Fairness and Privacy Guarantees in Federated Contextual Bandits

1. Proofs

Lemma 1 *(Lemma 5 in main paper) For the Fed-FairX-LinUCB, with high probability, the instantaneous regret for any agent i is bounded by,*

$$FR_t^i = \sum_{a \in \mathcal{D}} \left|\pi_t^i - \pi_*^i\right| \leq \frac{4L\sqrt{\beta_t}}{\gamma} \mathbb{E}_{a \sim \pi_t^i} \left\|x_t^i(a)\right\|_{(V_t^i)^{-1}}$$

Proof provided in Appendix.

Proof

$$FR_t^i = \sum_{a \in \mathcal{D}} \left| \frac{f^i(\theta^* x_t^i(a))}{\sum_{a' \in \mathcal{D}} f^i(\theta^* x_t^i(a'))} - \frac{f^i(\theta_t^i x_t^i(a))}{\sum_{a' \in \mathcal{D}} f^i(\theta_t^i x_t^i(a'))} \right|$$

$$= \sum_a \left| \frac{\begin{array}{l} f^i(\theta^* x_t^i(a)) \sum_{a'} f^i(\theta_t^i x_t^i(a')) \\ \quad - f^i(\theta_t^i x_t^i(a)) \sum_{a'} f^i(\theta^* x_t^i(a')) \end{array}}{\sum_{a'} f^i(\theta_t^i x_t^i(a')) \sum_{a'} f^i(\theta^* x_t^i(a'))} \right|$$

$$= \sum_a \frac{\left| \begin{array}{l} f^i(\theta^* x_t^i(a)) \sum_{a'} \left(f^i(\theta_t^i x_t^i(a')) - f^i(\theta^* x_t^i(a'))\right) \\ \quad + \left(f^i(\theta^* x_t^i(a)) - f^i(\theta_t^i x_t^i(a))\right) \sum_{a'} f^i(\theta^* x_t^i(a')) \end{array} \right|}{\sum_{a'} f^i(\theta_t^i x_t^i(a')) \sum_{a'} f^i(\theta^* x_t^i(a'))}$$

$$\leq \sum_a \frac{\begin{array}{l} \left|f^i(\theta^* x_t^i(a)) \sum_{a'} \left(f^i(\theta_t^i x_t^i(a')) - f^i(\theta^* x_t^i(a'))\right)\right| \\ \quad + \left|\left(f^i(\theta^* x_t^i(a)) - f^i(\theta_t^i x_t^i(a))\right) \sum_{a'} f^i(\theta^* x_t^i(a'))\right| \end{array}}{\sum_{a'} f^i(\theta_t^i x_t^i(a')) \sum_{a'} f^i(\theta^* x_t^i(a'))}$$

$$\leq \frac{2 \sum_a \left|f^i(\theta^* x_t^i(a)) - f^i(\theta_t^i x_t^i(a))\right|}{\sum_{a'} f^i(\theta_t^i x_t^i(a'))}$$

$$= 2 \sum_a \frac{\pi_t^i}{f^i(\theta_t^i x_t^i(a))} \left| \begin{array}{l} f^i(\theta^* x_t^i(a)) - f^i(\hat{\theta}_t^i x_t^i(a)) \\ \quad + f^i(\hat{\theta}_t^i x_t^i(a)) - f^i(\theta_t^i x_t^i(a)) \end{array} \right|$$

$$\leq \frac{2L}{\gamma}\mathbb{E}_{a\sim\pi_t^i}\left[\begin{array}{l}\left\|\theta^* - \hat{\theta}_t^i\right\|_{V_t^i}\left\|x_t^i(a)\right\|_{(V_t^i)^{-1}} \\ \quad + \left\|\hat{\theta}_t^i - \theta_t^i\right\|_{V_t^i}\left\|x_t^i(a)\right\|_{(V_t^i)^{-1}}\end{array}\right]$$

$$\leq \frac{4L\sqrt{\beta_t}}{\gamma}\mathbb{E}_{a\sim\pi_t^i}\left\|x_t^i(a)\right\|_{(V_t^i)^{-1}}$$

This completes the proof of Lemma 3. ■

Theorem 2 *(Theorem 6 in main text) With high probability, Fed-FairX-LinUCB achieves a fairness regret of*

$$O\left(\frac{4\nu L\sqrt{\beta_t}}{\gamma}\sqrt{mTd\log{(1+\frac{T}{d})} + m^2d^3\log^3{(1+\frac{T}{d})}}\right)$$

when $\| x_t^i(a) \|_2 \leq 1\ \forall a, t, i.$

Proof

Consider a hypothetical agent denoted by index 0 who plays in the following mT rounds - $(1,1),(1,2),\ldots(1,m),\ldots,(T,m)$ sequentially. Let the gram matrix for agent 0 till round (τ,j) be given by $V^0_{(\tau,j)} = mI + \sum_{i=1}^{i=m}\sum_{t=1}^{\tau-1}(x_t^i(a_t^i))(x_t^i(a_t^i))^T + \sum_{i=1}^{i=j}(x_\tau^i(a_\tau^i))(x_\tau^i(a_\tau^i))^T$. Substituting $U_1 = mI$ and $L = 1$ in Lemma 3, main text, we get,

$$\sum_{i=1}^{i=m}\sum_{t=1}^{T}\left\|x_t^i(a_t^i)\right\|^2_{(V^0_{(t,i)})^{-1}} \leq 2d\log\left(1+\frac{T}{d}\right)$$

