
Differential Privacy of Dirichlet Posterior Sampling

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Abstract

We study the inherent privacy of releasing a single sample from a Dirichlet posterior distribution. As a complement to the previous study that provides general theories on the differential privacy of posterior sampling from exponential families, this study focuses specifically on the Dirichlet posterior sampling and its privacy guarantees. With the notion of truncated concentrated differential privacy (tCDP), we are able to derive a simple privacy guarantee of the Dirichlet posterior sampling, which effectively allows us to analyze its utility in various settings. Specifically, we provide accuracy guarantees of the Dirichlet posterior sampling in Multinomial-Dirichlet sampling and private normalized histogram publishing.

1 Introduction

The Bayesian framework provides a way to perform statistical analysis by combining prior beliefs with real-life evidence. At a high level, the belief and the evidence are assumed to be described by probabilistic models. As we receive new data, our belief is updated accordingly via the Bayes' theorem, resulting in the so-called posterior belief. The posterior tells us how much we are uncertain about the model's parameters.

The Dirichlet distribution is usually chosen as the prior when performing Bayesian analysis on discrete variables, as it is a conjugate prior to the categorical and multinomial distributions. Specifically, Dirichlet distributions are often used in discrete mixture models, where a Dirichlet prior is put on the mixture weights [LW92; MMR05]. Such models have applications in NLP [PB98], biophysical systems [Hin15], accident analysis [de 06], and genetics [BHW00; PM01; CWS03]. In all of these studies, samplings from Dirichlet posteriors arise when performing Markov chain Monte Carlo methods for approximate Bayesian inference.

Dirichlet posterior sampling also appears in other learning tasks. For example, in Bayesian active learning, it arises in Gibbs sampling, which is used to approximate the posterior of the classifier over the labeled sample [NLYCC13]. In Thompson sampling for multi-armed bandits, one repeatedly draws a sample from the Dirichlet posterior of each arm, and picks the arm whose sample maximizes the reward [ZHGSY20; AAFK20; NIK20]. And in Bayesian reinforcement learning, state-transition probabilities are sampled from the Dirichlet posterior over past observed states [Str00; ORR13].

Dirichlet posterior sampling can also be used for data synthesis. Suppose that we have a histogram (x_1, \dots, x_d) of actual data. An approximate discrete distribution of this histogram can be obtained by drawing a sample \mathbf{Y} from $\text{Dirichlet}(x_1 + \alpha_1, \dots, x_d + \alpha_d)$, where $\alpha_1, \dots, \alpha_d$ are prior parameters. Then synthetic data is produced by repeatedly drawing from $\text{Multinomial}(\mathbf{Y})$. There are many studies on data synthesis that followed this approach [AV08; MKAGV08; RWZ14; PG14; SJGLY17].

34 In the above examples, the data that we integrate into these tasks might contain sensitive information.
 35 Thus it is important to ask: how much of the information is protected from the Dirichlet samplings?
 36 The goal of this study is to find an answer to this question.

37 The mathematical framework of differential privacy (DP) [DMNS06] allows us to quantify how much
 38 the privacy of the Dirichlet posterior sampling is affected by the prior parameters $\alpha_1, \dots, \alpha_d$. In the
 39 definition of DP, the privacy of a randomized algorithm is measured by how much its distribution
 40 changes upon perturbing a single data point of the input. Nonetheless, this notion might be too
 41 strict for the Dirichlet distribution, as a small perturbation of a near-zero parameter can cause a large
 42 distribution shift. Thus, it might be more appropriate to rely on one of several relaxed notions of
 43 DP, such as approximate differential privacy, Rényi differential privacy, or concentrated differential
 44 privacy. It is natural to wonder if the Dirichlet posterior sampling satisfies any of these definitions.

45 1.1 Overview of Our results

46 This study focuses on the privacy and utility of Dirichlet posterior sampling. In summary, we provide
 47 a closed-form privacy guarantee of the Dirichlet posterior sampling, which in turn allows us to
 48 effectively analyze its utility in various settings.

49 **§3 Privacy.** We study the role of the prior parameters in the privacy of the Dirichlet posterior
 50 sampling. Theorem 1 is our main result, where we provide a guaranteed upper bound for truncated
 51 concentrated differential privacy (tCDP) of the Dirichlet posterior sampling. In addition, we convert
 52 the tCDP guarantee into an approximate differential privacy guarantee in Corollary 2.

53 **§4 Utility.** Using the tCDP guarantee, we investigate the utility of Dirichlet posterior sampling
 54 applied in two specific applications:

- 55 • In Section 4.1, we consider one-time sampling from a Multinomial-Dirichlet distribution.
 56 But instead of directly sampling from this distribution, we sample from another distribution
 57 with larger prior parameters. The accuracy is then measured by the KL-divergence between
 58 the original and the private distributions.
- 59 • In Section 4.2, we use the Dirichlet posterior sampling for a private release of a normalized
 60 histogram. In this case, the accuracy is measured by the mean-squared error between the
 61 sample and the original normalized histogram.

62 In both tasks, we compute the sample size that guarantees the desired level of accuracy. In the case
 63 of private histogram publishing, we also compare the Dirichlet posterior sampling to the Gaussian
 64 mechanism.

65 1.2 Related work

66 There are several studies on the differential privacy of posterior sampling. Wang, Fienberg, and
 67 Smola [WFS15] showed that any posterior sampling with the log-likelihood bounded by B is $4B$ -
 68 differentially private. However, the likelihoods that we study are not bounded away from zero; they
 69 have the form $\prod_i p_i^{x_i}$ which becomes small when one of the p_i 's is close to zero. Dimitrakakis, Nelson,
 70 Zhang, Mitrokovtsa, and Rubinstein [DNZMR17] showed that if the condition on the log-likelihood is
 71 relaxed to the Lipschitz continuity with high probability, then one can obtain the approximate DP.
 72 Nonetheless, with the Dirichlet density, it is difficult to compute the probability of events in which
 73 the Lipschitz condition is satisfied.

74 In the case that the sufficient statistics \mathbf{x} has finite ℓ^1 -sensitivity, Foulds, Geumlek, Welling and
 75 Chaudhuri [FGWC16] suggested adding Laplace noises to \mathbf{x} . Suppose that \mathbf{y} is the output; they
 76 showed that sampling from $p(\theta|\mathbf{y})$ is differentially private and as asymptotically efficient as sampling
 77 from $p(\theta|\mathbf{x})$. However, for a small sample size, the posterior over the noisy statistics might be too
 78 far away from the actual posterior. Bernstein and Sheldon [BS18] thus proposed to approximate the
 79 joint distribution $p(\theta, \mathbf{x}, \mathbf{y})$ using Gibbs sampling, which is then integrated over \mathbf{x} to obtain a more
 80 accurate posterior over \mathbf{y} .

81 Geumlek, Song, and Chaudhuri [GSC17] were the first to study the posterior sampling with the
 82 RDP. Even though they provided a general framework to find (λ, ϵ) -RDP guarantees for exponential
 83 families, explicit forms of ϵ and the upper bound of λ were not given. In contrast, our tCDP guarantees
 84 of the Dirichlet posterior sampling imply an explicit expression for ϵ , and also an upper bound for λ .

85 The privacy of data synthesis via sampling from $\text{Multinomial}(\mathbf{Y})$, where \mathbf{Y} is a discrete distri-
 86 bution drawn from the Dirichlet posterior, was first studied by Machanavajjhala, Kifer, Abowd,
 87 Gehrke, and Vilhuber [MKAGV08]. They showed that the data synthesis is (ϵ, δ) -probabilistic DP,
 88 which implies (ϵ, δ) -approximate DP. However, as their privacy analysis includes the sampling from
 89 $\text{Multinomial}(\mathbf{Y})$, their privacy guarantee depends on the number of synthetic samples. In contrast,
 90 we show that the one-time sampling from the Dirichlet posterior is approximate DP, which by the
 91 post-processing property allows us to sample from $\text{Multinomial}(\mathbf{Y})$ as many times as we want while
 92 retaining the same privacy guarantee.

