A SCALABLE CONSTANT-FACTOR APPROXIMATION ALGORITHM FOR W_p OPTIMAL TRANSPORT

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ABSTRACT

Let (X, d) be a metric space and let μ, ν be discrete distributions supported on finite point sets $A, B \subseteq X$. For any $p \in [1, \infty]$, the W_p-distance between μ and ν, $W_p(\mu, \nu)$, is defined as the p-th root of the minimum cost of transporting the mass from μ to ν, where moving a unit of mass from $a \in A$ to $b \in B$ incurs a cost of $d^p(a, b)$. We give a (Las Vegas) randomized algorithm that always computes a $(4 + \varepsilon)$-approximate W_p optimal-transport (OT) plan in $O(n^2 + (n^{3/2}\varepsilon^{-1} \log^2 \Delta)^{1+o(1)})$ expected time, for all $p \in [1, \infty]$, where $\varepsilon > 0$ is an arbitrarily small constant and Δ is the spread of $A \cup B$. The best previous result achieved an $O(\log n)$-approximation in $O(pn^2)$ time, but only for constant p. Our algorithm significantly improves the approximation factor and, importantly, is the first quadratic-time method that extends to the W_∞-distance. In contrast, additive approximation methods such as Sinkhorn are efficient only for constant p and fail to handle $p = \infty$. Finally, we show that obtaining a relative approximation factor better than 2 in $O(n^2)$ time would resolve the long-standing open problem of computing a perfect matching in an arbitrary bipartite graph in quadratic time.

1 INTRODUCTION

Let μ and ν be discrete probability distributions supported on sets A and B, respectively, with $|A| + |B| = n$. For each pair $(a, b) \in A \times B$, let $\mathsf{d}(a, b)$ denote the ground distance between a and b. A *transport plan* is a function $\sigma : A \times B \to \mathbb{R}_{\geq 0}$ that assigns a mass to each pair (a, b) such that $\sum_{b \in B} \sigma(a, b) \leq \mu(a)$ and $\sum_{a \in A} \sigma(a, b) \leq \nu(b)$. Given a parameter $p \geq 1$, suppose the cost of moving a unit of mass from a point $a \in A$ to a point $b \in B$ is given by $\mathsf{d}(a, b)^p$. The W_p cost of any transport plan σ between μ and ν is defined as

$$w_p(\sigma) := \left(\sum_{a \in A, b \in B} \sigma(a, b) \times \mathsf{d}(a, b)^p \right)^{1/p}.$$

The W_p cost above for finite p extends naturally to the W_∞ cost of a transport plan σ, defined as

$$w_\infty(\sigma) := \lim_{p \to \infty} w_p(\sigma) = \max_{a,b \in A \times B \,:\, \sigma(a,b) > 0} \mathsf{d}(a, b).$$

In the W_p optimal transport (OT) problem, we wish to compute the transport plan σ^* that transports the entire mass and has the smallest W_p cost. We refer to the cost $w_p(\sigma^*)$ as the *W_p-distance* and denote it by $W_p(\mu, \nu)$. If μ and ν are uniform distributions, then we refer to the W_p OT problem as the W_p *matching* problem.

The W_p distances, for varying values of p, possess several appealing properties that make it favorable in applications. OT plans under the W_1 distance measure total displacement and are therefore useful to capture structural properties such as semantic relationships from word embeddings (Kusner et al. (2015)). OT plans arising from the W_2 distance have nice structural qualities such as monotonicity (Brenier (1991); Aurenhammer et al. (1998)) and translation invariance (Cohen & Guibas (1999)), and tend to preserve the geometry of the distributions. Furthermore, recent work in machine learning and topological data analysis uses W_∞ distance to establish consistency and convergence properties of topological summaries (Vishwanath et al. (2020); Damrich et al.

(2024)), and to design topological layers in neural networks (Kim et al. (2020)). Due to these favorable properties, W_p distances has been used in applications across machine learning (Chang et al. (2023); Chuang et al. (2022)), computer vision (Backurs et al. (2020); Lai et al. (2022)), and natural language processing (Alvarez-Melis & Jaakkola (2018); Yurochkin et al. (2019)).

From an algorithmic standpoint, the exact computation of the W_p distance between discrete distributions can be formulated as a minimum-cost flow (MCF) problem, which can be solved in $n^{2+o(1)}$ time using recent advances in MCF algorithms (Chen et al. (2022)). While these results mark important theoretical progress, they are highly complicated, making them unsuitable for practical implementations. Indeed, even the simpler task of designing a truly quadratic-time exact algorithm for deciding whether a dense graph admits a perfect matching remains a longstanding open problem in graph theory (Behnezhad et al. (2024)).

Because exact algorithms remain expensive, research has shifted toward scalable approximation methods. A seminal result by Charikar (2002) introduced an $O(\log n)$-approximation for W_1 by embedding the ground metric into a hierarchically well-separated tree; a greedy transport procedure on the tree yielded an exact solution in $O(n^2)$ time, producing an overall $O(\log n)$-approximation. This work opened the door to more refined methods, and subsequent efforts have developed near-linear-time $(1+\varepsilon)$-approximation algorithms under additional assumptions, such as when the ground distance is Euclidean in fixed dimensions (Agarwal et al. (2022; 2024); Fox & Lu (2023)), and more recently, sub-quadratic algorithms for higher-dimensional Euclidean settings (Andoni & Zhang (2023); Beretta et al. (2025)). However, none of these techniques extend naturally to the case $p \geq 2$. Building on this line of work, Lahn et al. (2025) recently presented a relative $O(\log n)$-approximation algorithm for any finite $p \geq 2$, with runtime $O\left(n^2 \log U \log \Delta \log n\right)$, where $\log U$ is the bit-length of the input probabilities and Δ is the spread of $A \cup B$ (the ratio of its largest to smallest nonzero pairwise distance).

One influential direction of work was introduced by Cuturi (2013), who proposed entropic regularization of OT. It guarantees solutions within an additive error of $\varepsilon\Delta$, where Δ denotes the maximum ground distance between points in $A \cup B$. Although weaker than a relative $(1+\varepsilon)$-approximation, it applies across all metrics and inspired a series of additive approximation algorithms, including parallelizable variants (Altschuler et al. (2017); Dvurechensky et al. (2018); Jambulapati et al. (2019); Lahn et al. (2019; 2023)). Nonetheless, their runtimes remain on the order of $n^2/\varepsilon^{O(1)}$ for W_1 and worsen to $n^2/\varepsilon^{O(p)}$ for larger p, with no extension to the case $p = \infty$.

Our results. In this paper, we present two constant-factor approximation algorithms for the W_p problem. These are the first truly quadratic-time (assuming $\Delta + U = 2^{O(n^{1/8})}$) approximation algorithms for the W_p problem over *any ground metric*, applicable to all $p \in [1, \infty]$, including $p = \infty$. This improves the $O(\log n)$-approximation of Lahn et al. (2025) to a constant factor, while extending the guarantee to every p.

Theorem 1.1. *Let μ and ν be two discrete distributions supported on a set of n points in an arbitrary metric space. Let $p \in [1, \infty]$ be a parameter, and let $\varepsilon > 0$ be an arbitrarily small constant. A $(4+\varepsilon)$-approximate OT plan under the W_p metric can be computed in $O(n^2 + (n^{3/2}\varepsilon^{-1} \log^2 \Delta \log U)^{1+o(1)})$ expected time, where Δ is the spread of the support set and U is the ratio of the max to min probability in μ or ν.*

Our main technical contribution is a technique for approximating $\mathsf{d}(\cdot,\cdot)^p$, inspired by Bourgain's multi-level sampling (Bourgain (1985)). This multi-level sampling has inspired a sequence of results based on clustering points, much like the approach we adopt in this paper. Broadly, these results fall into two categories: spanner constructions and distance oracles for metric spaces. The spanners (Har-Peled et al. (2023); Baswana & Sen (2007); Cohen (1998)) are typically designed to approximate the underlying metric, whereas in our setting we seek to approximate $\mathsf{d}^p(\cdot,\cdot)$, which is not a metric. This distinction forces us to design a clustering scheme and a *directed* spanner that is not strongly connected, but in which shortest paths nevertheless preserve $\mathsf{d}^p(\cdot,\cdot)$ within a factor of $(4+\varepsilon)^p$. Using a minimum-cost flow algorithm on the directed spanner we construct will imply Theorem 1.1.

