}(y_{k,i} = -1, \mathbf{x}_{k,i} \in D_{k,i}), \end{aligned}$$

and define the gradient as

$$\tilde{\mathbf{g}}_{k,i} := \left[-\bar{\eta} \tilde{\mathbf{x}}_{k,i}^+ + (1 - \bar{\eta}) \tilde{\mathbf{x}}_{k,i}^- \right] \mathbb{I}(\mathbf{x}_{k,i} \in D_{k,i}).$$

Then, we have

$$\mathbb{E}[\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{k,i} \rangle] \geq \mathbb{E}[\langle \mathbf{w}^*, -\mathbf{g}_{k,i} \rangle].$$

Proof. First, by Lemma 1, given $\mathbf{w}_{k,i}, m_{k,i}$, for $\forall \mathbf{x}_{k,i} \in \mathbb{R}^d$, we have

$$\mathbf{r}_{k,i} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i} \rangle) \mathbf{w}_{k,i} = \mathbf{x}_{k,i} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i} \rangle) \mathbf{w}_{k,i},$$

and

$$\mathbf{r}_{k,i} - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i} \rangle \mathbf{w}_{k,i} = \mathbf{x}_{k,i} - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i} \rangle \mathbf{w}_{k,i}.$$

Therefore, when $\mathbf{x}_{k,i} \in D_{k,i} = \{\mathbf{x} \mid 0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq b_k\}$, we have the following

$$\begin{aligned} \langle \mathbf{w}^*, \tilde{\mathbf{x}}_{k,i}^+ \rangle &= \langle \mathbf{w}^*, \mathbf{r}_{k,i} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i} \rangle) \mathbf{w}_{k,i} \rangle \mathbb{I}(y_{k,i} = 1, \mathbf{x}_{k,i} \in D_{k,i}) \\ &= \langle \mathbf{w}^*, \mathbf{x}_{k,i} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i} \rangle) \mathbf{w}_{k,i} \rangle \mathbb{I}(y_{k,i} = 1, \mathbf{x}_{k,i} \in D_{k,i}) \\ &= [\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i} \rangle) \langle \mathbf{w}^*, \mathbf{w}_{k,i} \rangle] \mathbb{I}(y_{k,i} = 1, \mathbf{x}_{k,i} \in D_{k,i}) \\ &\geq \langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle \mathbb{I}(y_{k,i} = 1, \mathbf{x}_{k,i} \in D_{k,i}), \end{aligned} \tag{25}$$

where the inequality holds since $\langle \mathbf{w}^*, \mathbf{w}_{k,i} \rangle > 0$ and $\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i} \rangle \mathbb{I}(\mathbf{x}_{k,i} \in D_{k,i}) \leq b_k$.

Similarly,

$$\begin{aligned} \langle \mathbf{w}^*, \tilde{\mathbf{x}}_{k,i}^- \rangle &= \langle \mathbf{w}^*, \mathbf{r}_{k,i} - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i} \rangle \mathbf{w}_{k,i} \rangle \mathbb{I}(y_{k,i} = -1, \mathbf{x}_{k,i} \in D_{k,i}) \\ &= \langle \mathbf{w}^*, \mathbf{x}_{k,i} - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i} \rangle \mathbf{w}_{k,i} \rangle \mathbb{I}(y_{k,i} = -1, \mathbf{x}_{k,i} \in D_{k,i}) \\ &= [\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i} \rangle \langle \mathbf{w}^*, \mathbf{w}_{k,i} \rangle] \mathbb{I}(y_{k,i} = -1, \mathbf{x}_{k,i} \in D_{k,i}) \\ &\leq \langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle \mathbb{I}(y_{k,i} = -1, \mathbf{x}_{k,i} \in D_{k,i}). \end{aligned} \tag{26}$$

where the inequality holds since $\langle \mathbf{w}^*, \mathbf{w}_{k,i} \rangle > 0$ and $\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i} \rangle \mathbb{I}(\mathbf{x}_{k,i} \in D_{k,i}) \geq 0$. Combing (25) and (26), we have

$$\begin{aligned} &\mathbb{E}[\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{k,i} \rangle] \\ &= \bar{\eta} \mathbb{E}[\langle \mathbf{w}^*, \tilde{\mathbf{x}}_{k,i}^+ \rangle] - (1 - \bar{\eta}) \mathbb{E}[\langle \mathbf{w}^*, \tilde{\mathbf{x}}_{k,i}^- \rangle] \\ &\geq \bar{\eta} \mathbb{E}[\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle \mathbb{I}(y_{k,i} = 1, \mathbf{x}_{k,i} \in D_{k,i})] - (1 - \bar{\eta}) \mathbb{E}[\langle \mathbf{w}^*, \mathbf{x}_{k,i} \rangle \mathbb{I}(y_{k,i} = -1, \mathbf{x}_{k,i} \in D_{k,i})] \\ &= \mathbb{E}[\langle \mathbf{w}^*, -\mathbf{g}_{k,i} \rangle]. \end{aligned}$$

\square

The following lemma shows that by pairwise comparing agents' responses under two different declared classifiers designed in Algorithm 4, we can unbiasedly estimate the proxy data desired by Lemma 18 and hence construct a gradient estimator $\hat{\mathbf{g}}_{k,i}$, accordingly.

Lemma 19. *Given $\tilde{\mathbf{x}}_{k,i}^+$, $\tilde{\mathbf{x}}_{k,i}^-$ and $\tilde{\mathbf{g}}_{k,i}$ defined in Lemma 18, for the proxy data $\hat{\mathbf{x}}_{k,i}^{(1,+)}$, $\hat{\mathbf{x}}_{k,i}^{(1,-)}$, $\hat{\mathbf{x}}_{k,i}^{(2,+)}$, $\hat{\mathbf{x}}_{k,i}^{(2,-)}$ and gradient estimator $\hat{\mathbf{g}}_{k,i}$ defined in Algorithm 4, we have*

$$\mathbb{E} \left[\hat{\mathbf{x}}_{k,i}^{(1,+)} - \hat{\mathbf{x}}_{k,i}^{(2,+)} \right] = \mathbb{E} \left[\tilde{\mathbf{x}}_{k,i}^+ \right] ,$$

and

$$\mathbb{E} \left[\hat{\mathbf{x}}_{k,i}^{(1,-)} - \hat{\mathbf{x}}_{k,i}^{(2,-)} \right] = \mathbb{E} \left[\tilde{\mathbf{x}}_{k,i}^- \right] .$$

Moreover,

$$\mathbb{E} [\langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle] = \mathbb{E} [\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{k,i} \rangle] .$$

Proof. For fixed normal vector $\mathbf{w}_{k,i}$ and bandwidth b_k , recall that $D_{k,i}^{(1)} = \{\mathbf{x} \mid 0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq \gamma + b_k\}$ and $D_{k,i}^{(2)} = \{\mathbf{x} \mid b_k \leq \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \leq \gamma + b_k\}$. Then, we can verify that $D_{k,i} = D_{k,i}^{(1)} / D_{k,i}^{(2)}$. By Corollary 4, we have

$$\begin{aligned} \tilde{\mathbf{x}}_{k,i}^+ &= [\mathbf{r}_{k,i} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i} = 1, \mathbf{x}_{k,i} \in D_{k,i}) \\ &= [\mathbf{x}_{k,i} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i} = 1, \mathbf{x}_{k,i} \in D_{k,i}) , \\ \tilde{\mathbf{x}}_{k,i}^- &= [\mathbf{r}_{k,i} - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i} \rangle \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i} = -1, \mathbf{x}_{k,i} \in D_{k,i}) \\ &= [\mathbf{x}_{k,i} - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i} \rangle \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i} = -1, \mathbf{x}_{k,i} \in D_{k,i}) , \\ \hat{\mathbf{x}}_{k,i}^{(1,+)} &= [\mathbf{r}_{k,i}^{(1)} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i}^{(1)} = 1, \gamma \leq \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle \leq \gamma + b_k) \\ &= [\mathbf{x}_{k,i}^{(1)} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(1)} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i}^{(1)} = 1, \mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)}) , \\ \hat{\mathbf{x}}_{k,i}^{(1,-)} &= [\mathbf{r}_{k,i}^{(1)} - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i}^{(1)} = -1, \gamma \leq \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle \leq \gamma + b_k) \\ &= [\mathbf{x}_{k,i}^{(1)} - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(1)} \rangle \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i}^{(1)} = -1, \mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)}) , \\ \hat{\mathbf{x}}_{k,i}^{(2,+)} &= [\mathbf{r}_{k,i}^{(2)} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(2)} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i}^{(2)} = 1, \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(2)} \rangle = \gamma + b_k) \\ &= [\mathbf{x}_{k,i}^{(2)} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i}^{(2)} = 1, \mathbf{x}_{k,i}^{(2)} \in D_{k,i}^{(2)}) , \\ \hat{\mathbf{x}}_{k,i}^{(2,-)} &= [\mathbf{r}_{k,i}^{(2)} - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(2)} \rangle \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i}^{(2)} = -1, \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(2)} \rangle = \gamma + b_k) \\ &= [\mathbf{x}_{k,i}^{(2)} - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \rangle \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i}^{(2)} = -1, \mathbf{x}_{k,i}^{(2)} \in D_{k,i}^{(2)}) . \end{aligned}$$

Also, $(\mathbf{x}_{k,i}^{(1)}, y_{k,i}^{(1)})$ and $(\mathbf{x}_{k,i}^{(2)}, y_{k,i}^{(2)})$ are drawn i.i.d. from \mathcal{D} , thus,

$$\begin{aligned} \mathbb{E} [\hat{\mathbf{x}}_{k,i}^{(1,+)} - \hat{\mathbf{x}}_{k,i}^{(2,+)}] &= \mathbb{E} \left[[\mathbf{x}_{k,i}^{(1)} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(1)} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i}^{(1)} = 1, \mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)}) \right] \\ &\quad - \mathbb{E} \left[[\mathbf{x}_{k,i}^{(2)} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i}^{(2)} = 1, \mathbf{x}_{k,i}^{(2)} \in D_{k,i}^{(2)}) \right] \\ &= \mathbb{E} \left[[\mathbf{x} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y = 1, \mathbf{x} \in D_{k,i}^{(1)}) \right] \\ &\quad - \mathbb{E} \left[[\mathbf{x} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y = 1, \mathbf{x} \in D_{k,i}^{(2)}) \right] \\ &= \mathbb{E} \left[[\mathbf{x} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y = 1, \mathbf{x} \in D_{k,i}) \right] \\ &= \mathbb{E} [\tilde{\mathbf{x}}_{k,i}^+] . \end{aligned}$$

Similarly, we can show that $\mathbb{E} [\hat{\mathbf{x}}_{k,i}^{(1,-)} - \hat{\mathbf{x}}_{k,i}^{(2,-)}] = \mathbb{E} [\tilde{\mathbf{x}}_{k,i}^-]$. Therefore,

$$\begin{aligned} \mathbb{E} [\langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle] &= \bar{\eta} \mathbb{E} [\langle \mathbf{w}^*, \hat{\mathbf{x}}_{k,i}^{(1,+)} - \hat{\mathbf{x}}_{k,i}^{(2,+)} \rangle] - (1 - \bar{\eta}) \mathbb{E} [\langle \mathbf{w}^*, \hat{\mathbf{x}}_{k,i}^{(1,-)} - \hat{\mathbf{x}}_{k,i}^{(2,-)} \rangle] \\ &= \bar{\eta} \mathbb{E} [\langle \mathbf{w}^*, \tilde{\mathbf{x}}_{k,i}^{(+)} \rangle] - (1 - \bar{\eta}) \mathbb{E} [\langle \mathbf{w}^*, \tilde{\mathbf{x}}_{k,i}^{(-)} \rangle] \\ &= \mathbb{E} [\langle \mathbf{w}^*, -\tilde{\mathbf{g}}_{k,i} \rangle] . \end{aligned}$$

□

Next, we establish the high probability bound of $\sum_{i=1}^{T_k} \mathbb{E} [\langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle | \mathcal{F}_{k,i-1}]$ by $\sum_{i=1}^{T_k} \langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle$.

