SUPPLEMENT TO "TIER BALANCING: TOWARDS DYNAMIC FAIRNESS OVER UNDERLYING CAUSAL FACTORS"

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TABLE OF CONTENTS: APPENDIX

A Detailed Discussions on Related Works **14**

A.1 Causal Notions of Fairness . . . 14

A.2 Fairness Inquires in Dynamic Settings . . . 14

B Additional Results, Technical Details, and Discussions **15**

B.1 Discussions on the Causal Modeling of Decision-Distribution Interplay . . . 15

B.2 When Tier Balancing is Initially Satisfied . . . 16

B.3 A Remark on Fact 3.2 . . . 17

B.4 Detailed Derivation of $\Delta_{\mathrm{STIR}}|_t^{t+1}$ (Section 3.2.1) . . . 18

B.5 Further Illustration on Assumption 3.5 and Assumption 3.6 . . . 20

B.6 Additional Experimental Results . . . 20

B.7 Potential Limitations of Our Work . . . 21

C Proof of Results **21**

C.1 Proof for Proposition 3.1 . . . 21

C.2 Proof for Theorem 4.1 . . . 22

C.3 Proof for Theorem 4.2 . . . 25

C.4 Proof for Theorem 4.3 . . . 31

LIST OF TABLES

1 When $\alpha_D > \alpha_Y$, compare cases of H_{t+1} with different values of A_t. . . . 37

2 When $\alpha_D < \alpha_Y$, compare cases of H_{t+1} with different values of A_t. . . . 38

3 When $\alpha_D = \alpha_Y = \alpha$, compare cases of H_{t+1} with different values of A_t. . . . 39

4 When $\alpha_D > \alpha_Y$, list possible instantiations of $\varphi_{t+1}(e_{t+1})$. . . . 40

5 When $\alpha_D < \alpha_Y$, list possible instantiations of $\varphi_{t+1}(e_{t+1})$. . . . 42

6 When $\alpha_D = \alpha_Y = \alpha$, list possible instantiations of $\varphi_{t+1}(e_{t+1})$. . . . 44

A DETAILED DISCUSSIONS ON RELATED WORKS

In this section, we provide detailed discussions on related works. In particular, considering our focus on providing a novel long-term fairness notion with the help of the detailed causal modeling of involved dynamics, we compare our work with previous literature on causal notions of fairness, as well as fairness inquiries in dynamic settings.

A.1 CAUSAL NOTIONS OF FAIRNESS

Various causal notions of algorithmic fairness have been proposed in the literature, for instance, fairness notions defined in terms of the (non-)existence of certain causal paths in the graph (Kamiran et al., 2013; Kilbertus et al., 2017; Zhang et al., 2017), fairness notions defined through estimating or bounding causal effects (Kusner et al., 2017; Chiappa, 2019; Wu et al., 2019; Mhasawade & Chunara, 2021), fairness notions defined with respect to statistics on certain factual/counterfactual groups (Imai & Jiang, 2020; Coston et al., 2020; Mishler et al., 2021). The proposed causal notions audit fairness in an instantaneous manner, i.e., the fairness inquires are with respect to a snapshot of reality, and the scope of consideration is limited to observed variables only. Our *Tier Balancing* notion has a built-in capacity to inquire fairness in the long-term and dynamic setting, which is very different from instantaneous causal fairness notions (beyond the fact that our notion encompasses latent causal factors).

While we can detect and measure discrimination based on previous (instantaneous) causal notions of fairness (e.g., the existence of certain causal paths or causal effects), eliminating such existence of causal paths or causal effects is a valid goal to achieve but might not be the means one should opt for. To begin with, there is no guarantee that eliminating a causal path or effect results in non-existence of such causal path or effect in the future under the interplay between decision-making and data dynamics. Furthermore, the data generating processes represented in the causal model might not be easily manipulable under the same timescale of decision-making (e.g., the ones governed by nature and/or the mode and structure of a society). One cannot expect that the manipulation on the causal model (for the purpose of enforcing fairness notions) directly translate to real-world changes in the underlying data generating processes.

Different from previous causal fairness notions, instead of directly "going against" the underlying data generating process (e.g., by eliminating certain causal path or causal effect), our *Tier Balancing* notion encourages "working with" the underlying data generating processes. With a detailed causal modeling of the decision-distribution interplay, *Tier Balancing* emphases on the possibility of inducing a future data distribution that is fair in the long run.

A.2 FAIRNESS INQUIRES IN DYNAMIC SETTINGS

Previous literature have considered dynamic fairness in specific practical scenarios, for instance, opportunity allocation in labor market (Hu & Chen, 2018), a pipeline consisting of college admission followed by hiring (Kannan et al., 2019), opportunity allocation in credit application (Liu et al., 2018), and resource allocation in predictive policing (Ensign et al., 2018). Different from previous literature, we present a detailed causal modeling of the decision-distribution interplay that is general enough to be applicable in various resource allocation problems (e.g., loan applications, hiring practices) while also being specific enough to encompass nuances in data dynamics for the particular practical scenario of interest.

In terms of the analyzing framework, closely related works have considered the one-step analysis (Liu et al., 2018; Kannan et al., 2019; Mouzannar et al., 2019; Zhang et al., 2019). However, previous works focus on the long-term effect of imposing certain fairness notions that are readily available, for example, *Demographic Parity* (Calders et al., 2009; Liu et al., 2018; Mouzannar et al., 2019) and *Equal Opportunity* (Hardt et al., 2016; Liu et al., 2018). In our work, we formulate a novel notion of long-term fairness, namely, *Tier Balancing*, and explore the possibility of providing a fairness notion that characterizes the dynamic nature of decision-distribution interplay through detailed causal modeling on both observed variables and latent causal factors.

In terms of the modeling choice for data dynamics, most closely related works model data dynamics using variants of Markov Decision Processes (MDPs) (Jabbari et al., 2017; Siddique et al., 2020;

Zhang et al., 2020; D'Amour et al., 2020; Wen et al., 2021; Zimmer et al., 2021; Ge et al., 2021). For example, Zhang et al. (2020) consider the partially observed Markov decision process (POMDP) model, and conduct evolution and equilibrium analysis with respect to *Demographic Parity* and *Equal Opportunity* notions of fairness. The dynamics are modeled through transition matrices on group-level qualification rates. Compared to the modeling of transition matrices in MDPs, our model is more fine-grained and on the individual level, answering the call for "richer and more complex modelings [of involved dynamics]" in previous literature (Hu & Chen, 2018).

Another closely related work is the one-step analysis on the impact of causal fairness notions on downstream utilities conducted by Nilforoshan et al. (2022). They consider a detailed causal modeling on the college admission running example and analyze previously proposed (instantaneous) causal fairness notions, namely, *Counterfactual Predictive Parity* (Coston et al., 2020), *Counterfactual Equalized Odds* (Coston et al., 2020), and *Conditional Principal Fairness* (Imai & Jiang, 2020). Our work is different in several ways: instead of utilizing a static graph, we focus on the decision-distribution interplay and explicitly capture both observed and latent variables along the temporal axis; different from analyzing one-step downstream consequence in terms of utility, we formulate a long-term fairness goal and investigate the challenges and opportunities revealed by the notion.

B Additional Results, Technical Details, and Discussions

In this section, we provide additional results, technical details, and discussions of our work. In Appendix B.1, we provide additional discussions on our causal modeling of the decision-distribution interplay; in Appendix B.2, we analyze the situation where *Tier Balancing* is initially attained; in Appendix B.3, we discuss the role of exogenous terms and provide a remark on Fact 3.2; in Appendix B.4, we present the detailed derivation of *Single-step Tier Imbalance Reduction* (STIR) term $\Delta_{\mathrm{STIR}}|_t^{t+1}$; in Appendix B.5, we illustrate the connection between Assumption 3.5 and Assumption 3.6; in Appendix B.6, we present additional experimental results; in Appendix B.7, we discuss potential limitations of our work.