Since the interior point method in Chen et al. (2022) does not admit a practical implementation, we also design a simple combinatorial algorithm for the W_p matching problem, i.e., when μ and ν are uniform distributions. A parallel line of work instead leverages such clusterings to de-

sign distance oracles, i.e., data structures that allow efficient querying of distances between two points (Thorup & Zwick (2005); Mendel & Naor (2007); Awerbuch et al. (1998)). However, these oracles cannot be used to answer bichromatic closest-pair (BCP) queries, which are central to our setting. To overcome this limitation, we tailor our clustering scheme to efficiently support both weighted nearest-neighbor (WNN) and weighted bichromatic closest-pair queries. Importantly, although answering pairwise-distance queries are expensive in our framework, our algorithms avoid them entirely and instead rely solely on BCP and WNN queries. Using these data structures, we obtain a simpler combinatorial algorithm to compute a $(4+\varepsilon)$-approximate W_p matching.

Theorem 1.2. *Let A and B be two point sets of size n each in an arbitrary metric space, and let $p \in [1,\infty]$ be a parameter. A $(4+\varepsilon)$-approximate W_p matching of A and B can be computed in $O(n^2\varepsilon^{-2}\log^2\Delta)$ expected time, and an $(8+\varepsilon)$-approximate W_p matching of A and B can be computed in $O(n^2 + n^{5/3}\varepsilon^{-2}\log^2\Delta)$ time.*

To our knowledge, there are no known practical and implementable approximation algorithms for the W_∞-matching problem that run in $o(n^{2.5})$ time.

Next, we establish conditional lower bounds that suggest our results cannot be significantly improved without a major breakthrough in the graph matching problem, namely, computing a perfect matching in any graph in $O(n^2)$ time.

Theorem 1.3. *If there exists a quadratic-time algorithm that achieves a $(2-\varepsilon)$-relative approximation or $\Delta/2-\varepsilon$ additive approximation for the W_∞-matching problem, where Δ is the diameter of the point set and $\varepsilon > 0$ is a constant, then a perfect matching in an arbitrary graph can be computed in $O(n^2)$ time if one exists.*

We conclude with a primitive implementation of our simple combinatorial algorithm alongside some experimental results suggesting that the algorithm computes good quality W_p-matchings for $p \in [1,\infty]$ in Section 4. While we prove that the approximation factor of our algorithm is $(4+\varepsilon)$ in the worst case, our experimental results indicate that our algorithm computes even better approximations of W_p-matchings in practice.

2 Distance Approximation and Proximity Queries

Let P be a set of points, and let $\mathsf{d}\colon P \times P \to \mathbb{R}_{\geq 0}$ be a metric. We describe a clustering based distance function that approximates $\mathsf{d}(\cdot,\cdot)$, similar to the methods for constructing k-spanners and distance oracles discussed in Section 1, and that can be represented using roughly $n^{3/2}$ space (as opposed to $O(n^2)$ space to store all pairwise distances), and we use it to construct a spanner and to maintain bichromatic closest pairs.

We present only a two-layered clustering in the main text. Similar to the prior works, this layered clustering approach can be generalized to a k-level clustering. Extending to k-level clustering has a reduced number of clusters in which any point is expected to participate, at the expense of an increased stretch factor of the data structure. We provide more details about the k-level clustering in Appendix B.

We begin with a few notations. Given a point $x \in P$ and a subset $Q \subseteq P$, the distance from x to Q is defined as $\mathsf{d}(x,Q) = \min_{q\in Q}\mathsf{d}(x,q)$. For a point $q \in P$ and subset $Q \subseteq P$, define the *Voronoi region* of q to be

$$V(q,Q) := \{y \in P \mid \mathsf{d}(y,q) < \mathsf{d}(y,Q)\}.$$

That is, $V(q,Q)$ consists of the points in P for which q is closer than any point in Q.

Two-layered clustering. We construct a two-layered clustering of points of P. Set $P_0 = P$. Next, we choose a subset $P_1 \subseteq P_0$ by sampling each point in P_0 independently with probability $n^{-1/2}$. The expected size of P_1 is $\mathbb{E}[|P_1|] = n^{1/2}$.

Let $\Delta = \max_{p,q\in P}\mathsf{d}(p,q)$ be the diameter of P. Without loss of generality, assume $\min_{p,q}\mathsf{d}(p,q) = 1$ implying the metric space (P,d) also has spread Δ. We choose $\varepsilon > 0$ to be a sufficiently small constant. Set $t = \lceil \log_{(1+\frac{\varepsilon}{4})}\Delta \rceil$, $r_0 = 0$, and $r_i = (1+\frac{\varepsilon}{4})^i$ for $1 \leq i \leq t$. We generate two types of clusters: (i) For each $q \in P_0 \setminus P_1$ and for every $i \leq t$,

define $C_q[i] = \{x \in V(q, P_1) \mid \mathsf{d}(x, q) \leq r_i\}$. (ii) For each $q \in P_1$ and for every $i \leq t$, define $C_q[i] = \{x \in P_0 \mid \mathsf{d}(x, q) \leq r_i\}$. We refer to i as the *index* of the cluster $C_q[i]$. Let $\mathcal{C} = \{C_q[i] \mid q \in P_0, i \leq t\}$ be the collection of all clusters. Note that a point $p \in P$ may belong to many clusters. The number of clusters that contain p is called the *degree* of p and is denoted as $\deg_{\mathcal{C}}(p)$. While the degree of any particular point may be as large as n in the worst case, we prove that the expected degree of each point in P is much smaller.

Lemma 2.1. $\mathbb{E}[\deg_{\mathcal{C}}(p)] = O(n^{1/2}\varepsilon^{-1}\log\Delta)$ *for all* $p \in P$.

Proof. We note that for any $0 \leq i \leq j \leq t$ and for any $q \in P$, $C_q[i] \subseteq C_q[j]$. There are at most $O(\varepsilon^{-1}\log\Delta)$ different values of i. Therefore it suffices to prove for any $p \in P$, the number of points $q \in P$ where $p \in C_q[t]$ is $O(n^{1/2})$ in expectation.

Fix an arbitrary $p \in P_0$. For points $q \in P_0 \setminus P_1$ we have that p can only participate in clusters centered at q if $\mathsf{d}(p, q) < \mathsf{d}(p, P_1)$. Let $w_1, \ldots, w_s$ be the points of P_0 ordered by non-decreasing distance to p. If $w_j \in P_0 \setminus P_1$ and $p \in C_{w_j}[t]$, then it must be the case that $w_1, \ldots, w_{j-1} \notin P_1$. We sampled the points P_1 independently from P_0 with probability $n^{-1/2}$, so we have

$$\Pr\left[p \in C_{w_j}[t]\right] \leq \prod_{t<j} \Pr\left[w_t \notin P_1\right] = \left(1 - \frac{1}{\sqrt{n}}\right)^j.$$

The expected number of points in $P_0 \setminus P_1$ with a cluster containing p is then

$$\sum_{s \leq n} \mathbb{1}(w_s \in P_0 \setminus P_1) \cdot \Pr\left[p \in C_{w_s}[t]\right] \leq \sum_{s \leq n} \left(1 - \frac{1}{\sqrt{n}}\right)^s \leq \sqrt{n}.$$

We additionally note that $\mathbb{E}[|P_1|] = \sqrt{n}$. Therefore, $\mathbb{E}[\deg_{\mathcal{C}}(p)] \leq 2\sqrt{n}$. □

Cluster-induced distance approximation. We define the distance function, $\mathsf{d}_{\mathcal{C}} \colon P \times P \to \mathbb{R}_{\geq 0}$ based on the clustering. For any pair of points $x, y \in P$, let i be the smallest index of a cluster that contains both x and y. Then we set $\mathsf{d}_{\mathcal{C}}(x, y) = 2r_i$.

Lemma 2.2. $\mathsf{d}(x, y) \leq \mathsf{d}_{\mathcal{C}}(x, y) < (4 + \varepsilon)\mathsf{d}(x, y)$.

