A Multiple min s-t cuts

Let G and G' be two neighboring graphs, and let $\widetilde{G}$ and $\widetilde{G}'$, respectively, be their modified versions constructed by Algorithm 1. Algorithm 1 outputs a min s-t cut in G. However, what happens if there are multiple min s-t cuts in $\widetilde{G}$ and the algorithm invoked on Line 4 breaks ties in a way that depends on whether a specific edge e appears in G or not? If it happens that e is the edge difference between G and G', then such a tie-breaking rule might reveal additional information about G and G'. We now outline how this can be bypassed.

Observe that if the random variables $X_{s,u}$ and $X_{t,u}$ were sampled by using infinite bit precision, then with probability 1 no two cuts would have the same value. So, consider a more realistic situation where edge-weights are represented by $O(\log n)$ bits, and assume that the least significant bit corresponds to the value 2^{-t}, for an integer $t \geq 0$. We show how to modify edge-weights by additional $O(\log n)$ bits that have extremely small values but will help obtain a unique min s-t cut. Our modification consists of two steps.

First step. All the bits corresponding to values from 2^{-t-1} to $2^{-t-2\log n}$ remain 0, while those corresponding to larger values remain unchanged. This is done so that even summing across all – but at most $\binom{n}{2}$ – edges it holds that no matter what the bits corresponding to values $2^{-t-2\log n-1}$ and less are, their total value is less than 2^{-t}. Hence, if the weight of cut C_1 is smaller than the weight of cut C_2 before the modifications we undertake, then C_1 has a smaller weight than C_2 after the modifications as well.

Second step. We first recall the celebrated Isolation lemma.

Lemma A.1 (Isolation lemma, [25]). *Let T and N be positive integers, and let $\mathcal{F}$ be an arbitrary nonempty family of subsets of the universe $\{1, \ldots, T\}$. Suppose each element $x \in \{1, \ldots, T\}$ in the universe receives an integer weight $g(x)$, each of which is chosen independently and uniformly at random from $\{1, \ldots, N\}$. The weight of a set $S \in \mathcal{F}$ is defined as $g(S) = \sum_{x \in S} g(x)$.*

Then, with probability at least $1 - T/N$ there is a unique set in $\mathcal{F}$ that has the minimum weight among all sets of $\mathcal{F}$.

We now apply Lemma A.1 to conclude our modification of the edge weights in $\widetilde{G}$. We let the universe $\{1, \ldots, T\}$ from that lemma be the following $2(n-2)$ elements $\mathcal{U} = \{(s, v) \mid v \in V(G) \setminus \{s, t\}\} \cup \{(t, v) \mid v \in V(G) \setminus \{s, t\}\}$. Then, we let $\mathcal{F}$ represent all min s-t cuts in $\widetilde{G}$, i.e., $S \subseteq \mathcal{U}$ belongs to $\mathcal{F}$ iff there is a min s-t cut C in $\widetilde{G}$ such that for each $(a, b) \in S$ the cut C contains $X_{a,b}$. So, by letting $N = 2n^2$, we derive that with probability $1 - 1/n$ it holds that no two cuts represented by $\mathcal{F}$ have the same minimum value **with respect to** g defined in Lemma A.1. To implement g in our modification of weights, we modify the bits of each $X_{s,v}$ and $X_{t,v}$ corresponding to values from $2^{-t-2\log n-1}$ to $2^{-t-2\log n-\log N}$ to be an integer between 1 and N chosen uniformly at random.

Only after these modifications, we invoke Line 4 of Algorithm 1. Note that the family of cuts $\mathcal{F}$ is defined only for the sake of analysis. It is not needed to know it algorithmically.

B Lower Bound for min s-t cut error

In this section, we prove our lower bound. Our high-level idea is similar to that of [8] for proving a lower bound for private algorithms for correlation clustering.

