Hopf Bifurcation Analysis using Zigzag Persistence

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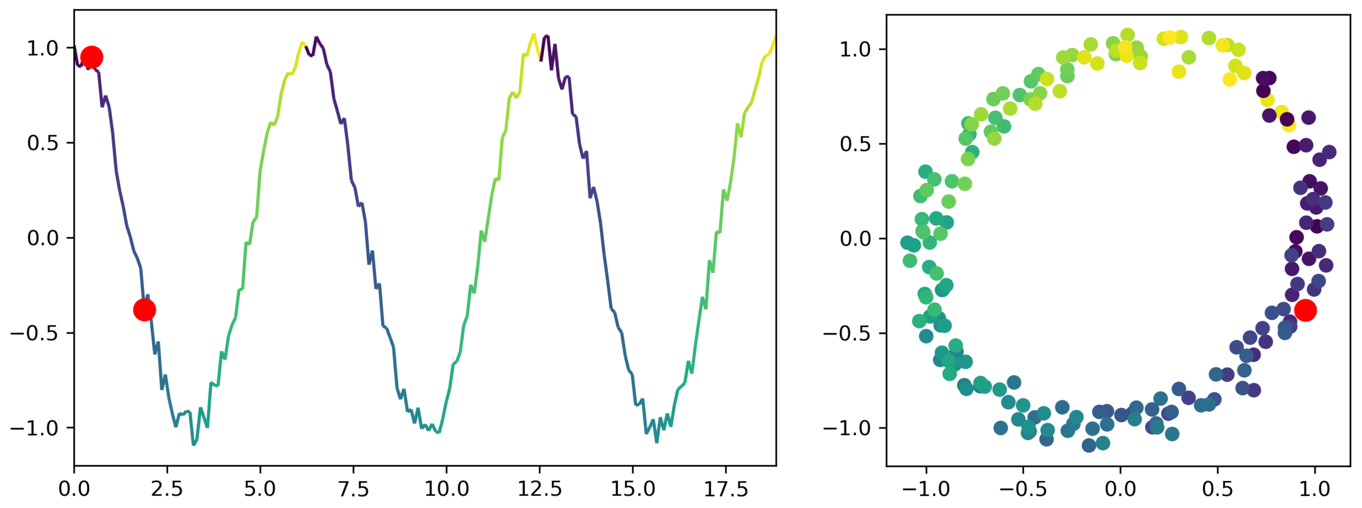
Using zigzag persistence, we can capture topological changes in the state space of the dynamical system caused by a Hopf bifurcation in only one persistence diagram. Here, we present Bifurcations using ZigZag (BuZZ), a one-step method to study and detect bifurcations using zigzag persistence.