Lemma 20. *At batch k of Algorithm 4, when $\|\mathbf{w}^* - \mathbf{w}_{k,i}\|_2 \leq r_k$ for $\forall i \in [T_k]$, then with probability at least $1 - 2/T^2$, we have*

$$\begin{aligned} \sum_{i=1}^{T_k} \mathbb{E} [\langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle | \mathcal{F}_{k,i-1}] &\leq \sum_{i=1}^{T_k} \langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle \\ &+ \frac{24}{\delta} \sqrt{1 + 2\rho(\gamma + b_k)} r_k \sqrt{T_k \ln T} + \frac{48}{\delta} r_k \ln T , \end{aligned}$$

where $\rho = \max \left\{ U_1 \exp(\delta), \frac{U_2 \exp(\delta)}{\delta} \right\}$.

Proof. First, we partition $\langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle$ into two parts, as the following:

$$\begin{aligned} &\langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle \\ &= \left\langle \mathbf{w}^*, \bar{\eta}(\hat{\mathbf{x}}_{k,i}^{(1,+)} - \hat{\mathbf{x}}_{k,i}^{(2,+)}) - (1 - \bar{\eta})(\hat{\mathbf{x}}_{k,i}^{(1,-)} - \hat{\mathbf{x}}_{k,i}^{(2,-)}) \right\rangle \\ &= \underbrace{\left\langle \mathbf{w}^*, \bar{\eta}\hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta})\hat{\mathbf{x}}_{k,i}^{(1,-)} \right\rangle}_{(a)} - \underbrace{\left\langle \mathbf{w}^*, \bar{\eta}\hat{\mathbf{x}}_{k,i}^{(2,+)} - (1 - \bar{\eta})\hat{\mathbf{x}}_{k,i}^{(2,-)} \right\rangle}_{(b)} . \end{aligned} \tag{27}$$

Since the randomness in part (a) arises from one sample, while the randomness in part (b) arises from another independent sample, parts (a) and (b) are independent. Therefore, we can bound them separately. We first discuss the high-probability tail bound of part (a).

We partition $\bar{\eta}\hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta})\hat{\mathbf{x}}_{k,i}^{(1,-)}$ into two orthonormal vectors. For notational convenience, we omit the subindex k, i and let $\mathbf{x}_{\parallel \mathbf{w}}$ denote the component of \mathbf{x} that is parallel to $\mathbf{w}_{k,i}$, *i.e.*, $\mathbf{x}_{\parallel \mathbf{w}} = \langle \mathbf{w}_{k,i}, \mathbf{x} \rangle \mathbf{w}_{k,i}$, and $\mathbf{x}_{\perp \mathbf{w}}$ denote the component of \mathbf{x} that is orthogonal to $\mathbf{w}_{k,i}$, *i.e.*, $\mathbf{x}_{\perp \mathbf{w}} = \mathbf{x} - \mathbf{x}_{\parallel \mathbf{w}}$. Then,

$$\begin{aligned} &\left\langle \mathbf{w}^*, \bar{\eta}\hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta})\hat{\mathbf{x}}_{k,i}^{(1,-)} \right\rangle \\ &= \underbrace{\left\langle \mathbf{w}^*, \left(\bar{\eta}\hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta})\hat{\mathbf{x}}_{k,i}^{(1,-)} \right)_{\parallel \mathbf{w}} \right\rangle}_{(a1)} + \underbrace{\left\langle \mathbf{w}^*, \left(\bar{\eta}\hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta})\hat{\mathbf{x}}_{k,i}^{(1,-)} \right)_{\perp \mathbf{w}} \right\rangle}_{(a2)} . \end{aligned} \tag{28}$$

Hence, we have to bound part (a1) and (a2) in (28), respectively.

To bound part (a1) in (28), we have

$$\begin{aligned} \left\langle \mathbf{w}^*, \hat{\mathbf{x}}_{\parallel \mathbf{w}}^{(1,+)} \right\rangle &= \left\langle \mathbf{w}^*, \mathbf{r}_{\parallel \mathbf{w}}^{(1)} + \left(b_k - \left\langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \right\rangle \right) \mathbf{w}_{k,i} \right\rangle \mathbb{I} \left(y_{k,i}^{(1)} = 1, \gamma \leq \left\langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \right\rangle \leq \gamma + b_k \right) \\ &= \left\langle \mathbf{w}^*, \mathbf{r}_{\parallel \mathbf{w}}^{(1)} + \left(b_k \mathbf{w}_{k,i} - \mathbf{r}_{\parallel \mathbf{w}}^{(1)} \right) \right\rangle \mathbb{I} \left(y_{k,i}^{(1)} = 1, \gamma \leq \left\langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \right\rangle \leq \gamma + b_k \right) \\ &= \langle \mathbf{w}^*, b_k \mathbf{w}_{k,i} \rangle \mathbb{I} \left(y_{k,i}^{(1)} = 1, \gamma \leq \left\langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \right\rangle \leq \gamma + b_k \right) \\ &\leq b_k , \end{aligned} \tag{29}$$

where the last inequality holds because $\langle \mathbf{w}^*, \mathbf{w}_{k,i} \rangle \leq \|\mathbf{w}^*\|_2 \|\mathbf{w}_{k,i}\|_2 \leq 1$. Similarly,

$$\begin{aligned} \langle \mathbf{w}^*, \hat{\mathbf{x}}_{\|\mathbf{w}\|}^{(1,-)} \rangle &= \langle \mathbf{w}^*, \mathbf{r}_{\|\mathbf{w}\|}^{(1)} - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle \mathbf{w}_{k,i} \rangle \mathbb{I}(y_{k,i}^{(1)} = -1, \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle = \gamma + b_k) \\ &= \langle \mathbf{w}^*, \mathbf{r}_{\|\mathbf{w}\|}^{(1)} - \mathbf{r}_{\|\mathbf{w}\|}^{(1)} \rangle \mathbb{I}(y_{k,i}^{(1)} = -1, \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle = \gamma + b_k) \\ &= 0. \end{aligned} \quad (30)$$

Combing (29) and (30), we can bound part (a1) as

$$\left\langle \mathbf{w}^*, \left(\bar{\eta} \hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta}) \hat{\mathbf{x}}_{k,i}^{(1,-)} \right)_{\|\mathbf{w}\|} \right\rangle \leq b_k. \quad (31)$$

Next, we bound part (a2) in (28). From Lemma 1, we get that

$$\mathbf{r}_{\perp \mathbf{w}}^{(1)} = \left(\mathbf{r}_{k,i}^{(1)} + \left(b_k - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle \right) \mathbf{w}_{k,i} \right)_{\perp \mathbf{w}} = \left(\mathbf{r}_{k,i}^{(1)} - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle \mathbf{w}_{k,i} \right)_{\perp \mathbf{w}} = \mathbf{x}_{\perp \mathbf{w}}^{(1)},$$

hence,

$$\begin{aligned} &\underbrace{\left\langle \mathbf{w}^*, \left(\bar{\eta} \hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta}) \hat{\mathbf{x}}_{k,i}^{(1,-)} \right)_{\perp \mathbf{w}} \right\rangle}_{(a2)} \\ &= \bar{\eta} \left\langle \mathbf{w}^*, \left(\mathbf{r}_{k,i}^{(1)} + \left(b_k - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle \right) \mathbf{w}_{k,i} \right)_{\perp \mathbf{w}} \right\rangle \mathbb{I}(y_{k,i}^{(1)} = 1, \gamma \leq \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle \leq \gamma + b_k) \\ &\quad - (1 - \bar{\eta}) \left\langle \mathbf{w}^*, \left(\mathbf{r}_{k,i}^{(1)} - \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle \mathbf{w}_{k,i} \right)_{\perp \mathbf{w}} \right\rangle \mathbb{I}(y_{k,i}^{(1)} = -1, \gamma \leq \langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle \leq \gamma + b_k) \\ &= \bar{\eta} \left\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle \mathbb{I}(y_{k,i}^{(1)} = 1, \mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)}) - (1 - \bar{\eta}) \left\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle \mathbb{I}(y_{k,i}^{(1)} = -1, \mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)}) \\ &= \left\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle \left(\frac{1}{2} y_{k,i}^{(1)} - \left(\frac{1}{2} - \bar{\eta} \right) \right) \mathbb{I}(\mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)}), \end{aligned}$$

where the second equality holds by Corollary 4.

Since $\left| \frac{1}{2} y_{k,i}^{(1)} - \left(\frac{1}{2} - \bar{\eta} \right) \right| \leq 1$, we only need to establish the high probability bound of $\left\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle \mathbb{I}(\mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)})$, for $a > b_k$ we have

$$\begin{aligned} &\mathbb{P} \left(\left| \left\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle \mathbb{I}(\mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)}) \right| \geq a - b_k \right) \\ &= \mathbb{P} \left(\left| \left\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle \right| \geq a - b_k, 0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(1)} \rangle \leq \gamma + b_k \right) \\ &= \mathbb{P} \left(\left| \left\langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{k,i}^{(1)} \right\rangle \right| \geq a - b_k, 0 \leq \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(1)} \rangle \leq \gamma + b_k \right), \end{aligned}$$

where the last inequality holds because $\left\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle = \left\langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle + \left\langle \mathbf{w}_{\|\mathbf{w}\|}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle = \left\langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle = \left\langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle + \left\langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{\|\mathbf{w}\|}^{(1)} \right\rangle = \left\langle \mathbf{w}_{\perp \mathbf{w}}^*, \mathbf{x}_{k,i}^{(1)} \right\rangle$.

Let $X := \left\langle \frac{\mathbf{w}_{\perp \mathbf{w}}^*}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}, \mathbf{x}_{k,i} \right\rangle$ and $Y := \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i} \rangle$. Then, (X, Y) forms a projection of $\mathbf{x}_{k,i}$ onto a 2-dimensional subspace V_2 spanned by $\frac{\mathbf{w}_{\perp \mathbf{w}}^*}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}$ and $\mathbf{w}_{k,i}$. Let ϕ_{V_2} denote the density of (X, Y) . By condition 2 of Assumption 2 $\phi_{V_2}(X, Y) \leq U_2 \exp(-\delta \|(X, Y)\|_2)$, thus, we can bound the above

probability by

$$\begin{aligned}
\mathbb{P}\left(\left|\left\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle \mathbb{I}\left(\mathbf{x} \in D_{k,i}^{(1)}\right)\right| \geq a - b_k\right) &= \int_{\frac{a-b_k}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}}^{+\infty} \int_0^{\gamma+b_k} \phi(X, Y) dX dY \\
&\leq U_2 \int_{\frac{a-b_k}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}}^{+\infty} \int_0^{\gamma+b_k} \exp(-\delta \sqrt{X^2 + Y^2}) dX dY \\
&\leq U_2(\gamma + b_k) \int_{\frac{a-b_k}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}}^{+\infty} \exp(-\delta X) dX dY \\
&= \frac{U_2(\gamma + b_k)}{\delta} \exp\left(-\delta \frac{a - b_k}{\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2}\right) \\
&\leq \frac{U_2(\gamma + b_k)}{\delta} \exp\left(-\delta \frac{a - b_k}{r_k}\right) \\
&\leq \frac{U_2 \exp(\delta)(\gamma + b_k)}{\delta} \exp\left(-\delta \frac{a}{r_k}\right),
\end{aligned}$$

where the third inequality holds since $\|\mathbf{w}^* - \mathbf{w}_{k,i}\| \leq r_k < \frac{\pi}{2}$, which implies $\|\mathbf{w}_{\perp \mathbf{w}}^*\|_2 \leq r_k$. The last inequality holds since $b_k < r_k$ by our setting.