Proof. We say two points x and y are *separated* by P_1 if there are points $a, b \in P_1$ such that $\mathsf{d}(x, a) < \mathsf{d}(x, y)$ and $\mathsf{d}(y, b) < \mathsf{d}(x, y)$. Without loss of generality, let $\mathsf{d}(x, a) \leq \mathsf{d}(y, a)$. Then $y \in C_a[i]$ for i such that $\mathsf{d}(y, a) \leq r_i = (1 + \frac{\varepsilon}{4})^i < (1 + \frac{\varepsilon}{4})\mathsf{d}(y, a)$. So we have

$$\mathsf{d}_{\mathcal{C}}(x, y) = 2r_i = 2\left(1 + \frac{\varepsilon}{4}\right)^i < 2\left(1 + \frac{\varepsilon}{4}\right)\mathsf{d}(y, a) \leq 4\left(1 + \frac{\varepsilon}{4}\right)\mathsf{d}(x, y).$$

If x and y are not separated then either $x \in C_y$ or $y \in C_x$. Without loss of generality, assume $x \in C_y[i]$. Then we have $\mathsf{d}_{\mathcal{C}}(x, y) = 2r_i = 2\left(1 + \frac{\varepsilon}{4}\right)^i < 2\left(1 + \frac{\varepsilon}{4}\right)\mathsf{d}(x, y)$. □

2.1 Proximity Queries

Next, we show that the clustering constructed above can be used for answering some proximity queries, which will be crucial for our OT plan computation.

Directed spanner. Let $A, B \subseteq P$ be two disjoint subsets of P, let $\mathsf{d} \colon P \times P \to \mathbb{R}_{\geq 0}$ be a metric, and let $p \in [1, \infty)$. We construct a graph $G = (V, E)$ and a set of edge weights $w_p \colon E \to \mathbb{R}_{\geq 0}$ such that the shortest path from any $a \in A$ to any $b \in B$ in the graph G with respect to weights w_p is approximately $\mathsf{d}^p(a, b)$.

For each cluster $C \in \mathcal{C}$, we create two vertices a_C, b_C. Set $V = A \cup B \cup \{a_C, b_C \mid C \in \mathcal{C}\}$. For each cluster $C \in \mathcal{C}$, we add the following three sets of edges to E:

(i) Add the edge $a_C \to b_C$ and set $w_p(a_C \to b_C) = (2r_i)^p$ if the index of C is i.

(ii) For every $a \in A \cap C$, we add the edge $a \to a_C$ and set $w_p(a \to a_C) = 0$.

(iii) For every $b \in B \cap C$, we add the edge $b_C \to b$ and set $w_p(b_C \to b) = 0$.

Clearly $|V| = O(n\varepsilon^{-1} \log \Delta)$ since $|\mathcal{C}| = O(n\varepsilon^{-1} \log \Delta)$. Since the expected degree of each point is $O(\sqrt{n}\varepsilon^{-1} \log \Delta)$, the expected number of edges is $O(n^{3/2}\varepsilon^{-1} \log \Delta)$. Define $\mathsf{d}_{G,p} \colon A \times B \to \mathbb{R}_{\geq 0}$ as the shortest path distance in G with respect to edge weights w_p.

Lemma 2.3. *The weighted graph G with weights w_p satisfies $\mathsf{d}^p(a,b) \leq \mathsf{d}_{G,p}(a,b) \leq (4+\varepsilon)^p \cdot \mathsf{d}^p(a,b)$ for all $a, b \in A \times B$ and for any $p \in [1, \infty)$.*

Weighted nearest neighbor. Let P be a point set, and let $A \subseteq P$. Given a weight function $w \colon A \to \mathbb{R}_{\geq 0}$, define the *weighted distance* $\mathsf{d}_w \colon A \times P \to \mathbb{R}$ as $\mathsf{d}_w(a,p) = \mathsf{d}_{\mathcal{C}}(a,p) - w(a)$. Our goal is to maintain the weighted nearest neighbor in A for every $p \in P$, i.e. $\mathrm{NN}_w(p) = \arg\min_{a \in A} \mathsf{d}_w(a,p)$, as points (of P) are inserted into and deleted from A.

We build the above clustering $\mathcal{C}$ on the entire set P. For each point $p \in A$, we store the list of clusters to which it belongs. For each $C \in \mathcal{C}$ such that $p \in P$, we maintain the points of $A \cap C$ in a max-heap using their weights. Let a_C be the point of $A \cap C$ stored at the root of the heap. If the index of C is i, we set $\phi_C = 2r_i - w(a_C)$. Next, we store the set $\mathsf{X} = \{(a_C, C) \mid C \in \mathcal{C}\}$ in a min-heap H using ϕ_C as the key. The first element a_C of the pair stored at the root of H is the desired nearest neighbor $\mathrm{NN}_w(p)$. Insertion or deletion of a point is straightforward. Omitting the details, we state that the expected update time is $O(n^{1/2}\varepsilon^{-1} \log(n) \log \Delta)$.

Dynamic bichromatic closest pair. Let P be a point set, and let $A, B \subseteq P$ be two disjoint point sets. Given a weight function $w \colon A \cup B \to \mathbb{R}_{\geq 0}$, we define the *weighted distance* $\mathsf{d}_w \colon A \times B \to \mathbb{R}$ as

$$\mathsf{d}_w(a,b) = \mathsf{d}_{\mathcal{C}}(a,b) - w(a) + w(b).$$

Our goal is to maintain $\mathrm{BCP}_w(A,B) = \arg\min_{(a,b) \in A \times B} \mathsf{d}_w(a,b)$ as points are inserted into and deleted from A and B. We only insert the points of P to A or B. We describe a simple data structure to maintain $\mathrm{BCP}_w(A,B)$.

As for NN queries, we build the above clustering $\mathcal{C}$ on the entire set P. For each $C \in \mathcal{C}$, we maintain the points of $B \cap C$ in a min-heap using their weights as the key, and we maintain the points of $A \cap C$ in a max-heap using their weights. Let a_C (resp. b_C) be the point of $A \cap C$ (resp. $B \cap C$) stored at the root of the heap. If the index of C is i, we set

$$\phi_C = 2r_i - w(a_C) + w(b_C).$$

Next, we store the set $\mathsf{X} = \{(a_C, b_C) \mid C \in \mathcal{C}\}$ in a min-heap H using ϕ_C as the key. The pair stored at the root of H is the desired pair $\mathrm{BCP}_w(A,B)$.

The following observation is critical to the design of the BCP data structure.

Lemma 2.4. *Let (a^*, b^*) be the pair stored at the root of H. Then $\mathsf{d}_w(a^*, b^*) = \min_{a,b \in A \times B} \mathsf{d}_w(a,b)$.*

A similar claim also appears in Lahn et al. (2025) as Lemma 2.3. We include the proof in Appendix C, for completeness. We also note that the BCP data structure can be updated efficiently.

Lemma 2.5. *Let P be a set of n points in a metric space. Let $A, B \subseteq P$ be two weighted point sets. A weighted BCP data structure under the distance function d_w can be maintained in $O(\sqrt{n}\varepsilon^{-1} \log \Delta \log n)$ expected time per insertion and deletion.*

3 Algorithms for W_p

In this section, we use the collection of clusters and data structures constructed in Section 2 to design two efficient algorithms for the optimal transport problem.

3.1 Minimum-Cost Flow Based Algorithm

Let μ and ν be discrete distributions with support sets A and B; let $|A| + |B| = n$. We compute an approximate W_p-OT as follows. First assume $p \geq 1$ is a finite value. Let $G = (V, E)$ be the directed graph constructed on $A \cup B$ described in Section 2.1, and let w_p be the corresponding

weight function on E. We add a source vertex s and sink vertex t to the graph G. We also add an edge $s \to a$ for every $a \in A$ with weight $w_p(s \to a) = 0$ and an edge $b \to t$ for every $b \in B$ with weight $w_p(b \to t) = 0$. This addition gives the graph a single source and single sink to run minimum cost flow. Next, we assign capacities to each edge as follows: For each cluster $C \in \mathcal{C}$ and corresponding edge $a_C \to b_C$ in E, assign a capacity of $u(a_C \to b_C) = 1$. Additionally, for each $a, b \in C$ we assign the capacity $u(a \to a_C) = u(b_C \to b) = 1$. Finally, for each $a \in A$ and $b \in B$, we assign the source and sink edge capacities as $u(s \to a) = \mu(a)$ and $u(b \to t) = \nu(b)$.

We compute the capacitated min-cost max- flow f^* in this directed graph using the algorithm by Chen et al. (2022) in $(n^{3/2}\epsilon^{-1}\log^2\Delta)^{1+o(1)}\log U$ expected time. Using the minimum cost flow f^*, we compute a transport plan σ where $w_p(\sigma) \le \left(\sum_{e\in E} f^*(e)w_p(e)\right)^{1/p}$ as follows. Initially, $\sigma(a,b) = 0$ for all $a \in A, b \in B$. While the total flow from s to t is positive, find any path $\pi = s \to a \to a_C \to b_C \to b \to t$ where $f^*(e) > 0$ for every edge e on the path π and increment $\sigma(a,b)$ by $\lambda = \min\{f^*(e) \mid e \in \pi\}$. Additionally decrement $f^*(e)$ by λ for every edge $e \in \pi$. We repeat until f^* is zero everywhere. This concludes the construction of the transport plan σ. It follows naturally from Lemma 2.3 that σ is an approximate transport plan with respect to the W_p distance. Since G has $O(n^{3/2}\varepsilon^{-1}\log\Delta)$ edges, the overall expected runtime of constructing σ from f^* is $O(n^{3/2}\varepsilon^{-1}\log\Delta)$. This proves Theorem 1.1 for $p \in [1,\infty)$.

