Hilbert geometry of the symmetric positive-definite bicone

Application to the geometry of the extended Gaussian family

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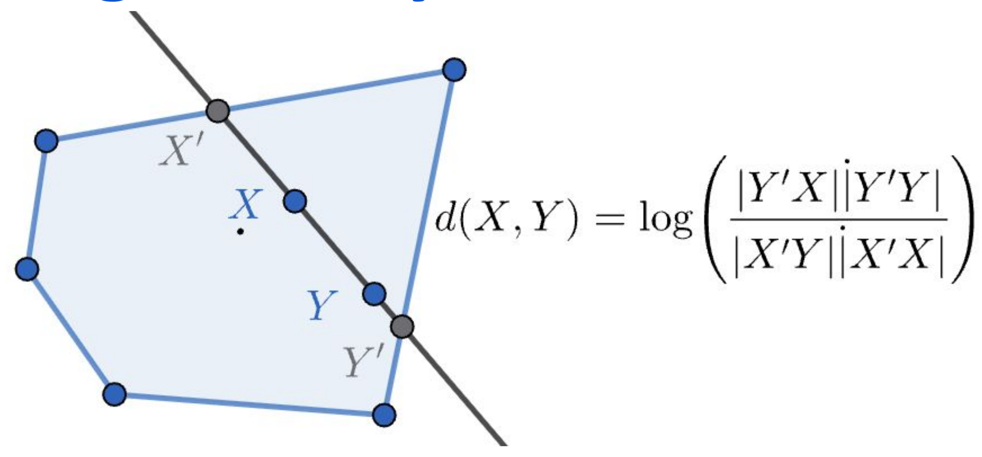
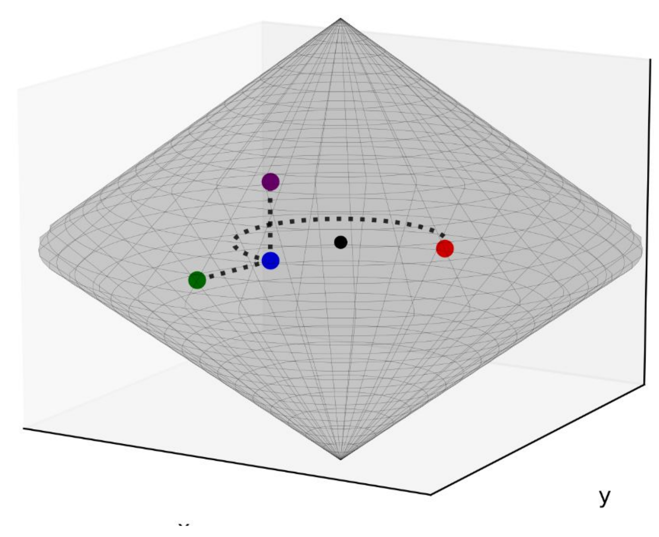
Our work in one (long) sentence

Our work finds a closed-form Hilbert metric for the open bicone of real symmetric positive-definite matrices, fully characterizes its invariance properties, and outlines potential applications for extended Gaussian distributions: i.e., distributions formed by completing the Gaussian family with degenerate covariance (supported on a subspace) or degenerate precision ("kind of uniform distribution on an affine subspace") cases.