93 The Dirichlet mechanism was first introduced by Gohari, Wu, Hawkins, Hale, and Topcu [GWHHT21].
 94 Originally, the Dirichlet mechanism takes a discrete distribution $\mathbf{p} := (p_1, \dots, p_d)$ and draws one
 95 sample $\mathbf{Y} \sim \text{Dirichlet}(rp_1, \dots, rp_d)$. Note the absence of the prior parameters, which makes \mathbf{Y} an
 96 unbiased estimator of \mathbf{p} . But this comes with a cost, as the worst case of privacy violation occurs
 97 when almost all of the parameters are close to zero. The authors avoided this issue by restricting
 98 the input space to a subset of the unit simplex, with some of the p_i 's bounded below by a fixed
 99 positive constant. This results in complicated expressions for the privacy guarantees as they involve
 100 a minimization problem over the restricted domain. In this study, we take a different approach by
 101 adding prior parameters to the Dirichlet mechanism. As a result, we obtain a biased algorithm that
 102 requires no assumption on the input space and has simpler forms of privacy guarantees.

103 1.3 Notations

104 We let $\mathbb{R}_{\geq 0}^d$ be the set of d -tuples of non-negative real numbers and $\mathbb{R}_{> 0}^d$ be the set of d -tuples of
 105 positive real numbers. We assume that all vectors are d -dimensional where $d \geq 2$. The notations for
 106 all vectors are always in bold. Specifically, $\mathbf{x} := (x_1, \dots, x_d) \in \mathbb{R}_{\geq 0}^d$ consists of sample statistics of
 107 the data and $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_d) \in \mathbb{R}_{> 0}^d$ consists of the prior parameters. The vector $\mathbf{p} := (p_1, \dots, p_d)$
 108 always satisfies $\sum_i p_i = 1$. The number of observations is always N . We also denote $x_0 := \sum_i x_i$
 109 and $\alpha_0 := \sum_i \alpha_i$. For any vectors \mathbf{x}, \mathbf{x}' and scalar $r > 0$, we write $\mathbf{x} + \mathbf{x}' := (x_1 + x'_1, \dots, x_d + x'_d)$
 110 and $r\mathbf{x} := (rx_1, \dots, rx_d)$. For any positive reals x and x' , the notation $x \propto x'$ means $x = Cx'$ for
 111 some constant $C > 0$, $x \approx x'$ means $cx' \leq x \leq Cx'$ for some $c, C > 0$, and $x \lesssim x'$ means $x \leq Cx'$
 112 for some $C > 0$. Lastly, $\|\mathbf{x}\|_\infty := \max_i |x_i|$ is the ℓ^∞ norm of \mathbf{x} .

113 2 Background

114 2.1 Privacy models

115 **Definition 2.1** (Pure and Approximate DP [DMNS06]). A randomized mechanism $M : \mathcal{X}^n \rightarrow \mathcal{Y}$
 116 is (ϵ, δ) -differentially private $((\epsilon, \delta)$ -DP) if for any datasets x, x' differing on a single entry, and all
 117 events $E \subset \mathcal{Y}$,

$$\mathbb{P}[M(x) \in E] \leq e^\epsilon \mathbb{P}[M(x') \in E] + \delta.$$

118 If M is $(\epsilon, 0)$ -DP, then we say that it is ϵ -differential privacy (ϵ -DP).

119 The term *pure differential privacy* (pure DP) refers to ϵ -differential privacy, while *approximate*
 120 *differential privacy* (approximate DP) refers to (ϵ, δ) -DP when $\delta > 0$.

121 In contrast to pure and approximate DP, the next definitions of differential privacy are defined in
 122 terms of the Rényi divergence between $M(x)$ and $M(x')$:

123 **Definition 2.2** (Rényi Divergence [Rén61]). Let P and Q be probability distributions. For $\lambda \in (1, \infty)$
 124 the Rényi divergence of order λ between P and Q is defined as

$$D_\lambda(P\|Q) := \frac{1}{\lambda - 1} \log \int P(y)^\lambda Q(y)^{1-\lambda} dy = \frac{1}{\lambda - 1} \log \left(\mathbb{E}_{y \sim P} \left[\frac{P(y)^{\lambda-1}}{Q(y)^{\lambda-1}} \right] \right)$$

125 **Definition 2.3** (tCDP and zCDP [BDRS18; BS16]). A randomized mechanism $M : \mathcal{X}^n \rightarrow \mathcal{Y}$ is
 126 ω -truncated ρ -concentrated differentially private $((\rho, \omega)$ -tCDP) if for any datasets x, x' differing on a
 127 single entry and for all $\lambda \in (1, \omega)$,

$$D_\lambda(M(x) \| M(x')) \leq \lambda \rho.$$

128 If M is (ρ, ∞) -tCDP, then we say that it is ρ -zero-concentrated differential privacy (ρ -zCDP).

129 Note that both tCDP and zCDP have the composition and post-processing properties. Intuitively, ρ con-
 130 trols the expectation and standard deviation of the privacy loss random variable: $Z = \log \frac{P[M(x)=Y]}{P[M(x')=Y]}$,
 131 where Y has density $M(x)$, and ω controls the number of standard deviations for which Z concen-
 132 trates like a Gaussian. A smaller ρ and larger ω correspond to a stronger privacy guarantee. It turns
 133 out that tCDP implies approximate DP:

134 **Lemma 1** (From tCDP to Approximate DP [BDRS18]). *Let $\delta > 0$. If M is a (ρ, ω) -tCDP mechanism,*
 135 *then it also satisfies (ε, δ) -DP with*

$$\varepsilon = \begin{cases} \rho + 2\sqrt{\rho \log(1/\delta)} & \text{if } \log(1/\delta) \leq (\omega - 1)^2 \rho \\ \rho\omega + \frac{\log(1/\delta)}{\omega - 1} & \text{if } \log(1/\delta) > (\omega - 1)^2 \rho \end{cases}.$$

136 2.2 Dirichlet distribution

137 For $\alpha \in \mathbb{R}_{>0}^d$, the Dirichlet distribution $\text{Dirichlet}(\alpha)$ is a continuous distribution of d -dimensional
 138 probability vectors i.e. vectors whose coordinate sum is equal to 1. The density function of $\mathbf{Y} \sim$
 139 $\text{Dirichlet}(\alpha)$ is given by:

$$p(\mathbf{y}) = \frac{1}{B(\alpha)} \prod_{i=1}^d y_i^{\alpha_i - 1},$$

140 where $B(\alpha)$ is the *beta function*, which can be written in terms of the gamma function:

$$B(\alpha) = \frac{\prod_i \Gamma(\alpha_i)}{\Gamma(\sum_i \alpha_i)}. \quad (1)$$

141 2.3 Dirichlet posterior sampling

142 We consider the prior $\text{Dirichlet}(\alpha)$ and the likelihood of the form $p(\mathbf{x}|\mathbf{y}) \propto \prod_{i=1}^d y_i^{x_i}$ where
 143 $\mathbf{x} \in \mathbb{R}_{\geq 0}^d$ consists of sample statistics of the dataset. The *Dirichlet posterior sampling* is a one-time
 144 sampling:

$$\mathbf{Y} \sim \text{Dirichlet}(\mathbf{x} + \alpha).$$

145 There is a modification of the sampling which introduces a concentration parameter $r > 0$, and
 146 instead we sample from $\text{Dirichlet}(r\mathbf{x} + \alpha)$ [GSC17; GWHHT21]. Smaller values of r make the
 147 sampling more private, and larger values of r make \mathbf{Y} a closer approximation of \mathbf{x} . Even though the
 148 case $r = 1$ is the main focus of this study, our main privacy results can be easily extended to other
 149 values of r as we will see at the end of Section 3.1.

150 Consider a special case where $\mathbf{x} = \mathbf{p}$ is an empirical distribution derived from the dataset, and we
 151 want \mathbf{Y} to be a private approximation of \mathbf{p} ; the sampling $\mathbf{Y} \sim \text{Dirichlet}(r\mathbf{p} + \alpha)$ is called the
 152 *Dirichlet mechanism* [GWHHT21]. It is interesting to note that the Dirichlet mechanism is a form of
 153 the exponential mechanism [MT07]: let $r > 0$ be the privacy parameter, $\text{Dirichlet}(\alpha)$ be the prior,
 154 and the negative KL-divergence be the score function of the exponential mechanism. Then the output
 155 \mathbf{Y} of this mechanism is distributed according to the following density function:

$$\begin{aligned} \frac{\exp(-r \text{D}_{\text{KL}}(\mathbf{p}, \mathbf{y})) \prod_i y_i^{\alpha_i - 1}}{\int \exp(-r \text{D}_{\text{KL}}(\mathbf{p}, \mathbf{y})) \prod_i y_i^{\alpha_i - 1} d\mathbf{y}} &\propto \exp\left(r \sum_{i, p_i \neq 0} p_i \log(y_i/p_i)\right) \prod_i y_i^{\alpha_i - 1} \\ &\propto \prod_{i, p_i \neq 0} y_i^{rp_i} \prod_i y_i^{\alpha_i - 1} = \prod_i y_i^{rp_i + \alpha_i - 1}, \end{aligned}$$

156 which is exactly the density function of $\text{Dirichlet}(r\mathbf{p} + \alpha)$.