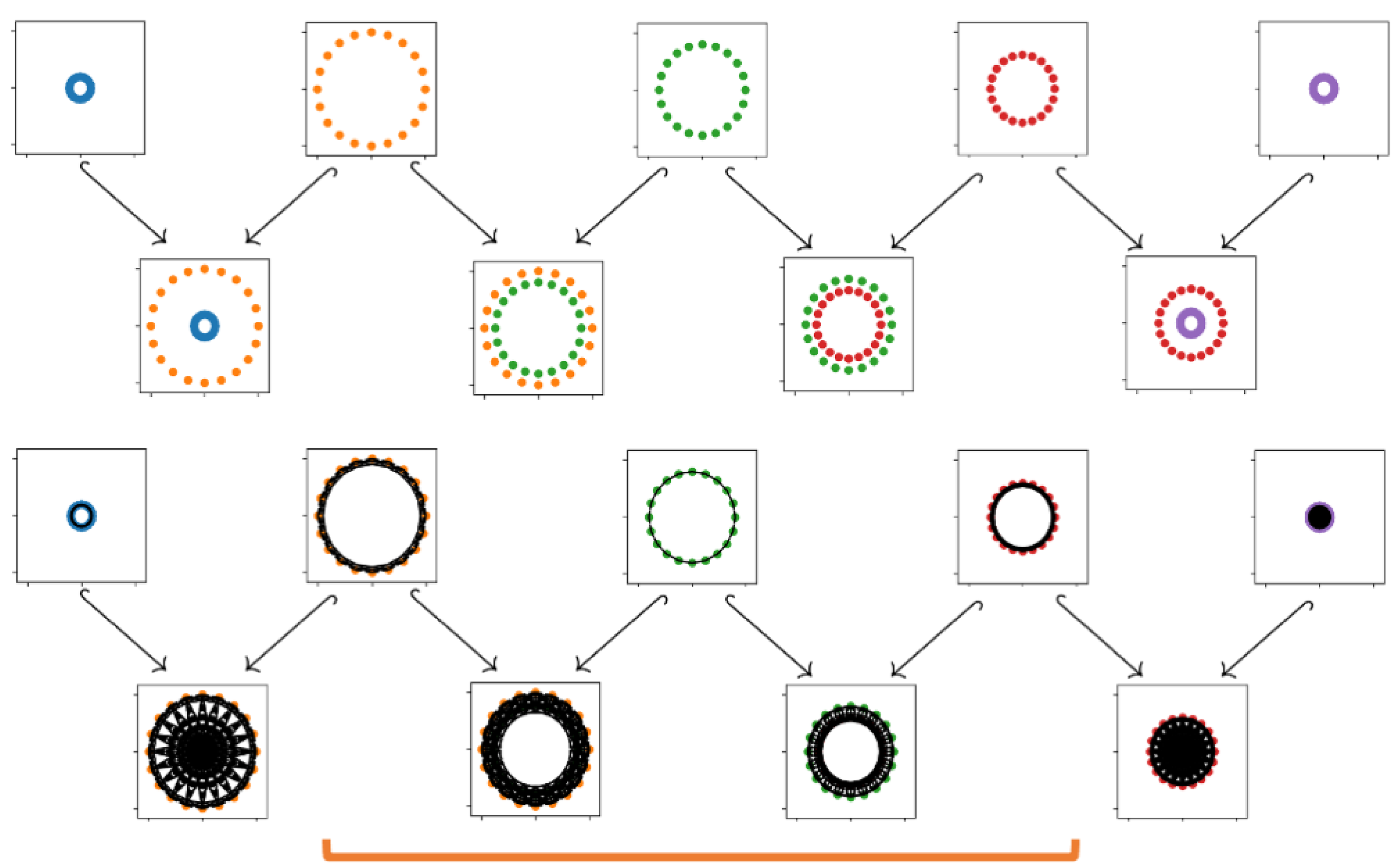
Time Delay Embedding

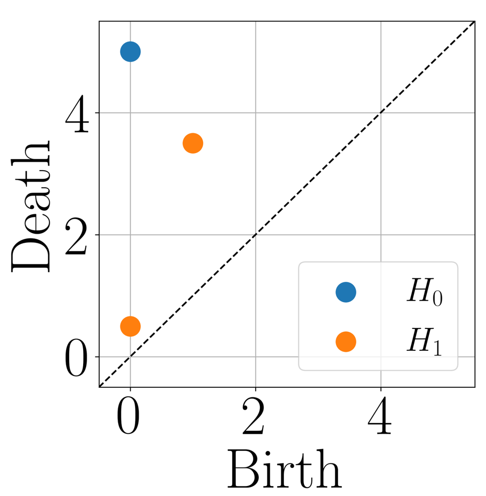
Given a time series, $[x_1, \ldots, x_n]$, a choice of dimension d and lag τ, the delay embedding is the point cloud, $\{\mathbf{x}_i := (x_i, x_{i+\tau}, \ldots, x_{i+(d-1)\tau})\} \subset \mathbb{R}^d$.