Let the communication in the original algorithm occur at rounds $T_1, T_2, \ldots, T_{p-1}$. Let $\Psi_k = mI + \sum_{i=1}^{i=m}\sum_{t=1}^{T_k}(x_t^i(a_t^i))(x_t^i(a_t^i))^T$ be the synchronised gram matrix after communication round k. Then $\det\Psi_0 = (m)^d$ and $\det\Psi_p \leq \left(\frac{\mathrm{tr}(\Psi_p)}{d}\right)^d \leq (m + mT/d)^d$. Thus, for any $\nu > 1$, $\log_\nu\left(\frac{\det(\Psi_p)}{\det(\Psi_0)}\right) \leq d\log_\nu(1+\frac{T}{d})$. Let event E represent the set of rounds when $1 \leq \frac{\det(\Psi_k)}{\det(\Psi_{k-1})} \leq \nu$ is true. Then, in all but $\lceil d\log_\nu(1+\frac{T}{d})\rceil$ rounds E is true.

For any $T_{k-1} \leq t \leq T_k$, when E is true,

$$\begin{aligned}
fr_t^i &\leq \frac{4L\sqrt{\beta_t}}{\gamma}\mathbb{E}_{a\sim\pi_t^i}\left\|x_t^i(a)\right\|_{(V_t^i)^{-1}} \\
&\leq \frac{4L\sqrt{\beta_t}}{\gamma}\mathbb{E}_{a\sim\pi_t^i}\left\|x_t^i(a)\right\|_{(V^0_{(t,i)})^{-1}}\sqrt{\frac{\det V^0_{(t,i)}}{\det V_t^i}} \\
&\leq \frac{4L\sqrt{\beta_t}}{\gamma}\mathbb{E}_{a\sim\pi_t^i}\left\|x_t^i(a)\right\|_{(V^0_{(t,i)})^{-1}}\sqrt{\frac{\det \Psi_k}{\det \Psi_{k-1}}} \\
&\leq \frac{4\nu L\sqrt{\beta_t}}{\gamma}\mathbb{E}_{a\sim\pi_t^i}\left\|x_t^i(a)\right\|_{(V^0_{(t,i)})^{-1}}
\end{aligned}$$

Here, second last equation follows because $V_t^i \succeq \Psi_{k-1}$ and $\Psi_k \succeq V_{(t,i)}^0$. Now, using Lemma 4, main text,

$$\sum_{i=1}^{m}\sum_{t\in E} fr_t^i \leq \sum_{i=1}^{m}\sum_{t\in E} \frac{4\nu L\sqrt{\beta_t}}{\gamma}\mathbb{E}_{a\sim\pi_t^i}\left\|x_t^i(a)\right\|_{(V_{(t,i)}^0)^{-1}}$$
$$\leq \frac{4\nu L\sqrt{\beta_T}}{\gamma}\left(\sum_{i=1}^{m}\sum_{t=1}^{T}\left\|x_t^i(a_t^i)\right\|_{(V_{(t,i)}^0)^{-1}} + \sqrt{2mT\log(4/\delta)}\right)$$
$$\leq \frac{4\nu L\sqrt{mT\beta_T}}{\gamma}\left(\sqrt{d\log(1+\frac{T}{d})} + \sqrt{2\log(4/\delta)}\right)$$

Now, let us consider any period $t \in [T_{k-1}, T_k]$, where E does not hold and $t_k = T_k - T_{k-1}$ represent the length of the interval. Fairness regret during this period is given by,

$$FR([T_{k-1}, T_k]) \leq \frac{4L\sqrt{\beta_T}}{\gamma}\sum_{i=1}^{m}\sum_{t=T_{K-1}}^{T_k}\mathbb{E}_{a\sim\pi_t^i}\left\|x_t^i(a)\right\|_{(V_t^i)^{-1}}$$
$$\leq \frac{4L\sqrt{\beta_T}}{\gamma}\left(\sum_{i=1}^{m}\sum_{t=T_{K-1}}^{T_k}\left\|x_t^i(a_t^i)\right\|_{(V_t^i)^{-1}} + m\sqrt{2t_k\log(4/\delta)}\right) \quad \text{(Using Lemma 4, main text)}$$
$$\leq \frac{4L\sqrt{\beta_T}}{\gamma}\left(\sum_{i=1}^{m}\sqrt{t_k\log_\nu\frac{\det V_{T_{k-1}+t_k}^i}{\det V_{T_{k-1}}^i}} + m\sqrt{2t_k\log(4/\delta)}\right)$$

We know that $\forall$ agents, $t_k \leq \frac{T}{md^2\log^2(1+T/d)}+1$ (otherwise there be a communication round), thus $FR([T_{k-1}, T_k]) \leq \frac{4L\sqrt{\beta_T}}{\gamma}$ $\left(\sqrt{\frac{m(T+md^2\log^2(1+\frac{T}{d}))}{d\log(1+\frac{T}{d})}} + \sqrt{\frac{2m(T+md^2\log^2(1+\frac{T}{d}))}{d^2\log^2(1+\frac{T}{d})}\log(\frac{4}{\delta})}\right)$.
Using the fact that E does not hold true in at most in $\lceil d\log_\nu(1+\frac{T}{d})\rceil$ rounds, we get

$$FR(T) \leq O\left(\frac{4\nu L\sqrt{\beta_T}}{\gamma}\sqrt{mTd\log(1+T/d) + m^2d^3\log^3(1+T/d)}\right)$$

■