Theorem 1.2. *Any (ϵ, δ)-differential private algorithm for min s-t cut on n-node graphs requires expected additive error of at least $n/20$ for any $\epsilon \leq 1$ and $\delta \leq 0.1$.*

Proof. For the sake of contradiction, let $\mathcal{A}$ be a (ϵ, δ)-differential private algorithm for min s-t cut that on any input n-node graph outputs an s-t cut that has expected additive error of less than $n/20$. We construct a set of 2^n graphs S and show that $\mathcal{A}$ cannot have low expected cost on all of the graphs on this set while preserving privacy.

The node set of all the graphs in S are the same and consist of $V = \{s, t, v_1, \ldots, v_n\}$ where s and t are the terminals of the graph and $n > 30$. For any $\tau \in \{0, 1\}^n$, let G_τ be the graph on node set V

with the following edges: For any $1 \le i \le n$, if $\tau_i = 1$, then there is an edge between s and v_i. If $\tau = 0$, then there is an edge between t and v_i. Note that v_i is attached to exactly one of the terminals s and t. Moreover, the min s-t cut of each graph G_τ is zero.

Algorithm $\mathcal{A}$ determines for each i if v_i is on the s-side of the output cut or the t-side. The contribution of each node v_i to the total error is the number of edges attached to v_i that are in the cut. We denote this random variable in graph G_τ by $e_\tau(v_i)$. Since there are no edges between any two non-terminal nodes in any of the graphs G_τ, the total error of the output is the sum of these individual errors, i.e., $\sum_{i=0}^n e_\tau(v_i)$. Let $\bar{e}_\tau(v_i)$ be the expected value of $e_\tau(v_i)$ over the outputs of $\mathcal{A}$ given G_τ.

Let $p_\tau^{(i)}$ be the marginal probability that v_i is on the s-side of the output s-t cut in G_τ. If $\tau_i = 0$, then v_i is connected to t and so $\bar{e}_\tau(v_i) = p_\tau^{(i)}$. If $\tau_i = 1$, then v_i is connected to s and so $\bar{e}_\tau(v_i) = 1 - p_\tau^{(i)}$. By the assumption that $\mathcal{A}$ has a low expected error on every input, we have that for any $\tau \in \{0,1\}^n$,

$$(n+2)/20 > \sum_{i,\tau_i=0} p_\tau^{(i)} + \sum_{i,\tau_i=1} (1 - p_\tau^{(i)}) \tag{3}$$

Let S_i be the set of $\tau \in \{0,1\}^n$ such that $\tau_i = 0$, and $\bar{S}_i$ be the complement of S_i, so that $\tau \in \bar{S}_i$ if $\tau_i = 1$. Note that $|S_i| = |\bar{S}_i| = 2^{n-1}$. Fix some i, and for any $\tau \in \{0,1\}^n$, let τ' be the same as τ except for the i-th entry being different, i.e., for all $j \neq i$, $\tau_j = \tau'_j$, and $\tau_i \neq \tau'_i$. Since G_τ and $G_{\tau'}$ only differ in two edges, from $\mathcal{A}$ being (ϵ, δ)-differentially private for any j we have $p_\tau^{(j)} \le e^{2\epsilon} \cdot p_{\tau'}^{(j)} + \delta$. So for any i, j we have

$$\sum_{\tau \in S_i} p_\tau^{(j)} \le \sum_{\tau \in \bar{S}_i} (e^{2\epsilon} p_\tau^{(j)} + \delta) \tag{4}$$

From Eq. (3) we have

$$\begin{aligned}
2^n \cdot 0.05(n+2) &> \sum_{\tau \in \{0,1\}^n} \sum_{i:\tau_i=1} (1 - p_\tau^{(i)}) \\
&= \sum_{i=1}^n \sum_{\tau \in S_i} (1 - p_\tau^{(i)}) \\
&\ge \sum_{i=1}^n \sum_{\tau \in \bar{S}_i} (1 - [e^{2\epsilon} p_\tau^{(i)} + \delta]) \\
&= n 2^{n-1}(1-\delta) - e^{2\epsilon} \sum_{i=1}^n \sum_{\tau \in \bar{S}_i} p_\tau^{(i)}
\end{aligned}$$