Since $\left|\left(\frac{1}{2}y_{k,i}^{(1)} - \left(\frac{1}{2} - \bar{\eta}\right)\right)\right| \leq 1$, for $a \geq b_k$, we have

$$\begin{aligned}
&\mathbb{P}\left(\left|\left\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle \left(\frac{1}{2} - \left(\frac{1}{2} - \bar{\eta}\right)y_{k,i}^{(1)}\right) \mathbb{I}\left(\mathbf{x}_{k,i}^{(1)} \in D^{(1)}\right)\right| \geq a - b_k\right) \\
&\leq \mathbb{P}\left(\left|\left\langle \mathbf{w}^*, \mathbf{x}_{\perp \mathbf{w}}^{(1)} \right\rangle \mathbb{I}\left(\mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)}\right)\right| \geq a - b_k\right) \leq \frac{U_2 \exp(\delta)(\gamma + b_k)}{\delta} \exp\left(-\delta \frac{a}{r_k}\right).
\end{aligned} \tag{32}$$

Combing (32) and (31), we get that for $a \geq b_k$,

$$\begin{aligned}
\mathbb{P}\left(\left|\left\langle \mathbf{w}^*, \bar{\eta}\hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta})\hat{\mathbf{x}}_{k,i}^{(1,-)} \right\rangle\right| \geq a\right) &\leq \mathbb{P}\left(\left|\left\langle \mathbf{w}^*, \left(\bar{\eta}\hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta})\hat{\mathbf{x}}_{k,i}^{(1,-)}\right)_{\perp \mathbf{w}} \right\rangle\right| \geq a - b_k\right) \\
&\leq \frac{U_2 \exp(\delta)(\gamma + b_k)}{\delta} \exp\left(-\delta \frac{a}{r_k}\right).
\end{aligned} \tag{33}$$

For $0 < a < b_k$,

$$\begin{aligned}
\mathbb{P}\left(\left|\left\langle \mathbf{w}^*, \bar{\eta}\hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta})\hat{\mathbf{x}}_{k,i}^{(1,-)} \right\rangle\right| \geq a\right) &\leq \mathbb{P}\left(\left|\left\langle \mathbf{w}^*, \bar{\eta}\hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta})\hat{\mathbf{x}}_{k,i}^{(1,-)} \right\rangle\right| > 0\right) \\
&\leq \mathbb{P}\left(\mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)}\right) \\
&\leq U_1(\gamma + b_k) \\
&\leq U_1 \exp(\delta)(\gamma + b_k) \exp\left(-\delta \frac{a}{r_k}\right).
\end{aligned} \tag{34}$$

Where the third inequality holds by Lemma 4 property (a) and the last equality holds since $a < b_k < r_k$.

Combing (33) and (34), we establish probability tail bound of (a) in (27): for $\forall a > 0$ and $\rho = \max\left\{U_1 \exp(\delta), \frac{U_2 \exp(\delta)}{\delta}\right\}$, we have

$$\mathbb{P}\left(\left|\left\langle \mathbf{w}^*, \bar{\eta}\hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta})\hat{\mathbf{x}}_{k,i}^{(1,-)} \right\rangle\right| \geq a\right) \leq \rho(\gamma + b_k) \exp\left(-\delta \frac{a}{r_k}\right).$$

Following the same technique, we can bound part (b) of (27) for $a > 0$ by:

$$\mathbb{P}\left(\left|\left\langle \mathbf{w}^*, \bar{\eta}\hat{\mathbf{x}}_{k,i}^{(2,+)} - (1 - \bar{\eta})\hat{\mathbf{x}}_{k,i}^{(2,-)} \right\rangle\right| \geq a\right) \leq \rho\gamma \exp\left(-\delta \frac{a}{r_k}\right).$$

By Lemma 9, part (a) and part (b) in (27) are $\left(\frac{6}{\delta}\sqrt{1+2\rho(\gamma+b_k)}r_k, \frac{6}{\delta}r_k\right)$ -subexponential, thus, by Lemma 7, we get that with probability at least $1 - 1/T^2$,

$$\begin{aligned} & \sum_{i=1}^{T_k} \mathbb{E} \left[\left\langle \mathbf{w}^*, \bar{\eta} \tilde{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta}) \tilde{\mathbf{x}}_{k,i}^{(1,-)} \right\rangle \middle| \mathcal{F}_{k,i-1} \right] \\ & \leq \sum_{i=1}^{T_k} \left\langle \mathbf{w}^*, \bar{\eta} \tilde{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta}) \tilde{\mathbf{x}}_{k,i}^{(1,-)} \right\rangle \\ & \quad + \frac{12}{\delta} \sqrt{1+2\rho(\gamma+b_k)}r_k \sqrt{T_k \ln T} + \frac{24}{\delta} r_k \ln T, \end{aligned} \quad (35)$$

and with probability at least $1 - 1/T^2$,

$$\begin{aligned} & \sum_{i=1}^{T_k} \mathbb{E} \left[\left\langle \mathbf{w}^*, \bar{\eta} \tilde{\mathbf{x}}_{k,i}^{(2,+)} - (1 - \bar{\eta}) \tilde{\mathbf{x}}_{k,i}^{(2,-)} \right\rangle \middle| \mathcal{F}_{k,i-1} \right] \\ & \geq \sum_{i=1}^{T_k} \left\langle \mathbf{w}^*, \bar{\eta} \tilde{\mathbf{x}}_{k,i}^{(2,+)} - (1 - \bar{\eta}) \tilde{\mathbf{x}}_{k,i}^{(2,-)} \right\rangle \\ & \quad - \frac{12}{\delta} \sqrt{1+2\rho(\gamma+b_k)}r_k \sqrt{T_k \ln T} - \frac{24}{\delta} r_k \ln T. \end{aligned} \quad (36)$$

Taking the union bound of (35) and (36), we get that with probability at least $1 - 2/T^2$,

$$\begin{aligned} & \sum_{i=1}^{T_k} \mathbb{E}[\langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle | \mathcal{F}_{k,i-1}] \leq \sum_{i=1}^{T_k} \langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle \\ & \quad + \frac{24}{\delta} \sqrt{1+2\rho(\gamma+b_k)}r_k \sqrt{T_k \ln T} + \frac{48}{\delta} r_k \ln T. \end{aligned}$$

□

We then show that by our construction of $\hat{\mathbf{g}}_{k,i}$, $\sum_{i=1}^{T_k} \langle -\mathbf{w}_{k,i}, \hat{\mathbf{g}}_{k,i} \rangle$ also has a high probability upper bound.

Lemma 21 (High probability bound of $\sum_{i=1}^{T_k} \langle -\mathbf{w}_{k,i}, \hat{\mathbf{g}}_{k,i} \rangle$). *At batch k , with probability at least $1 - 2/T^2$, we have*

$$\sum_{i=1}^{T_k} \langle -\mathbf{w}_{k,i}, \hat{\mathbf{g}}_{k,i} \rangle \leq \bar{\eta} b_k \sum_{i=1}^{T_k} \mathbb{P}(\mathbf{x} \in D_{k,i}) + 2\bar{\eta} b_k \sqrt{T_k \ln T}.$$

Proof. Since $\hat{\mathbf{g}}_{k,i} = -\bar{\eta}(\hat{\mathbf{x}}_{k,i}^{(1,+)} - \hat{\mathbf{x}}_{k,i}^{(2,+)}) + (1 - \bar{\eta})(\hat{\mathbf{x}}_{k,i}^{(1,-)} - \hat{\mathbf{x}}_{k,i}^{(2,-)})$, and we have the following

$$\begin{aligned} \langle \mathbf{w}_{k,i}, \hat{\mathbf{x}}_{k,i}^{(1,+)} \rangle &= b_k \mathbb{I}(y_{k,i}^{(1)} = 1) \mathbb{I}(\mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)}) \\ \langle \mathbf{w}_{k,i}, \hat{\mathbf{x}}_{k,i}^{(2,+)} \rangle &= b_k \mathbb{I}(y_{k,i}^{(2)} = 1) \mathbb{I}(\mathbf{x}_{k,i}^{(2)} \in D_{k,i}^{(2)}) \\ \langle \mathbf{w}_{k,i}, \hat{\mathbf{x}}_{k,i}^{(1,-)} \rangle &= \langle \mathbf{w}_{k,i}, \hat{\mathbf{x}}_{k,i}^{(2,-)} \rangle = 0 \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E}[\langle \mathbf{w}_{k,i}, -\hat{\mathbf{g}}_{k,i} \rangle] &= \bar{\eta} b_k \mathbb{E} \left[\mathbb{I}(y = 1, \mathbf{x} \in D_{k,i}^{(1)}) - \mathbb{I}(y = 1, \mathbf{x} \in D_{k,i}^{(2)}) \right] \\ &= \bar{\eta} b_k \mathbb{P}(y = 1, \mathbf{x} \in D_{k,i}) \\ &= \bar{\eta} b_k \mathbb{P}(y = 1 | \mathbf{x} \in D_{k,i}) \mathbb{P}(\mathbf{x} \in D_{k,i}). \end{aligned}$$

Also, since $\hat{\mathbf{x}}_{k,i}^{(1,+)}$ and $\hat{\mathbf{x}}_{k,i}^{(1,-)}$ are calculated by one sample while $\hat{\mathbf{x}}_{k,i}^{(2,+)}$ and $\hat{\mathbf{x}}_{k,i}^{(2,-)}$ are calculated by another independent sample, we can reformulate $\langle \mathbf{w}_{k,i}, -\hat{\mathbf{g}}_{k,i} \rangle$ as the following:

$$\langle \mathbf{w}_{k,i}, -\hat{\mathbf{g}}_{k,i} \rangle = \underbrace{\langle \mathbf{w}_{k,i}, \bar{\eta} \tilde{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta}) \tilde{\mathbf{x}}_{k,i}^{(1,-)} \rangle}_{(a)} - \underbrace{\langle \mathbf{w}_{k,i}, \bar{\eta} \tilde{\mathbf{x}}_{k,i}^{(2,+)} - (1 - \bar{\eta}) \tilde{\mathbf{x}}_{k,i}^{(2,-)} \rangle}_{(b)}, \quad (37)$$

where (a) and (b) in (37) are independent. Thus, we establish the high probability bound of part (a) and (b) respectively.