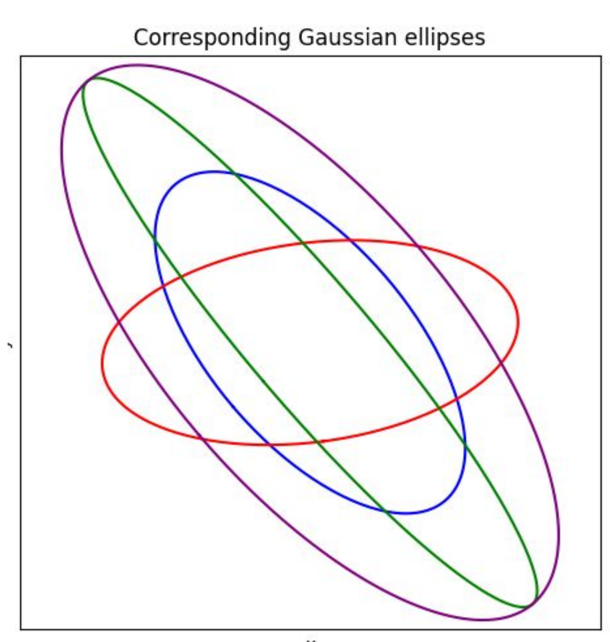
Hilbert geometry

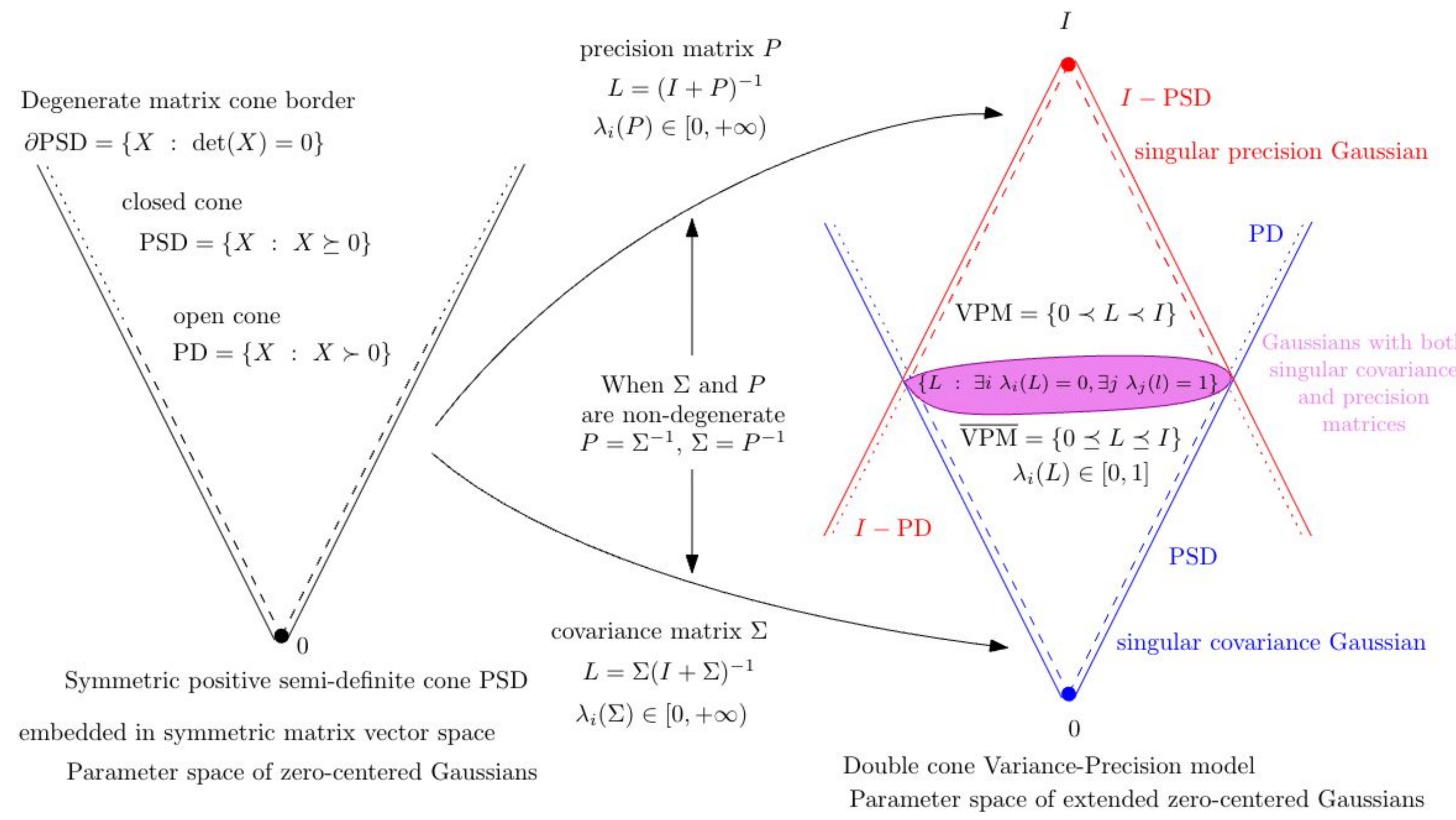


Variance-Precision Model



James' parametrization of the space of 2x2 symmetric positive- definite matrices.





Hilbert distance in VPM

Theorem (Hilbert distance on $\mathrm{VPM}(n)$). *Given two matrices $A, B \in \mathrm{VPM}(n)$,*

$$d_H(A,B) = \log \frac{\max(\lambda_{\max}, \mu_{\max})}{\min(\lambda_{\min}, \mu_{\min})}$$

where

$$\lambda_{\min} = \lambda_{\min}(B^{-1}A), \qquad \lambda_{\max} = \lambda_{\max}(B^{-1}A),$$