Where the last inequality comes from Eq. (4). Using Eq. (3) again, we have that $\sum_{i=1}^n \sum_{\tau \in \bar{S}_i} p_\tau^{(i)} < 2^n \cdot 0.05(n+2)$, so we have that $2^n \cdot 0.05(n+2) > n2^{n-1}(1-\delta) - e^{2\epsilon}(2^n \cdot 0.05(n+2))$. Dividing by 2^n we have

$$0.05(n+2)(1+e^{2\epsilon}) > n(1-\delta)/2.$$

Now since $\epsilon \le 1$, $\delta \le 0.1$, and $e^2 < 7.4$ we get that

$$0.05 \cdot 8.4(n+2) > 0.45n$$

Hence we have $n < 28$ which is a contradiction to $n > 30$. □

C Omitted Proofs

C.1 Proof of Claim 1

By definition, we have

$$\frac{f_{\text{Lap}}(t+\tau)}{f_{\text{Lap}}(t)} = \frac{\frac{\epsilon}{2}\exp(-\epsilon|t+\tau|)}{\frac{\epsilon}{2}\exp(-\epsilon|t|)} = \exp(-\epsilon|t+\tau| + \epsilon|t|) \le \exp(\tau\epsilon). \tag{5}$$

Also by definition, it holds $F_{\text{Lap}}(t+\tau) = \int_{-\infty}^{t+\tau} f_{\text{Lap}}(x)dx$. Using Eq. (5) we derive

$$F_{\text{Lap}}(t+\tau) \le \exp(\tau\epsilon) \int_{-\infty}^{t+\tau} f_{\text{Lap}}(x-\tau)dx = \exp(\tau\epsilon) \int_{-\infty}^{t} f_{\text{Lap}}(x)dx = \exp(\tau\epsilon) F_{\text{Lap}}(t).$$

C.2 Proof of Lemma 3.1

To prove the lower-bound, we observe that if $x < \alpha$ and $y < \beta$, then $x < \alpha + \tau$ and $y < \beta + \tau$ as well. Hence, it trivially holds $P(\alpha+\tau, \beta+\tau, \gamma) \geq P(\alpha, \beta, \gamma)$ and hence

$$\frac{P(\alpha+\tau, \beta+\tau, \gamma)}{P(\alpha, \beta, \gamma)} \geq 1.$$

We now analyze the upper-bound. For the sake of brevity, in the rest of this proof, we use F to denote F_{Lap} and f to denote f_{Lap}. We consider three cases depending on parameters α, β, γ.

Case $\gamma \geq \alpha + \beta + 2\tau$. In this case we have $\Pr[x+y < \gamma | x < \alpha, y < \beta] = 1 = \Pr[x+y < \gamma | x < \alpha+\tau, y < \beta+\tau]$. So we have that

$$\begin{aligned} P(\alpha, \beta, \gamma) &= \Pr[x+y<\gamma | x<\alpha, y<\beta] \cdot \Pr[x<\alpha, y<\beta] \\ &= \Pr[x<\alpha, y<\beta] \\ &= F(\alpha)F(\beta) \end{aligned}$$

Similarly, $P(\alpha+\tau, \beta+\tau, \gamma) = F(\alpha+\tau)F(\beta+\tau)$. Now using Claim 1, we obtain that $\frac{P(\alpha+\tau,\beta+\tau,\gamma)}{P(\alpha,\beta,\gamma)} \leq e^{2\tau\epsilon}$.