B.1 Discussions on the Causal Modeling of Decision-Distribution Interplay

In Appendix B.1.1, we provide additional details of the involved dynamics in the causal modeling of the decision-distribution interplay. In Appendix B.1.2, we discuss the relation between the practical scenarios and the modeled dynamics.

B.1.1 Additional Modeling Details of the Involved Dynamics

We use $X_{t,i}$'s to represent three different patterns (instead of the number of count) of variables with respect to how observed features are caused by the protected feature A_t and the latent causal factor H_t. There are three types of observed features: (1) features that only have the latent causal factor H_t as the case, e.g., $X_{t,1}$, (2) features that have both the latent causal factor H_t and the protected feature A_t as cause, e.g., $X_{t,2}$, and (3) features that only have the protected feature A_t as the cause, e.g,. $X_{t,3}$. For conciseness, we omit features that are not relevant to the practical scenario of interest, i.e., variables that are not causally relevant to (H_t, A_t). One can replace $X_{t,i}$'s with the actual number of additional features together with the causal relations among them in specific practical scenarios.

At every time step $T = t$, the decision-making strategy D_t is trained on the joint distribution $(A_t, X_{t,i}, Y_t^{(\mathrm{obs})})$. However, when making the decision, D_t only takes $(A_t, X_{t,i})$ as input. Since we are modeling causal relations in data generating processes, we only include a directed edge in the DAG if there is a causal relation between variables. Therefore, the data generating process of D_t does not involve an edge between $Y_t^{(\mathrm{obs})}$ and D_t.

B.1.2 The Practical Scenarios of Interest

As we can see from previous literature (discussed in Appendix A.2), the modeling choices are closely related to the practical scenarios of interest, and therefore, can be very different in terms of modeling details of the involved dynamics in long-term and dynamic settings.

Our causal modeling of repetitive resource application and allocation keeps track of individual-level situation changes, and enables informative and principled analysis on the decision-distribution interplay in different practical scenarios. For example, in credit application (e.g., Liu et al. 2018), the agents are clients and the latent causal factor (tier) can be individual's socio-economic status or creditworthiness; in predictive policing (e.g., Ensign et al. 2018), the agents are neighborhoods and the latent tier can be neighborhood's safety ratings; in the dual market pipeline (e.g., temporary labor markets followed by the permanent labor market considered in Hu & Chen 2018) or the admission-followed-by-hiring pipeline (e.g., Kannan et al. 2019), the agents are applicants who subject to a sequence of decisions and the latent tier can be the relevant qualification for the school program and the job.

However, when the decision received by the individual is once in a lifetime (or at least very long time compared to the timescale of the decision-making), repeated application and allocation of resource may not be a suitable modeling choice. For example, college admission decisions are made on a yearly basis but an individual does not repeatedly apply for college every single year (Mouzannar et al., 2019). In this case, if we focus on the decision made by a specific college, it is more natural to study changes in the population in terms of the group-level qualification profiles (Mouzannar et al., 2019). As another example, in the context of health care (e.g., Mhasawade & Chunara 2021), when the resource takes the form of the medical treatment for the purpose of improving health outcome, not all treatment requires regular doses and therefore, repeated allocation modeled on the individual-level may not be an optimal choice. One can, for instance, resort to the modeling at the level of subgroups as an alternative (Mhasawade & Chunara, 2021).

Considering the difference in semantics of fairness in various practical scenarios, previous literature has pointed out that there is in general no one-size-fits-all solution for algorithmic fairness (e.g., Kearns & Roth 2019). By presenting a detailed causal modeling for the decision-distribution interplay, we do not intend to provide a general framework to encompass long-term fairness considerations in all practical scenarios. Instead, we would like to demonstrate the opportunities and challenges and hope our work can inspire further research.

B.2 When Tier Balancing is Initially Satisfied

In the paper we have presented possibility and impossibility results to achieve, or get closer to, the long-term fairness goal when *Tier Balancing* is not initially satisfied. It is natural to wonder what we should do if we find out that *Tier Balancing* happen to be satisfied during fairness audit. In fact, as we shall see in Proposition B.1, if *Tier Balancing* is satisfied as the initial condition, under the specified dynamics, one can use *Demographic Parity* (Calders et al., 2009) decision-making strategy to maintain the status of satisfying *Tier Balancing*. This indicates that when *Tier Balancing* is satisfied (as a lucky initial condition, or as a result of K-step interventions), we have at least one way to maintain the fair state of affairs.

Proposition B.1. *When Tier Balancing is initially satisfied, i.e., $H_t \perp\!\!\!\perp A_t$, under Fact 3.2, Assumption 3.3, and Assumption 3.4, as well as the specified dynamics, the Demographic Parity decision-making strategy, i.e., $D_t \perp\!\!\!\perp A_t$, can ensure Tier Balancing still holds true for the next time step, i.e., $H_{t+1} \perp\!\!\!\perp A_{t+1}$.*

Proof. To begin with, since $H_t \perp\!\!\!\perp A_t$, by Fact 3.2, H_t is not a function of A_t. As a direct result, $Y_t^{(\text{ori})}$ is also not a function of A_t (since the distribution of $Y_t^{(\text{ori})}$ is fully determined by the value of $H_t = h_t$). Besides, since D_t satisfies *Demographic Parity*, $D_t \perp\!\!\!\perp A_t$, and therefore by Fact 3.2, D_t is not a function of A_t.

According to Assumption 3.3, under the specified dynamics, H_{t+1} is fully determined by $(H_t, D_t, Y_t^{(\text{ori})})$, among which none of them is a function of A_t. Then, we have H_{t+1} is not a function of A_t.

Recall that in the specified dynamics, the same group of agents repetitively apply for credit with the entire group unchanged. According to Assumption 3.4, A_{t+1} is an identical copy of A_t. Therefore we have H_{t+1} is not a function of A_{t+1}, i.e., $H_{t+1} \perp\!\!\!\perp A_{t+1}$. $\square$

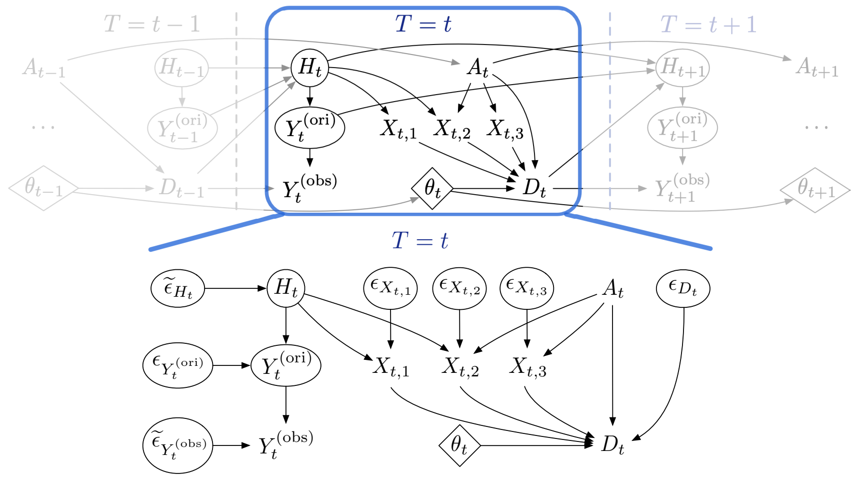


Figure 4: The causal modeling of the decision-distribution interplay. The circle indicates that the corresponding variable is unobserved. We use diamond to denote the underlying causal factor and explicitly indicate the (potential) non-stationary nature of the decision-making strategies across time.