157 2.4 Polygamma functions

158 In most of this study, we take advantage of several nice properties of the log-gamma function and its
 159 derivatives. Specifically, $\psi(x) := \frac{d}{dx} \log \Gamma(x)$ is concave and increasing, while its derivative $\psi'(x)$ is
 160 positive, convex, and decreasing. In addition, ψ' can be approximated by the reciprocals:

$$\frac{1}{x} + \frac{1}{2x^2} < \psi'(x) < \frac{1}{x} + \frac{1}{x^2},$$

161 which implies that $\psi'(x) \approx \frac{1}{x^2}$ as $x \rightarrow 0$ and $\psi'(x) \approx \frac{1}{x}$ as $x \rightarrow \infty$.

162 3 Main privacy results

163 3.1 Truncated concentrated differential privacy

164 **Theorem 1.** Let $\alpha \in \mathbb{R}_{>0}^d$ and $\alpha_m := \min_i \alpha_i$. Let $\gamma \in (0, \alpha_m)$. Let $\Delta_2, \Delta_\infty > 0$ be constants that
 165 satisfy $\sum_i (x_i - x'_i)^2 \leq \Delta_2^2$ and $\max_i |x_i - x'_i| \leq \Delta_\infty$ whenever $\mathbf{x}, \mathbf{x}' \in \mathbb{R}_{\geq 0}^d$ are sample statistics
 166 of any two datasets differing on a single entry. The one-time sampling from Dirichlet($\mathbf{x} + \alpha$) is
 167 (ρ, ω) -tCDP, where $\omega = \frac{\gamma}{\Delta_\infty} + 1$ and

$$\rho = \frac{1}{2} \Delta_2^2 \psi'(\alpha_m - \gamma). \quad (2)$$

168 Note that (ρ, ∞) -tCDP is not obtainable, as the ratio between two Dirichlet densities blows up as
 169 $\omega \rightarrow \infty$. We provide here a short proof of the Theorem 1 and the full proof in Appendix A

170 *proof.* Consider any $\lambda \in \left(1, \frac{\gamma}{\Delta_\infty} + 1\right)$. Let $\mathbf{u} := \mathbf{x} + \alpha$ and $\mathbf{u}' := \mathbf{x}' + \alpha'$. Let $P(\mathbf{y})$ be the density
 171 of Dirichlet(\mathbf{u}) and $P'(\mathbf{y})$ be the density of Dirichlet(\mathbf{u}'). A quick calculation shows that:

$$\mathbb{E}_{\mathbf{y} \sim P(\mathbf{y})} \left[\frac{P(\mathbf{y})^{\lambda-1}}{P'(\mathbf{y})^{\lambda-1}} \right] = \frac{B(\mathbf{u}')^{\lambda-1}}{B(\mathbf{u})^{\lambda-1}} \cdot \frac{B(\mathbf{u} + (\lambda-1)(\mathbf{u} - \mathbf{u}'))}{B(\mathbf{u})}. \quad (3)$$

172 We take the logarithm on both sides and apply the second-order Taylor expansion to the following
 173 $G(u_i, u'_i)$ and $H(u_i, u'_i)$ terms that appear on the right-hand side. As a result, there exist ξ between
 174 $u_i + (\lambda-1)(u_i - u'_i)$ and u_i , and ξ' between u_i and u'_i such that

$$\begin{aligned} G(u_i, u'_i) &:= (\lambda-1)(\log \Gamma(u'_i) - \log \Gamma(u_i)) \\ &= -(\lambda-1)(x_i - x'_i)\psi(u_i) + \frac{1}{2}(\lambda-1)(x_i - x'_i)^2\psi'(\xi') \end{aligned} \quad (4)$$

$$\begin{aligned} H(u_i, u'_i) &:= \log \Gamma(u_i + (\lambda-1)(u_i - u'_i)) - \log \Gamma(u_i) \\ &= (\lambda-1)(x_i - x'_i)\psi(u_i) + \frac{1}{2}(\lambda-1)^2(x_i - x'_i)^2\psi'(\xi), \end{aligned} \quad (5)$$

175 Note that ψ' is increasing. If $x_i > x'_i$, then ξ and ξ' are bounded below by $u'_i \geq \alpha_m$. On the
 176 other hand, if $x_i \leq x'_i$, then ξ and ξ' are bounded below by $u_i - (\lambda-1)|u_i - u'_i|$. The condition
 177 $\lambda < \frac{\gamma}{\Delta_\infty} + 1$ guarantees that $u_i - (\lambda-1)|u_i - u'_i| > \alpha_m - \gamma$. All cases considered, we have

$$\begin{aligned} G(u_i, u'_i) + H(u_i, u'_i) &\leq \frac{1}{2}((\lambda-1) + (\lambda-1)^2)(x_i - x'_i)^2\psi'(\alpha_m - \gamma) \\ &= \frac{1}{2}\lambda(\lambda-1)(x_i - x'_i)^2\psi'(\alpha_m - \gamma). \end{aligned}$$

178 Denoting $u_0 := \sum_i u_i$ and $u'_0 := \sum_i u'_i$, the same argument shows that $G(u_0, u'_0) + H(u_0, u'_0) > 0$.
 179 Therefore,

$$\begin{aligned} D_\lambda(P(\mathbf{y}) \| P'(\mathbf{y})) &= \frac{1}{\lambda-1} \left(\sum_i (G(u_i, u'_i) + H(u_i, u'_i)) - G(u_0, u'_0) - H(u_0, u'_0) \right) \\ &< \frac{1}{\lambda-1} \sum_i (G(u_i, u'_i) + H(u_i, u'_i)) \\ &\leq \frac{1}{2}\lambda \sum_i (x_i - x'_i)^2\psi'(\alpha_m - \gamma) \leq \frac{1}{2}\lambda \Delta_2^2 \psi'(\alpha_m - \gamma). \quad \square \end{aligned}$$

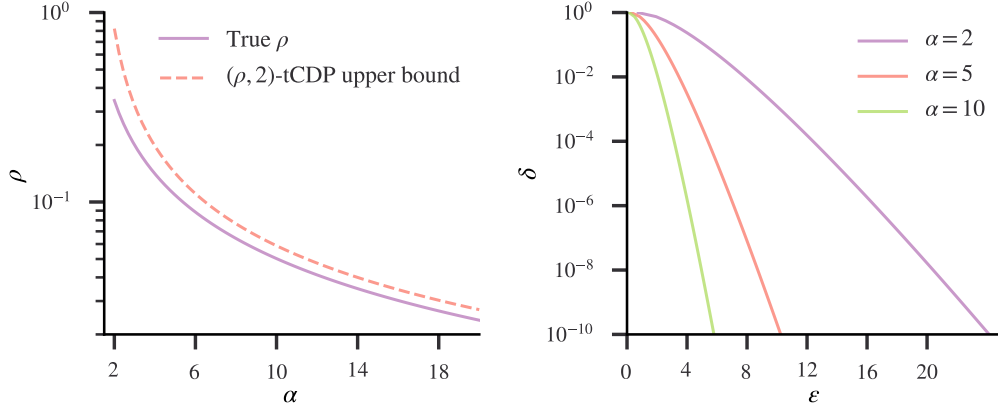


Figure 1: Left: the actual values of $\rho = \frac{1}{2} D_2(P\|P')$ and the worst case $(\rho, 2)$ -tCDP guarantees (2) at $\Delta_2^2 = \Delta_\infty = 1$. Here, P and P' are Dirichlet posterior densities over $\mathbf{x} = (11, 8, 65, 25, 38, 0)$, $\mathbf{x}' = (11, 8, 65, 25, 38, 1)$, and $\alpha = (\alpha, \dots, \alpha)$. Right: comparison between (ϵ, δ) -DP guarantees of the Dirichlet posterior samplings (8) with different uniform priors: $\alpha = (\alpha, \dots, \alpha)$.