For part (a), $\left| \left\langle \mathbf{w}_{k,i}, \bar{\eta} \hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta}) \hat{\mathbf{x}}_{k,i}^{(1,-)} \right\rangle \right| \leq \bar{\eta} b_k$, so by Lemma 8, with probability at least $1 - 1/T^2$,

$$\sum_{i=1}^{T_k} \left\langle \mathbf{w}_{k,i}, \bar{\eta} \hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta}) \hat{\mathbf{x}}_{k,i}^{(1,-)} \right\rangle \leq \sum_{i=1}^{T_k} \mathbb{E} \left[\left| \left\langle \mathbf{w}_{k,i}, \bar{\eta} \hat{\mathbf{x}}_{k,i}^{(1,+)} - (1 - \bar{\eta}) \hat{\mathbf{x}}_{k,i}^{(1,-)} \right\rangle \right| \mathcal{F}_{k,i-1} \right] + \bar{\eta} b_k \sqrt{T_k \ln T}.$$

Similarly, for part (b), with probability at least $1 - 1/T^2$,

$$\sum_{i=1}^{T_k} \left\langle \mathbf{w}_{k,i}, \bar{\eta} \hat{\mathbf{x}}_{k,i}^{(2,+)} - (1 - \bar{\eta}) \hat{\mathbf{x}}_{k,i}^{(2,-)} \right\rangle \geq \sum_{i=1}^{T_k} \mathbb{E} \left[\left| \left\langle \mathbf{w}_{k,i}, \bar{\eta} \hat{\mathbf{x}}_{k,i}^{(2,+)} - (1 - \bar{\eta}) \hat{\mathbf{x}}_{k,i}^{(2,-)} \right\rangle \right| \mathcal{F}_{k,i-1} \right] - \bar{\eta} b_k \sqrt{T_k \ln T}. \quad (38)$$

Combing part (a) and (b) above, and take the union bound, we get that with probability at least $1 - 2/T^2$,

$$\begin{aligned} \sum_{i=1}^{T_k} \langle -\mathbf{w}_{k,i}, \hat{\mathbf{g}}_{k,i} \rangle &\leq \mathbb{E} \left[\sum_{i=1}^{T_k} \langle -\mathbf{w}_{k,i}, \hat{\mathbf{g}}_{k,i} \rangle \right] + 2\bar{\eta} b_k \sqrt{T_k \ln T} \\ &= \bar{\eta} b_k \sum_{i=1}^{T_k} \Pr(y = 1 | \mathbf{x} \in D_{k,i}) \Pr(\mathbf{x} \in D_{k,i}) + 2\bar{\eta} b_k \sqrt{T_k \ln T} \\ &\leq \bar{\eta} b_k \sum_{i=1}^{T_k} \Pr(\mathbf{x} \in D_{k,i}) + 2\bar{\eta} b_k \sqrt{T_k \ln T}. \end{aligned} \quad (39)$$

□

The following lemma shows a high probability upper bound of $\sum_{i=1}^{T_k} \|\hat{\mathbf{g}}_{k,i}\|_q^2$.

Lemma 22 (High probability bound of $\sum_{i=1}^{T_k} \|\hat{\mathbf{g}}_{k,i}\|_q^2$). *In Algorithm 4, with probability at least $1 - 2/T^2$,*

$$\sum_{i=1}^{T_k} \|\hat{\mathbf{g}}_{k,i}\|_\infty^2 \leq \frac{144}{Q^2} T_k (\ln T)^2.$$

Proof. For $q = \ln(8d) > 2$, $\|\hat{\mathbf{g}}_{k,i}\|_q \leq 2\|\hat{\mathbf{g}}_{k,i}\|_\infty$, hence, we only need to establish the high probability bound of $\|\hat{\mathbf{g}}_{k,i}\|_\infty$. By our construction of $\hat{\mathbf{g}}_{k,i}$,

$$\begin{aligned} \|\hat{\mathbf{g}}_{k,i}\|_\infty &= \left\| -\bar{\eta}(\hat{\mathbf{x}}_{k,i}^{(1,+)} - \hat{\mathbf{x}}_{k,i}^{(2,+)}) + (1 - \bar{\eta})(\hat{\mathbf{x}}_{k,i}^{(1,-)} - \hat{\mathbf{x}}_{k,i}^{(2,-)}) \right\|_\infty \\ &\leq \bar{\eta} \left\| \hat{\mathbf{x}}_{k,i}^{(1,+)} \right\|_\infty + (1 - \bar{\eta}) \left\| \hat{\mathbf{x}}_{k,i}^{(1,-)} \right\|_\infty + \bar{\eta} \left\| \hat{\mathbf{x}}_{k,i}^{(2,+)} \right\|_\infty + (1 - \bar{\eta}) \left\| \hat{\mathbf{x}}_{k,i}^{(2,-)} \right\|_\infty \\ &= \bar{\eta} \left\| [\mathbf{x}_{k,i}^{(1)} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(1)} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i} = 1) \mathbb{I}(\mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)}) \right\|_\infty \\ &\quad + (1 - \bar{\eta}) \left\| (\mathbf{x}_{k,i}^{(1)} - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(1)} \rangle \mathbf{w}_{k,i}) \mathbb{I}(y_{k,i} = -1) \mathbb{I}(\mathbf{x}_{k,i}^{(1)} \in D_{k,i}^{(1)}) \right\|_\infty \\ &\quad + \bar{\eta} \left\| [\mathbf{x}_{k,i}^{(2)} + (b_k - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \rangle) \mathbf{w}_{k,i}] \mathbb{I}(y_{k,i} = 1) \mathbb{I}(\mathbf{x}_{k,i}^{(2)} \in D_{k,i}^{(2)}) \right\|_\infty \\ &\quad + (1 - \bar{\eta}) \left\| (\mathbf{x}_{k,i}^{(2)} - \langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \rangle \mathbf{w}_{k,i}) \mathbb{I}(y_{k,i} = -1) \mathbb{I}(\mathbf{x}_{k,i}^{(2)} \in D_{k,i}^{(2)}) \right\|_\infty \\ &\leq \left\| \mathbf{x}_{k,i}^{(1)} \right\|_\infty + \left\| \mathbf{x}_{k,i}^{(2)} \right\|_\infty + 2(\gamma + b_k). \end{aligned}$$

From Lemma 13, we get that, with probability at least $1 - \frac{1}{T_k T^2}$,

$$\left\| \mathbf{x}_{k,i}^{(1)} \right\|_\infty \leq \frac{1}{Q} (1 + \ln(d T_k T^2)), \quad (40)$$

and with probability at least $1 - \frac{1}{T_k T^2}$,

$$\|\mathbf{x}_{k,i}^{(2)}\|_\infty \leq \frac{1}{Q} (1 + \ln(dT_k T^2)) . \quad (41)$$

Taking the union bound of (40),(41), we get that with probability at least $1 - \frac{2}{T_k T^2}$,

$$\|\hat{\mathbf{g}}_{k,i}\|_\infty \leq \frac{2}{Q} + \frac{2}{Q} \ln(dT_k T^2) + 2(\gamma + b_k) .$$

Taking union bound over all iterations $i \in [T_k]$, we have that with probability at least $1 - 2/T^2$,

$$\sum_{i=1}^{T_k} \|\hat{\mathbf{g}}_{k,i}\|_\infty^2 \leq T_k \left(\frac{2}{Q} + \frac{2}{Q} \ln(dT_k T^2) + 2(\gamma + b_k) \right)^2 \leq T_k \left(\frac{2}{Q} \ln(T^3) \right)^2 = \frac{36}{Q^2} T_k (\ln T)^2 .$$

Thus we conclude that with probability at least $1 - 2/T^2$,

$$\sum_{i=1}^{T_k} \|\hat{\mathbf{g}}_{k,i}\|_q^2 \leq 4 \sum_{i=1}^{T_k} \|\hat{\mathbf{g}}_{k,i}\|_\infty^2 \leq \frac{144}{Q^2} T_k (\ln T)^2 .$$

□

Given the starting angle of batch k as $\theta_k = \frac{\pi}{2^{k+1}}$, the following lemma establishes the high probability upper bound of average of $f_{k,i}(\mathbf{w}_{k,i})$.

Lemma 23. *In Algorithm 4, at every batch $k \in \{1, 2, \dots, K\}$, if $\theta(\mathbf{w}^*, \mathbf{w}_k) \leq \theta_k$, there exists some constants $c_5, c_6, c_7 > 0$, when setting bandwidth $b_k = c_5 \frac{1-2\bar{\eta}}{2^k}$, iteration number $T_k = c_6 \frac{(\gamma+1)d \ln d \ln T}{(1-2\bar{\eta})^4} \cdot 4^k$, step size $\alpha_k = c_7 \frac{\sqrt{d\theta_k}}{T_k \ln T}$, then with probability at least $1 - \frac{6}{T^2}$, the following holds:*

$$\frac{1}{T_k} \sum_{i=1}^{T_k} f_{k,i}(\mathbf{w}_{k,i}) \leq \frac{L_2 R^2 \theta_k}{12800 U_1} . \quad (42)$$

Proof. Combing Lemma 10, Lemma 18 and Lemma 19, we have $(1 - 2\bar{\eta}) f_{k,i}(\mathbf{w}_{k,i}) \mathbb{P}(\mathbf{x} \in D_{k,i}) \leq \mathbb{E}[\langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle]$, hence, it suffices to upper bound $\sum_{i=1}^{T_k} \mathbb{E}[\langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle]$.

First, we upper bound $\sum_{i=1}^{T_k} \mathbb{E}[\langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle]$ by $\sum_{i=1}^{T_k} \langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle$.

By our setting of constraint set, $\cos(\theta(\mathbf{w}_{k,i}, \mathbf{w}_k)) = \langle \mathbf{w}_{k,i}, \mathbf{w}_k \rangle \geq \cos \theta_k$, since $\cos \theta$ is decreasing in $\theta \in [0, \pi]$, hence, we have $\theta(\mathbf{w}_{k,i}, \mathbf{w}_k) \leq \theta_k$, thus, $\|\mathbf{w}_{k,i} - \mathbf{w}_k\|_2 \leq \theta(\mathbf{w}_{k,i}, \mathbf{w}_k) \leq \theta_k$. Also, $\|\mathbf{w}_k - \mathbf{w}^*\|_2 \leq \theta(\mathbf{w}_k, \mathbf{w}^*) \leq \theta_k$, by Lemma 12, $\|\mathbf{w}_{k,i} - \mathbf{w}^*\|_2 \leq 2\theta_k$. According to Lemma 20, set $r_k = 2\theta_k$, then with probability at least $1 - 2/T^2$, the following holds:

$$\begin{aligned} \sum_{i=1}^{T_k} \mathbb{E}[\langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle | \mathcal{F}_{k,i-1}] &\leq \sum_{i=1}^{T_k} \langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle \\ &+ \frac{48}{\delta} \sqrt{1 + 2\rho(\gamma + b_k)} \theta_k \sqrt{T_k \ln T} + \frac{96}{\delta} \theta_k \ln T , \end{aligned} \quad (43)$$

Next, we move on to upper bound $\sum_{i=1}^{T_k} \langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle$ through a nonstandard regret analysis of online mirror decent. Let $B(\mathbf{v}_1, \mathbf{v}_2) := \frac{1}{2(p-1)} \|\mathbf{v}_1 - \mathbf{v}_2\|_p^2$ denotes the Bregman divergence w.r.t. $\frac{1}{2(p-1)} \|\cdot\|_p^2$, where $p = \frac{\ln(8d)}{\ln(8d)-1}$. In each iteration i , the regularizer, $B(\cdot, \mathbf{w}_{k,i-1})$ is 1-strongly convex with respect to $\|\cdot\|_p$ [see Shalev-Shwartz [2007]]. From the analysis of online mirror descent [see Orabona [2023], Lemma 6.9], with step size α_k , we have

$$\langle \alpha_k \hat{\mathbf{g}}_{k,i}, \mathbf{w}_{k,i} - \mathbf{w}^* \rangle \leq B(\mathbf{w}^*, \mathbf{w}_{k,i}) - B(\mathbf{w}^*, \mathbf{w}_{k,i+1}) + \frac{\alpha_k^2}{2} \|\hat{\mathbf{g}}_{k,i}\|_q^2 .$$

Where $q = \ln(8d) > 2$. Summing the above equality over $i \in [T_k]$, we get

$$\sum_{i=1}^{T_k} \langle \alpha_k \hat{\mathbf{g}}_{k,i}, \mathbf{w}_{k,i} - \mathbf{w}^* \rangle \leq B(\mathbf{w}^*, \mathbf{w}_{k,1}) - B(\mathbf{w}^*, \mathbf{w}_{k,T_k+1}) + \frac{\alpha_k^2}{2} \sum_{i=1}^{T_k} \|\hat{\mathbf{g}}_{k,i}\|_q^2 .$$

Dividing both sides by α_k , and moving $\sum_{i=1}^{T_k} \langle \mathbf{w}_{k,i}, \hat{\mathbf{g}}_{k,i} \rangle$ to RHS, we get

$$\begin{aligned} \sum_{i=1}^{T_k} \langle \mathbf{w}^*, -\hat{\mathbf{g}}_k \rangle &\leq \frac{1}{\alpha_k} [B(\mathbf{w}^*, \mathbf{w}_{k,1}) - B(\mathbf{w}^*, \mathbf{w}_{k,T_k+1})] + \sum_{i=1}^{T_k} \langle \mathbf{w}_{k,i}, -\hat{\mathbf{g}}_{k,i} \rangle + \frac{\alpha_k}{2} \sum_{i=1}^{T_k} \|\hat{\mathbf{g}}_{k,i}\|_q^2 \\ &\leq \frac{1}{\alpha_k} B(\mathbf{w}^*, \mathbf{w}_{k,1}) + \sum_{i=1}^{T_k} \langle \mathbf{w}_{k,i}, -\hat{\mathbf{g}}_{k,i} \rangle + \frac{\alpha_k}{2} \sum_{i=1}^{T_k} \|\hat{\mathbf{g}}_{k,i}\|_q^2. \end{aligned} \quad (44)$$

Now we need to bound the three terms in RHS of (44) respectively.