B.3 A Remark on Fact 3.2

Let us first present the definition of a functional causal model (Spirtes et al., 1993; Pearl, 2009):

Definition B.2 (Functional Causal Model). We can represent a causal model with a tuple $(E, V, \mathbf{F})$ such that:

(1) V is a set of observed variables involved in the system of interest;

(2) E is a set of exogenous variables that we cannot directly observe but contains the background information representing all other causes of V and jointly follows a distribution $P(E)$;

(3) $\mathbf{F}$ is a set of functions (also known as structural equations) $\{f_1, f_2, \ldots, f_n\}$ where each f_i corresponds to one variable $V_i \in V$ and is a mapping $E \cup V \setminus \{V_i\} \rightarrow V_i$.

The triplet $(E, V, \mathbf{F})$ is known as the functional causal model (FCM). We can also capture causal relations among variables via a directed acyclic graph (DAG), where nodes (vertices) represent variables and edges represent functional relations between variables and the corresponding direct causes (i.e., observed parents and unobserved exogenous terms).

For the purpose of illustration, in Figure 4 we present the DAG (at time step $T = t$) with the exogenous terms E_t explicitly modeled, where E_t is the concatenation of individual exogenous terms:

$$E_t = \Big(\widetilde{\epsilon}_{H_t}, \widetilde{\epsilon}_{Y_t^{(\mathrm{obs})}}, \epsilon_{Y_t^{(\mathrm{ori})}}, \epsilon_{X_{t,i}}, \epsilon_{D_t}\Big). \tag{B.1}$$

We use the $\widetilde{\cdot}$ symbol on certain exogenous noise terms, e.g., $\widetilde{\epsilon}_{H_t}$ and $\widetilde{\epsilon}_{Y_t^{(\mathrm{obs})}}$, to denote the fact that the corresponding variables are affected by previous time step ($T = t-1$), and such influence are encapsulated into exogenous terms from the standpoint of current time step ($T = t$). For example, the influence from the randomness in D_{t-1} (when $T = t-1$) on current $Y_t^{(\mathrm{obs})}$ is encapsulated into an exogenous term $\widetilde{\epsilon}_{Y_t^{(\mathrm{obs})}}$ when $T = t$.

As we can see from Figure 4, (A_t, E_t) are root causes of all other variables $(H_t, X_{t,i}, Y_t^{(\mathrm{ori})}, Y_t^{(\mathrm{obs})}, D_t)$. Applying Definition B.2, we can utilize the functional causal model and represent each variable with a function (the structural equation) of its direct causes (including observed parents and unobserved exogenous terms). Then, we can iteratively replace variables with its corresponding structural equation and eventually represent variables in $(H_t, X_{t,i}, Y_t^{(\mathrm{ori})}, Y_t^{(\mathrm{obs})}, D_t)$ with functions of *only* root causes (A_t, E_t), as summarized in Fact 3.2.

The noise terms E_t are the unobserved exogenous terms that signify the unique characteristics of an individual. The utilization of such uniqueness of individual can be found in the estimation of counterfactual causal effect by making use of the posterior distribution of exogenous noise terms conditioning on the observed features, e.g., Kusner et al. (2017).

B.4 Detailed Derivation of $\Delta_{\text{STIR}}|_t^{t+1}$ (Section 3.2.1)

In this section, we provide the derivation detail of the *Single-step Tier Imbalance Reduction* (STIR) term:

$$\Delta_{\text{STIR}}|_t^{t+1} := \mathbb{E}\left[\, |f_{t+1}(0, E_{t+1}) - f_{t+1}(1, E_{t+1})| \,\right] - \mathbb{E}\left[\, |f_t(0, E_t) - f_t(1, E_t)| \,\right] \tag{B.2}$$

Firstly, in Appendix B.4.1, we characterize the conditional joint density of $\big(f_T(0, E_T), f_T(1, E_T)\big)$. Then, in Appendix B.4.2, we focus on the situation changes of each individual from $T = t$ to $T = t + 1$ induced by the specified dynamics. Finally, in Appendix B.4.3, we can calculate the expectation in Equation B.2 by aggregating situation changes for each individual from $T = t$ to $T = t + 1$.

B.4.1 Characterizing Conditional Joint Density

We can view $f_t(0, E_t)$ and $f_t(1, E_t)$ as two dependent random variables. Given combinations of D_t and Y_t, we can define their conditional joint probability density when $E_t = \epsilon$ as $q_t\big(f_t(0, \epsilon), f_t(1, \epsilon) \mid d, d', y, y'\big)$ and calculate it as following:

$$\begin{aligned}
&q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid d, d', y, y'\big) \\
:=\ & q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid g_t^D(0,\epsilon) = d, g_t^D(1,\epsilon) = d', g_t^{Y^{(\text{ori})}}(0,\epsilon) = y, g_t^{Y^{(\text{ori})}}(1,\epsilon) = y'\big) \\
=\ & \int_{\xi \in \mathcal{E}} \mathbb{1}\{f_t(0,\xi) = f_t(0,\epsilon), f_t(1,\xi) = f_t(1,\epsilon)\} \\
&\quad \cdot p_t\big(E_t = \xi \mid g_t^D(0,\epsilon) = d, g_t^D(1,\epsilon) = d', g_t^{Y^{(\text{ori})}}(0,\epsilon) = y, g_t^{Y^{(\text{ori})}}(1,\epsilon) = y'\big) d\xi,
\end{aligned} \tag{B.3}$$

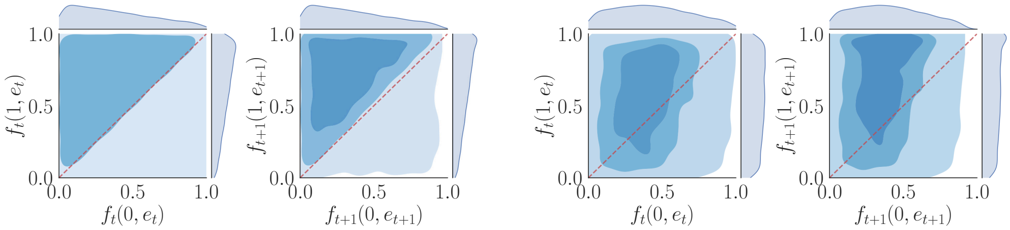
where $\mathbb{1}\{\cdot\}$ is the indicator function, and the subscript t of the conditional probability densities (e.g., $q_t(\cdot)$ and $p_t(\cdot)$) indicates that they (might) change over time with different time step $T = t$. The functional form of f_t can be convoluted and it is not necessarily the case that $f_t(0, \cdot)$ and $f_t(1, \cdot)$ are injective mappings $\mathcal{E} \to (0, 1]$. Therefore, for the purpose of generality, in Equation B.3 we explicitly introduce the identity function $\mathbb{1}\{f_t(0, \xi) = f_t(0, \epsilon), f_t(1, \xi) = f_t(1, \epsilon)\}$ when characterizing the conditional joint density q_t.

B.4.2 Capturing Situation Changes for An Individual

For a specific individual (j), given the value of individual's exogenous terms $E_t^{(j)} = e_t^{(j)}$, let us denote the difference between $f_t(0, e_t^{(j)})$ and $f_t(1, e_t^{(j)})$ as $\varphi_t(e_t^{(j)}) := f_t(0, e_t^{(j)}) - f_t(1, e_t^{(j)})$, and the sum of $f_t(0, e_t^{(j)})$ and $f_t(1, e_t^{(j)})$ as $\eta_t(e_t^{(j)}) := f_t(0, e_t^{(j)}) + f_t(1, e_t^{(j)})$. We introduce $\varphi_t(\cdot)$ and $\eta_t(\cdot)$ for the conciseness of notation, and we can always map $\big(\varphi_t(\cdot), \eta_t(\cdot)\big)$ back to $\big(f_t(0, \cdot), f_t(1, \cdot)\big)$ via a coordinate transformation:

$$\begin{bmatrix} f_t(0, e_t^{(j)}) \\ f_t(1, e_t^{(j)}) \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} \varphi_t(e_t^{(j)}) \\ \eta_t(e_t^{(j)}) \end{bmatrix}. \tag{B.4}$$