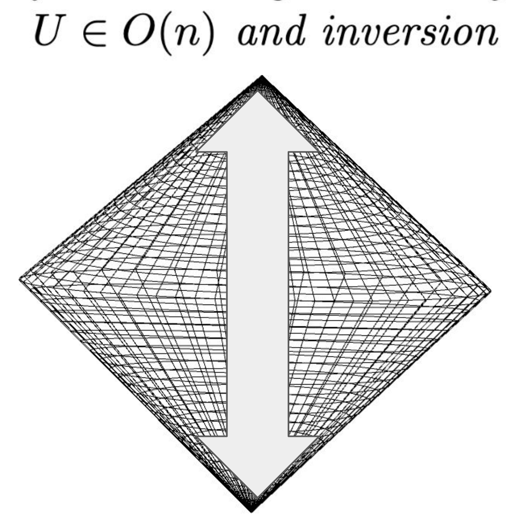
are the minimal and maximal eigenvalues of the $B^{-1}A$ matrix, and

$$\mu_{\min} = \lambda_{\min}((I-B)^{-1}(I-A)), \qquad \mu_{\max} = \lambda_{\max}((I-B)^{-1}(I-A)),$$

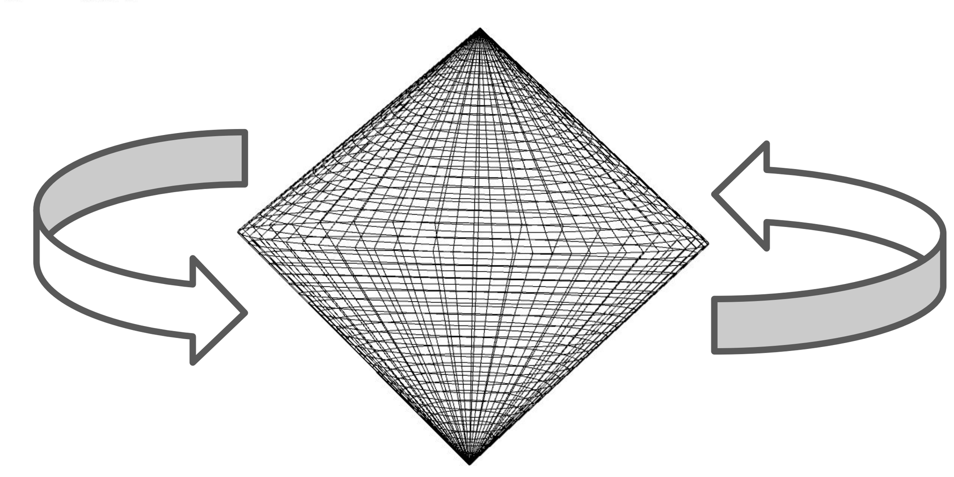
are the minimal and maximal eigenvalues of the $(I-B)^{-1}(I-A)$ matrix.

Classification of isometries of VPM

Theorem (Classification of VPM isometries). *The group of isometries of* $\mathrm{VPM}(n)$ *for $n > 1$ is generated by conjugation by orthonormal matrices $X \mapsto U^T X U$ for $U \in O(n)$ and inversion $X \mapsto I - X$.*



Inversion



Conjugation

Comparison with AIRM

Affine-Invariant Riemannian distance:

$$\rho(Q_1, Q_2) = \sqrt{\sum_{i=1}^{n} \log^2 \lambda_i(Q_1 Q_2^{-1})}.$$

Comparison of the AIRM vs Hilbert VPM distances. By $\mathrm{Mob}(Q_1,Q_2)$ we denote the Möbius transformation $\mathrm{Mob}(Q_1,Q_2) = (I-Q_1)^{-1}(I-Q_2)$.

	AIRM distance	Hilbert VPM distance
Eigenvalues:	$\{\lambda_i(Q_1Q_2^{-1})\}_{1\le i\le n}$	$\lambda_1(Q_1Q_2^{-1}), \lambda_n(Q_1Q_2^{-1})$ $\lambda_1(\mathrm{Mob}(Q_1,Q_2)), \lambda_n(\mathrm{Mob}(Q_1,Q_2))$
Invariance under a map:	$X \mapsto X^{-1}$	$X \mapsto I - X$
Invariance under congruence:	$\mathrm{GL}(n)$	$O(n)$