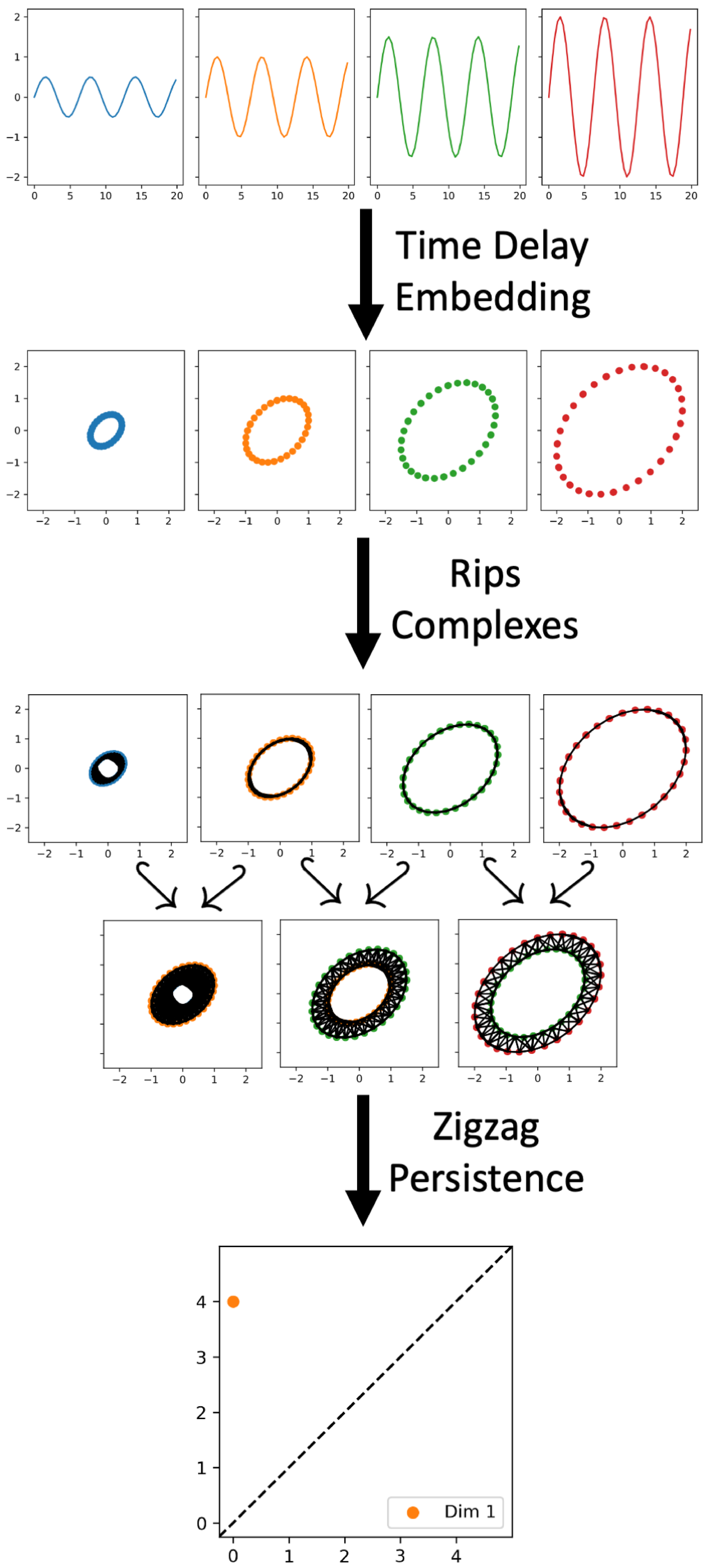
Zigzag Persistent Homology

Standard persistent homology requires a collection of simplicial complexes with inclusions, $\mathcal{K}_1 \hookrightarrow \mathcal{K}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{K}_n$. Zigzag persistent homology is a generalization of standard persistent homology where the inclusion maps can go in either direction. Specifically, we consider a sequence of point clouds and their unions.