For $p = \infty$, we proceed as follows. We maintain the same source and sink vertices s, t as well as the same edge capacities as above. We then compute a sequence of maximum flows instead of a single minimum cost flow in G, and perform binary search on the radii of the clusters. By construction, there are at most $O(\varepsilon^{-1}\log\Delta)$ different values of r_i. For a fixed $1 \le i \le O(\varepsilon^{-1}\log\Delta)$, define the graph G_i to be the graph G with all edges of cost $w_1(a_C \to b_c) > 2r_i$ removed. Compute a maximum flow f_i in G_i from s to t in $(n^{3/2}\varepsilon^{-1}\log\Delta)^{1+o(1)}\log U$ expected time using the algorithm of Chen et al. (2022). If $\sum_{a\in A} f_i(s \to a) = 1$, then conclude that $W_\infty(\mu,\nu) \le 2r_i$ and decrease i. Otherwise, conclude that $W_\infty(\mu,\nu) > 2r_i$ and increase i.

Let i^* be the smallest value of i such that $\sum_{a\in A} f_i(s \to x) = 1$. Then we compute a transport plan σ from f_{i^*} as above in the case when $p < \infty$. Initially, $\sigma(a,b) = 0$ for all $a \in A, b \in B$. While the total flow from s to t is positive, find any path $\pi = s \to a \to a_C \to b_C \to b \to t$ where $f_{i^*}(e) > 0$ for every edge e on the path π and increment $\sigma(a,b)$ by $\lambda = \min\{f_{i^*}(e) \mid e \in \pi\}$. Additionally decrement $f_{i^*}(e)$ by λ for every edge $e \in \pi$. We repeat until f_{i^*} is zero everywhere. This concludes the construction of the transport plan σ. It follows naturally from Lemma 2.3 that σ is an approximate transport plan with respect to the W_∞ distance. Similar to the algorithm for finite p, we observe that the overall expected runtime of constructing σ from f_{i^*} is $O(n^{3/2}\varepsilon^{-1}\log\Delta)$. This proves Theorem 1.1 for $p = \infty$.

3.2 A Simpler Matching Algorithm

We present a significantly simpler, combinatorial algorithm that runs in $\tilde{O}(n^2)$ time and computes a minimum–cost matching under the W_p metric between two point sets A and B, each of size n. The algorithm selects an appropriate parameter δ and simulates a single scale of the Gabow–Tarjan cost–scaling framework for bipartite matching, with all steps executed efficiently via a bichromatic closest–pair data structure. Given the parameter δ, we begin by defining scaled costs as a scaled version of the p-th power of the proxy distance $\hat{c}(a,b) = \left\lceil \frac{1}{\delta}\, \mathrm{d}_{\mathcal{C}}^p(a,b) \right\rceil$. The algorithm proceeds with these integer costs $\hat{c}(a,b)$.

Matchings and augmenting paths A *matching* M is a collection of vertex–disjoint edges. A vertex not incident to any edge of M is said to be *free*. A matching is *perfect* if no vertex is free. Given a matching M, an *alternating path* is a path whose edges alternate between those in M and those outside M. An *augmenting path* is an alternating path whose two endpoints are free. Augmenting along such a path flips the membership of its edges in M, thereby increasing the size of the matching by one.

1-feasible matching. Each vertex $v \in A \cup B$ is assigned an integer dual variable $y(v)$. A matching M and dual weights $y(\cdot)$ are 1-*feasible* if

$$y(a) + y(b) \;\le\; \hat{c}(a,b) + 1 \qquad \text{for all } (a,b) \in A \times B, \tag{1}$$

$$y(a) + y(b) \;=\; \hat{c}(a,b) \qquad \text{for all } (a,b) \in M. \tag{2}$$

We define the *slack* of an edge (a, b) with respect to a matching M and dual weights $y(\cdot)$ as

$$s(a,b) \;=\; \begin{cases} 0, & \text{if } (a,b) \in M, \\ \hat{c}(a,b) - y(a) - y(b) + 1, & \text{if } (a,b) \notin M. \end{cases}$$

An edge is *admissible* if $s(a, b) = 0$, and the set of admissible edges forms the *admissible graph*

We initialize the matching $M = \emptyset$ and set all dual weights to zero, i.e., $y(v) = 0$ for every $v \in A \cup B$. Note that (M, y) is 1-feasible. Let $B_F = B$ be the free vertices of B with resepct to M. The algorithm maintains a 1-feasible pair (M, y) consisting of a matching M and dual weights $y(\cdot)$, and executes iterations. Each iteration has the *dual adjustment* step, which builds an augmenting path of admissible edges, and *augmentation* step, which computes a maximal set of vertex-disjoint augmenting paths and augments the matching along these paths to increase the size of the matching. Next, we describe the dual adjustment and the augmentation steps.

Dual adjustment via BCP-based Hungarian Search. The Hungarian search procedure runs a Dijkstra-style shortest path search using slacks as edge lengths. This search is implemented using bichromatic closest pair (BCP) queries, described in Section 2.1, with implicit dual updates. The search maintains a tree. Let $U \subseteq B$: the set of vertices of B already added to this search tree and let $V \subseteq A$ be the set of vertices of A not yet added to the search tree. Initially, U contains all free vertices in B, each with distance label $\ell_b = 0$, and $V = A$. Define effective weights

$$w(b) = y(b) - \ell_b \quad \text{for } b \in U, \qquad w(a) = y(a) \quad \text{for } a \in V.$$

At each iteration, select the edge

$$(a,b) = \arg\min_{a' \in V,\, b' \in U} \big\{ s(a',b') + \ell_{b'} \big\} = \arg\min_{a' \in V,\, b' \in U} \big\{ \mathsf{d}^p(a',b') - w(a') - w(b') \big\}.$$

This minimization is exactly a BCP query, which we answer using the data structure described in Section 2.1.

Remove a from V, set $\ell_a = \ell_b + s(a, b)$, and add it to the search tree. If a is free, a shortest augmenting path has been found. Otherwise, let b' be its matched partner; set $\ell_{b'} = \ell_a$, update $w(b') = y(b') - \ell_{b'}$, and insert b' into U.

This procedure simulates Hungarian search procedure without explicitly updating all duals: offsets are stored in the effective weights and automatically incorporated by BCP queries. When the search terminates at a free vertex $a^\star \in A$, let $\Delta = \ell_{a^\star}$. The dual weights are then updated as

$$y(a) \leftarrow y(a) + \ell(a) \quad \text{for all } a \in S, \qquad y(b) \leftarrow y(b) - \ell(b) \quad \text{for all } b \in T,$$

where $S \subseteq A$ and $T \subseteq B$ are the sets of vertices reached. In practice, these updates are never carried out explicitly. Instead, the effective weights $w(\cdot)$ store the necessary offsets, and BCP queries automatically incorporate them.

Augmentation step using weighted nearest neighbor. Once the dual adjustment phase reaches a free vertex in A, the search guarantees that there is at least one augmenting path in the admissible graph. The algorithm then finds a maximal set of vertex-disjoint augmenting paths by conducting a sequence of partial depth-first search (DFS): Start a DFS from each free point of B in a sequential manner. Let X be the set of points of A that have not yet been visited by any DFS. Initially $X = A$. The DFS alternates between unmatched admissible edges from B to A and matched edges from A back to B. Whenever the DFS is at a vertex $u \in B$, the next admissible edge can be retrieved by a *weighted nearest neighbor query*:

$$a = \arg\min_{a' \in X} \big\{ \mathsf{d}^p(u,a') - y(u) - y(a') \big\}.$$

We use the data structure of Section 2.1 to answer the NN query. We then check whether (u, a) is admissible. Thus, a single nearest neighbor query suffices to reveal the next admissible edge. a is removed from X. If a is matched to b'', then we extend the alternating path by adding (a, b'') to it and the DFS continues from b''. If the DFS reaches a free vertex a in A, an augmenting path is

identified. The algorithm then starts a DFS from a different free point of B. This step terminates when a DFS has been executed from each of the free points of B.