The guaranteed upper bound (2) is independent of the sample statistics. As a result, the bound applies even in worst settings i.e., when $x_i = 0$ and $x'_i = \Delta_\infty$, or vice versa, for some i . As we can see in Figure 1, the upper bound is a close approximation to the actual value of ρ when $x_6 = 0$ and $x'_6 = 1$. However, being a sample independent bound, the difference becomes substantial when all x_i 's are large. There is one way to get around this issue: if there is no privacy violation in assuming that the sample statistics are always bounded below by some threshold τ , then we can incorporate the threshold into the prior (thus $\psi'(\alpha_m - \gamma)$ in (2) is replaced by $\psi'(\alpha_m + \tau - \gamma)$).

The parameter γ allows us to adjust the moment bound ω as desired. Even though a higher ω usually leads to a better privacy guarantee, there are two downsides to picking γ close to α_m in this case. First, note that ρ contains $\psi'(\alpha_m - \gamma)$; as $\gamma \rightarrow \alpha_m$, the value of ρ diverges to ∞ , leading to a weaker privacy guarantee instead. Second, as the Taylor approximation (5) is accurate when u_i is close to $u_i + (\lambda - 1)(u_i - u'_i)$, having a large value of λ would push the guaranteed upper bound away from the actual privacy loss. Thus it is recommended to pick γ so that $\gamma/\Delta_\infty \geq 1$ and $\alpha_m - \gamma \gg 0$. Alternatively, we can choose the value of γ that minimizes ϵ when converting from tCDP to (ϵ, δ) -DP using Lemma 1—this method will be explored in the next subsection.

Theorem 1 can be easily applied to sampling from $\text{Dirichlet}(r\mathbf{x} + \alpha)$. Replacing \mathbf{x} with $r\mathbf{x}$, we have Δ_2 replaced by $r\Delta_2$ and Δ_∞ replaced by $r\Delta_\infty$. Consequently, the sampling is $(\rho, \frac{\gamma}{r\Delta_\infty} + 1)$ -tCDP, where $\rho = \frac{1}{2} r^2 \Delta_2^2 \psi'(\alpha_m - \gamma)$. In Appendix D, we analyze the scaling of r in conjunction with α_m at a fixed privacy budget ρ .

3.2 Approximate differential privacy

We now convert the tCDP guarantee to an approximate DP guarantee. Let $\delta \in (0, 1)$. Using Lemma 1, the Dirichlet posterior sampling with $\text{Dirichlet}(\alpha)$ as the prior is (ϵ, δ) -DP with

$$\epsilon = \begin{cases} \rho(\gamma) + 2\sqrt{\rho(\gamma)\log(1/\delta)} & \text{if } \log(1/\delta) \leq \gamma^2 \rho(\gamma) / \Delta_\infty^2 \\ \rho(\gamma) \left(\frac{\gamma}{\Delta_\infty} + 1 \right) + \frac{\log(1/\delta) \Delta_\infty}{\gamma} & \text{if } \log(1/\delta) > \gamma^2 \rho(\gamma) / \Delta_\infty^2 \end{cases}, \quad (6)$$

where $\rho(\gamma) = \frac{1}{2} \Delta_2^2 \psi'(\alpha_m - \gamma)$.

We try to minimize ϵ by adjusting the value of γ . First, we consider the case $\log(1/\delta) \leq \gamma^2 \rho(\gamma) / \Delta_\infty^2$. Since $\rho(\gamma)$ is a strictly increasing function of γ , both $\rho(\gamma) + 2\sqrt{\rho(\gamma)\log(1/\delta)}$ and $\gamma^2 \rho(\gamma) / \Delta_\infty^2$ are both strictly increasing function of γ . Therefore, ϵ is minimized at the minimum possible value of γ in this case, that is, at the unique γ_M that satisfies $\log(1/\delta) = \gamma_M^2 \rho(\gamma_M) / \Delta_\infty^2 = \frac{1}{2} \gamma_M^2 \Delta_2^2 \psi'(\alpha_m - \gamma_M) / \Delta_\infty^2$.

Now we consider the second case, when $\gamma < \gamma_M$. As $\rho(\gamma)$ is an increasing positive convex function of γ , the function

$$f(\gamma) := \frac{1}{2}\Delta_2^2\psi'(\alpha_m - \gamma)\left(\frac{\gamma}{\Delta_\infty} + 1\right) + \frac{\log(1/\delta)\Delta_\infty}{\gamma}; \quad \gamma \in (0, \gamma_M], \quad (7)$$

is also convex in γ , and thus has a unique minimizer $\gamma_m \in (0, \gamma_M]$. Comparing to the first case, we have $f(\gamma_m) \leq f(\gamma_M) = \rho(\gamma_M) + 2\sqrt{\rho(\gamma_M)\log(1/\delta)}$. We then conclude that $\varepsilon = f(\gamma_m)$.

Theorem 2. Let $\alpha \in \mathbb{R}_{>0}^2$ and denote $\alpha_m = \min_i \alpha_i$. Let $\Delta_2, \Delta_\infty > 0$ be constants that satisfy $\sum_i (x_i - x'_i)^2 \leq \Delta_2^2$ and $\max_i |x_i - x'_i| \leq \Delta_\infty$ whenever $\mathbf{x}, \mathbf{x}' \in \mathbb{R}_{\geq 0}^d$ are sample statistics of any two datasets differing on a single entry. For any $\delta \in (0, 1)$, let γ_M be the solution to the equation $\log(1/\delta) = \frac{1}{2}\gamma^2\Delta_2^2\psi'(\alpha_m - \gamma)/\Delta_\infty^2$. The one-time sampling from $\text{Dirichlet}(\mathbf{x} + \alpha)$ is (ε, δ) -DP, where

$$\varepsilon = \min_{\gamma \in (0, \gamma_M]} f(\gamma). \quad (8)$$

Figure 1 shows how δ decays as a function of ε at three different values of α_m .

4 Utility

Using the results from the previous section, we analyze the Dirichlet posterior sampling's utility in two specific tasks.

4.1 Multinomial-Dirichlet sampling

Suppose that we are observing N trials, each of which has d possible outcomes. For each $i \in \{1, \dots, d\}$, let x_i be the number of times the i -th outcome was observed. Then we have the multinomial likelihood $p(\mathbf{x}|\mathbf{y}) \propto \prod_i y_i^{x_i}$. From this, we sample from the Dirichlet posterior:

$$\mathbf{Y} \sim \text{Dirichlet}(\mathbf{x} + \alpha). \quad (9)$$

Suppose that we want to sample from a true distribution $P_{\mathbf{X}} \sim \text{Dirichlet}(\mathbf{x} + \alpha)$, but for privacy reasons, we instead sample from $Q_{\mathbf{X}} \sim \text{Dirichlet}(\mathbf{x} + \alpha')$ where $\alpha'_i > \alpha_i$ for all i . The utility of the privacy scheme is then measured by the KL-divergence between $P_{\mathbf{X}}$ and $Q_{\mathbf{X}}$. Assuming that \mathbf{x} is an observation of $\text{Multinomial}(\mathbf{p})$, the following Theorem tells us that, on average, the KL-divergence is small when the sample size is large, and the p_i 's are evenly distributed.