Let us consider the connection between $\varphi_{t+1}(e_{t+1}^{(j)}) = f_{t+1}(0, e_{t+1}^{(j)}) - f_{t+1}(1, e_{t+1}^{(j)})$ in the time step $T = t + 1$ and $\varphi_t(e_t^{(j)}) = f_t(0, e_t^{(j)}) - f_t(1, e_t^{(j)})$ in the time step $T = t$. We use different time step subscripts for the exogenous terms, e.g., $e_{t+1}^{(j)}$ in $\varphi_{t+1}(e_{t+1}^{(j)})$ and $e_t^{(j)}$ in $\varphi_t(e_t^{(j)})$, since it is not necessarily the case that $e_{t+1}^{(j)} = e_t^{(j)}$, even if we are focusing on the same individual from $T = t$ to $T = t + 1$. Nevertheless, for the given functional forms of $f_t, g_t^D, g_t^{Y^{(\text{ori})}}$, the combination of $(d^{(j)}, d'^{(j)}, y^{(j)}, y'^{(j)})$, the value of exogenous term $e_t^{(j)}$ in the initial situation of the current one-step analysis (when $T = t$), and the hyperparameters (α_D, α_Y), we can uniquely derive the value of



(a) the example of distribution change when the initial condition satisfies the quantitative assumption

(b) the example of distribution change when the initial condition satisfies the qualitative assumption

Figure 5: An illustration of the connection between qualitative and quantitative assumptions in terms of the one-step update of the conditional joint distribution $q_T\big(f_T(0, e_T), f_T(1, e_T) \mid d, d', y, y'\big)$ (when $y < y'$, and from $T = t$ to $T = t + 1$).

$\varphi_{t+1}(e_{t+1}^{(j)}) = f_{t+1}(0, e_{t+1}^{(j)}) - f_{t+1}(1, e_{t+1}^{(j)})$, and list all possible instantiations of $\varphi_{t+1}(e_{t+1}^{(j)})$ in Table 4 (if $\alpha_D > \alpha_Y$), Table 5 (if $\alpha_D < \alpha_Y$), and Table 6 (if $\alpha_D = \alpha_Y$).

Let us denote such mapping from $\varphi_t(e_t^{(j)})$ to $\varphi_{t+1}(e_{t+1}^{(j)})$ with the function G_t. For the purpose of simplifying notations, we can omit the superscript (j) if without ambiguity, since the value of exogenous terms e_T signify the unique characteristics of an individual:

$$\varphi_{t+1}(e_{t+1}) \coloneqq f_{t+1}(0, e_{t+1}) - f_{t+1}(1, e_{t+1}) = G_t(f_t, g_t^D, g_t^{Y^{(\text{ori})}}; d, d', y, y', e_t, \alpha_D, \alpha_Y). \tag{B.5}$$

Notice that the value of the function G_t *only* relies on the information available at time step $T = t$.

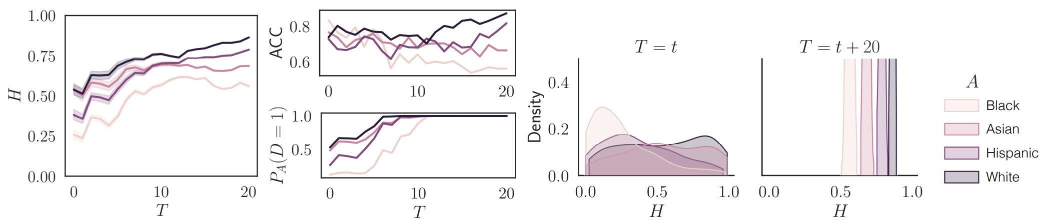
B.4.3 Aggregating Individual-Level Situation Changes

We can calculate *Single-step Tier Imbalance Reduction*, i.e., the term $\Delta_{\text{STIR}}|_t^{t+1}$, as following:

$$\begin{aligned}
\Delta_{\text{STIR}}|_t^{t+1} &\coloneqq \mathbb{E}\left[\, |f_{t+1}(0, E_{t+1}) - f_{t+1}(1, E_{t+1})| \,\right] - \mathbb{E}\left[\, |f_t(0, E_t) - f_t(1, E_t)| \,\right] \\
&\overset{(i)}{=} \mathbb{E}\left[\, |\varphi_{t+1}(E_{t+1})| - |\varphi_t(E_t)| \,\right] \\
&\overset{(ii)}{=} \mathbb{E}\Big\{ \mathbb{E}\Big[\, |\varphi_{t+1}(\xi)| - |\varphi_t(\epsilon)| \;\Big|\; \underbrace{E_{t+1} = \xi,}_{\text{The value of exogenous terms of an individual take value } \xi \text{ at } T = t+1.} \\
&\qquad \underbrace{E_t = \epsilon,}_{\text{The value of exogenous terms of an individual take value } \epsilon \text{ at } T = t.} \\
&\qquad \underbrace{\varphi_{t+1}(\xi) = G_t(f_t, g_t^D, g_t^{Y^{(\text{ori})}}; d, d', y, y', \epsilon, \alpha_D, \alpha_Y)}_{\substack{\text{This is to make sure that we are keeping track of the same individual in the sense that,} \\ \varphi_{t+1}(\xi) \text{ when } T = t+1 \text{ is indeed a valid instantiation from } \varphi_t(\epsilon) \text{ when } T = t. \\ \text{If } \varphi_{t+1}(\cdot) \text{ is not a valid instantiation from } \varphi_t(\cdot)\text{, the contribution to the expectation is 0.}}} \Big] \Big\} \\
&\overset{(iii)}{=} \sum_{d, d', y, y' \in \{0,1\}} P_t(d, d', y, y') \cdot \int_{\epsilon \in \mathcal{E}} \int_{\xi \in \mathcal{E}} q_t\big(f_t(0, \epsilon), f_t(1, \epsilon) \mid d, d', y, y'\big) \\
&\qquad \cdot \big(|\varphi_{t+1}(\xi)| - |\varphi_t(\epsilon)|\big) \cdot \mathbb{1}\{\varphi_{t+1}(\xi) = G_t(f_t, g_t^D, g_t^{Y^{(\text{ori})}}; d, d', y, y', \epsilon, \alpha_D, \alpha_Y)\} d\xi d\epsilon,
\end{aligned} \tag{B.6}$$

where the equality (i) is based on the definition of $\varphi_t(\cdot)$ and $\varphi_{t+1}(\cdot)$; the equality (ii) is derived from the Law of Iterated Expectation, keeping track of individual-level situation changes in the inner conditional expectation; the equality (iii) is the aggregation of individual-level situation changes by plugging in the conditional joint density q_t calculated in Appendix B.4.1, joint probability P_t, and the individual-level situation changes discussed in Appendix B.4.2.

The indicator function $\mathbb{1}\{\varphi_{t+1}(\xi) = G_t(f_t, g_t^D, g_t^{Y^{(\text{ori})}}; d, d', y, y', \epsilon, \alpha_D, \alpha_Y)\}$ makes sure that we are keeping track of the same individual (whose exogenous noise term equals to ϵ at time t) before and after the one-step dynamic, even if his/her exogenous noise term equals to ξ at time $t + 1$, and that ϵ might not be equal to ξ.



(a) the trend of the changing tier H at different step (b) the accuracy (top) and approval rate (bottom) for different groups (c) the distribution of the underlying hidden tier H among groups at time step $T = t$ (left) and $T = t + 20$ (right)

Figure 6: Illustration of the interplay between decision with accuracy-oriented predictors and the data dynamics (20 steps) on the credit score data set. Panel (a) and (b) present the step-by-step tracks of update in tier, accuracy, and approval rates for different groups; panel (c) presents group-conditioned distributions of tier before (left) and after (right) 20 steps of interventions. The legend is shared across panel (a), (b), and (c).