Case $\gamma < \alpha + \beta$. We write $P(\alpha, \beta, \gamma)$ as follows.

$$\begin{aligned} P(\alpha, \beta, \gamma) &= \int_{-\infty}^{\beta} \int_{-\infty}^{\min(\alpha, \gamma-y)} f_x(x|y) dx f_y(y) dy \\ &= \int_{-\infty}^{\beta} F(\min(\alpha, \gamma-y)) f(y) dy \\ &= F(\alpha) \int_{-\infty}^{\gamma-\alpha} f(y) dy + \int_{\gamma-\alpha}^{\beta} F(\gamma-y) f(y) dy \\ &= F(\alpha)F(\gamma-\alpha) + \int_{\gamma-\alpha}^{\beta} F(\gamma-y) f(y) dy \end{aligned} \tag{6}$$

Similar to Eq. (6) we have

$$P(\alpha+\tau, \beta+\tau, \gamma) = F(\alpha+\tau)F(\gamma-\alpha-\tau) + \int_{\gamma-\alpha-\tau}^{\beta+\tau} F(\gamma-y) f(y) dy \tag{7}$$

Now we rewrite Eq. (6) as follows to obtain a lower bound on $P(\alpha, \beta, \gamma)$.

$$\begin{aligned} P(\alpha, \beta, \gamma) &= F(\alpha)F(\gamma-\alpha-2\tau) + \int_{\gamma-\alpha-2\tau}^{\gamma-\alpha} F(\alpha) f(y) dy + \int_{\gamma-\alpha}^{\beta} F(\gamma-y) f(y) dy \\ &\geq F(\alpha)F(\gamma-\alpha-2\tau) + e^{-2\tau\epsilon} \int_{\gamma-\alpha-2\tau}^{\beta} F(\gamma-y) f(y) dy \end{aligned} \tag{8}$$

In obtaining the inequality, we used the fact that if $y \in [\gamma-\alpha-2\tau, \gamma-\alpha]$ then $0 \leq (\gamma-y)-\alpha \leq 2\tau$ and so by Claim 1 we have $F(\alpha) \geq e^{-2\tau\epsilon} F(\gamma-y)$.

Now we compare the two terms of Eq. (8) with Eq. (7). By Claim 1 we have that $F(\alpha)F(\gamma-\alpha-2\tau) \geq e^{-2\tau\epsilon} F(\alpha+\tau)F(\gamma-\alpha-\tau)$ and $\int_{\gamma-\alpha-2\tau}^{\beta} F(\gamma-y)f(y)dy \geq e^{-\tau\epsilon} \int_{\gamma-\alpha-\tau}^{\beta+\tau} F(\gamma-y)f(y)dy$. So we have $P(\alpha, \beta, \gamma) \geq e^{-3\tau\epsilon} P(\alpha+\tau, \beta+\tau, \gamma)$.

Case $\alpha + \beta \leq \gamma < \alpha + \beta + 2\tau$. Then

$$
\begin{aligned}
P(\alpha, \beta, \gamma) &= \int_{-\infty}^{\beta} \int_{-\infty}^{\min(\alpha, \gamma - y)} f_x(x|y) dx f_y(y) dy \\
&= \int_{-\infty}^{\beta} F(\min(\alpha, \gamma - y)) f(y) dy \\
&= F(\alpha) F(\beta) \qquad (9) \\
&\geq e^{-2\tau\epsilon} F(\alpha + \tau) F(\beta + \tau) \qquad (10) \\
&= e^{-2\tau\epsilon} (F(\alpha + \tau) F(\beta - \tau) + F(\alpha + \tau)(F(\beta + \tau) - F(\beta - \tau)) \\
&\geq e^{-4\tau\epsilon} (F(\alpha + \tau) F(\beta + \tau) + F(\alpha + \tau)(F(\beta + \tau) - F(\beta - \tau)). \qquad (11)
\end{aligned}
$$

Note that Eq. (9) is obtained since for any $y \leq \beta$ we have $\alpha \leq \gamma - y$. Eq. (10) and Eq. (11) are both obtained using Claim 1. One can easily verify that Eq. (7) for $P(\alpha + 1, \beta + 1, \gamma)$ holds in this case as well. Using the fact that $\gamma - \alpha - \tau \leq \beta + \gamma$ and F being a non-decreasing function, we further lower-bound Eq. (7) as

$$
\begin{aligned}
P(\alpha + \tau, \beta + \tau, \gamma) \leq & F(\alpha + \tau) F(\beta + \tau) + \int_{\gamma - \alpha - \tau}^{\beta + \tau} F(\gamma - y) f(y) dy \\
\leq & F(\alpha + \tau) F(\beta + \tau) + F(\alpha + \tau) \int_{\gamma - \alpha - \tau}^{\beta + \tau} f(y) dy \\
= & F(\alpha + \tau) F(\beta + \tau) + F(\alpha + \tau)(F(\beta + \tau) - F(\gamma - \alpha - \tau)) \\
\leq & F(\alpha + \tau) F(\beta + \tau) + F(\alpha + \tau)(F(\beta + \tau) - F(\beta - \tau)). \qquad (12)
\end{aligned}
$$