Extension to the boundary

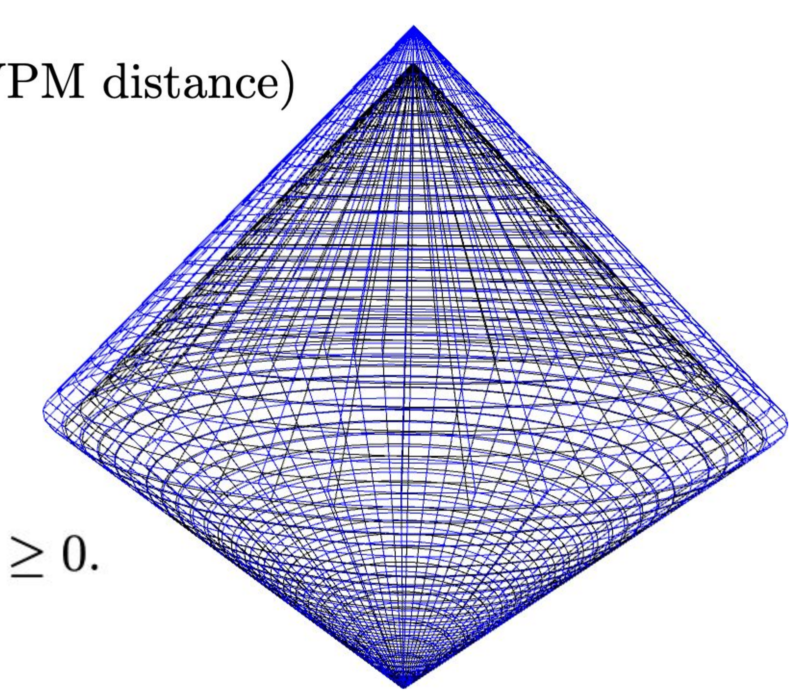
Proposition (Lower bounding Hilbert VPM distance)

$$\forall \epsilon > 0, \forall S_1, S_2 \in \mathrm{VPM}(n)$$

$$d_H(S_1,S_2) \ge d_{H,\epsilon}(S_1,S_2).$$

where:

$$\overline{\mathrm{VPM}}_\epsilon = \{-\epsilon I \preceq X \preceq (1+\epsilon) I\}, \quad \epsilon \ge 0.$$

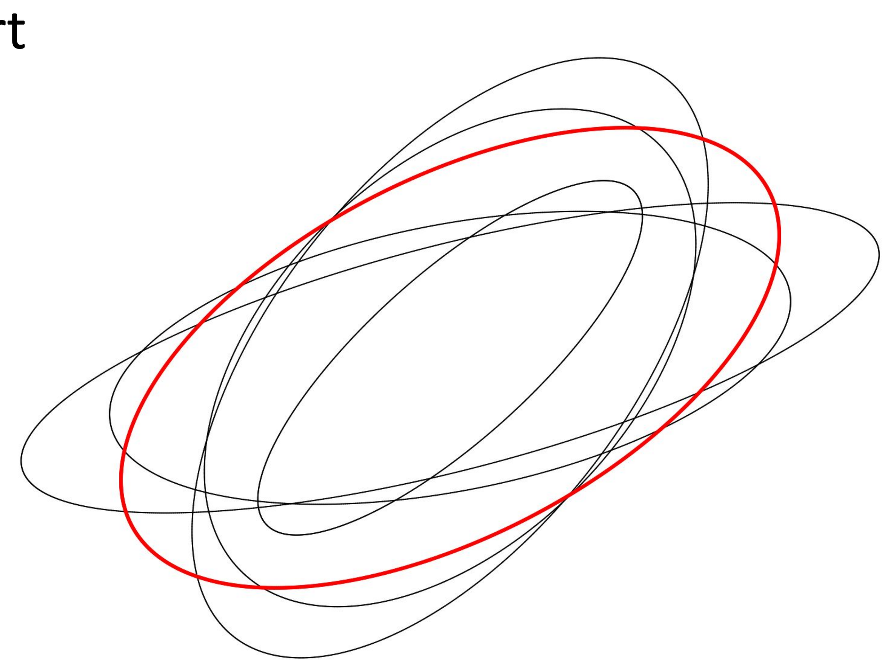


For $S_1, S_2 \in \partial\overline{\mathrm{VPM}}(n)$, $d_H(S_1,S_2) = +\infty$ but $d_{H,\epsilon}(S_1,S_2) < +\infty$.

Smallest Enclosing Ball

Straight-line geodesics in Hilbert geometry allow for easy implementation of various geometric primitives.

Here, an example implementation of Badoiu and Clarkson iterative geodesic-cut algorithm for approximating Smallest Enclosing Ball.



Extension to non-centered Gaussians

Using Calvo-Oller embedding, we can map non-centered Gaussians into positive-definite matrices, and thus, into VPM.

$$(\mu, \Sigma) \mapsto \in \Sigma_\mu^+ = \begin{bmatrix} \Sigma + \mu\mu^\top & \mu \\ \mu^\top & 1 \end{bmatrix} \subset \mathrm{PD}(n+1).$$

If you'd like to read more, see the full paper (arXiv:2508.14369) QR code →