Finally, all discovered augmenting paths are flipped simultaneously to update the matching. For each augmenting path and for every vertex $b \in B$ lying on it, reduce the dual weight by one: $y(b) \leftarrow y(b) - 1$. This correction guarantees that all newly matched edges remain tight under the 1-feasible condition.

The algorithm alternates between dual adjustment and augmentation until all vertices are matched. The final pair (M, y) is a perfect matching and remains 1-feasible throughout the execution.

Efficiency. We select the parameter δ so that the edge costs become integers and the optimal cost is scaled to $\Theta(n/\varepsilon)$. Scaling by δ preserves the true optimum, while rounding introduces at most an additive error of n. Moreover, the 1-feasible matching produced is itself within $+n$ of the rounded optimum (Gabow & Tarjan (1989)). Hence, the total deviation is at most $2n$, and whenever the rounded optimum is at least $2n/\varepsilon$, the resulting solution is guaranteed to be within a $(1+\varepsilon)$ factor of the true optimum.

Gabow and Tarjan showed that if the costs are integers and the value of the optimal solution is $O(n/\varepsilon)$, then a single scale of their algorithm converges in $O(\sqrt{n/\varepsilon})$ phases. In particular, when the optimal solution has value $2n/\varepsilon$, the algorithm terminates in $O(\sqrt{n/\varepsilon})$ phases, and combined with the error bounds above, produces a $(1+\varepsilon)$-approximation.

Each of the two steps—dual adjustment and augmentation—of a phase can be implemented using efficient geometric data structures. The dual adjustment step builds a weighted bichromatic closest pair (BCP) data structure on U and V and performs dynamic updates as U and V change. Since points are only added to U and deleted from V, the total number of updates cannot exceed $2n$, and therefore the total time spent in this step is bounded by $O(n)$ queries to the BCP data structure. The augmentation step builds a weighted nearest neighbor (WNN) data structure on all points of A, and as the points of A are visited by a depth-first search they are deleted from the structure. Thus, the augmentation step can also be implemented using $O(n)$ queries to the WNN data structure. Each of these queries is supported in $\tilde{O}(\sqrt{n})$ time (see Section 2.1), so the overall execution time per phase is $\tilde{O}(n^2)$.

Next, we describe how to choose a δ so that the optimal cost with the rounded costs is scaled to $\Theta(n/\varepsilon)$. Since the optimal cost can take values between 1 and $n\Delta$, we consider a sequence of scales

$$\delta_i \;=\; (1+\varepsilon)^i \cdot \frac{\varepsilon}{n}, \qquad i = 1, 2, \ldots, \left\lceil \log_{(1+\varepsilon)}(n\Delta) \right\rceil .$$

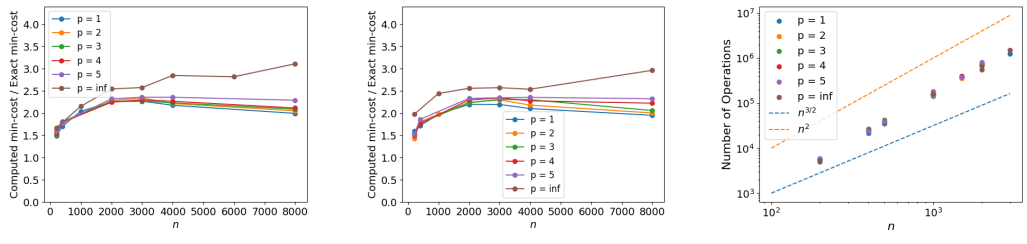
For each δ_i, we execute a single scale of the algorithm, each of which runs in $\tilde{O}(n^2)$ time. Among all executions that terminate within this bound, we return the matching of smallest cost. This proves Theorem 1.2.

4 Empirical Evaluation

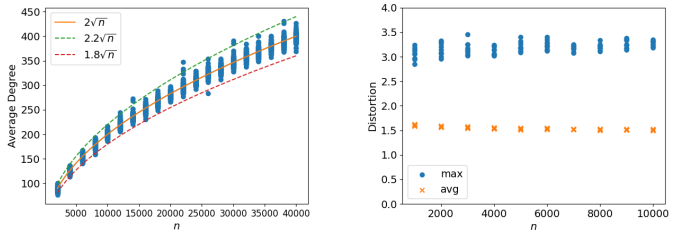
This section contains an empirical evaluation of the clustering method from Section 2 as well as the approximation factor obtained from the primal–dual algorithm of Section 3.2. Computations were performed on a computer with an 8-core Apple M1 CPU with 16GB RAM. Samples are drawn from uniform and truncated normal distributions on the unit cube in up to 10 dimensions.

Cluster distance accuracy. We first evaluate the quality of the clustering by comparing the induced cluster distances to the ground metric. For each value of n, we measure both the maximum and the average distortion across all pairs. Figure (1e) confirms that the worst-case distortion never exceeds the theoretical $(4+\varepsilon)$-approximation guarantee of Lemma 2.2. More importantly, the average distortion is often substantially smaller, typically close to a factor of 2. This suggests that in practice the effective approximation factor is significantly tighter than the worst-case analysis.

Clustering efficiency. Next, we examine the degree, ie. the number of clusters each point participates in. Figure (1d) shows that the observed averages closely track the theoretical bound of



(a) Approximation Ratio (Normal) (b) Approximation Ratio (Uniform) (c) Algorithm Efficiency



(d) Average Degree (e) Max and Average Distortion

Figure 1: Empirical evaluation of the 2-layer clustering and W_p-matching algorithm.

Lemma 2.1 for dimension $d \leq 10$, and both distributions. This indicates that the two-layer clustering is both space-efficient and stable across different settings.

Algorithm accuracy. To evaluate the accuracy of the primal–dual matching algorithm, we compare the computed matching cost to that obtained using the exact distance matrix. Figures (1b) and (1a) report the approximation ratio across values of $p \in \{1, 2, 3, 4, 5, \infty\}$. The ratios consistently remain well within the theoretical $(4 + \varepsilon)$ factor, with typical values close to 1.5–2, again suggesting that the empirical performance is considerably better than the worst-case analysis. This trend is stable across both uniform and normal distributions.

Algorithm efficiency. We measure efficiency by the number of bichromatic closest pair (BCP) queries, which dominate the running time. As shown in Figure (1c), the query counts scale as predicted and remain nearly identical across all choices of p. Combined with the $\tilde{O}(n^2)$ per-query complexity, this provides strong empirical evidence that the algorithm runs in quadratic time and scales smoothly with problem size.

Summary. Overall, the experiments demonstrate that the proposed method is both theoretically grounded and empirically robust. While the theoretical analysis guarantees only a $(4 + \varepsilon)$ approximation, the observed approximation ratios are consistently much smaller, indicating that the algorithm is practically near-optimal. The clustering step is efficient in both time and space, and its distortions are far below the worst-case bound. Taken together, these results suggest that our approach is a practical alternative to additive methods such as Sinkhorn, particularly in regimes where existing techniques either fail to apply (e.g., $p = \infty$) or require higher-than-quadratic time.

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A Conditional Hardness of the W_∞-Matching Problem

In this section, we prove a conditional hardness result for approximating the W_∞ distance.

Suppose we are given the bipartite graph $G = (V, E)$, where $V = V_1 \cup V_2$ with $|V_1|, |V_2| = n$, $V_1 \cap V_2 = \varnothing$, and $E \subseteq V_1 \times V_2$. We will reduce this problem of determining if G contains a perfect matching to computing an approximate W_∞-distance.

Construct the following metric ρ on V. For each distinct $v_1, v_2 \in V$, if $(v_1, v_2) \in E$ or $(v_2, v_1) \in E$ then define $\rho(v_1, v_2) = 1$. Otherwise, define $\rho(v_1, v_2) = 2$. For completeness, one can define $\rho(v, v) = 0$ for all $v \in V$. The finite metric space (V, ρ) can be constructed in $O(n^2)$ time, given G. It is easy to see that (V, ρ) is a metric space: (i) by definition, $\rho(v, v) = 0$ and $\rho(v, v') > 0$ for all $v' \neq v$; (ii) if $\rho(v_1, v_2) = 1$ for some $v_1 \neq v_2$, then either (v_1, v_2) or (v_2, v_1) is in E, in which case $\rho(v_2, v_1) = 1$; (iii) $\rho(v_1, v_2) \leq 2$ and $\rho(v_1, v_3) + \rho(v_3, v_2) \geq 1 + 1 = 2$ for all $v_1, v_2 \in V$ and $v_3 \neq v_1, v_2$, implying triangle inequality in combination with observation (i) to prove the degenerate case $v_3 = v_1$ or $v_3 = v_2$.