First, we bound $B(\mathbf{w}^*, \mathbf{w}_{k,1}) = B(\mathbf{w}^*, \mathbf{w}_k)$.

$$B(\mathbf{w}^*, \mathbf{w}_{k,1}) = \frac{\|\mathbf{w}^* - \mathbf{w}_k\|_p^2}{2(p-1)} \stackrel{(a)}{\leq} \frac{\|\mathbf{w}^* - \mathbf{w}_k\|_1^2}{2(p-1)} \stackrel{(b)}{\leq} \frac{d\|\mathbf{w}^* - \mathbf{w}_k\|_2^2}{2(p-1)} \stackrel{(c)}{\leq} \frac{d \ln(8d)\theta_k^2}{2}. \quad (45)$$

Where inequality (a) holds by the fact that $\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_1$ for all $p > 1$ and $\mathbf{x} \in \mathbb{R}^d$. Inequality (b) holds since $\|\mathbf{x}\|_1 \leq \sqrt{d}\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^d$. Inequality (c) holds since $\|\mathbf{w}^* - \mathbf{w}_k\|_2 \leq \theta(\mathbf{w}^*, \mathbf{w}_k) \leq \theta_k$.

Next, we bound $\sum_{i=1}^{T_k} \langle -\mathbf{w}_{k,i}, \hat{\mathbf{g}}_{k,i} \rangle$. By Lemma 21, we have that with probability at least $1 - 2/T^2$,

$$\sum_{i=1}^{T_k} \langle -\mathbf{w}_{k,i}, \hat{\mathbf{g}}_{k,i} \rangle \leq \bar{\eta} b_k \sum_{i=1}^{T_k} \mathbb{P}(\mathbf{x} \in D_{k,i}) + 2\bar{\eta} b_k \sqrt{T_k \ln T}. \quad (46)$$

Finally, we bound $\sum_{i=1}^{T_k} \|\hat{\mathbf{g}}_{k,i}\|_q^2$. By Lemma 22, we have with probability at least $1 - 2/T^2$,

$$\sum_{i=1}^{T_k} \|\hat{\mathbf{g}}_{k,i}\|_\infty^2 \leq \frac{144}{Q^2} T_k (\ln T)^2. \quad (47)$$

Combining (43), (45), (46) and (47), and take the union bound, we get that with probability at least $1 - 6/T^2$,

$$\begin{aligned} &(1 - 2\bar{\eta}) \sum_{i=1}^{T_k} f_{k,i}(\mathbf{w}_{k,i}) \mathbb{P}(\mathbf{x} \in D_{k,i}) \\ &\leq \sum_{i=1}^{T_k} \mathbb{E}[\langle \mathbf{w}^*, -\hat{\mathbf{g}}_{k,i} \rangle] \\ &\leq \frac{1}{\alpha_k} \frac{d \ln(8d)\theta_k^2}{2} + \bar{\eta} b_k \sum_{i=1}^{T_k} \mathbb{P}(\mathbf{x} \in D_{k,i}) + 2\bar{\eta} b_k \sqrt{T_k \ln T} + \frac{72}{Q^2} \alpha_k T_k (\ln T)^2 \\ &\quad + \frac{48}{\delta} \sqrt{1 + 2\rho(\gamma + b_k)} \theta_k \sqrt{T_k \ln T} + \frac{96}{\delta} \theta_k \ln T. \end{aligned} \quad (48)$$

By Lemma 4, property (a), and the fact that $b_k \leq b_1 < R$, we have $L_1 b_k \leq \mathbb{P}(\mathbf{x} \in D_{k,i}) \leq U_1 b_k$, dividing both sides of (48) by $(1 - 2\bar{\eta}) L_1 T_k b_k$, we get:

$$\begin{aligned} &\frac{1}{T_k} \sum_{i=1}^{T_k} f_{k,i}(\mathbf{w}_{k,i}) \\ &\leq \frac{1}{\alpha_k} \frac{d \ln(8d)\theta_k^2}{2(1 - 2\bar{\eta}) L_1 T_k b_k} + \frac{U_1 \bar{\eta}}{(1 - 2\bar{\eta}) L_1} b_k + \frac{2\bar{\eta}}{(1 - 2\bar{\eta}) L_1} \sqrt{\frac{\ln T}{T_k}} + \frac{72\alpha_k}{Q^2(1 - 2\bar{\eta}) L_1 b_k} (\ln T)^2 \\ &\quad + \frac{48}{\delta(1 - 2\bar{\eta}) L_1} \sqrt{1 + 2\rho(\gamma + b_k)} \frac{\theta_k}{b_k} \sqrt{\frac{\ln T}{T_k}} + \frac{96}{\delta(1 - 2\bar{\eta}) L_1} \frac{\theta_k \ln T}{b_k T_k}. \end{aligned} \quad (49)$$

By our setting, $b_k = \frac{(1-2\bar{\eta})L_1 L_2 R^2}{38400U_1^2} \theta_k = c_5 \frac{1-2\bar{\eta}}{2^k}$, $T_k = \frac{576\pi^2(1+2\rho(\gamma+b_1))d \ln(8d)(\ln T)^2}{\delta^2 c_5^2 (1-2\bar{\eta})^2 b_k^2} = c_6 \frac{(\gamma+1)d \ln d (\ln T)^2}{(1-2\bar{\eta})^4} 4^k$, $\alpha_k = \frac{Q\sqrt{d \ln(8d)}\theta_k}{12\sqrt{T_k} \ln T} = c_7 \frac{\sqrt{d \ln d} \theta_k}{\sqrt{T_k} \ln T}$, then we get our proof. \square

Based on Lemma 23, we establish the main theoretical guarantee of Algorithm 4 in Proposition 4.

Proposition 4. *For some constants $c_5, c_6, c_7 > 0$, when Algorithm 4 runs with an initial vector \mathbf{w}_k satisfying $\theta(\mathbf{w}^*, \mathbf{w}_k) \leq \theta_k = \pi/2^{k+1}$, bandwidth $b_k = c_5(1 - 2\bar{\eta})2^{-k}$ for $T_k = c_64^k(\gamma + 1)d \ln d(\ln T)^2/(1 - 2\bar{\eta})^4$ iterations with step size $\alpha_k = c_7\sqrt{d \ln d}\theta_k/(\sqrt{T_k} \ln T)$, its output \mathbf{w}_{k+1} satisfies $\theta(\mathbf{w}^*, \mathbf{w}_{k+1}) \leq \theta_{k+1} = \frac{\theta_k}{2}$ with probability at least $1 - 6/T^2$.*

Proof. For the given unit vector \mathbf{w}_k that satisfy $\theta(\mathbf{w}^*, \mathbf{w}_k) \leq \theta_k \leq \frac{\pi}{4}$, we have

$$\|\mathbf{w}^* - \mathbf{w}_k\|_2 \leq 2 \sin\left(\frac{\theta(\mathbf{w}^*, \mathbf{w}_k)}{2}\right) \leq 2 \sin\left(\frac{\theta_k}{2}\right) \leq \theta_k.$$

The first inequality holds since $\|\mathbf{w}^*\|_2 = \|\mathbf{w}_k\|_2 = 1$. The second inequality holds since $\sin x$ is increasing in $x \in [0, \frac{\pi}{2}]$, the last inequality holds since $\sin x \leq x$ for all $0 \leq x \leq \frac{\pi}{2}$.

By our choice of \mathcal{W}_k , for every iteration i , we have $\cos(\theta(\mathbf{w}_{k,i}, \mathbf{w}_k)) = \langle \mathbf{w}_{k,i}, \mathbf{w}_k \rangle \geq \cos(\theta_k)$, thus, since $\cos x$ is decreasing for $0 \leq x \leq \pi$, then $\theta(\mathbf{w}_{k,i}, \mathbf{w}_k) \leq \theta_k$, hence,

$$\theta(\mathbf{w}^*, \mathbf{w}_{k,i}) \leq \theta(\mathbf{w}^*, \mathbf{w}_k) + \theta(\mathbf{w}_k, \mathbf{w}_{k,i}) \leq 2\theta_k, \quad (50)$$

where the first inequality holds by Lemma 12. By Lemma 10, with probability at least $1 - \frac{6}{T^2}$, $\frac{1}{T_k} \sum_{i=1}^{T_k} f_{k,i}(\mathbf{w}_{k,i}) \leq \frac{L_2 R^2 \theta_k}{12800U_1}$. Let $A_k := \{i \in [T_k] \mid f_{k,i}(\mathbf{w}_{k,i}) \geq \frac{L_2}{160U_1} R^2 \theta_k\}$. Thus,

$$\frac{L_2 R^2 \theta_k}{12800U_1} \geq \frac{1}{T_k} \sum_{i=1}^{T_k} f_{k,i}(\mathbf{w}_{k,i}) \geq \frac{L_2 R^2 \theta_k}{160U_1} \frac{|A_k|}{T_k}.$$

From the above inequality and we get $\frac{|A_k|}{T_k} \leq \frac{1}{80}$, and thus $\frac{|\bar{A}_k|}{T_k} \geq \frac{79}{80}$. By Corollary 3, the iterations $i' \in \bar{A}_k$ satisfy $\theta(\mathbf{w}^*, \mathbf{w}_{k,i'}) \leq \frac{\theta_k}{5}$. Other iterations $i' \in A_k$ satisfy $\theta(\mathbf{w}^*, \mathbf{w}_{k,i}) \leq 2\theta_k$ by (50). Therefore,

$$\begin{aligned} \frac{1}{T_k} \sum_{i=1}^{T_k} \cos(\theta(\mathbf{w}^*, \mathbf{w}_{k,i})) &\geq \cos\left(\frac{1}{5}\theta\right) \times \frac{|\bar{A}_k|}{T_k} + \cos(2\theta) \times \frac{|A_k|}{T_k} \\ &\geq \left(1 - \frac{1}{50}\theta^2\right) \times \frac{79}{80} + \left(1 - \frac{1}{2} \times 4\theta^2\right) \times \frac{1}{80} \\ &\geq 1 - \frac{1}{50}\theta^2 - \frac{1}{2} \times \frac{1}{20}\theta^2 \\ &\geq 1 - \frac{1}{20}\theta^2 \\ &= 1 - \frac{1}{5} \times \left(\frac{1}{2}\theta\right)^2 \\ &\geq \cos\left(\frac{\theta}{2}\right), \end{aligned}$$

where the second inequality utilizes the fact that $\cos x \geq 1 - \frac{1}{2}x^2$ and the last inequality holds since $\cos x \leq 1 - \frac{1}{5}x^2$ for $0 \leq x \leq \frac{\pi}{2}$. By the concavity of $\cos(\theta(\mathbf{w}^*, \cdot))$ when $\theta(\mathbf{w}^*, \mathbf{w}_{k,i}) \leq \frac{\pi}{2}$, we have

$$\cos(\theta(\mathbf{w}^*, \mathbf{w}_{k+1})) = \cos\left(\theta\left(\mathbf{w}^*, \frac{1}{T_k} \sum_{i=1}^{T_k} \mathbf{w}_{k,i}\right)\right) \geq \frac{1}{T_k} \sum_{i=1}^{T_k} \cos(\theta(\mathbf{w}^*, \mathbf{w}_{k,i})) \geq \cos\left(\frac{\theta_k}{2}\right).$$

Since $\cos x$ is decreasing in $x \in [0, \pi]$, we have $\theta(\mathbf{w}^*, \mathbf{w}_{k+1}) \leq \frac{\theta_k}{2}$. \square

A.6 Proofs for Section 4

In this section, we outline the proof of Theorem 1, which is the key theorem of this paper.