Theorem 3. Let $\mathbf{p} := (p_1, \dots, p_d)$ where $p_i > 0$ for all i and $\sum_i p_i = 1$. Define a random variable $\mathbf{X} \sim \text{Multinomial}(\mathbf{p})$. Let $P_{\mathbf{X}} \sim \text{Dirichlet}(\mathbf{X} + \alpha)$ and $Q_{\mathbf{X}} \sim \text{Dirichlet}(\mathbf{X} + \alpha')$ where $\alpha'_i \geq \alpha_i \geq 1$ for all i . The following estimate holds:

$$\mathbb{E}_{\mathbf{X}}[\text{D}_{\text{KL}}(P_{\mathbf{X}}\|Q_{\mathbf{X}})] \leq \frac{1}{N+1} \sum_i (\alpha'_i - \alpha_i)^2 \cdot \frac{1}{p_i}. \quad (10)$$

The proof is given in Appendix B. Let us consider a simple privacy scheme where we fix $s > 0$ and let $\alpha'_i = \alpha_i + s$ for all i . Thus (10) becomes:

$$\mathbb{E}_{\mathbf{X}}[\text{D}_{\text{KL}}(P_{\mathbf{X}}\|Q_{\mathbf{X}})] \leq \frac{G(\mathbf{p})s^2}{N+1}, \quad (11)$$

where $G(\mathbf{p}) := \sum_i 1/p_i$. Now we take into account the privacy parameters. Let $\rho = \Delta_2^2\psi'(\alpha_m - \gamma)$ and $\rho' = \Delta_2^2\psi'(\alpha'_m - \gamma)$, where $\alpha_m = \min_i \alpha_i$, $\alpha'_m = \min_i \alpha'_i$, and $\gamma < \alpha_m$. Here, we approximate the values of $\psi'(\alpha_m - \gamma)$ and $\psi'(\alpha'_m - \gamma)$ under two regimes:

High-privacy regime: $\alpha'_m - \gamma > 1$. We have $\psi'(\alpha'_m - \gamma) \approx 1/(\alpha'_m - \gamma)$, which implies $\alpha'_m - \gamma \approx \Delta_2^2/\rho'$. We also have $\alpha_m - \gamma \approx \Delta_2^2/\rho$ for $\alpha_m - \gamma \geq 1$ and $\alpha_m - \gamma > (\alpha_m - \gamma)^2 \approx \Delta_2^2/\rho$ for $\alpha - \gamma < 1$. Thus we have the following bound for the right-hand side of (11):

$$\frac{G(\mathbf{p})s^2}{N+1} = \frac{G(\mathbf{p})(\alpha'_m - \alpha_m)^2}{N+1} \lesssim \frac{\Delta_2^4 G(\mathbf{p})}{N+1} \left(\frac{1}{\rho'} - \frac{1}{\rho}\right)^2 < \frac{\Delta_2^4 G(\mathbf{p})}{\rho'^2(N+1)}. \quad (12)$$

Consequently, we have $\text{D}_{\text{KL}}(P\|Q) < \epsilon$ for $N = \Omega\left(\frac{\Delta_2^4 G(\mathbf{p})}{\rho'^2 \epsilon}\right)$.

242 **Low-privacy regime:** $1 > \alpha'_m - \gamma > 0$. This is similar as above, except we have $\alpha'_m - \gamma \approx$
 243 $\Delta_2/\rho'^{1/2}$ and $\alpha_m - \gamma \approx \Delta_2/\rho'^{1/2}$. Similar computation as (12) shows that $D_{\text{KL}}(P\|Q) < \epsilon$ when
 244 $N = \Omega\left(\frac{\Delta_2^2 G(\mathbf{p})}{\rho' \epsilon}\right)$.

245 We observe that, in both regimes, the sample size scales faster with respect to ϵ with a higher value of
 246 $G(\mathbf{p})$, which is associated with a higher number of outcomes d , and more concentrated multinomial
 247 parameter \mathbf{p} ; this agrees with the result of our simulation in Appendix C. Moreover, for small ρ' the
 248 sample size scales as $1/\rho'^2$, while for large ρ' the sample size scales as $1/\rho'$.

249 4.2 Private normalized histograms

250 Let $\mathbf{x} = (x_1, \dots, x_d)$ be a histogram of N observations and $\mathbf{p} := \mathbf{x}/N$. We can privatize \mathbf{p} by
 251 sampling a probability vector: $\mathbf{Y} \sim \text{Dirichlet}(\mathbf{x} + \boldsymbol{\alpha})$. Note that \mathbf{Y} is a biased estimator of \mathbf{p} .
 252 Denoting $\alpha_0 := \sum_i \alpha_i$, the bias of each component of \mathbf{Y} is given by $\mathbb{E}[\mathbf{Y}] - \mathbf{p}$. Hence,

$$|\text{Bias}(Y_i)| = \left| \frac{x_i + \alpha_i}{N + \alpha_0} - p_i \right| = \frac{|x_i \alpha_0 - N \alpha_i|}{N(N + \alpha_0)} \leq \frac{N \alpha_0}{N(N + \alpha_0)} = \frac{\alpha_0}{N + \alpha_0}.$$

253 Since $Y_i \sim \text{Beta}(x_i + \alpha_i, N + \alpha_0 - x_i - \alpha_i)$ is $\frac{1}{4(N + \alpha_0 + 1)}$ -sub-Gaussian [MA17], we have,

$$\begin{aligned} \mathbb{P}[|Y_i - p_i| > t + |\text{Bias}(Y_i)|] &\leq \mathbb{P}[|Y_i - \mathbb{E}[Y_i]| + |\text{Bias}(Y_i)| > t + |\text{Bias}(Y_i)|] \\ &= \mathbb{P}[|Y_i - \mathbb{E}[Y_i]| > t] \\ &\leq 2e^{-2t^2(N + \alpha_0 + 1)}. \end{aligned}$$

254 With the union bound, we plug in $t = \sqrt{\frac{\log(2d/\beta)}{2(N + \alpha_0 + 1)}}$, for any $\beta \in (0, 1)$, to obtain the following
 255 accuracy guarantee of the private normalized histogram:

256 **Theorem 4.** Let $\mathbf{Y} \sim \text{Dirichlet}(\mathbf{x} + \boldsymbol{\alpha})$, where $\mathbf{x} \in \mathbb{R}_{\geq 0}^d$ and $\boldsymbol{\alpha} \in \mathbb{R}_{> 0}^d$, and $\mathbf{p} := \mathbf{x}/N$. For any
 257 $\beta \in (0, 1)$, with probability at least $1 - \beta$, the following inequality holds:

$$\|\mathbf{Y} - \mathbf{p}\|_\infty \leq \sqrt{\frac{\log(2d/\beta)}{2(N + \alpha_0 + 1)}} + \frac{\alpha_0}{N + \alpha_0}. \quad (13)$$

258 Given $\epsilon > 0$, we use (13) to find a lower bound for N that gives $\|\mathbf{Y} - \mathbf{p}\|_\infty < \epsilon$ w.p. $1 - \beta$ when
 259 \mathbf{Y} is sampled with ρ -tCDP. For simplicity, we consider a uniform prior: $\alpha_i = \alpha > 0$ for all i .
 260 Thus, $\rho = \frac{1}{2}\Delta_2^2\psi'(\alpha - \gamma)$, where γ might be chosen according to Corollary 2. We consider the two
 261 following regimes:

262 **High-privacy regime:** $\alpha - \gamma > 1$. In this case, $\psi'(\alpha - \gamma) \approx 1/(\alpha - \gamma)$. From $\rho = \frac{1}{2}\Delta_2^2\psi'(\alpha - \gamma)$,
 263 we have $\alpha \approx \Delta_2^2/2\rho + \gamma$. Replacing α_0 by $d\alpha$ in (13) yields the sample size:

$$N = \Omega\left(\frac{\log(2d/\beta)}{\epsilon^2} + \frac{d}{\epsilon}\left(\frac{\Delta_2^2}{2\rho} + \gamma\right)\right), \quad (14)$$

264 for the desired accuracy.