- Persistence points indicate features that are homologically equivalent through the zigzag
- Coordinates of the points indicate the index of the point cloud where a feature appears and disappears

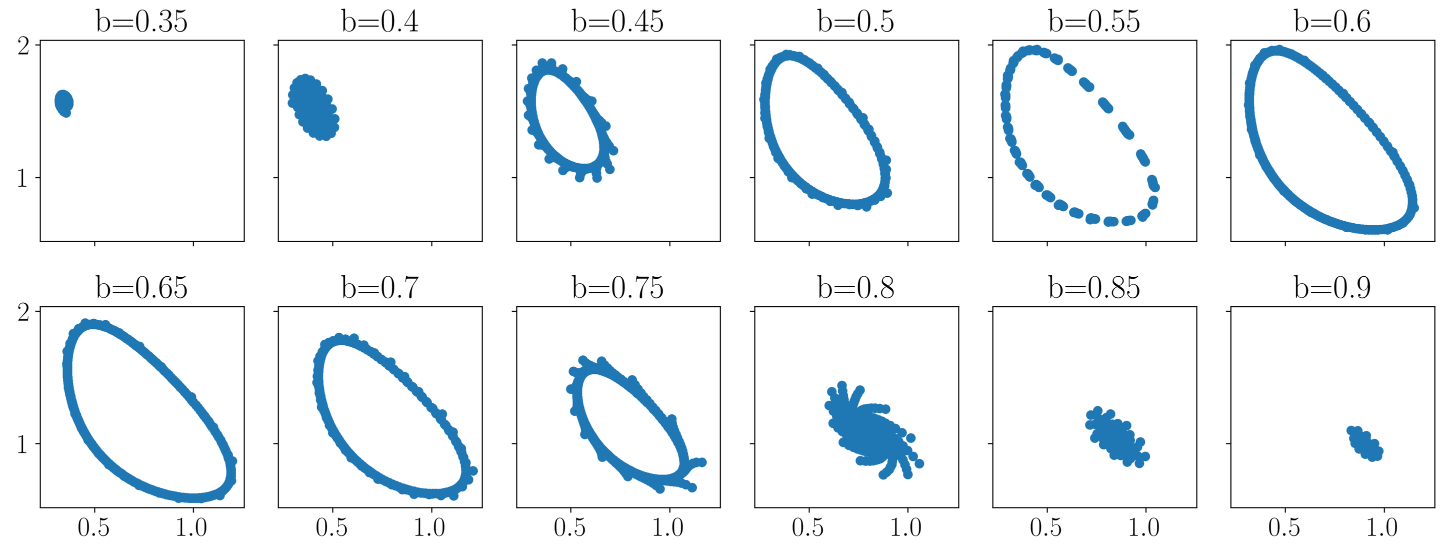


Paper: `https://bit.ly/33s4CXt`
Code: `https://github.com/sarahtymochko/BuZZ`

Example: Sel'kov Model

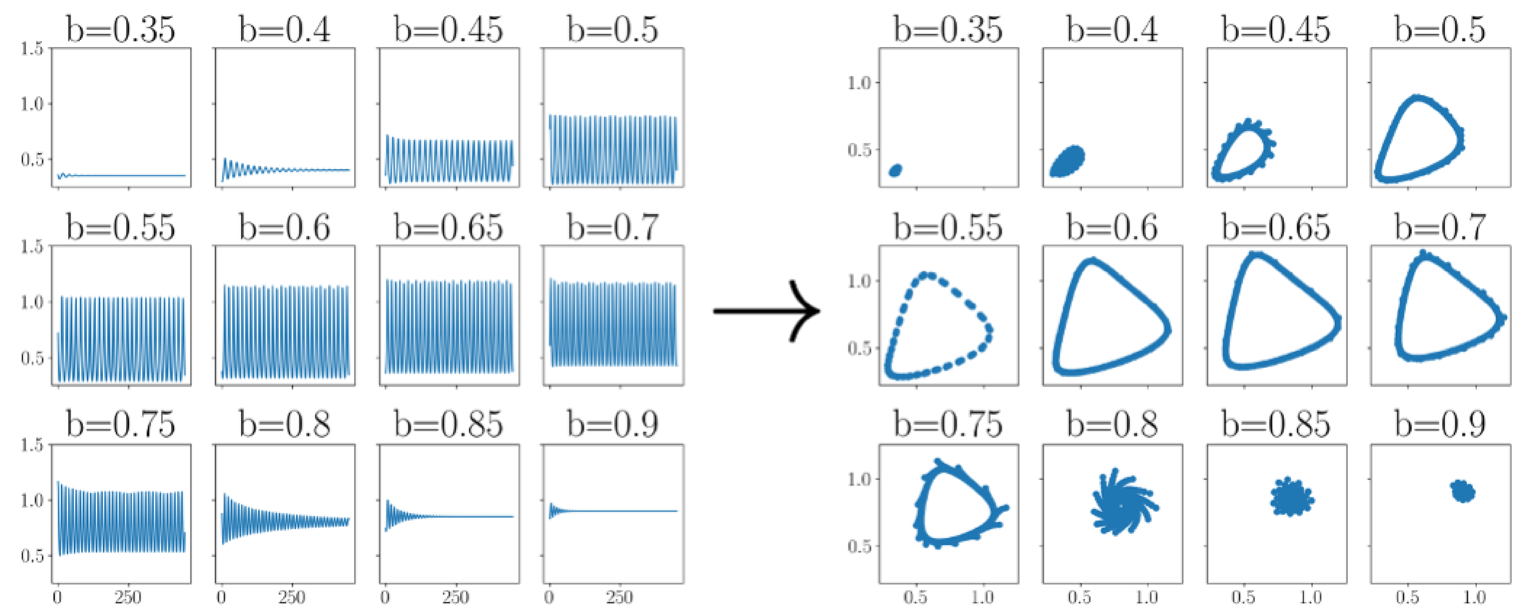
The Sel'kov model is a model for glycolysis, a process of breaking down sugar for energy. This model is defined by the system of differential equations,

$$\frac{dx}{dt} = -x + ay + x^2y \qquad \frac{dy}{dt} = b - ay - x^2y.$$

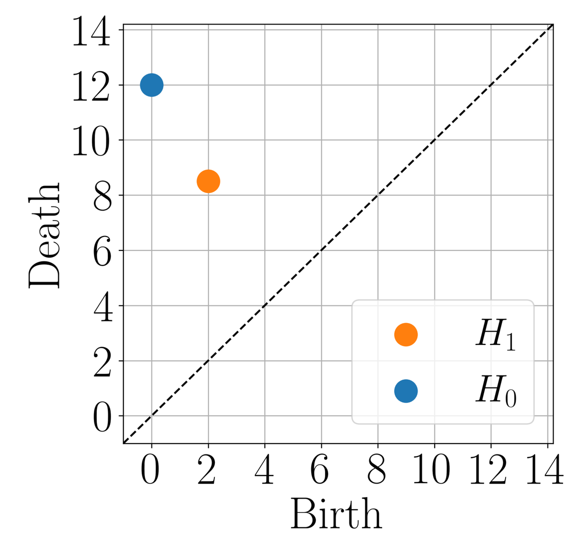


Can we detect for which values of b there is a Hopf bifurcation in the Sel'kov model for glycolysis?

- Fix $a = 0.1$ and vary $b \in \{0.35, 0.4, \ldots, 0.9\}$
- Generate time series corresponding to x-coordinates and compute the time delay embeddings



- Applying our method, we get a 1-dimensional persistence point $(2, 8.5)$ which corresponds to $0.45 \leq b \leq 0.75$
- The Sel'kov model has a Hopf bifurcation between the parameter values $0.4 \leq b \leq 0.8$, so our method is picking up approximately that range.



Acknowledgements & References

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[1] Gunnar Carlsson and Vin de Silva. Zigzag persistence. *Foundations of Computational Mathematics*, 10(4):367–405, 2010.

[2] Sarah Tymochko, Elizabeth Munch, and Firas A. Khasawneh. Using zigzag persistent homology to detect Hopf bifurcations in dynamical systems. *Algorithms*, 13(11):278, 2020.