Eqs. (11) and (12) conclude the analysis of this case as well.

C.3 Proof of Theorem 4.1

We use induction on k to prove the theorem. Suppose that Algorithm 2 outputs C_{ALG}. We show that the C_{ALG} is a multiway k-cut and that the value of C_{ALG} is at most $w(E(\overline{V})) + \sum_{i=1}^{k} \delta(V_i) + 2 \log(k) e(n)$. We will first perform the analysis of approximation assuming that $\mathcal{A}$ provides the stated approximation deterministically, and at the end of this proof, we will take into account that the approximation guarantee holds with probability $1 - \alpha$.

Base case: $k = 1$. If $k = 1$, then $C_{ALG} = \emptyset$, and so it is a multiway 1-cut and $w(C_{ALG}) = 0 \leq \delta(V_1) + w(E(\overline{V}))$.

Inductive step: $k \geq 2$. So suppose that $k \geq 2$. Hence, $k' \geq 1$ and $k - k' \geq 1$, where k' is defined on Line 4. Let (A, B) be the s-t cut obtained in Line 6, where $\widetilde{G}_1$ is the graph induced on A and $\widetilde{G}_2$ is the graph induced on B. Since the only terminals in $\widetilde{G}_1$ are $s_1, \ldots, s_{k'}$, we have that $V_1 \cap A, \ldots, V_{k'} \cap A$ is a partial multiway k'-cut on $\widetilde{G}_1$. By the induction hypothesis, the cost of the multiway cut that Algorithm 2 finds on $\widetilde{G}_1$ is at most $w(E(\overline{V} \cap A)) + \sum_{i=1}^{k'} \delta_{\widetilde{G}_1}(V_i \cap A) + 2 \log(k') e(|A|)$. Similarly, by considering the partial multiway $(k - k')$ cut $V_{k'+1} \cap B, \ldots, V_k \cap B$ on $\widetilde{G}_2$, the cost of the multiway cut that Algorithm 2 finds on $\widetilde{G}_2$ is at most $w(E(\overline{V} \cap B)) + \sum_{i=k'+1}^{k} \delta_{\widetilde{G}_2}(V_i \cap B) + 2 \log(k - k') e(|B|)$. So the total cost $w(C_{ALG})$ of the multiway cut that Algorithm 2 outputs is at most

$$
\begin{aligned}
w(C_{ALG}) \leq & \; w(E(\overline{V} \cap A)) + w(E(\overline{V} \cap B)) \qquad (13) \\
& + \sum_{i=1}^{k'} \delta_{\widetilde{G}_1}(V_i \cap A) + \sum_{i=k'+1}^{k} \delta_{\widetilde{G}_2}(V_i \cap B) \\
& + w(E(A, B)) \\
& + 2 \log(k') e(|A|) + 2 \log(k - k') e(|B|)
\end{aligned}
$$

First note that C_{ALG} is a multiway k-cut: this is because by induction the output of the algorithm on $\widetilde{G}_1$ is a multiway k'-cut and the output of the algorithm on $\widetilde{G}_2$ is a multiway $(k - k')$-cut. Moreover,

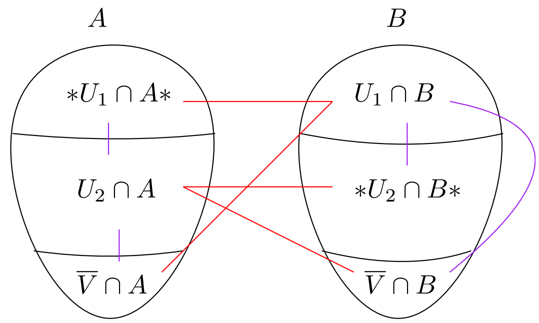

Figure 2: Node subsets of graph G. The subsets in asterisks have terminals in them. Red edges indicate the left-hand side edges in Eq. (14), and purple edges indicate the edges on the right-hand side in Eq. (14).