265 **Low-privacy regime:** $\alpha - \gamma < 1$. This is the same as above, except now we have $\psi'(\alpha - \gamma) \approx$
 266 $1/(\alpha - \gamma)^2$, which implies $\alpha \approx \Delta_2/(2\rho)^{1/2} + \gamma$. The sample size that guarantees the desired
 267 accuracy is:

$$N = \Omega\left(\frac{\log(2d/\beta)}{\epsilon^2} + \frac{d}{\epsilon}\left(\frac{\Delta_2}{\sqrt{2\rho}} + \gamma\right)\right). \quad (15)$$

268 Let us compare this result to the Gaussian mechanism, which adds a noise $\mathbf{Z} \sim N(0, \sigma^2 I_d)$ to the
 269 normalized histogram \mathbf{p} directly. Thus the ℓ_2 -sensitivity in this case is Δ_2/N . We have that the
 270 Gaussian mechanism is ρ -zCDP where $\rho = \frac{\Delta_2^2}{2N^2\sigma^2}$ [BS16]. Using the same argument as above, with
 271 probability at least $1 - \beta$, the following inequality holds for all i :

$$\|\mathbf{Z}\|_\infty \leq \sqrt{\frac{\log(2d/\beta)\Delta_2^2}{N^2\rho}}. \quad (16)$$

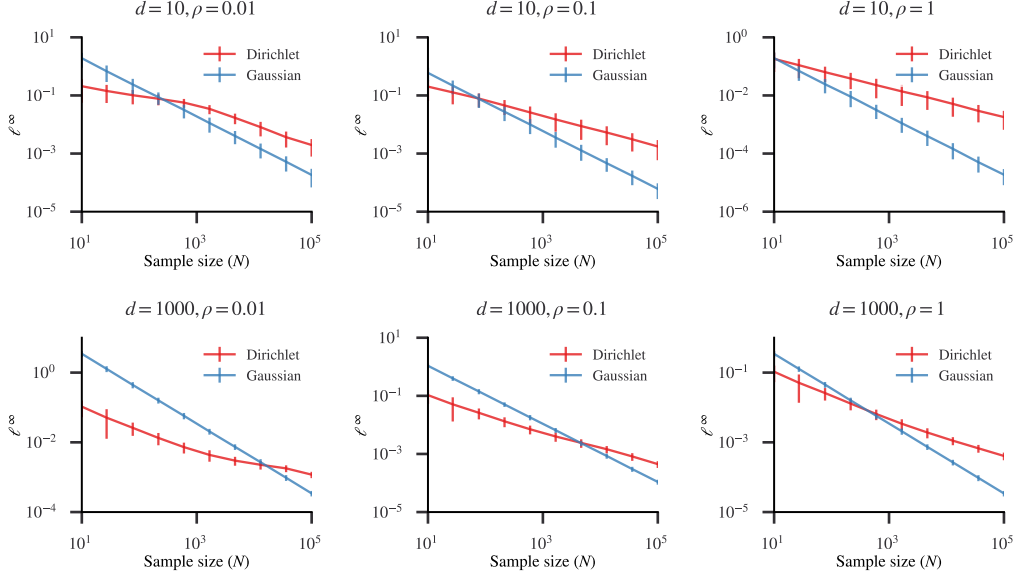


Figure 2: The ℓ^∞ -accuracy, as a function of N , of Dirichlet posterior sampling ($\gamma = 1$) and Gaussian mechanisms for private normalized histograms ($\Delta_2^2 = 2$ and $\Delta_\infty = 1$). For each N , d and ρ , we generated the inputs $\mathbf{x}_1, \dots, \mathbf{x}_{200}$, where $\mathbf{x}_k \sim \text{Multinomial}(\mathbf{q}_k)$ and $\mathbf{q}_k \sim \text{Dirichlet}(5, \dots, 5)$.

Hence, the sample size of $N = \Omega\left(\sqrt{\log(2d/\beta)\Delta_2^2/\rho\epsilon^2}\right)$ guarantees the desired accuracy. Comparing this to (14), if we assume $\epsilon < 1$, the AM-GM inequality tells us that

$$\frac{\log(2d/\beta)}{\epsilon^2} + \frac{d\Delta_2^2}{\rho\epsilon} > \frac{\log(2d/\beta)}{\epsilon^2} + \frac{\Delta_2^2}{\rho} \geq 2\sqrt{\frac{\log(2d/\beta)\Delta_2^2}{\rho\epsilon^2}}. \quad (17)$$

The inequality (17) implies that the Gaussian mechanism requires less sample than the Dirichlet mechanism in order to guarantee the same level of accuracy. The Gaussian mechanism is also better in the low-privacy regime as the ρ in (15) satisfies $\sqrt{\rho} < \rho$ and $\Delta_2 \approx \Delta_2^2$, leading to the same inequality (17). Nonetheless, the decay in (16) is linear in d , while that in (13) has $\alpha_0 = d\alpha$ in the denominators. This observation suggests that, when \mathbf{x} is a sparse histogram i.e. when $N \leq d$, the ℓ^∞ -accuracy of the Dirichlet mechanism is smaller than that of the Gaussian mechanism. This conclusion is supported by our simulation in Figure 2. We see that the ℓ^∞ -accuracy of the Dirichlet mechanism is smaller than that of the Gaussian mechanism for small N when $d = 1000$. The code for all experiments in this study can be found in the supplemental material.

Potential negative societal impacts

It is important to note that, when ρ becomes unacceptably large (e.g., $\rho = 10^4$), the sampling is far away from being private. Thus any organization that deploys the posterior sampling on sensitive data must not vacuously refer to this study and claim that its algorithm is private. It is the organization's responsibility to fully publish the prior parameters, and educate its users/customers on differential privacy and how the privacy guarantees are calculated.

It is desirable that differentially private algorithms are accurate for the task at hand, especially when the data is used for important decision-making. Thus, one needs to make sure that there is enough sample to achieve the desired level of accuracy. For a large differentially private system, privacy budgets need to be allocated to the parts that require accurate outputs.

Lastly, one must be careful with the choice of prior parameters; if a uniform prior is used, smaller groups will suffer a relatively larger statistical bias. As a result, private statistics of small populations (such as ethnic or racial minorities) will be relatively less accurate. One way to get around this issue is to (privately) impose larger prior parameters on larger populations.

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405 Checklist

- 406 1. For all authors...
- 407 (a) Do the main claims made in the abstract and introduction accurately reflect the paper’s
- 408 contributions and scope? [Yes] We gave a simple guaranteed upper bound of tCDP (2)
- 409 for the Dirichlet posterior sampling and illustrated how it can be used to derive accuracy
- 410 guarantees in Section 4.
- 411 (b) Did you describe the limitations of your work? [Yes] We discussed a limitation of
- 412 the guaranteed upper bound of tCDP in the paragraph following Theorem 1. We also
- 413 described a situation under which the Gaussian mechanism is preferable to the Dirichlet
- 414 posterior sampling at the end of Section 4.2.

- 415 (c) Did you discuss any potential negative societal impacts of your work? [Yes] See the
 416 section on potential negative societal impacts at the end of the paper.
- 417 (d) Have you read the ethics review guidelines and ensured that your paper conforms to
 418 them? [Yes]
- 419 2. If you are including theoretical results...
- 420 (a) Did you state the full set of assumptions of all theoretical results? [Yes]
 421 (b) Did you include complete proofs of all theoretical results? [No] The proofs of all
 422 theorems are given in the main paper, except that of Theorem 3 which is given in
 423 Appendix B.
- 424 3. If you ran experiments...
- 425 (a) Did you include the code, data, and instructions needed to reproduce the main experi-
 426 mental results (either in the supplemental material or as a URL)? [Yes] The code and
 427 the instructions for our simulations are included in the supplemental material.
- 428 (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they
 429 were chosen)? [Yes] We specified the details of our simulations in the figures' captions.
- 430 (c) Did you report error bars (e.g., with respect to the random seed after running experi-
 431 ments multiple times)? [Yes] We reported the error bars in Figure 2
- 432 (d) Did you include the total amount of compute and the type of resources used (e.g.,
 433 type of GPUs, internal cluster, or cloud provider)? [N/A] Our experiments are not
 434 computationally intensive.
- 435 4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
- 436 (a) If your work uses existing assets, did you cite the creators? [N/A]
 437 (b) Did you mention the license of the assets? [N/A]
 438 (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
 439
- 440 (d) Did you discuss whether and how consent was obtained from people whose data you're
 441 using/curating? [N/A]
 442 (e) Did you discuss whether the data you are using/curating contains personally identifiable
 443 information or offensive content? [N/A]
- 444 5. If you used crowdsourcing or conducted research with human subjects...
- 445 (a) Did you include the full text of instructions given to participants and screenshots, if
 446 applicable? [N/A]
 447 (b) Did you describe any potential participant risks, with links to Institutional Review
 448 Board (IRB) approvals, if applicable? [N/A]
 449 (c) Did you include the estimated hourly wage paid to participants and the total amount
 450 spent on participant compensation? [N/A]