Additionally define the distributions $\mu_1 \colon V_1 \to [0, 1]$ and $\mu_2 \colon V_2 \to [0, 1]$ by $\mu_1(v_1) = \mu_2(v_2) = \frac{1}{n}$ for all $v_1 \in V_1$ and all $v_2 \in V_2$. Assume we are given an approximation algorithm $\mathcal{A}$ for the W_∞ distance, and let $\sigma_{\mathcal{A}}$ denote the transport plan from μ_1 to μ_2 computed by algorithm $\mathcal{A}$.

Since $\rho(v_1, v_2) \in \{1, 2\}$ for all $v_1, v_2 \in V_1 \times V_2$, it must also be the case that

$$w_\infty(\sigma) := \max_{x,y:\sigma(x,y)>0} \rho(x, y) \in \{1, 2\}$$

for any transport plan σ and therefore $W_\infty(\mu_1, \mu_2) \in \{1, 2\}$. We use this observation to prove the following crucial relationship between $W_\infty(\mu_1, \mu_2)$ and the maximum cardinality matching in G.

Lemma A.1. *A perfect matching exists in G if and only if $W_\infty(\mu_1, \mu_2) = 1$.*

Proof. If $W_\infty(\mu_1, \mu_2) = \min_\sigma w_p(\sigma) = 2$, then it is impossible to construct a transport plan σ where every edge has distance 1 and therefore at least one edge (u, v) where $\sigma(u, v) > 0$ satisfies $\rho(u, v) = 2$. By definition, the metric ρ is equal to 2 if and only if the pair is not an edge in G. Therefore, any matching must have size at most $n - 1$.

If $W_\infty(\mu_1, \mu_2) = 1$, then it is possible to construct a transport plan σ where every edge has a distance 1, and by standard network flow theory this transport plan σ is a convex combination of matchings in $(V_1 \cup V_2, V_1 \times V_2)$. By definition of ρ, we note $\rho(v_1, v_2) = 1$ if and only if $(v_1, v_2) \in E$. We conclude that any matching M where $\sigma(v_1, v_2) > 0$ for every $(v_1, v_2) \in M$ is also a perfect matching in G. □

Therefore, it suffices to determine if either $W_\infty(\mu_1, \mu_2) = 1$ or $W_\infty(\mu_1, \mu_2) = 2$, and extract any matching from the resulting transport plan in $O(n^2)$ time in the former case.

We briefly describe this (standard) procedure of how to construct a perfect matching M in $V_1 \times V_2$ using $\sigma_{\mathcal{A}}$ in $O(n^2)$ time such that for every $(v_1, v_2) \in M$, we have $\sigma_{\mathcal{A}}(v_1, v_2) > 0$. To do this, we iteratively choose an arbitrary edge $(u, v) \in V_1 \times V_2$ such that $\sigma(u, v) > 0$, add (u, v) to M, remove u from V_1 and v from V_2. Repeat until M is a perfect matching. Since σ is known to be a convex combination of perfect matchings, we conclude that the algorithm always constructs a perfect matching.

What follows is an observation that even when given relatively weak approximation algorithms for the ∞-Wasserstein distance, it is possible to distinguish between these two cases. We first prove this for the case that $\mathcal{A}$ is a relative approximation algorithm.

Lemma A.2. *Suppose for some $\varepsilon > 0$ there exists an algorithm running in $O(n^2)$ time which computes a $(2 - \varepsilon)$-approximate transport plan under the W_∞ metric between two discrete probability distributions with supports of size at most n. Then, for any bipartite graph G, one can compute a perfect matching in G or conclude that none exists in $O(n^2)$ time.*

Proof. We assume the existence of an algorithm $\mathcal{A}$ which, when given an input finite metric space (V, ρ) and distributions μ, ν, computes a transport plan σ such that $w_\infty(\sigma) \leq (2 - \varepsilon) \cdot W_\infty(\mu, \nu)$ in $O(n^2)$ time.

First suppose that $w_\infty(\sigma_\mathcal{A}) = 2$. By the approximation guarantee of the algorithm $\mathcal{A}$, we conclude that $W_\infty(\mu_1, \mu_2) \geq \frac{2}{2-\varepsilon} > 1$ and therefore $W_\infty(\mu_1, \mu_2) = 2$. Now suppose that $w_\infty(\sigma_\mathcal{A}) = 1$. Then we have immediately found a minimizing transport plan and conclude that $W_\infty(\mu_1, \mu_2) = 1$. We conclude that if $\mathcal{A}$ is a $(2-\varepsilon)$-approximation algorithm then necessarily $w_\infty(\sigma_\mathcal{A}) = W_\infty(\mu_1, \mu_2)$ for the constructed input metric space (V, ρ) and distributions μ_1, μ_2. The result follows after a simple application of Lemma A.1. □

We emphasize that the metric space used to prove Lemma A.2 has spread $\Delta = 2$. Therefore, Lemma A.2 is useful even if the algorithm $\mathcal{A}$ has $O(n^2 f(\Delta))$ runtime for any function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ independent of n. Next, we prove the analogous claim for the case where $\mathcal{A}$ is an additive approximation algorithm.

Lemma A.3. *Suppose for some $\varepsilon > 0$ there exists an algorithm running in $O(n^2)$ time which computes a $(\frac{\Delta}{2} - \varepsilon)$-additive approximate transport plan under the W_∞ metric between two discrete probability distributions with supports of size at most n in a metric space with diameter Δ. Then, for any bipartite graph G, one can compute a perfect matching in G or conclude that none exists in $O(n^2)$ time.*

Proof. We assume the existence of an algorithm $\mathcal{A}$ which, when given an input finite metric space (V, ρ) and distributions μ, ν, computes a transport plan σ such that $w_\infty(\sigma) \leq W_\infty(\mu, \nu) + \left(\frac{\Delta}{2} - \varepsilon\right)$ in $O(n^2)$ time.

First suppose that $w_\infty(\sigma_\mathcal{A}) = 2$. By the approximation guarantee of the algorithm $\mathcal{A}$, we conclude that $W_\infty(\mu_1, \mu_2) \geq w_\infty(\sigma_\mathcal{A}) - \left(\frac{\Delta}{2} - \varepsilon\right) = 2 - (1 - \varepsilon) > 1$ and therefore $W_\infty(\mu_1, \mu_2) = 2$. Now suppose that $w_\infty(\sigma_\mathcal{A}) = 1$. Then we have immediately found a minimizing transport plan and conclude that $W_\infty(\mu_1, \mu_2) = 1$. We conclude that if $\mathcal{A}$ is a $\left(\frac{\Delta}{2} - \varepsilon\right)$-approximation algorithm then necessarily $w_\infty(\sigma_\mathcal{A}) = W_\infty(\mu_1, \mu_2)$ for the constructed input metric space (V, ρ) and distributions μ_1, μ_2. The result follows after a simple application of Lemma A.1. □

Then a simple combination of Lemmas A.2 and A.3 implies Theorem 1.3.

B k-Level Clustering

In Section 2, we used a two-layered clustering to approximate d up to multiplicative factor 4 using $O(n^{3/2})$ space. This approach can be generalized to a k-level clustering. Extending to k-level clustering has the benefit of a reduced number of clusters in which any point in P_0 is expected to participate, at the expense of an increase in the stretch factor of the data structure.

In this section, we describe a clustering based distance function that can be constructed in $O(n^2)$ time, approximates d up to a factor of $(4+\varepsilon)(k-1)$ and that can be represented using $O(kn^{1+1/k}\varepsilon^{-1}\log\Delta)$ space. Similar to Section 2.1, we can use this clustering to construct a spanner and maintain (weighted) bichromatic closest pairs. Then once we have those data structures, the algorithms in Section 3 work in the same manner. Applying the combinatorial algorithm in Section 3.2 to the k-level clustering for $k = 3$ gives the $(8+\varepsilon)$-approximation result in Theorem 1.2.

k-layered clustering. We now construct a k-level clustering of points of P. Let $P_0 = P$. For $i = 1, \ldots, k-1$, we next choose a subset $P_i \subseteq P_{i-1}$ by sampling each point in P_{i-1} independently with probability $\theta = n^{-1/k}$. The expected size of P_i is $\mathbb{E}[|P_i|] = n\theta^i = n^{1-i/k}$ for each $i \leq k-1$.