Theorem 1. For any instance of our online strategic classification problem with noise level $\bar{\eta}$, maximum manipulation distance γ , and feature dimension d , the expected regret of classifiers $\tilde{\mathbf{h}}$ from Algorithm 1 over T cycles satisfies

$$\mathbb{E}[\text{Reg}(\tilde{\mathbf{h}}; T)] = O\left(d \ln d \times (\ln T)^2 / (1 - 2\bar{\eta})^8 + \sqrt{(\gamma + 1)d \ln d \times T \ln T / (1 - 2\bar{\eta})^2}\right).$$

Proof. To derive the regret bound in Theorem 1, we decompose the total regret $\text{Reg}(\tilde{\mathbf{h}}; T)$ as defined in (1) into two parts according to pure exploration phase versus exploration-exploitation phase, and then we move on to decompose the regret in exploration-exploitation phase according to certain events. We upper bound each of these parts separately. First, we define the two phases and the events used in the regret decomposition.

Definition 4. Define the set $\mathcal{T}_{PE} := \{t \in [T] \mid 0 < t \leq 2T_{\text{init}} + T_0\}$ as the pure exploration phase, where T_{init} and T_0 are number of iterations in Algorithm 2 and Algorithm 3. Define the set $\mathcal{T}_{EE} := \{t \in [T] \mid 2T_{\text{init}} + T_0 < t \leq T\}$ as the exploration-exploitation phase. Define the event $\varepsilon_{\text{init}} := \{\langle \bar{\mathbf{w}}_0, \mathbf{w}^* \rangle \geq c_0(1 - 2\bar{\eta})\}$, where c_0 is a constant defined in Proposition 2. For $k \in \{0, 1, 2, \dots, K\}$, define the event $\varepsilon_k := \{\theta(\mathbf{w}_{k+1}, \mathbf{w}^*) \leq \frac{\pi}{2^{k+2}}\}$. Define the event $\varepsilon := \varepsilon_{\text{init}} \cap \varepsilon_0 \cap_{k \in [K]} \varepsilon_k$ as the “clean event”.

In Definition 4, the pure exploration phase \mathcal{T}_{PE} corresponds to all cycles in the Initialization and Refinement Algorithm, and the exploration-exploitation phase \mathcal{T}_{EE} corresponds to all cycles in Enhancement Algorithm. By Proposition 2, Proposition 3 and Proposition 4, the events defined in Definition 4 satisfy the following properties:

$$\mathbb{P}(\varepsilon_{\text{init}}) \geq 1 - \frac{2}{T^2}, \quad (51)$$

$$\mathbb{P}(\varepsilon_0 \mid \varepsilon_{\text{init}}) \geq 1 - \frac{3}{T^2}, \quad (52)$$

$$\mathbb{P}(\varepsilon_k \mid \varepsilon_{k-1}, \varepsilon_{k-2}, \dots, \varepsilon_0, \varepsilon_{\text{init}}) = \mathbb{P}(\varepsilon_k \mid \varepsilon_{k-1}) \geq 1 - \frac{6}{T^2}, \quad \forall k \in [K]. \quad (53)$$

Hence, taking the union bound by (51), (52) and (53), we get that the probability of clean event satisfy

$$\mathbb{P}(\varepsilon) = \mathbb{P}\left(\varepsilon_{\text{init}} \cap \varepsilon_0 \cap_{k \in [K]} \varepsilon_k\right) \geq \left(1 - \frac{2}{T^2}\right) \times \left(1 - \frac{3}{T^2}\right) \times \left(1 - \frac{6}{T^2}\right)^K \geq 1 - \frac{6}{T}.$$

Then, we decompose the total regret as

$$\text{Reg}(\tilde{\mathbf{h}}; T) = \text{Reg}(\tilde{\mathbf{h}}; \mathcal{T}_{PE}) + \text{Reg}(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \varepsilon) + \text{Reg}(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \bar{\varepsilon}), \quad (54)$$

where $\bar{\varepsilon}$ denotes the complement of ε , and the three parts of (54) is represented as the following:

$$\begin{aligned} \text{Reg}(\tilde{\mathbf{h}}; \mathcal{T}_{PE}) &= \sum_{t \in \mathcal{T}_{PE}} \text{Err}(\tilde{h}_t) - |\mathcal{T}_{PE}| \times \text{Err}(\tilde{h}^*), \\ \text{Reg}(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \varepsilon) &= \sum_{t \in \mathcal{T}_{EE}} \left(\text{Err}(\tilde{h}_t) - \text{Err}(\tilde{h}^*) \right) \mathbb{I}(\varepsilon), \\ \text{Reg}(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \bar{\varepsilon}) &= \sum_{t \in \mathcal{T}_{EE}} \left(\text{Err}(\tilde{h}_t) - \text{Err}(\tilde{h}^*) \right) \mathbb{I}(\bar{\varepsilon}). \end{aligned}$$

The first term in (54) denotes the expected regret incurred during the pure exploration phase. The second term captures the expected regret incurred during the exploration-exploitation phase given that the clean event holds. The last term characterizes the expected regret incurred during the exploration-exploitation phase given that the clean event does not hold.

Now we upper bound the three parts of (54) respectively. For the first term $\text{Reg}(\tilde{\mathbf{h}}; \mathcal{T}_{PE})$, the regret incurred in a single cycle is at most 1, and the length of pure exploration phase is $|\mathcal{T}_{PE}| = 2T_{\text{init}} + T_0$, then, the expected total regret during these time can be upper bounded by

$$\mathbb{E}[\text{Reg}(\tilde{\mathbf{h}}; \mathcal{T}_{PE})] \leq \sum_{t \in \mathcal{T}_{PE}} 1 \leq 2T_{\text{init}} + T_0 = O\left(\frac{1}{(1 - 2\bar{\eta})^8} d \ln d (\ln T)^2\right), \quad (55)$$

where the last equality holds by our setting that $T_{\text{init}} = O\left(\frac{1}{(1-2\eta)^2} \ln T\right)$ and $T_0 = O\left(\frac{1}{(1-2\eta)^8} d \ln d (\ln T)^2\right)$.

Then, we upper bound the expectation of the second term $\text{Reg}(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \varepsilon)$ which is the cumulative regret incurred under clean event during the Enhancement procedure. Let $\text{Reg}_k(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \varepsilon)$ denote the regret in each batch $k \in \{1, 2, \dots, K\}$ during this procedure under “clean event” *i.e.*, $\text{Reg}(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \varepsilon) = \sum_{k=1}^K \text{Reg}_k(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \varepsilon)$, then we only need to upper bound $\mathbb{E} \left[\text{Reg}_k(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \varepsilon) \right]$ for each batch $k \in \{1, 2, \dots, K\}$, which is characterized as

$$\begin{aligned}
& \mathbb{E} \left[\text{Reg}_k(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \varepsilon) \right] \\
&= \sum_{i=1}^{T_k} \sum_{j=1}^2 \mathbb{E} \left[\text{Err}(\tilde{h}_{k,i}^{(j)}) - \text{Err}(\tilde{h}^*) \mid \varepsilon \right] \mathbb{P}(\varepsilon) \\
&\leq \sum_{i=1}^{T_k} \sum_{j=1}^2 \mathbb{E} \left[\text{Err}(\tilde{h}_{k,i}^{(j)}) - \text{Err}(\tilde{h}^*) \mid \varepsilon \right] \\
&\leq \sum_{i=1}^{T_k} \mathbb{E} \left[\mathbb{I} \left(\text{sgn} \left(\langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(1)} \rangle \right) - \gamma \right) \neq y_{k,i}^{(1)} \right] - \mathbb{I} \left(\text{sgn} \left(\langle \mathbf{w}^*, \mathbf{r}_{k,i}^{(1,*)} \rangle \right) - \gamma \right) \neq y_{k,i}^{(1)} \mid \varepsilon \right] \\
&\quad + \sum_{i=1}^{T_k} \mathbb{E} \left[\mathbb{I} \left(\text{sgn} \left(\langle \mathbf{w}_{k,i}, \mathbf{r}_{k,i}^{(2)} \rangle \right) - \gamma - b_k \right) \neq y_{k,i}^{(2)} \right] - \mathbb{I} \left(\text{sgn} \left(\langle \mathbf{w}^*, \mathbf{r}_{k,i}^{(2,*)} \rangle \right) - \gamma \right) \neq y_{k,i}^{(2)} \mid \varepsilon \right] \\
&= \sum_{i=1}^{T_k} \mathbb{E} \left[\mathbb{I} \left(\text{sgn} \left(\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(1)} \rangle \right) \neq y_{k,i}^{(1)} \right) - \mathbb{I} \left(\text{sgn} \left(\langle \mathbf{w}^*, \mathbf{x}_{k,i}^{(1)} \rangle \right) \neq y_{k,i}^{(1)} \right) \mid \varepsilon \right] \\
&\quad + \sum_{i=1}^{T_k} \mathbb{E} \left[\mathbb{I} \left(\text{sgn} \left(\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \rangle \right) - b_k \right) \neq y_{k,i}^{(2)} \right] - \mathbb{I} \left(\text{sgn} \left(\langle \mathbf{w}^*, \mathbf{x}_{k,i}^{(2)} \rangle \right) \neq y_{k,i}^{(2)} \right) \mid \varepsilon \right] .
\end{aligned} \tag{56}$$

Where $\mathbf{r}_{k,i}^{(1,*)}$ and $\mathbf{r}_{k,i}^{(2,*)}$ are the counterfactual agent responses under the optimal classifier. The first equality holds by the fact that $\mathbb{E}[X \mathbb{I}(\varepsilon)] = \mathbb{E}[X \mid \mathbb{I}(\varepsilon) = 1] \mathbb{P}(\varepsilon) + \mathbb{E}[0 \mid \mathbb{I}(\varepsilon) = 0] \mathbb{P}(\bar{\varepsilon}) = \mathbb{E}[X \mid \varepsilon] \mathbb{P}(\varepsilon)$. The inequality holds since $0 \leq \mathbb{P}(\varepsilon) \leq 1$. The last equality holds by Proposition 1.

For the first term within the summation in RHS of (56), we have

$$\begin{aligned}
& \mathbb{E} \left[\mathbb{I} \left(\text{sgn} \left(\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(1)} \rangle \right) \neq y_{k,i}^{(1)} \right) - \mathbb{I} \left(\text{sgn} \left(\langle \mathbf{w}^*, \mathbf{x}_{k,i}^{(1)} \rangle \right) \neq y_{k,i}^{(1)} \right) \mid \varepsilon \right] \\
&\leq \mathbb{P}(\text{sgn}(\langle \mathbf{w}^*, \mathbf{x} \rangle) \neq \text{sgn}(\langle \mathbf{w}_{k,i}, \mathbf{x} \rangle) \mid \varepsilon) \\
&\leq \frac{c_{10}}{2^k} ,
\end{aligned} \tag{57}$$

where c_{10} is a positive constant. The first inequality holds by triangular inequality, the last equality holds since by Lemma 4 property (b), and Proposition 4.