$E(A,B) \in C_{ALG}$. So the union of these cuts and $E(A,B)$ is a k-cut, and since each terminal is in exactly one partition, it is a multiway k-cut.

Now we prove the value guarantees. Let $U_1 = V_1 \cup \ldots \cup V_{k'}$ and $U_2 = V_{k'+1} \cup \ldots \cup V_k$. So $U = U_1 \cup U_2 = V_1 \cup \ldots \cup V_k$ is the set of nodes that are in at least one partition. Recall that $\overline{V} = V \setminus U$ is the set of nodes that are not in any partition.

Consider the following cut that separates $\{s_1, \ldots, s_{k'}\}$ from $\{s_{k'+1}, \ldots, s_k\}$: Let $A' = [U_1 \cap A] \cup [U_1 \cap B] \cup [\overline{V} \cap A]$. Let $B' = [U_2 \cap B] \cup [U_2 \cap A] \cup [\overline{V} \cap B]$. Since (A,B) is a *min* cut that separates $\{s_1, \ldots, s_{k'}\}$ from $\{s_{k'+1}, \ldots, s_k\}$ with additive error $e(n)$, we have $w(E(A,B)) \leq w(E(A',B'))+e(n)$. Note that $A = [U_1 \cap A] \cup [U_2 \cap A] \cup [\overline{V} \cap A]$ and $B = [U_1 \cap B] \cup [U_2 \cap B] \cup [\overline{V} \cap B]$. So turning (A,B) into (A',B') is equivalent to switching $U_2 \cap A$ and $U_1 \cap B$ between A and B. So we have that

$$\begin{aligned}
&w(E(U_2 \cap A, U_2 \cap B)) + w(E(U_2 \cap A, \overline{V} \cap B)) + w(E(U_1 \cap B, U_1 \cap A)) + w(E(U_1 \cap B, \overline{V} \cap A)) \\
&\leq \qquad (14) \\
&w(E(U_2 \cap A, U_1 \cap A)) + w(E(U_2 \cap A, \overline{V} \cap A)) + w(E(U_1 \cap B, U_2 \cap B)) + w(E(U_1 \cap B, \overline{V} \cap B)) \\
&+ e(n)
\end{aligned}$$

Eq. (14) is illustrated in Figure 2. Using Eq. (14), we obtain that

$$\begin{aligned}
w(E(A,B)) &= w(E(U_2 \cap A, U_2 \cap B)) + w(E(U_2 \cap A, \overline{V} \cap B)) \\
&+ w(E(U_1 \cap B, U_1 \cap A)) + w(E(U_1 \cap B, \overline{V} \cap A)) \\
&+ w(E(U_2 \cap A, U_1 \cap B)) + w(E([U_1 \cap A] \cup [\overline{V} \cap A], [U_2 \cap B] \cup [\overline{V} \cap B])) \\
&\leq w(E(U_2 \cap A, U_1 \cap A)) + w(E(U_2 \cap A, \overline{V} \cap A)) \\
&+ w(E(U_1 \cap B, U_2 \cap B)) + w(E(U_1 \cap B, \overline{V} \cap B)) \\
&+ w(E(U_2 \cap A, U_1 \cap B)) + w(E([U_1 \cap A] \cup [\overline{V} \cap A], [U_2 \cap B] \cup [\overline{V} \cap B])) \\
&+ e(n)
\end{aligned}$$