The fact that we are keeping track of the same individual also justifies the practice of only integrating over (conditional) densities with subscript t, e.g., $q_t(\cdot)$ and $P_t(\cdot)$, instead of both t and $t+1$. To see this from a different angle, keeping track of situation changes of each individual (when comparing $\varphi_{t+1}(e_{t+1})$ with $\varphi_t(e_t)$) also alleviates us from the trouble of estimating (conditional) densities that involve future information. At the time step t, we do not know the densities $q_{t+1}\big(f_{t+1}(0, E_{t+1}), f_{t+1}(1, E_{t+1}) \mid d, d', y, y'\big)$ and $P_{t+1}(d, d', y, y')$ since they involve future information D_{t+1} and Y_{t+1} at the standpoint of time step $T = t$.

B.5 Further Illustration on Assumption 3.5 and Assumption 3.6

In this subsection, we provide further illustrations of the connection between Assumption 3.5 and Assumption 3.6. In Figure 5 we present the one-step update of the conditional joint distribution $q_T\big(f_T(0, e_T), f_T(1, e_T) \mid d, d', y, y'\big)$ from $T = t$ to $T = t + 1$ (we present the case when $y < y'$ as an example). For panel (a) and (b), the joint distribution of $\big(f_T(0, E_T), f_T(1, E_T)\big)$ is plotted before and after one-step dynamics, with quantitative and qualitative assumptions respectively. The distributions are color-coded, the deeper the color, the larger the value of the joint density.

Compared to the qualitative assumption (Assumption 3.5, illustrated in Figure 5b), the quantitative assumption (Assumption 3.6, illustrated in Figure 5a) is just a special case, with quantitative characteristics built-in for technical purposes (we will make use of Assumption 3.6 in the proofs for Theorem 4.2 and Theorem 4.3 in Appendix C). From the illustrations in Figure 5, we can also see that the behaviors of the one-step update of conditional joint density under qualitative and quantitative assumptions are similar, with deeper color patterns occurring on the upper-left corner, indicating similar changes in the corresponding conditional joint densities $q_T\big(f_T(0, e_T), f_T(1, e_T) \mid d, d', y, y'\big)$ (when $y < y'$, and from $T = t$ to $T = t + 1$).

B.6 Additional Experimental Results

In this section, we present additional experimental results on the preprocessed FICO credit score data set (Board of Governors of the Federal Reserve System (US), 2007; Hardt et al., 2016). Similar to the experiment summarized in Figure 3, we convert the cumulative distribution function (CDF) of group-wise TransRisk scores into group-wise density distributions of the credit score, and use them as the initial tier distributions for different groups.

We consider utility-maximizing decision-making strategies, i.e., the decision-making policy is accuracy oriented and there is no explicit fairness consideration. In Figure 6 we present the summary of a 20-step interplay between decision with accuracy-oriented predictors and the underlying data generating process on the credit score data set. The accuracy-oriented decision-making strategy is retrained after each one-step data dynamics. From Figure 6(a), there is no obvious evidence that the gap between step-by-step tracks of tiers for different groups is decreasing over time. This observation aligns with our theoretical analysis (Theorem 4.2) and simulation results (Figure 2) for perfect predictors.

B.7 Potential Limitations of Our Work

In this subsection, we discuss potential limitations of our work.

B.7.1 Specified Dynamics vs. Causal Discovery

In this paper, we present a detailed causal modeling of decision-distribution interplay on DAG (Section 2.1) and formulate the dynamic fairness notion, *Tier Balancing*, that captures the long-term fairness goal over the underlying causal factor.

The research of causal discovery, where the goal is to discover the causal relations among variables (Spirtes et al., 1993; Shimizu et al., 2006; Zhang & Hyvärinen, 2009; Zhang et al., 2011), is a highly relevant area but is out of the scope of our paper. Our *Tier Balancing* notion of dynamic fairness, as well as our analyzing framework, does not rely on a causal model derived from causal discovery. As we discussed in the comparison with previous literature in dynamic fairness studies (Appendix A), our causal model is richer and more complex, which provides the potential of a more principled reasoning of the essential decision-distribution interplay in the pursuit of long-term fairness. We acknowledge the fact that it is nice to have the ability to discover the underlying causal model of the involved dynamics, which would provide further refinements of our dynamic modeling based on the specific practical scenario of interest. Causal discovery can act as the icing on the cake, but not a necessary component, of our analysis.

B.7.2 The Number and Dimension of Latent Causal Factors

In the causal modeling of decision-distribution interplay we present in the paper, we consider one latent causal factor that carries on the influence of current decision to future distributions. Recent developments in the identification of causal structures that involve (more than one) latent factors (Xie et al., 2020; Adams et al., 2021; Kivva et al., 2021; Xie et al., 2022) provide not only a theoretical justification, but also an indication of the potential, for our effort in exploring long-term fairness inquires over latent causal factors. We believe that our detailed causal modeling of decision-distribution interplay (on both observed variables and latent causal factors) and our formulation of *Tier Balancing* notion of long-term fairness act as an important first step.

C Proof of Results

In this section, we provide proofs for results presented in the paper. For better readability, we provide an additional *Proof (sketch)* before proving Theorem 4.1 (proof in Appendix C.2), Theorem 4.2 (proof in Appendix C.3), and Theorem 4.3 (proof in Appendix C.4), respectively.

C.1 Proof for Proposition 3.1

Proposition. *At time step $T = t$, for any $H_t = h_t \in (0, 1]$, under the specified dynamics, among the population where ground truth is actually observable, i.e., $Y_t^{(obs)}$ is not undefined, we have:*

$$Y_t^{(obs)} \sim \text{Bernoulli}(h_t).$$

Proof. To begin with, according the d-separation relation among D_{t-1}, H_t, and $Y_t^{(\text{ori})}$ on Figure 1, we notice that $Y_t^{(\text{ori})} \perp\!\!\!\perp D_{t-1} \mid H_t$. Therefore we have:

$$\begin{aligned}
&Y_t^{(\text{ori})} \sim \text{Bernoulli}(h_t), \\
&P(Y_t^{(\text{ori})} = 1 \mid H_t = h_t) = h_t, \\
&P(D_{t-1} = 1 \mid H_t = h_t) = d(h_t),
\end{aligned}$$

where $d(\cdot)$ is a function $d : (0, 1] \to [0, 1]$.

Notice that there is no claim that D_{t-1} can be uniquely determined by a function of only h_t. We only represent the conditional probability mass $P(D_{t-1} = 1 \mid H_t = h_t)$ with a function of h_t without specifying the exact functional form. In fact, as we shall see in the later part of this proof, the exact functional form of $d(\cdot)$ does not affect the validity of the result.

Since $Y_t^{\text{(obs)}}$ is in fact $Y_t^{\text{(ori)}}$ masked by D_{t-1}, i.e., $Y_t^{\text{(obs)}}$ is observable only when $D_{t-1} = 1$ and is undefined when $D_{t-1} = 0$, we have:

$$P(D_{t-1} = 0, Y_t^{\text{(obs)}} = 0 \mid H_t = h_t) = P(D_{t-1} = 0, Y_t^{\text{(obs)}} = 1 \mid H_t = h_t) = 0.$$

This indicates that among the population where $Y_t^{\text{(obs)}}$ is not undefined (the population itself may change at different time step), $\forall y \in \{0, 1\}$:

$$\begin{aligned}
&P(Y_t^{\text{(obs)}} = y \mid H_t = h_t)\\
=&P(D_{t-1} = 0, Y_t^{\text{(obs)}} = y \mid H_t = h_t) + P(D_{t-1} = 1, Y_t^{\text{(obs)}} = y \mid H_t = h_t)\\
=&P(D_{t-1} = 1, Y_t^{\text{(obs)}} = y \mid H_t = h_t)\\
=&P(D_{t-1} = 1, Y_t^{\text{(ori)}} = y \mid H_t = h_t).
\end{aligned}$$

