On sparse connectivity, adversarial robustness, and a novel model of the artificial neuron: supplementary materials

Anonymous ECCV submission

Paper ID 0006

1 Appendix A: proofs of theorems 1 and 2

Theorem 1: L_∞-nonexpansive AND problem. $\exists!\ f(x,y) = min(x,y)$ such that following holds:

C1 $f(x,y)$ is defined for $x,y \in [0,1]$
C2 $f(0,0) = f(0,1) = f(1,0) = 0, \quad f(1,1) = 1$
C3 $a{\le}A, \quad b{\le}B \implies f(a,b){\le}f(A,B)$ (monotonicity)
C4 $|f(a+\Delta a, b+\Delta b) - f(a,b)| \le max(|\Delta a|, |\Delta b|)$

Proof. We will prove Theorem 1 by demonstrating that conditions C1...C4 constrain $f(x,y)$ in such a way that the only possible solution is $f(x,y) = min(x,y)$.

The monotonicity condition C3 combined with C2 means that

$$\forall\ y{\in}[0,1] \quad f(0,y) = 0 \tag{1}$$

Conditions C3 (monotonicity) and C4 (nonexpansivity), when combined with 'reference' function values $f(0,0)$ and $f(1,1)$, constrain f along line connecting these two points:

$$|f(y,y) - f(0,0)| \le |y| \quad \implies f(y,y) - 0 \le y \quad \implies f(y,y) \le y \tag{2}$$

$$|f(1,1) - f(y,y)| \le |1-y| \quad \implies 1 - f(y,y) \le 1 - y \quad \implies f(y,y) \ge y \tag{3}$$

As result, we have

$$\forall\ y{\in}[0,1] \quad f(y,y) = y \tag{4}$$

Similarly to the previous paragraph, nonexpansivity condition C4 combined with equations 1 and 4 constrains $f(x,y)$ along line connecting points $(0,y)$ and (y,y), i.e. $\forall\ 0{\le}x{\le}y{\le}1$ following holds:

$$|f(x,y) - f(0,y)| \le |x| \quad \implies f(x,y) - 0 \le x \quad \implies f(x,y){\le}x \tag{5}$$

$$|f(y,y) - f(x,y)| \le |y-x| \quad \implies y - f(x,y) \le y - x \quad \implies f(x,y){\ge}x \tag{6}$$

As result, we have

$$\forall\ 0{\le}x{\le}y{\le}1 \quad f(x,y) = x = min(x,y)$$

Due to the symmetry of the problem, it is obvious that the following also holds:

$$\forall\ 0{\le}y{\le}x{\le}1 \quad f(x,y) = y = min(x,y)$$

So, finally,

$$\forall x,y \in [0,1] \quad f(x,y) = min(x,y)$$

what was to be shown.

Theorem 2: L_∞-nonexpansive OR problem. $\exists!\ g(x,y) = max(x,y)$ such that following holds:

C1 $g(x,y)$ is defined for $x,y \in [0,1]$
C2 $g(0,0) = 0, \quad g(0,1) = g(1,0) = g(1,1) = 1$
C3 $a{\le}A, \quad b{\le}B \implies g(a,b){\le}g(A,B)$ (monotonicity)
C4 $|g(a+\Delta a, b+\Delta b) - g(a,b)| \le max(|\Delta a|, |\Delta b|)$

Proof. Similarly to the previous proof, we will prove Theorem 2 by demonstrating that conditions C1...C4 constrain $g(x,y)$ in such a way that the only possible solution is $g(x,y) = max(x,y)$.

The monotonicity condition C3, when combined with 'reference' function values $g(1,0)$ and $g(1,1)$, constrains g along line connecting these two points:

$$\forall\ y{\in}[0,1] \quad g(1,y) = 1 \tag{7}$$

Similarly, conditions C3 (monotonicity) and C4 (nonexpansivity), when combined with 'reference' function values $g(0,0)$ and $g(1,1)$, constrain g along line connecting these two points:

$$|g(y,y) - g(0,0)| \le |y| \implies g(y,y) - 0 \le y \implies g(y,y) \le y \tag{8}$$

$$|g(1,1) - g(y,y)| \le |1-y| \implies 1 - g(y,y) \le 1 - y \implies g(y,y) \ge y \tag{9}$$

As result, we have

$$\forall\ y{\in}[0,1] \quad g(y,y) = y \tag{10}$$

Similarly to previous paragraphs, nonexpansivity condition C4 combined with equations 7 and 10 constrains $g(x, y)$ along line connecting points (y, y) and $(1, y)$, i.e. $\forall\ 0 \leq y \leq x \leq 1$ following holds:

$$|g(1, y) - g(x, y)| \leq |1 - x| \implies 1 - g(x, y) \leq 1 - x \implies g(x, y) \geq x \tag{11}$$

$$|g(x, y) - g(y, y)| \leq |x - y| \implies g(x, y) - y \leq x - y \implies g(x, y) \leq x \tag{12}$$

As result, we have

$$\forall\ 0 \leq y \leq x \leq 1 \quad g(x, y) = x = max(x, y)$$

Due to the symmetry of the problem, it is obvious that the following also holds:

$$\forall\ 0 \leq x \leq y \leq 1 \quad g(x, y) = y = max(x, y)$$

So, finally,

$$\forall x, y \in [0, 1] \quad g(x, y) = max(x, y)$$

what was to be shown.