For the second term, we have

$$\begin{aligned}
& \mathbb{E} \left[\mathbb{I} \left(\text{sgn} \left(\left\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \right\rangle - b_k \right) \neq y_{k,i}^{(2)} \right) - \mathbb{I} \left(\text{sgn} \left(\left\langle \mathbf{w}^*, \mathbf{x}_{k,i}^{(2)} \right\rangle \right) \neq y_{k,i}^{(2)} \right) \mid \varepsilon \right] \\
&= \mathbb{E} \left[\mathbb{I} \left(\text{sgn} \left(\left\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \right\rangle - b_k \right) \neq y_{k,i}^{(2)} \right) - \mathbb{I} \left(\text{sgn} \left(\left\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \right\rangle \right) \neq y_{k,i}^{(2)} \right) \mid \varepsilon \right] \\
&\quad + \mathbb{E} \left[\mathbb{I} \left(\text{sgn} \left(\left\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \right\rangle \right) \neq y_{k,i}^{(2)} \right) - \mathbb{I} \left(\text{sgn} \left(\left\langle \mathbf{w}^*, \mathbf{x}_{k,i}^{(2)} \right\rangle \right) \neq y_{k,i}^{(2)} \right) \mid \varepsilon \right] \\
&\leq \mathbb{E} \left[\mathbb{I} \left(\text{sgn} \left(\left\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \right\rangle - b_k \right) \neq \text{sgn} \left(\left\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \right\rangle \right) \right) \mid \varepsilon \right] \\
&\quad + \mathbb{E} \left[\mathbb{I} \left(\text{sgn} \left(\left\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \right\rangle \right) \neq \text{sgn} \left(\left\langle \mathbf{w}^*, \mathbf{x}_{k,i}^{(2)} \right\rangle \right) \right) \mid \varepsilon \right] \\
&= \mathbb{P} \left(0 \leq \left\langle \mathbf{w}_{k,i}, \mathbf{x}_{k,i}^{(2)} \right\rangle < b_k \right) + \mathbb{P} \left(\text{sgn} \left(\left\langle \mathbf{w}^*, \mathbf{x} \right\rangle \right) \neq \text{sgn} \left(\left\langle \mathbf{w}_{k,i}, \mathbf{x} \right\rangle \right) \mid \varepsilon \right) \\
&\leq U_1 b_k + \frac{c_{10}}{2^k} \leq \frac{c_{11}(1-2\bar{\eta})}{2^k} + \frac{c_{10}}{2^k},
\end{aligned} \tag{58}$$

where $c_{11} > 0$ is a positive constant. The first inequality holds by triangle inequality, the last inequality holds by Lemma 4 property (a) and (b), and Proposition 4.

Summing (57) and (58) over $[T_k]$, and then we can upper bound (56) by

$$\begin{aligned}
\mathbb{E} \left[\text{Reg}_k(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \varepsilon) \right] &\leq \sum_{i=1}^{T_k} O(1) \cdot \frac{1}{2^k} + O(1) \cdot \frac{1-2\eta}{2^k} \\
&= O(1) \cdot \frac{T_k}{2^k} + O(1) \cdot \frac{(1-2\bar{\eta})T_k}{2^k} \\
&= 2^k O \left(\frac{1}{(1-2\eta)^4} (\gamma+1) d \ln d (\ln T)^2 \right).
\end{aligned}$$

Since $T = |\mathcal{T}_{PE}| + |\mathcal{T}_{EE}|$ and $|\mathcal{T}_{PE}| = O(d \ln d (\ln T)^2)$, then $|\mathcal{T}_{EE}| = O(T)$. Also, the exploration-exploitation phase corresponds to all cycles run in Algorithm 4, hence $2 \sum_{k=1}^K T_k = |\mathcal{T}_{EE}| = O(T)$. By $T_k = 4^k O \left(\frac{1}{(1-2\eta)^4} (\gamma+1) d \ln d (\ln T)^2 \right)$, we get the total number of batches as $K = \log_4 \left(O \left(\frac{(1-2\eta)^4 T}{(\gamma+1) d \ln d (\ln T)^2} \right) \right)$. Then, we can upper bound the cumulative regret during the exploration-exploitation phase under “clean event” as

$$\begin{aligned}
\mathbb{E} \left[\text{Reg}(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \varepsilon) \right] &= \sum_{k=1}^K \mathbb{E} \left[\text{Reg}_k(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \varepsilon) \right] \\
&= \sum_{k=1}^{\log_4 \left(O \left(\frac{(1-2\eta)^4 T}{(\gamma+1) d \ln d (\ln T)^2} \right) \right)} 2^k O \left(\frac{1}{(1-2\eta)^4} (\gamma+1) d \ln d (\ln T)^2 \right) \\
&= O \left(\frac{1}{(1-2\eta)^2} \sqrt{(\gamma+1) d \ln d T \ln T} \right).
\end{aligned} \tag{59}$$

Finally, we upper bound the third term in (54) as

$$\begin{aligned}
\mathbb{E} \left[\text{Reg}(\tilde{\mathbf{h}}; \mathcal{T}_{EE}, \bar{\varepsilon}) \right] &= \sum_{t \in \mathcal{T}_{EE}} \mathbb{E} \left[\left(\text{Err}(\tilde{h}_t) - \text{Err}(\tilde{h}^*) \right) \mathbb{I}(\bar{\varepsilon}) \right] \\
&\leq \mathbb{E} \left[\sum_{t \in \mathcal{T}_{EE}} \mathbb{I}(\bar{\varepsilon}) \right] = |\mathcal{T}_{EE}| \mathbb{P}(\bar{\varepsilon}) \leq T \cdot \frac{6}{T} = 6.
\end{aligned} \tag{60}$$

By combining the upper bounds of (55), (59), and (60), we finish the proof. \square

A.7 Proofs for Appendix A.1

Before proving Lemma 3, we need to first prove some intermediate lemmas. In the following lemma, we show that any log-concave distributed random vector with zero mean and positive definite

covariance matrix can be linearly transformed into a new random vector with isotropic log-concave distribution.

Lemma 24. *For any random vector $\mathbf{x} \in \mathbb{R}^d$ that follows a log-concave distribution with $\mathbb{E}[\mathbf{x}] = \mathbf{0}$ and $\mathbb{E}[\mathbf{x}\mathbf{x}^T] = \Sigma$, where Σ is positive definite, the transformed random vector $\mathbf{z} = \Sigma^{-\frac{1}{2}}\mathbf{x}$ that follows an isotropic log-concave distribution.*

Proof. We first prove that $\mathbf{z} = \Sigma^{-\frac{1}{2}}\mathbf{x}$ is isotropic. Since $\mathbb{E}[\mathbf{x}\mathbf{x}^T] = \Sigma$, Σ is positive definite, then $\mathbb{E}[\mathbf{z}\mathbf{z}^T] = \Sigma^{-\frac{1}{2}}\mathbb{E}[\mathbf{x}\mathbf{x}^T]\left(\Sigma^{-\frac{1}{2}}\right)^T = \Sigma^{-\frac{1}{2}}\Sigma\left(\Sigma^{-\frac{1}{2}}\right)^T = I$. Also, since $\mathbb{E}[\mathbf{x}] = \mathbf{0}$, then $\mathbb{E}[\mathbf{z}] = \Sigma^{-\frac{1}{2}}\mathbb{E}[\mathbf{x}] = \mathbf{0}$.

Next, we show that the probability density function of \mathbf{z} is log-concave. Suppose the corresponding probability density functions of \mathbf{x} and \mathbf{z} are $\phi_{\mathbf{x}}(\cdot)$ and $\phi_{\mathbf{z}}(\cdot)$, respectively. Since $\phi_{\mathbf{x}}(\cdot)$ is log-concave, then for $\forall \alpha \in [0, 1]$ and $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$,

$$\alpha \ln(\phi_{\mathbf{x}}(\mathbf{x}_1)) + (1 - \alpha) \ln(\phi_{\mathbf{x}}(\mathbf{x}_2)) \leq \ln(\phi_{\mathbf{x}}(\alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2)).$$

Then, for $\forall \alpha \in [0, 1], \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^d$,

$$\begin{aligned} & \alpha \ln(\phi_{\mathbf{z}}(\mathbf{z}_1)) + (1 - \alpha) \ln(\phi_{\mathbf{z}}(\mathbf{z}_2)) \\ &= \alpha \ln\left(\det\left(\Sigma^{\frac{1}{2}}\right) \phi_{\mathbf{x}}\left(\Sigma^{\frac{1}{2}}\mathbf{z}_1\right)\right) + (1 - \alpha) \ln\left(\det\left(\Sigma^{\frac{1}{2}}\right) \phi_{\mathbf{x}}\left(\Sigma^{\frac{1}{2}}\mathbf{z}_2\right)\right) \\ &= \alpha \ln\left(\phi_{\mathbf{x}}\left(\Sigma^{\frac{1}{2}}\mathbf{z}_1\right)\right) + (1 - \alpha) \ln\left(\phi_{\mathbf{x}}\left(\Sigma^{\frac{1}{2}}\mathbf{z}_2\right)\right) + \ln\left(\det\left(\Sigma^{\frac{1}{2}}\right)\right) \\ &\leq \ln\left(\phi_{\mathbf{x}}\left(\alpha\Sigma^{\frac{1}{2}}\mathbf{z}_1 + (1 - \alpha)\Sigma^{\frac{1}{2}}\mathbf{z}_2\right)\right) + \ln\left(\det\left(\Sigma^{\frac{1}{2}}\right)\right) \\ &= \ln(\phi_{\mathbf{z}}(\alpha\mathbf{z}_1 + (1 - \alpha)\mathbf{z}_2)). \end{aligned}$$

Thus, $\phi_{\mathbf{z}}(\cdot)$ is isotropic log-concave. \square

The next lemma outlines the relationship between the eigenvalues of the covariance matrices for a high-dimensional random variable before and after it is projected onto a lower-dimensional subspace.

Lemma 25. *Let \mathbf{x} be an arbitrary d -dimensional random variable with a positive definite covariance matrix Σ , whose maximum and minimum eigenvalues are $\bar{\lambda}$ and $\underline{\lambda}$, respectively. Let $V_{d'}$ an arbitrary d' -dimensional subspace with $d' \leq d$. Let $\mathbf{x}_{V_{d'}}$ denote the projection of \mathbf{x} onto $V_{d'}$ with covariance matrix $\Sigma_{V_{d'}}$, whose maximum and minimum eigenvalues are $\bar{\lambda}_{V_{d'}}$ and $\underline{\lambda}_{V_{d'}}$, respectively. Then, $\bar{\lambda}_{V_{d'}} \leq \bar{\lambda}$ and $\underline{\lambda}_{V_{d'}} \geq \underline{\lambda}$.*

Proof. Let $P \in \mathbb{R}^{d \times d'}$ denote the projection matrix of \mathbf{x} , i.e., $P^T P = I$ and $P^T \mathbf{x} = \mathbf{x}_{V_{d'}}$, then, $\Sigma_{V_{d'}} = P^T \Sigma P$. Hence, by definition of maximum eigenvalue, we have

$$\begin{aligned} \bar{\lambda}_{V_{d'}} &= \max_{\mathbf{v} \neq \mathbf{0}, \mathbf{v} \in \mathbb{R}^{d'}} \frac{\mathbf{v}^T \Sigma_{V_{d'}} \mathbf{v}}{\mathbf{v}^T \mathbf{v}} = \max_{\mathbf{v} \neq \mathbf{0}, \mathbf{v} \in \mathbb{R}^{d'}} \frac{\mathbf{v}^T P^T \Sigma P \mathbf{v}}{\mathbf{v}^T \mathbf{v}} \\ &= \max_{\mathbf{v} \neq \mathbf{0}, \mathbf{v} \in \mathbb{R}^{d'}} \frac{\mathbf{v}^T P^T \Sigma P \mathbf{v}}{\mathbf{v}^T P^T P \mathbf{v}} \leq \max_{\mathbf{u} \neq \mathbf{0}, \mathbf{u} \in \mathbb{R}^d} \frac{\mathbf{u}^T \Sigma \mathbf{u}}{\mathbf{u}^T \mathbf{u}} = \bar{\lambda}, \end{aligned}$$

where $\mathbf{u} = P\mathbf{v}$.

$$\begin{aligned} \underline{\lambda}_{V_{d'}} &= \min_{\mathbf{v} \neq \mathbf{0}, \mathbf{v} \in \mathbb{R}^{d'}} \frac{\mathbf{v}^T \Sigma_{V_{d'}} \mathbf{v}}{\mathbf{v}^T \mathbf{v}} = \min_{\mathbf{v} \neq \mathbf{0}, \mathbf{v} \in \mathbb{R}^{d'}} \frac{\mathbf{v}^T P^T \Sigma P \mathbf{v}}{\mathbf{v}^T \mathbf{v}} \\ &= \min_{\mathbf{v} \neq \mathbf{0}, \mathbf{v} \in \mathbb{R}^{d'}} \frac{\mathbf{v}^T P^T \Sigma P \mathbf{v}}{\mathbf{v}^T P^T P \mathbf{v}} \geq \min_{\mathbf{u} \neq \mathbf{0}, \mathbf{u} \in \mathbb{R}^d} \frac{\mathbf{u}^T \Sigma \mathbf{u}}{\mathbf{u}^T \mathbf{u}} = \underline{\lambda}, \end{aligned}$$

where $\mathbf{u} = P\mathbf{v}$. \square

Now we are ready to prove Lemma 3.