So we conclude that

$$\begin{aligned}
w(E(A,B)) \leq\, & w(E(U_1 \cap B, [U_2 \cap B] \cup [\overline{V} \cap B] \cup [U_2 \cap A])) \qquad (15) \\
&+ w(E(U_1 \cap A, [U_2 \cap B] \cup [\overline{V} \cap B])) \\
&+ w(E(U_2 \cap A, [U_1 \cap A] \cup [\overline{V} \cap A])) \\
&+ w(E(U_2 \cap B, \overline{V} \cap A)) \\
&+ w(E(\overline{V} \cap A, \overline{V} \cap B)) \\
&+ e(n)
\end{aligned}$$

We substitute $w(E(A,B))$ in Eq. (13) using Eq. (15). Recall that $U_1 = \cup_{i=1}^{k'} V_i$, $\delta_{\widetilde{G}_1}(V_i \cap A) = w(E(V_i \cap A, A \setminus V_i))$ and $\delta_G(V_i) = w(E(V_i, V \setminus V_i))$. For any $i \in \{1, \ldots, k'\}$, we have that

$E(V_i \cap B, [U_2 \cap B] \cup [\overline{V} \cap B] \cup [U_2 \cap A])$ and $E(V_i \cap A, [U_2 \cap B] \cup [\overline{V} \cap B])$ are both disjoint from $E(V_i \cap A, A \setminus V_i)$. Moreover all these three terms appear in $E(V_i, V \setminus V_i)$. So we have

$$\begin{aligned} & w(E(U_1 \cap B, [U_2 \cap B] \cup [\overline{V} \cap B] \cup [U_2 \cap A])) \\ & + w(E(U_1 \cap A, [U_2 \cap B] \cup [\overline{V} \cap B])) + \sum_{i=1}^{k'} \delta_{\widetilde{G}_1}(V_i \cap A) \\ & \leq \sum_{i=1}^{k'} \delta_G(V_i) \end{aligned}$$

Note that the first two terms above are the first two terms in Eq. (15). Similarly, we have

$$w(E(U_2 \cap A, [U_1 \cap A] \cup [\overline{V} \cap A])) + w(E(U_2 \cap B, \overline{V} \cap A)) + \sum_{i=k'+1}^{k} \delta_{\widetilde{G}_2}(V_i \cap B) \leq \sum_{i=k'+1}^{k} \delta_G(V_i) \quad (16)$$

Note that the first two terms above are the third and forth terms in Eq. (15). Finally $w(E(\overline{V} \cap A)) + w(E(\overline{V} \cap B)) + w(E(\overline{V} \cap A, \overline{V} \cap B)) \leq w(E(\overline{V}))$. So, we upper-bound Eq. (13) as

$$\begin{aligned} w(C_{ALG}) \leq & \sum_{i=1}^{k} \delta_G(V_i) + w(E(\overline{V})) \\ & + e(n) + 2\log(k')e(|A|) + 2\log(k-k')e(|B|). \end{aligned}$$

Since $k' = \lfloor k/2 \rfloor$ and $k - k' = \lceil k/2 \rceil$, we have that $k' \leq \lfloor \frac{k+1}{2} \rfloor$ and $k - k' \leq \lfloor \frac{k+1}{2} \rfloor$. Moreover, since $e = cn/\epsilon$ for $\epsilon > 0$ and $c \geq 0$, we have that $e(|A|) + e(|B|) \leq e(|A| + |B|) = e(n)$. Therefore,

$$\begin{aligned} & e(n) + 2\log(k')e(|A|) + 2\log(k-k')e(|B|) \\ \leq & e(n)(1 + 2\log\left(\left\lfloor \frac{k+1}{2} \right\rfloor)\right) \leq 2\log(k)e(n). \end{aligned}$$

The above inequality finishes the approximation proof.

The success probability. As proved by Lemma 4.1, the min s-t cut computations by Algorithm 2 can be seen as invocations of a min s-t cut algorithm on $O(\log k)$ many n-node graphs; in this claim, we use $\mathcal{A}$ to compute min s-t cuts. By union bound, each of those $O(\log k)$ invocations output the desired additive error by probability at least $1 - \alpha O(\log k)$.