Set $t = \lceil \log_{(1+\frac{\varepsilon}{4})} \Delta \rceil$. Let $r_0 = 0$ and $r_i = (1+\frac{\varepsilon}{4})^i$ for $1 \leq i \leq t$. For each $0 \leq l < k-1$, $q \in P_l \setminus P_{l+1}$ and $1 \leq i \leq t$, define the cluster $C_q[i]$ as

$$C_q[i] = \{x \in V(q, P_{l+1}) \mid \mathsf{d}(x, q) \leq r_i\}.$$

Finally for each $q \in P_{k-1}$ and $1 \leq i \leq t$, define the cluster $C_q[i]$ as

$$C_q[i] = \{x \in P_0 \mid \mathsf{d}(x, q) \leq r_i\}.$$

Let $\mathcal{C} = \{C_q[i] \mid q \in P, i \leq t\}$ be the collection of all clusters. Again define the *degree* of a point $p \in P$ as $\deg_\mathcal{C}(p) := |\{C \in \mathcal{C} \mid p \in C\}|$. In an analogous manner to Lemma 2.1, we prove that the expected degree of each point in P is small.

Lemma B.1. $\mathbb{E}[\deg_{\mathcal{C}}(p)] = O(kn^{1/k}\varepsilon^{-1}\log\Delta)$ *for all* $p \in P$.

Proof. Let $x \in P_0$ and fix $i < k-1$. Let $w_1, \ldots, w_l$ be the elements of P_i in order of non-decreasing distances to x. If $x \in C_{w_j}$ then $\mathsf{d}(x, w_j) < \mathsf{d}(x, P_{i+1})$. So it must be the case that $w_1, \ldots, w_{j-1} \notin P_{i+1}$. Note that $\theta = \Pr[x \in P_{i+1} \mid x \in P_i]$. Then,

$$\Pr[x \in C_{w_j}] \leq \prod_{t<j} \Pr[w_t \notin P_{i+1} \mid w_t \in P_i] = (1-\theta)^{(j-1)}.$$

The expected number of clusters that contain x at level $i < k-1$ is

$$\sum_{t \leq l} \Pr[x \in C_{w_t}] \leq \sum_{t \leq l} (1-\theta)^t \leq \theta^{-1} = n^{1/k}.$$

For level $i = k-1$, the point x is contained in every cluster C_w for $w \in P_{k-1}$. The expected number of clusters at level $i = k-1$ is

$$E[|P_{k-1}|] = n\Pr[x \in P_{k-1}] = n\theta^{k-1} = n^{1/k}$$

Therefore the total expected number of clusters containing x is $kn^{1/k}$. □

Cluster-induced distance approximation. As in Section 2, for any cluster $C = C_q[i] \in \mathcal{C}$ and any pair of points $x, y \in C_q[i]$, define the *cluster-induced distance* between x and y as

$$\mathsf{d}_C(x, y) = 2r_i.$$

Then define $\mathcal{C}(x, y) = \{C \in \mathcal{C} \mid x, y \in C\}$ and $\mathsf{d}_{\mathcal{C}}(x, y) = \min_{C \in \mathcal{C}(x,y)} \mathsf{d}_C(x, y)$. We prove that this minimum cluster-induced distance approximates $\mathsf{d}(x, y)$ within a factor or $(4+\varepsilon)(k-1)$.

Lemma B.2. $\mathsf{d}(x, y) \leq \mathsf{d}_{\mathcal{C}}(x, y) \leq (4+\varepsilon)(k-1) \cdot \mathsf{d}(x, y)$

Proof. First we show that if x and y are separated by P_l, i.e. if there exist $\alpha, \beta \in P_l$ such that $\mathsf{d}(x, \alpha) < \mathsf{d}(x, y)$ and $\mathsf{d}(y, \beta) < \mathsf{d}(x, y)$, then $\mathsf{d}(y, P_l) \geq \mathsf{d}(y, P_{l+1}) - 2\mathsf{d}(x, y)$. Let α and β be the closest points to x and y in P_{l+1}, respectively. Let a and b be the closest points to x and y in P_l respectively. Then

$$\begin{aligned}
\mathsf{d}(x, a) &\geq \mathsf{d}(y, a) - \mathsf{d}(x, y) && \text{[triangle inequality]} \\
&\geq \mathsf{d}(y, \beta) - \mathsf{d}(x, y) && \text{[definition of } \beta\text{]} \\
&\geq \mathsf{d}(x, \beta) - 2\mathsf{d}(x, y) && \text{[triangle inequality]} \\
&\geq \mathsf{d}(x, \alpha) - 2\mathsf{d}(x, y) && \text{[definition of } \alpha\text{]}.
\end{aligned}$$

Let x be inserted in round l. Then $y \notin C_x$ if and only if $\mathsf{d}(y, P_{l+1}) \leq \mathsf{d}(x, y)$. Let C_z be the cluster containing both x and y that minimizes $\mathsf{d}(y, z)$. Assume that C_z was inserted in round j. Note that $j < l$ and $l - j \leq k-2$. Then by the claim above,

$$\begin{aligned}
\mathsf{d}(y, z) &\leq \mathsf{d}(y, P_{l+1}) + 2(l-j)\mathsf{d}(x, y) \\
&\leq (2(k-2)+1)\mathsf{d}(x, y).
\end{aligned}$$

Then by the triangle inequality it follows that $\mathsf{d}(x, z) \leq 2(k-1)\mathsf{d}(x, y)$. Note that $\mathsf{d}(x, z) \leq r_i < (1+\frac{\varepsilon}{4})\mathsf{d}(x, z)$, where C_z has minimum index i. Therefore $\mathsf{d}_{\mathcal{C}}(x, y) = 2r_i \leq 2(1+\frac{\varepsilon}{4})\mathsf{d}(x, z) \leq 4(k-1)(1+\frac{\varepsilon}{4})\mathsf{d}(x, y)$. □

C Missing Details from the Main Text

We omitted many of the proofs of technical lemmas from the main text for the sake of space. In this section, we conclude with any missing details.

C.1 Details from Section 2.1

We first prove that the shortest path distance $\mathsf{d}_{G,p}$ in the directed graph G with weights w_p approximates $\mathsf{d}(\cdot,\cdot)^p$.

Proof of Lemma 2.3. Let $p \in [1,\infty)$ be an arbitrarily chosen value, and let $a \in A, b \in B$ be arbitrary points. We note by construction of G, for any path π from a to b in the graph G, there exists a unique edge $e = (a_C \to b_C) \in E$ for some cluster $C \in \mathcal{C}$ such that $w(e) > 0$.

To prove the upper bound, we show that there exists a path where the edge $e = (a_C \to b_C)$ satisfies $w_p(e) \le (4+\varepsilon)^p \cdot \mathsf{d}^p(a,b)$. By Lemma 2.2, observe that there exists a cluster $C \in \mathcal{C}$ with index i where $a, b \in C$ and $2r_i \le (4+\varepsilon)\cdot \mathsf{d}(a,b)$. Since $a \in C$, we observe that the edge $a \to a_C$ exists in G. Similarly, since $b \in C$, the edge $b_C \to b$ exists in G. Therefore, the path $\pi = a \to a_C \to b_C \to b$ is a path from a to b in G. The weight of the edge $a_C \to b_C$ is $w_p(a_C \to b_C) = (2r_i)^p \le ((4+\varepsilon)\cdot \mathsf{d}(a,b))^p = (4+\varepsilon)^p \cdot d^p(a,b)$.

To prove the lower bound, we show that for any such path π from a to b in G, $w_p(e) \ge d^p(a,b)$. Suppose $\pi = a \to a_C \to b_C \to b$ is a path from a to b in G. By construction, observe that if the path $\pi = a \to a_C \to b_C \to b$ is a path in G for some cluster $C \in \mathcal{C}$ with index i, then both a and b are contained in C. Since a and b are contained in C, there exists a point $q \in C$ (in particular, choose the center of the cluster $C_q[i]$) such that $\mathsf{d}(a,q) \le r_i$ and $\mathsf{d}(b,q) \le r_i$. Conclude that $\mathsf{d}(a,b) \le \mathsf{d}(a,q) + \mathsf{d}(q,b) \le 2r_i$ by triangle inequality and therefore $\mathsf{d}^p(a,b) \le (2r_i)^p = w_p(a_C \to b_C)$. □

We next prove that the global heap H constructed for the BCP data structure will contain the weighted bichromatic closest pair at its root.