Then, when $d(h_t) \in (0, 1)$, we can calculate the following probability:

$$\begin{aligned}
&P(Y_t^{\text{(obs)}} = 1 \mid H_t = h_t)\\
=&\frac{P(Y_t^{\text{(obs)}} = 1 \mid H_t = h_t)}{P(Y_t^{\text{(obs)}} = 1 \mid H_t = h_t) + P(Y_t^{\text{(obs)}} = 0 \mid H_t = h_t)}\\
=&\frac{P(D_{t-1} = 1, Y_t^{\text{(ori)}} = 1 \mid H_t = h_t)}{P(D_{t-1} = 1, Y_t^{\text{(ori)}} = 1 \mid H_t = h_t) + P(D_{t-1} = 1, Y_t^{\text{(ori)}} = 0 \mid H_t = h_t)}\\
=&\frac{d(h_t)h_t}{d(h_t)h_t + d(h_t)(1 - h_t)}\\
=&h_t\\
=&P(Y_t^{\text{(ori)}} = 1 \mid H_t = h_t);
\end{aligned}$$

when $d(h_t) = 1$, this indicates that if $H_t = h_t$, we know for sure that this individual received a positive decision in the previous time step (when $T = t - 1$), and we have $Y_t^{\text{(ori)}} = Y_t^{\text{(obs)}}$ by definition; when $d(h_t) = 0$, this indicates that if $H_t = h_t$, we know for sure that this individual did not receive a positive decision in the previous time step (when $T = t - 1$), and in this case $Y_t^{\text{(obs)}}$ is undefined.

Therefore, among the population where ground truth is actually observable, i.e., $Y_t^{\text{(obs)}}$ is not undefined, we have:

$$Y_t^{\text{(obs)}} \sim \text{Bernoulli}(h_t).$$

$\square$

C.2 Proof for Theorem 4.1

Theorem. *Let us consider the general situation where both D_t and $Y_t^{(ori)}$ are dependent with A_t, i.e., $D_t \not\perp\!\!\!\perp A_t, Y_t^{(ori)} \not\perp\!\!\!\perp A_t$. Then under Fact 3.2, Assumption 3.3, and Assumption 3.4, as well as the specified dynamics, when $H_t \not\perp\!\!\!\perp A_t$, only if at least one of the following conditions holds true for all $e_t \in \mathcal{E}$ can we possibly attain $H_{t+1} \perp\!\!\!\perp A_{t+1}$:*

(1) The ratio $\frac{f_t(0,e_t)}{f_t(1,e_t)}$ has a specific domain of value:

$$\frac{f_t(0, e_t)}{f_t(1, e_t)} = \frac{1 \pm \alpha_D \pm \alpha_Y}{1 \pm \alpha_D \pm \alpha_Y};$$

(2) Positive (negative) labels only appear in the advantaged (disadvantaged) group, and the decision for everyone is positive (if $\alpha_D > \alpha_Y$):

$$\begin{cases}
f_t(0, e_t) \in [\frac{1}{1+\alpha_D-\alpha_Y}, 1],\\
f_t(1, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1],\\
g_t^{Y^{(ori)}}(0, e_t) = 0, g_t^{Y^{(ori)}}(1, e_t) = 1,\\
g_t^D(0, e_t) = g_t^D(1, e_t) = 1;
\end{cases}$$

(3) *Negative (positive) labels only appear in the advantaged (disadvantaged) group, and the decision for everyone is positive (if $\alpha_D > \alpha_Y$):*

$$\begin{cases} f_t(0, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1], \\ f_t(1, e_t) \in [\frac{1}{1+\alpha_D-\alpha_Y}, 1], \\ g_t^{Y^{(ori)}}(0, e_t) = 1, g_t^{Y^{(ori)}}(1, e_t) = 0, \\ g_t^D(0, e_t) = g_t^D(1, e_t) = 1; \end{cases}$$

(4) *Everyone has a positive label, but the positive decision is exclusive to the advantaged group (if $\alpha_D < \alpha_Y$):*

$$\begin{cases} f_t(0, e_t) \in [\frac{1}{1-\alpha_D+\alpha_Y}, 1], \\ f_t(1, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1], \\ g_t^{Y^{(ori)}}(0, e_t) = g_t^{Y^{(ori)}}(1, e_t) = 1, \\ g_t^D(0, e_t) = 0, g_t^D(1, e_t) = 1; \end{cases}$$

(5) *Everyone has a positive label, but the positive decision is exclusive to the disadvantaged group (if $\alpha_D < \alpha_Y$):*

$$\begin{cases} f_t(0, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1], \\ f_t(1, e_t) \in [\frac{1}{1-\alpha_D+\alpha_Y}, 1], \\ g_t^{Y^{(ori)}}(0, e_t) = g_t^{Y^{(ori)}}(1, e_t) = 1, \\ g_t^D(0, e_t) = 1, g_t^D(1, e_t) = 0. \end{cases}$$

Proof (sketch). In order to see the exact condition under which it is possible to achieve $H_{t+1} \perp\!\!\!\perp A_{t+1}$, we consider the necessary and sufficient condition such that $H_{t+1} = f_{t+1}(A_{t+1}, E_{t+1})$ is not a function of A_{t+1}. This, together with Fact 3.2, Assumption 3.3, and Assumption 3.4, indicates that we need to consider the condition under which $H_{t+1} = \min\big\{1, f_t(A_t, E_t)\big[1 + \alpha_D(2D_t - 1) + \alpha_Y(2Y_t^{\text{(ori)}} - 1)\big]\big\}$ is not a function of A_t.

Since both D_t and $Y_t^{\text{(ori)}}$ are binary, we can exhaustively consider all value combinations of D_t and $Y_t^{\text{(ori)}}$, and list every possible value H_{t+1} can take in each case in Table 1 (if $\alpha_D > \alpha_Y$), Table 2 (if $\alpha_D < \alpha_Y$), or Table 3 (if $\alpha_D = \alpha_Y$). By exhaustively going through possible cases, we can have a full picture of the update of H_{t+1} based on $(H_t, Y_t^{\text{(ori)}}, D_t)$, and then derive conditions under which H_{t+1} is *not* a function of A_t, i.e., we have the conditions under which it is possible to attain $H_{t+1} \perp\!\!\!\perp A_{t+1}$. $\square$

Proof (full). In order to see the exact condition under which it is possible to achieve $H_{t+1} \perp\!\!\!\perp A_{t+1}$, we consider the necessary and sufficient condition such that $H_{t+1} = f_{t+1}(A_{t+1}, E_{t+1})$ is not a function of A_{t+1}. By Fact 3.2, Assumption 3.3, and Assumption 3.4, it is necessary and sufficient to consider the condition under which $H_{t+1} = \min\big\{1, f_t(A_t, E_t)\big[1 + \alpha_D(2D_t - 1) + \alpha_Y(2Y_t^{\text{(ori)}} - 1)\big]\big\}$ is not a function of A_t.

Considering the fact that both D_t and $Y_t^{\text{(ori)}}$ are binary, we can compare the values of H_{t+1} when $A_t = 0$ and $A_t = 1$ for all possible combinations of D_t and $Y_t^{\text{(ori)}}$. For any fixed $e_t \in \mathcal{E}$, we can list all the cases in Table 1 (if $\alpha_D > \alpha_Y$), Table 2 (if $\alpha_D < \alpha_Y$), or Table 3 (if $\alpha_D = \alpha_Y$), and see if for all $e_t \in \mathcal{E}$, there is no difference in the value of H_{t+1} between the cases when $A_t = 0$ and $A_t = 1$.

From Table 1, Table 2, and Table 3, we can see that if and only the following hold true can we achieve $H_{t+1} \perp\!\!\!\perp A_{t+1}$: for every $e_t \in \mathcal{E}$, whenever the joint probability $P\big(g_t^D(0, e_t) = d, g_t^{Y^{\text{(ori)}}}(0, e_t) = y, g_t^D(1, e_t) = d', g_t^{Y^{\text{(ori)}}}(1, e_t) = y'\big)$ is nonzero, the last two columns of the corresponding row(s) in the table, i.e., the exact values of H_{t+1}, need to match. For example, when $\alpha_D > \alpha_Y$, if we know $P\big(g_t^D(0, e_t) = 0, g_t^{Y^{\text{(ori)}}}(0, e_t) = 0, g_t^D(1, e_t) = 0, g_t^{Y^{\text{(ori)}}}(1, e_t) = 0\big) \neq 0$, we need the last two columns of Case (i) of Table 1 to equal to each other, i.e., we need $f_t(0, e_t) = f_t(1, e_t)$ to hold true.

