Using zigzag persistence, we can capture topological changes in the state space of the dynamical system caused by a Hopf bifurcation in only one persistence diagram. Here, we present Bifurcations using ZigZag (BuZZ), a one-step method to study and detect bifurcations using zigzag persistence.

**Time Delay Embedding**

Given a time series, \([x_1, \ldots, x_n]\), a choice of dimension \(d\) and lag \(\tau\), the delay embedding is the point cloud, \(\{x_i := (x_i, x_{i+\tau}, \ldots, x_{i+(d-1)\tau})\} \subseteq \mathbb{R}^d\).

**Zigzag Persistent Homology**

Standard persistent homology requires a collection of simplicial complexes with inclusions, \(K_1 \hookrightarrow K_2 \hookrightarrow \ldots \hookrightarrow K_n\). Zigzag persistent homology is a generalization of standard persistent homology where the inclusion maps can go in either direction. Specifically, we consider a sequence of point clouds and their unions.

- Persistence points indicate features that are homologically equivalent through the zigzag
- Coordinates of the points indicate the index of the point cloud where a feature appears and disappears

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**Example: Sel’kov Model**

The Sel’kov model is a model for glycolysis, a process of breaking down sugar for energy. This model is defined by the system of differential equations,

\[
\begin{align*}
\frac{dx}{dt} &= -x + ay + x^2y \\
\frac{dy}{dt} &= b - ay - x^2y.
\end{align*}
\]

Can we detect for which values of \(b\) there is a Hopf bifurcation in the Sel’kov model for glycolysis?

- Fix \(a = 0.1\) and vary \(b \in \{0.35, 0.4, \ldots, 0.9\}\)
- Generate time series corresponding to \(x\)-coordinates and compute the time delay embeddings
- Applying our method, we get a 1-dimensional persistence point \((2, 8.5)\) which corresponds to \(0.45 \leq b \leq 0.75\)
- The Sel’kov model has a Hopf bifurcation between the parameter values \(0.4 \leq b \leq 0.8\), so our method is picking up approximately that range.

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