Proof of Lemma 2.4. Let (a^*, b^*) be the pair stored at the root of H, with key $\phi_{C^*} = \mathsf{d}_w(a^*, b^*)$. Then $\phi_{C^*} = \min_{C\in\mathcal{C}} \phi_C$. Then we have the following,

$$\begin{aligned}\min_{C\in\mathcal{C}} \phi_C &= \min_{C\in\mathcal{C}}\{2r_i - \max_{(a,b)\in C}\{w(a) - w(b)\}\} \\ &= \min_{(a,b)\in A\times B}\{\mathsf{d}_C(a,b) - w(a) + w(b)\} \\ &= \min_{(a,b)\in A\times B} \mathsf{d}_w(a,b).\end{aligned}$$

This completes the proof. □

Finally, we prove that updates to the dynamic BCP data structure can be done in $O(\sqrt{n}\varepsilon^{-1}\log n \log \Delta)$ time.

Proof of Lemma 2.5. Without loss of generality assume $p \in A$. There are two steps to insert (or delete) a point p into (from) the BCP$_w$ data structure. For each cluster $C \in \mathcal{C}$ that contains p,

1. First, we add (delete) p to (from) the max-heap of C, and update the edge (a_C, b_C) for the cluster.
2. Next, for each updated cluster, update the key $\phi_C = 2r_i - w(a_C) + w(b_C)$ in the min-heap H.

By Lemma 2.1, the expected degree of p is $O(n^{1/2}\varepsilon^{-1}\log\Delta)$. Both steps take $O(\log n)$ time. So each update takes an expected $O(n^{1/2}\varepsilon^{-1}\log\Delta\log n)$ time. □

We briefly note that the details of correctness and time complexity for the weighted nearest neighbor data structure in Section 2.1 were also omitted. The analysis of the described data structure is nearly identical to that of the dynamic BCP data structure: nearest neighbors are stored at the root of the heap, where the weight is most extreme, and insertion or deletion will modify every cluster containing the point.

C.2 Details from Section 3.1

We omitted a formal cost analysis for the transport plan computed by the minimum-cost flow based algorithm from the main text since intuitively the fact that shortest paths in G approximate d^p should imply a minimum cost flow will also approximate W_p. In this section, we give a formal cost analysis of the algorithm in Section 3.1 for completeness. We first prove the case when p is finite.

Proof of Theorem 1.1 for $p < \infty$. By standard analysis of network flow algorithms, since the capacity of each edge $s \to x$ is $\mu(x)$ and the capacity of each edge $y \to t$ is $\nu(y)$, we conclude that the resulting σ is a transport plan. Moreover, given f^*, the transport plan σ can be constructed in $O(|E|)$ time since every $s-t$ path in G has constant length and at every iteration the flow along at least one edge is decremented to zero. We conclude with a cost analysis of σ.

First, we prove the lower bound. By construction of the graph G, we note that any path from s to t must be of the form $s \to a \rightsquigarrow b \to t$ where $a \rightsquigarrow b$ is a path in G from $a \in A$ to $b \in B$. By Lemma 2.3, conclude that $\sum_{e \in a \rightsquigarrow b} w_p(e) \geq \mathsf{d}_{G,p}(a,b) \geq \mathsf{d}^p(a,b)$ for any $a \in A, b \in B$ and any path $a \rightsquigarrow b$ in G. Moreover, every edge of the form $s \to a$ and $b \to t$ has cost zero. This implies

$$W_p(\mu,\nu) \leq w_p(\sigma) = \left(\sum_{a,b \in A \times B} \sigma(a,b) \cdot \mathsf{d}^p(a,b) \right)^{1/p} \leq \left(\sum_{e \in E} f^*(e) \cdot w_p(e) \right)^{1/p}.$$

We now prove the upper bound. Suppose we are given the optimal transport plan σ^*. We construct a flow f with cost at most $(4+\epsilon) \cdot w_p(\sigma^*)$. For each $x, y \in A \times B$, by Lemma 2.3 there exists a path $x \rightsquigarrow y$ in G such that $\sum_{e \in x \rightsquigarrow y} w_p(e) \leq (4+\epsilon) \cdot d(x,y)$. Let $\pi(x,y)$ denote this specific path for each pair x, y.

Initially, set f to be zero everywhere. Then for each pair x, y where $\sigma^*(x,y) > 0$, we increment flow f on every edge along $s \to x$, $y \to t$ and every edge of $\pi(x,y)$ by $\sigma^*(x,y)$. The resulting flow is a max flow since the capacity on every edge $s \to x$ leaving s is $\mu(x)$ and the capacity on every edge $y \to t$ entering t is $\nu(y)$. Moreover by construction of f and the fact that every edge $s \to x$ and $y \to t$ has cost zero, the cost of the flow f is

$$\begin{aligned}
\left(\sum_{e \in E} f(e) \cdot w_p(e) \right)^{1/p} &= \left(\sum_{x,y : \sigma^*(x,y) > 0} \sigma^*(x,y) \cdot \mathsf{d}_{G,p}(x,y) \right)^{1/p} \\
&\leq \left(\sum_{x,y : \sigma^*(x,y) > 0} \sigma^*(x,y) \cdot ((4+\epsilon)^p \cdot \mathsf{d}^p(x,y)) \right)^{1/p} \\
&\leq (4+\epsilon) \cdot w_p(\sigma^*) = (4+\epsilon) \cdot W_p(\mu,\nu).
\end{aligned}$$

We note that the minimum cost flow must have a cost no more than f. □

Then the cost analysis for when $p = \infty$ follows in a similar manner.

Proof of Theorem 1.1 for $p = \infty$. By standard analysis of network flow algorithms, since the capacity of each edge $s \to x$ is $\mu(x)$ and the capacity of each edge $y \to t$ is $\nu(y)$, we conclude that the resulting σ is a transport plan. Moreover, given f_{i^*}, the transport plan σ can be constructed in $O(|E|)$ time since every $s-t$ path in G has constant length and at every iteration the flow along at least one edge is decremented to zero. We conclude with a cost analysis of σ.

First, we prove the lower bound. By construction of the graph G, we note that any path from s to t must be of the form $s \to a \rightsquigarrow b \to t$ where $a \rightsquigarrow b$ is a path in G from $a \in A$ to $b \in B$. By Lemma 2.3, conclude that $\sum_{e \in a \rightsquigarrow b} w_1(e) \geq \mathsf{d}_{G,1}(a,b) \geq \mathsf{d}(a,b)$ for any $a \in A, b \in B$ and any path $a \rightsquigarrow b$ in G. Moreover, every edge of the form $a \to a_C$, $b_C \to b$, $s \to a$ and $b \to t$ has cost zero. This implies

$$W_\infty(\mu,\nu) \leq w_\infty(\sigma) = \max_{a,b \in A \times B \,:\, \sigma(a,b) > 0} \mathsf{d}(a,b) \leq \max_{e \in E \,:\, f_{i^*}(e) > 0} w_1(e).$$

We now prove the upper bound. Suppose we are given the optimal transport plan σ^*. We construct a flow f with cost at most $(4+\epsilon)\cdot w_\infty(\sigma^*)$. For each $x, y \in A \times B$, by Lemma 2.3 there exists a path $x \rightsquigarrow y$ in G such that $\sum_{e \in x \rightsquigarrow y} w_1(e) \le (4+\epsilon)\cdot \mathsf{d}(x,y)$. Let $\pi(x,y)$ denote this specific path for each pair x, y.

Initially, set f to be zero everywhere. Then for each pair x, y where $\sigma^*(x,y) > 0$, we increment flow f on every edge along $s \to x$, $y \to t$ and every edge of $\pi(x,y)$ by $\sigma^*(x,y)$. The resulting flow is a max flow since the capacity on every edge $s \to x$ leaving s is $\mu(x)$ and the capacity on every edge $y \to t$ entering t is $\nu(y)$. Moreover by construction of f and the fact that every edge $s \to x$, $y \to t$, $x \to a_C$ and $b_C \to y$ has cost zero, the cost of the flow f is

$$\begin{aligned}\max_{e \in E \,:\, f(e) > 0} w_1(e) &= \max_{x,y \in A \times B \,:\, \sigma^*(x,y) > 0} \mathsf{d}_{G,1}(x,y) \\ &\le \max_{x,y \in A \times B \,:\, \sigma^*(x,y) > 0} (4+\varepsilon)\cdot \mathsf{d}(x,y) \\ &\le (4+\epsilon)\cdot w_\infty(\sigma^*) = (4+\epsilon)\cdot W_\infty(\mu,\nu).\end{aligned}$$

We note that the cost of f is at most $(1+\varepsilon)$ times that of f_{i^*} since i^* is the smallest index where f_{i^*} is a max flow, i.e. has total flow 1, by definition. Rescaling ε by some constant gives the desired result. □