451 A Proof of Theorem 1

452 *proof.* Consider any $\lambda \in \left(1, \frac{\gamma}{\Delta_\infty} + 1\right)$. Let $\mathbf{u} := \mathbf{x} + \boldsymbol{\alpha}$ and $\mathbf{u}' := \mathbf{x}' + \boldsymbol{\alpha}'$. Let $P(\mathbf{y})$ be the
 453 density of Dirichlet(\mathbf{u}) and $P'(\mathbf{y})$ be the density of Dirichlet(\mathbf{u}'). To compute the Rényi divergence
 454 between $P(\mathbf{y})$ and $P'(\mathbf{y})$, we start with:

$$\begin{aligned} \mathbb{E}_{\mathbf{y} \sim P(\mathbf{y})} \left[\frac{P(\mathbf{y})^{\lambda-1}}{P'(\mathbf{y})^{\lambda-1}} \right] &= \frac{B(\mathbf{u}')^{\lambda-1}}{B(\mathbf{u})^{\lambda-1}} \mathbb{E}_{\mathbf{y} \sim P(\mathbf{y})} \left[\mathbf{y}^{(\lambda-1)(\mathbf{u}-\mathbf{u}')} \right] \\ &= \frac{B(\mathbf{u}')^{\lambda-1}}{B(\mathbf{u})^{\lambda-1}} \cdot \frac{B(\mathbf{u} + (\lambda-1)(\mathbf{u}-\mathbf{u}'))}{B(\mathbf{u})}. \end{aligned} \quad (18)$$

455 We write the beta functions in terms of gamma functions using (1): The ratio can be expressed in
 456 terms of gamma functions:

$$\frac{B(\mathbf{u}')}{B(\mathbf{u})} = \frac{\prod_i \Gamma(u'_i) / \Gamma(\sum_i u'_i)}{\prod_i \Gamma(u_i) / \Gamma(\sum_i u_i)} = \frac{\Gamma(u_0)}{\Gamma(u'_0)} \prod_i \frac{\Gamma(u'_i)}{\Gamma(u_i)},$$

457 where $u_0 := \sum_i u_i$ and $u'_0 := \sum_i u'_i$. Similarly,

$$\frac{B(\mathbf{u} + (\lambda - 1)(\mathbf{u} - \mathbf{u}'))}{B(\mathbf{u})} = \frac{\Gamma(\sum_i u_i)}{\Gamma(\sum_i u_i + (\lambda - 1) \sum_i (u_i - u'_i))} \prod_i \frac{\Gamma(u_i + (\lambda - 1)(u_i - u'_i))}{\Gamma(u_i)}.$$

458 Taking the logarithm on both side of (18), we need to find an upper bound of:

$$\log \mathbb{E}_{\mathbf{y} \sim P(\mathbf{y})} \left[\frac{P(\mathbf{y})^{\lambda-1}}{P'(\mathbf{y})^{\lambda-1}} \right] = \sum_i (G(u_i, u'_i) + H(u_i, u'_i)) - G(u_0, u'_0) - H(u_0, u'_0), \quad (19)$$

459 where

$$\begin{aligned} G(u_i, u'_i) &:= (\lambda - 1)(\log \Gamma(u'_i) - \log \Gamma(u_i)) \\ H(u_i, u'_i) &:= \log \Gamma(u_i + (\lambda - 1)(u_i - u'_i)) - \log \Gamma(u_i), \end{aligned}$$

460 and similarly for $G(u_0, u'_0)$ and $H(u_0, u'_0)$. Using the second-order Taylor expansion, there exist
 461 there exist ξ between $u_i + (\lambda - 1)(u_i - u'_i)$ and u_i , and ξ' between u_i and u'_i such that

$$\begin{aligned} G(u_i, u'_i) &= -(\lambda - 1)(x_i - x'_i)\psi(u_i) + \frac{1}{2}(\lambda - 1)(x_i - x'_i)^2\psi'(\xi') \\ H(u_i, u'_i) &= (\lambda - 1)(x_i - x'_i)\psi(u_i) + \frac{1}{2}(\lambda - 1)^2(x_i - x'_i)^2\psi'(\xi), \end{aligned}$$

462 We will try to find an upper bound of both $\psi'(\xi)$ and $\psi'(\xi')$. Note that ψ' is increasing. If $x_i > x'_i$,
 463 then $u'_i < u_i < u_i + (\lambda - 1)(u_i - u'_i)$. Thus both ξ and ξ' are bounded below by $u'_i \geq \alpha_m$. On the
 464 other hand, if $x_i \leq x'_i$, then $u_i + (\lambda - 1)(u_i - u'_i) \leq u_i \leq u'_i$. In this case, ξ and ξ' are bounded
 465 below by:

$$\begin{aligned} u_i + (\lambda - 1)(u_i - u'_i) &= x_i + \alpha_i + (\lambda - 1)(x'_i - x_i) \\ &\geq \alpha_m - (\lambda - 1)\Delta_\infty \\ &> \alpha_m - \gamma, \end{aligned}$$

466 where the last inequality follows from the condition $\lambda < \frac{\gamma}{\Delta_\infty} + 1$. Therefore, $\psi'(\xi)$ and $\psi'(\xi')$ are
 467 both bounded above by $\psi'(\alpha_m - \gamma)$. Consequently,

$$\begin{aligned} G(u_i, u'_i) + H(u_i, u'_i) &\leq \frac{1}{2}((\lambda - 1) + (\lambda - 1)^2)(x_i - x'_i)^2\psi'(\alpha_m - \gamma) \\ &= \frac{1}{2}\lambda(\lambda - 1)(x_i - x'_i)^2\psi'(\alpha_m - \gamma). \end{aligned}$$

468 The same argument can be used to show that, there exist ξ_0 and ξ'_0 such that:

$$G(u_0, u'_0) + H(u_0, u'_0) = \frac{1}{2}(\lambda - 1)(u_0 - u'_0)^2\psi'(\xi'_0) + \frac{1}{2}(\lambda - 1)^2(u_0 - u'_0)^2\psi'(\xi_0) > 0.$$

469 Therefore, continuing from (19),

$$\begin{aligned} D_\lambda(P(\mathbf{y}) \| P'(\mathbf{y})) &= \frac{1}{\lambda - 1} \left(\sum_i (G(u_i, u'_i) + H(u_i, u'_i)) - G(u_0, u'_0) - H(u_0, u'_0) \right) \\ &< \frac{1}{\lambda - 1} \sum_i (G(u_i, u'_i) + H(u_i, u'_i)) \\ &\leq \frac{1}{2}\lambda \sum_i (x_i - x'_i)^2\psi'(\alpha_m - \gamma) \\ &\leq \frac{1}{2}\lambda \Delta_2^2 \psi'(\alpha_m - \gamma). \end{aligned}$$

470 In conclusion, the Dirichlet posterior sampling is $\left(\frac{1}{2}\lambda \Delta_2^2 \psi'(\alpha_m - \gamma), \frac{\gamma}{\Delta_\infty} + 1 \right)$ -tCDP. \square

471 B Proof of Theorem 3

472 *Proof.* For notational convenience, define $\alpha_0 := \sum_i \alpha_i$, $\alpha'_0 := \sum_i \alpha'_i$ and $X_0 = \sum_i X_i$. Moreover,
 473 we define $\delta_i = \alpha'_i - \alpha$ and $\delta_0 = \sum_i \delta_i$. The KL-divergence can be explicitly written as follows
 474 [Pen01]:

$$\begin{aligned} D_{\text{KL}}(P_{\mathbf{X}} \| Q_{\mathbf{X}}) &= \log \Gamma(X_0 + \alpha_0) - \sum_i \log \Gamma(X_i + \alpha_i) \\ &\quad - \log \Gamma(X_0 + \alpha'_0) + \sum_i \log \Gamma(X_i + \alpha'_i) \\ &\quad - \sum_i (\alpha'_i - \alpha_i) (\psi(X_i + \alpha_i) - \psi(X_0 + \alpha_0)). \end{aligned} \quad (20)$$

475 Using the second-order Taylor approximation, there exists $C \in [X_0 + \alpha_0, X_0 + \alpha'_0]$ such that

$$\begin{aligned} &\log \Gamma(X_0 + \alpha_0) - \log \Gamma(X_0 + \alpha'_0) + (\alpha'_0 - \alpha_0) \psi(X_0 + \alpha_0) \\ &= -\frac{1}{2} (\alpha'_0 - \alpha_0)^2 \psi'(C) \leq 0, \end{aligned}$$