Without further assumptions on the joint distribution of the data, we do not know which combination of (d, y, d', y') will result in a nonzero joint probability:

$$P\big(g_t^D(0,e_t)=d, g_t^{Y^{(\text{ori})}}(0,e_t)=y, g_t^D(1,e_t)=d', g_t^{Y^{(\text{ori})}}(1,e_t)=y'\big)\neq 0.$$

However, considering the fact that $\sum_{d,d'\in\mathcal{D},y,y'\in\mathcal{Y}} P\big(g_t^D(0,e_t)=d, g_t^{Y^{(\text{ori})}}(0,e_t)=y, g_t^D(1,e_t)=d', g_t^{Y^{(\text{ori})}}(1,e_t)=y'\big)=1$ holds for all $e_t\in\mathcal{E}$, we do know that for any fixed $e_t\in\mathcal{E}$, there is at least one possible instantiation of (d^*, y^*, d'^*, y'^*) such that:

$$P\big(g_t^D(0,e_t)=d^*, g_t^{Y^{(\text{ori})}}(0,e_t)=y^*, g_t^D(1,e_t)=d'^*, g_t^{Y^{(\text{ori})}}(1,e_t)=y'^*\big)\neq 0. \tag{C.1}$$

Let us first consider situations where $\alpha_D>\alpha_Y$ and focus on Table 1. The analysis on situations where $\alpha_D<\alpha_Y$ (i.e., Table 2) or $\alpha_D=\alpha_Y$ (i.e, Table 3), is of the same flavor and therefore we omit the detail in the proof.

To begin with, we can observe that not every entry of the last two columns explicitly keeps the $\min\{\cdot,1\}$ operator. On the one hand, since $\alpha_D>\alpha_Y$ ($\alpha_D,\alpha_Y\in[0,\frac{1}{2})$, as of Assumption 3.3), we have $(1-\alpha_D\pm\alpha_Y)\in(0,1)$ and $f_t(a_t,e_t)(1-\alpha_D\pm\alpha_Y)\in(0,1)$ (since $H_t=f_t(a_t,e_t)\in(0,1]$); therefore, we do not need to keep the $\min\{\cdot,1\}$ operator explicit, for instance, in the second to last column of Case (v - viii). On the other hand, when the coefficients $(1+\alpha_D\pm\alpha_Y)>1$ we are not sure if $f_t(a_t,e_t)(1+\alpha_D\pm\alpha_Y)$ exceed 1; therefore, we need to keep the $\min\{\cdot,1\}$ operator explicit, for instance, in the last column of Case (v - viii).

Besides, if only one entry of the last two columns explicitly has the $\min\{\cdot,1\}$ operator, it is equivalent to require that the terms themselves (before applying the operator) are equal (since the one without the $\min\{\cdot,1\}$ operator is known to be within the $(0,1)$ interval). For instance, Case (ix) requires that $\min\{f_t(0,e_t)(1+\alpha_D-\alpha_Y),1\}=f_t(1,e_t)(1-\alpha_D-\alpha_Y)$, which is equivalent to requiring $f_t(0,e_t)(1+\alpha_D-\alpha_Y)=f_t(1,e_t)(1-\alpha_D-\alpha_Y)$.

Furthermore, if both entries of the last two columns explicitly has the $\min\{\cdot,1\}$ operator, the exact condition of matching the last two columns depends on the actual value of $f_t(0,e_t)$ and $f_t(1,e_t)$. For instance, Case (xv) requires that $\min\{f_t(0,e_t)(1+\alpha_D+\alpha_Y),1\}=\min\{f_t(1,e_t)(1+\alpha_D-\alpha_Y),1\}$, which could be equivalent to one of the following conditions (recall that $1+\alpha_D\pm\alpha_Y>1$):

- if we have $f_t(0,e_t)\in[\frac{1}{1+\alpha_D+\alpha_Y},1]$ and $f_t(1,e_t)\in[\frac{1}{1+\alpha_D-\alpha_Y},1]$, we require $1=1$, which trivially holds true;
- if we have $f_t(0,e_t)\in(0,\frac{1}{1+\alpha_D+\alpha_Y})$ and $f_t(1,e_t)\in[\frac{1}{1+\alpha_D-\alpha_Y},1]$, we require $f_t(0,e_t)(1+\alpha_D+\alpha_Y)=1$, which cannot hold true;
- if we have $f_t(0,e_t)\in[\frac{1}{1+\alpha_D+\alpha_Y},1]$ and $f_t(1,e_t)\in(0,\frac{1}{1+\alpha_D-\alpha_Y})$, we require $1=f_t(1,e_t)(1+\alpha_D-\alpha_Y)$ which cannot hold true;
- if we have $f_t(0,e_t)\in(0,\frac{1}{1+\alpha_D+\alpha_Y})$ and $f_t(1,e_t)\in(0,\frac{1}{1+\alpha_D-\alpha_Y})$, we require $\frac{f_t(0,e_t)}{f_t(1,e_t)}=\frac{1+\alpha_D-\alpha_Y}{1+\alpha_D+\alpha_Y}$.

Recall that without further assumptions on the data distribution, we do not know which row(s) of the table correspond to a nonzero probability $P\big(g_t^D(0,e_t)=d, g_t^{Y^{(\text{ori})}}(0,e_t)=y, g_t^D(1,e_t)=d', g_t^{Y^{(\text{ori})}}(1,e_t)=y'\big)$. As a result, in general, we do not know which set of requirements we should enforce for each $e_t\in\mathcal{E}$. Therefore, we cannot derive a necessary and sufficient condition for attaining $H_{t+1}\perp\!\!\!\perp A_{t+1}$ in general cases. Nevertheless, we can summarize the previous analysis and derive the necessary condition of attaining $H_{t+1}\perp\!\!\!\perp A_{t+1}$, i.e., only if at least one of the following conditions holds true for all $e_t\in\mathcal{E}$ can we possibly attain $H_{t+1}\perp\!\!\!\perp A_{t+1}$:

(1) The ratio $\frac{f_t(0,e_t)}{f_t(1,e_t)}$ has a specific domain of value:

$$\frac{f_t(0,e_t)}{f_t(1,e_t)}=\frac{1\pm\alpha_D\pm\alpha_Y}{1\pm\alpha_D\pm\alpha_Y};$$

(2) Positive (negative) labels only appear in the advantaged (disadvantaged) group, and the decision for everyone is positive (if $\alpha_D > \alpha_Y$):

$$\begin{cases} f_t(0, e_t) \in [\frac{1}{1+\alpha_D-\alpha_Y}, 1], \\ f_t(1, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1], \\ g_t^{Y^{\text{(ori)}}}(0, e_t) = 0, g_t^{Y^{\text{(ori)}}}(1, e_t) = 1, \\ g_t^D(0, e_t) = g_t^D(1, e_t) = 1; \end{cases}$$

(3) Negative (positive) labels only appear in the advantaged (disadvantaged) group, and the decision for everyone is positive (if $\alpha_D > \alpha_Y$):

$$\begin{cases} f_t(0, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1], \\ f_t(1, e_t) \in [\frac{1}{1+\alpha_D-\alpha_Y}, 1], \\ g_t^{Y^{\text{(ori)}}}(0, e_t) = 1, g_t^{Y^{\text{(ori)}}}(1, e_t) = 0, \\ g_t^D(0, e_t) = g_t^D(1, e_t) = 1; \end{cases}$$