Lemma 3. Let $\mathbf{x} \in \mathbb{R}^d$ ($d \geq 2$) have zero mean and a log-concave distribution. Suppose the eigenvalues of its covariance matrix $\Sigma = \mathbb{E}[\mathbf{x}\mathbf{x}^\top]$ are all bounded within $[\underline{\lambda}, \bar{\lambda}]$ for some positive constants $\underline{\lambda}, \bar{\lambda}$, then the distribution of \mathbf{x} satisfies the regularity conditions in assumption 2, with parameters $L_1 = \frac{\beta_1(1)}{\sqrt{\underline{\lambda}}}$, $L_2 = \frac{\beta_1(2)}{\bar{\lambda}}$, $R = \frac{1}{9}\sqrt{\bar{\lambda}}$, $U_1 = \frac{1}{\sqrt{\underline{\lambda}}}$, $U_2 = \frac{\beta_2(2)}{\bar{\lambda}}$, $\delta = \frac{\beta_3(2)}{\sqrt{\underline{\lambda}}}$, $Q = \sqrt{\bar{\lambda}}$ for $\beta_1(1), \beta_1(2), \beta_2(2), \beta_3(2)$ given in Lemma 2.

Proof for Lemma 3. For arbitrary 1-dimensional subspace $V_1 \subseteq \mathbb{R}^1$ and 2-dimensional subspace $V_2 \subseteq \mathbb{R}^2$, let \mathbf{x}_{V_1} and \mathbf{x}_{V_2} denote the projected vectors on V_1 and V_2 with covariance matrices Σ_{V_1} and Σ_{V_2} , respectively. Let $\bar{\lambda}_{V_1}$ and $\bar{\lambda}_{V_2}$ denote the maximum eigenvalues of Σ_{V_1} , Σ_{V_2} and $\underline{\lambda}_{V_1}$ and $\underline{\lambda}_{V_2}$ denote the minimum eigenvalues of Σ_{V_1} , Σ_{V_2} , respectively. Then by Lemma 25, we have $\bar{\lambda} \geq \bar{\lambda}_{V_1}$, $\bar{\lambda} \geq \bar{\lambda}_{V_2}$, $\underline{\lambda} \leq \underline{\lambda}_{V_1}$ and $\underline{\lambda} \leq \underline{\lambda}_{V_2}$. Let $\mathbf{z}_{V_1} = \Sigma_{V_1}^{-\frac{1}{2}} \mathbf{x}_{V_1}$, $\mathbf{z}_{V_2} = \Sigma_{V_2}^{-\frac{1}{2}} \mathbf{x}_{V_2}$. Then, \mathbf{z}_{V_1} and \mathbf{z}_{V_2} have isotropic log-concave densities, denoted as $\phi_{\mathbf{z}_{V_1}}$ and $\phi_{\mathbf{z}_{V_2}}$, respectively.

We first determine L_1 , L_2 and R prescribed in Assumption 2, Condition 1. For all $\|\mathbf{x}_{V_1}\|_2 \leq \frac{1}{9}\sqrt{\bar{\lambda}}$, \mathbf{x}_{V_1} 's probability density function $\phi_{\mathbf{x}_{V_1}}(\cdot)$ satisfies

$$\phi_{\mathbf{x}_{V_1}}(\mathbf{x}_{V_1}) = \phi_{\mathbf{z}_{V_1}}\left(\Sigma_{V_1}^{-\frac{1}{2}} \mathbf{x}_{V_1}\right) \det\left(\Sigma_{V_1}^{-\frac{1}{2}}\right) \geq \frac{\beta_1(1)}{\sqrt{\bar{\lambda}_{V_1}}} \geq \frac{\beta_1(1)}{\sqrt{\bar{\lambda}}}. \quad (61)$$

Now $\|\mathbf{z}_{V_1}\|_2 = \left\|\Sigma_{V_1}^{-\frac{1}{2}} \mathbf{x}_{V_1}\right\|_2 \leq \frac{1}{\sqrt{\underline{\lambda}_{V_1}}} \|\mathbf{x}_{V_1}\|_2 \leq \frac{1}{\sqrt{\underline{\lambda}}} \cdot \frac{1}{9}\sqrt{\bar{\lambda}} = \frac{1}{9}$. Then, the first inequality in (61) holds by Lemma 2, property (b) and $\det\left(\Sigma_{V_1}^{-\frac{1}{2}}\right) \geq \frac{1}{\sqrt{\bar{\lambda}_{V_1}}}$. The second inequality holds by Lemma 25.

Similarly, for all $\|\mathbf{x}_{V_2}\|_2 \leq \frac{1}{9}\sqrt{\bar{\lambda}}$, \mathbf{x}_{V_2} 's probability density function $\phi_{\mathbf{x}_{V_2}}(\cdot)$ satisfies

$$\phi_{\mathbf{x}_{V_2}}(\mathbf{x}_{V_2}) = \phi_{\mathbf{z}_{V_2}}\left(\Sigma_{V_2}^{-\frac{1}{2}} \mathbf{x}_{V_2}\right) \det\left(\Sigma_{V_2}^{-\frac{1}{2}}\right) \geq \frac{\beta_1(2)}{\bar{\lambda}_{V_2}} \geq \frac{\beta_1(2)}{\bar{\lambda}}. \quad (62)$$

Now $\|\mathbf{z}_{V_2}\|_2 = \left\|\Sigma_{V_2}^{-\frac{1}{2}} \mathbf{x}_{V_2}\right\|_2 \leq \frac{1}{\sqrt{\underline{\lambda}_{V_2}}} \|\mathbf{x}_{V_2}\|_2 \leq \frac{1}{\sqrt{\underline{\lambda}}} \cdot \frac{1}{9}\sqrt{\bar{\lambda}} = \frac{1}{9}$. Then, the first inequality in (62) holds by Lemma 2, property (b) and $\det\left(\Sigma_{V_2}^{-\frac{1}{2}}\right) \geq \frac{1}{\bar{\lambda}_{V_2}}$. The second inequality holds by Lemma 25.

Combining (61) and (62), we have $L_1 = \frac{\beta_1(1)}{\sqrt{\bar{\lambda}}}$, $L_2 = \frac{\beta_1(2)}{\bar{\lambda}}$, $R = \frac{1}{9}\sqrt{\bar{\lambda}}$.

Then, we determine U_1 , U_2 and δ prescribed in Assumption 2, Condition 2. For $\phi_{\mathbf{x}_{V_1}}(\cdot)$, we have

$$\phi_{\mathbf{x}_{V_1}}(\mathbf{x}_{V_1}) = \phi_{\mathbf{z}_{V_1}}\left(\Sigma_{V_1}^{-\frac{1}{2}} \mathbf{x}_{V_1}\right) \det\left(\Sigma_{V_1}^{-\frac{1}{2}}\right) \leq \frac{1}{\sqrt{\underline{\lambda}_{V_1}}} \leq \frac{1}{\sqrt{\underline{\lambda}}},$$

where the first inequality holds by Lemma 2, property (c) and $\det\left(\Sigma_{V_1}^{-\frac{1}{2}}\right) \leq \frac{1}{\sqrt{\underline{\lambda}_{V_1}}}$. The second inequality holds by Lemma 25. Thus, $U_1 = \frac{1}{\sqrt{\underline{\lambda}}}$.

For $\phi_{\mathbf{x}_{V_2}}(\cdot)$, we have

$$\begin{aligned} \phi_{\mathbf{x}_{V_2}}(\mathbf{x}_{V_2}) &= \phi_{\mathbf{z}_{V_2}}\left(\Sigma_{V_2}^{-\frac{1}{2}} \mathbf{x}_{V_2}\right) \det\left(\Sigma_{V_2}^{-\frac{1}{2}}\right) \leq \frac{\beta_2(2) \exp\left(-\beta_3(2) \left\|\Sigma_{V_2}^{-\frac{1}{2}} \mathbf{x}_{V_2}\right\|_2\right)}{\underline{\lambda}_{V_2}} \\ &\leq \frac{\beta_2(2) \exp\left(-\frac{\beta_3(2)}{\sqrt{\bar{\lambda}_{V_2}}} \|\mathbf{x}_{V_2}\|_2\right)}{\underline{\lambda}_{V_2}} \\ &\leq \frac{\beta_2(2) \exp\left(-\frac{\beta_3(2)}{\sqrt{\bar{\lambda}}} \|\mathbf{x}_{V_1}\|_2\right)}{\underline{\lambda}}, \end{aligned}$$

where the first inequality holds by Lemma 2, property (d) and $\det\left(\Sigma_{V_2}^{-\frac{1}{2}}\right) \leq \frac{1}{\underline{\lambda}_{V_2}}$. The second inequality holds by $\left\|\Sigma_{V_2}^{-\frac{1}{2}} \mathbf{x}_{V_2}\right\|_2 \geq \frac{1}{\sqrt{\bar{\lambda}_{V_2}}} \|\mathbf{x}_{V_2}\|_2$. The last inequality holds by Lemma 25.

Thus, we have $U_2 = \frac{\beta_2(2)}{\underline{\lambda}}, \delta = \frac{\beta_3(2)}{\sqrt{\underline{\lambda}}}$.

Finally, we determine Q prescribed in Assumption 2, Condition 3. For arbitrary $\mathbf{w} \in \mathbb{B}^d$, $\langle \mathbf{w}, \mathbf{x} \rangle$ forms a 1-dimensional random variable whose probability density function is log-concave. Let $\mathbf{x}_{V_1'} := \langle \mathbf{w}, \mathbf{x} \rangle$ denote the projected random variable and $\sigma_{V_1'}^2 = \mathbb{E} \left[\mathbf{x}_{V_1'}^T \mathbf{x}_{V_1'} \right]$ denote its variance. Then, by Lemma 2 property (a), for every $t > 0$,

$$\mathbb{P}(|\mathbf{z}| > t) \leq e^{-\sigma_{V_1'} t+1} \leq e^{-\sqrt{\underline{\lambda}} t+1},$$

where the second inequality holds by $\sigma_{V_1'}^2 \geq \underline{\lambda}_{V_1'} \geq \underline{\lambda}$. Thus $Q = \sqrt{\underline{\lambda}}$. □