476 since ψ' is positive. Hence, $D_{\text{KL}}(P_{\mathbf{X}} \| Q_{\mathbf{X}})$ is bounded by the rest of the terms in (20). In other
 477 words,

$$D_{\text{KL}}(P_{\mathbf{X}} \| Q_{\mathbf{X}}) \leq \sum_i (\log \Gamma(X_i + \alpha'_i) - \log \Gamma(X_i + \alpha_i) - \delta_i \psi(X_i + \alpha_i)).$$

478 We apply the Taylor approximation again and use $\psi'(X) \leq \frac{1}{X} + \frac{1}{X^2}$. With $X_i + \alpha_i \geq 1$, we have

$$\begin{aligned} &\log \Gamma(X_i + \alpha'_i) - \log \Gamma(X_i + \alpha_i) - \delta_i \psi(X_i + \alpha_i) \\ &\leq \frac{1}{2} \delta_i^2 \psi'(X_i + \alpha_i) \\ &\leq \frac{\delta_i^2}{2(X_i + \alpha_i)} + \frac{\delta_i^2}{2(X_i + \alpha_i)^2} \\ &\leq \frac{\delta_i^2}{X_i + \alpha_i}, \end{aligned}$$

479 from which we take the sum in i to obtain the upper bound of the KL-divergence. Since $\alpha_i \geq 1$ for
 480 all i , we have

$$\mathbb{E}[D_{\text{KL}}(P_{\mathbf{X}} \| Q_{\mathbf{X}})] \leq \sum_i \delta_i^2 \cdot \mathbb{E}\left[\frac{1}{X_i + \alpha_i}\right] \leq \sum_i \delta_i^2 \cdot \mathbb{E}\left[\frac{1}{X_i + 1}\right].$$

481 Here, X_i is distributed as $\text{Binomial}(N, p_i)$. The desired estimate follows from the following inequal-
 482 ity regarding the reciprocal of a binomial variable from [CS72]:

$$\mathbb{E}\left[\frac{1}{X_i + 1}\right] = \frac{1 - (1 - p_i)^{N+1}}{(N+1)p_i} < \frac{1}{(N+1)p_i}.$$

483

□

484 C Additional experiment on Multinomial-Dirichlet sampling

485 We consider the task of privatizing the count data \mathbf{x} , with $\Delta_2^2 = 2$ and $\Delta_\infty = 1$. The base
 486 density is $P_{\mathbf{x}} \sim \text{Dirichlet}(\mathbf{x} + (1, \dots, 1))$. For each $\rho \in \{0.01, 0.1, 1\}$, we choose α that yields
 487 privacy guarantee ρ for the Dirichlet posterior sampling at $\gamma = 2$. We also consider the Gaussian
 488 mechanism: let $M_\sigma(\mathbf{x}) = \max\{\mathbf{x} + \mathbf{Z}, 0\}$ where $\mathbf{Z} \sim N(0, \sigma^2 I_d)$. Here, we choose $\sigma^2 = \Delta_2^2/2\rho =$
 489 $1/\rho$ so that M is ρ -zCDP. By the post-processing property of zCDP mechanisms, sampling from
 490 $\text{Dirichlet}(M_\sigma(\mathbf{x}) + (1, \dots, 1))$ is also ρ -zCDP.

491 For each $\eta \in \{0.1, 1, 10\}$, we generated $\mathbf{x}_1, \dots, \mathbf{x}_{200}$, where $\mathbf{x}_k \sim \text{Multinomial}(\mathbf{q}_k)$ and
 492 $\mathbf{q}_k \sim \text{Dirichlet}(\eta, \dots, \eta)$. Note that \mathbf{x}_k 's are more evenly distributed for a larger value of η

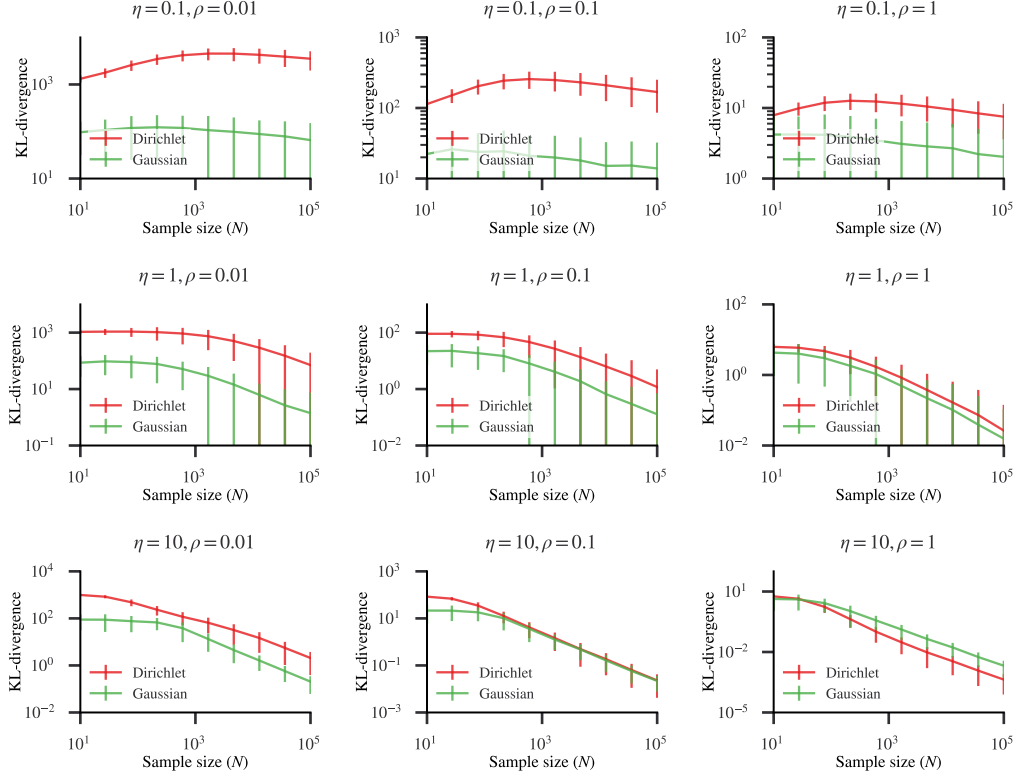


Figure 3: The KL-divergence between the base Multinomial-Dirichlet distribution and the private distributions with privacy guarantee $\rho \in \{0.01, 0.1, 1\}$, as a function of sample size (N). Data generated with a smaller η is more concentrated, while that with larger η is more evenly distributed.

and more concentrated for a smaller value of η . Let $P_{\mathbf{x}_k}^\alpha \sim \text{Dirichlet}(\mathbf{x}_k + (\alpha, \dots, \alpha))$ and $P_{\mathbf{x}_k}^\sigma \sim \text{Dirichlet}(M_\sigma(\mathbf{x}_k) + (1, \dots, 1))$. Figure 3 shows the averages and two standard deviations of $D_{\text{KL}}(P_{\mathbf{x}_k} \| P_{\mathbf{x}_k}^\alpha)$ and $D_{\text{KL}}(P_{\mathbf{x}_k} \| P_{\mathbf{x}_k}^\sigma)$. We observe that the average KL-divergence decreases much faster for higher values of ρ and η , which agrees with our analysis in Section 4.1. In addition, the plots show that the Dirichlet mechanism is almost always inferior to the Gaussian mechanism, where they become close when \mathbf{x}_k 's are more evenly distributed. Note that the zCDP guarantees of the Gaussian mechanism might be better than ρ due to the addition of priors; This aspect requires further investigation.

D Scaling between r and α_m

For $\alpha_m - \gamma \geq 1$, we have $\psi'(\alpha_m - \gamma) \approx 1/(\alpha_m - \gamma)$, and so α_m scales as r^2 . This creates a tradeoff situation between the bias and the variance of the Dirichlet sampling: for example, suppose that we start with $x_i = 10$, $r = 1$ and $\alpha_i = \alpha_m = 1$ for all i . Then $\alpha_i = \frac{1}{10}rx_i$. If we increase r to 4, then α_i has to be 16 to keep ρ roughly the same. Even though the variance of the sample $\mathbf{Y} \sim \text{Dirichlet}(r\mathbf{x} + \alpha)$ is reduced by a factor of $1/4$, the bias has increased as the prior is now $\alpha_i = \frac{16}{40}rx_i$. Therefore, one must choose r that gives the right balance between the bias and the variance.