(4) Everyone has a positive label, but the positive decision is exclusive to the advantaged group (if $\alpha_D < \alpha_Y$):

$$\begin{cases} f_t(0, e_t) \in [\frac{1}{1-\alpha_D+\alpha_Y}, 1], \\ f_t(1, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1], \\ g_t^{Y^{\text{(ori)}}}(0, e_t) = g_t^{Y^{\text{(ori)}}}(1, e_t) = 1, \\ g_t^D(0, e_t) = 0, g_t^D(1, e_t) = 1; \end{cases}$$

(5) Everyone has a positive label, but the positive decision is exclusive to the disadvantaged group (if $\alpha_D < \alpha_Y$):

$$\begin{cases} f_t(0, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1], \\ f_t(1, e_t) \in [\frac{1}{1-\alpha_D+\alpha_Y}, 1], \\ g_t^{Y^{\text{(ori)}}}(0, e_t) = g_t^{Y^{\text{(ori)}}}(1, e_t) = 1, \\ g_t^D(0, e_t) = 1, g_t^D(1, e_t) = 0. \end{cases}$$

□

C.3 Proof for Theorem 4.2

Theorem. *Let us consider the general situation where both D_t and $Y_t^{\textit{(ori)}}$ are dependent with A_t, i.e., $D_t \not\perp\!\!\!\perp A_t, Y_t^{\textit{(ori)}} \not\perp\!\!\!\perp A_t$. Under Fact 3.2, Assumption 3.3, Assumption 3.4, and Assumption 3.6, as well as the specified dynamics, when $H_t \not\perp\!\!\!\perp A_t$, the perfect predictor does not have the potential to get closer to the long-term fairness goal after one-step intervention, i.e.,*

$$D_t = Y_t^{\textit{(ori)}} \implies \Delta_{\text{STIR}}^{\textit{(Perfect Predictor)}}\big|_t^{t+1} > 0.$$

Proof (sketch). The goal is to calculate if it is possible for *Single-step Tier Imbalance Reduction* $\Delta_{\text{STIR}}|_t^{t+1}$ to be smaller than 0 when using perfect predictors. As defined in Equation 5, $\Delta_{\text{STIR}}|_t^{t+1}$ is a weighted aggregation (integration followed by summation) of $|\varphi(e_{t+1})| - |\varphi(e_t)|$. The quantitative analysis involves three key components: instantiations of $\varphi_{t+1}(e_{t+1})$, the knowledge/assumptions on $q_t\big(f_t(0, \epsilon), f_t(1, \epsilon) \mid d, d', y, y'\big)$, and characteristics of $P_t(d, d', y', y')$.

For the first component, we can list all possible instantiations of $\varphi_{t+1}(e_{t+1})$ in Table 4 (if $\alpha_D > \alpha_Y$), Table 5 (if $\alpha_D < \alpha_Y$), and Table 6 (if $\alpha_D = \alpha_Y$), respectively. For the second component, we can introduce a quantitative assumption on $q_t\big(f_t(0, \epsilon), f_t(1, \epsilon) \mid d, d', y, y'\big)$ (Assumption 3.6). For the third component, we need to exploit the characteristic of the predictor of interest to gain further insight into the joint distribution $P_t(d, d', y, y')$. For perfect predictors, we have $P_t(d, d', y, y')$ satisfies Equation 8 (as we have discussed in Section 4.2.1).

For the purpose of calculating the value of $\Delta_{\text{STIR}}|_t^{t+1}$, the proof contains two steps: (1) exhaustively derive the value of $|\varphi(e_{t+1})| - |\varphi(e_t)|$ after one-step dynamics in all possible cases, and (2) aggregate the difference $|\varphi(e_{t+1})| - |\varphi(e_t)|$ with the help of the additional knowledge/assumptions on $q_t\big(f_t(0, \epsilon), f_t(1, \epsilon) \mid d, d', y, y'\big)$ and $P_t(d, d', y, y')$. □

Proof (full). For the perfect predictor $D_t = Y_t^{(\text{ori})}$, among all possible instantiations of $\varphi_{t+1}(e_{t+1})$ as listed in Table 4, Table 5, and Table 6, not every case corresponds to a nonzero $P_t(d, d', y, y')$ and therefore may not contribute to the computation of $\Delta_{\text{STIR}}|_t^{t+1}$ as detailed in Equation 5. By applying Equation 8 we only need to consider Case (i), Case (vi), Case (xi), and Case (xvi) in Table 4 (if $\alpha_D > \alpha_Y$), Table 5 (if $\alpha_D < \alpha_Y$), and Table 6 (if $\alpha_D = \alpha_Y$), respectively. We list all possible values of $|\varphi(e_{t+1})| - |\varphi(e_t)|$ for each of the aforementioned cases (the result applies to scenarios where $\alpha_D > \alpha_Y$, $\alpha_D < \alpha_Y$, or $\alpha_D = \alpha_Y$).

When $(d, d', y, y') = (0, 1, 0, 1)$, i.e., for Case (vi):

(vi.1.1) $|\varphi(e_{t+1})| - |\varphi(e_t)| = -(\alpha_D + \alpha_Y)\big(f_t(0, e_t) + f_t(1, e_t)\big) < 0$

$$\text{if we have } \begin{cases} f_t(0, e_t) \in (0, 1], f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y}) \\ f_t(1, e_t) \leq \tan\big(\arctan \frac{1}{\alpha_D+\alpha_Y} - \frac{\pi}{4}\big) \end{cases} ;$$

(vi.1.2) $|\varphi(e_{t+1})| - |\varphi(e_t)| = (\alpha_D + \alpha_Y - 2)f_t(0, e_t) + (\alpha_D + \alpha_Y + 2)f_t(1, e_t) < 0$

$$\text{if we have } \begin{cases} f_t(0, e_t) \in (0, 1], f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y}) \\ f_t(1, e_t) < \tan\big(\arctan \frac{2}{\alpha_D+\alpha_Y} - \frac{\pi}{4}\big) \\ f_t(1, e_t) \geq \tan\big(\arctan \frac{1}{\alpha_D+\alpha_Y} - \frac{\pi}{4}\big) \end{cases} ;$$

(vi.1.3) $|\varphi(e_{t+1})| - |\varphi(e_t)| = (\alpha_D + \alpha_Y - 2)f_t(0, e_t) + (\alpha_D + \alpha_Y + 2)f_t(1, e_t) > 0$

$$\text{if we have } \begin{cases} f_t(0, e_t) \in (0, 1], f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y}) \\ f_t(1, e_t) < f_t(0, e_t) \\ f_t(1, e_t) \geq \tan\big(\arctan \frac{2}{\alpha_D+\alpha_Y} - \frac{\pi}{4}\big) \end{cases} ;$$

(vi.1.4) $|\varphi(e_{t+1})| - |\varphi(e_t)| = (\alpha_D + \alpha_Y)\big(f_t(0, e_t) + f_t(1, e_t)\big) > 0$

$$\text{if we have } \begin{cases} f_t(0, e_t) \in (0, 1], f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y}) \\ f_t(1, e_t) \geq f_t(0, e_t) \end{cases} ;$$

(vi.2.1) $|\varphi(e_{t+1})| - |\varphi(e_t)| = 1 - (2 - \alpha_D - \alpha_Y)f_t(0, e_t) + f_t(1, e_t) > 0$

$$\text{if we have } \begin{cases} f_t(0, e_t) \in (0, 1], f_t(1, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1] \\ f_t(1, e_t) < f_t(0, e_t) \end{cases} ;$$

(vi.2.2) $|\varphi(e_{t+1})| - |\varphi(e_t)| = 1 + (\alpha_D + \alpha_Y)f_t(0, e_t) - f_t(1, e_t) > 0$

$$\text{if we have } \begin{cases} f_t(0, e_t) \in (0, 1], f_t(1, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1] \\ f_t(1, e_t) \geq f_t(0, e_t) \end{cases} .$$

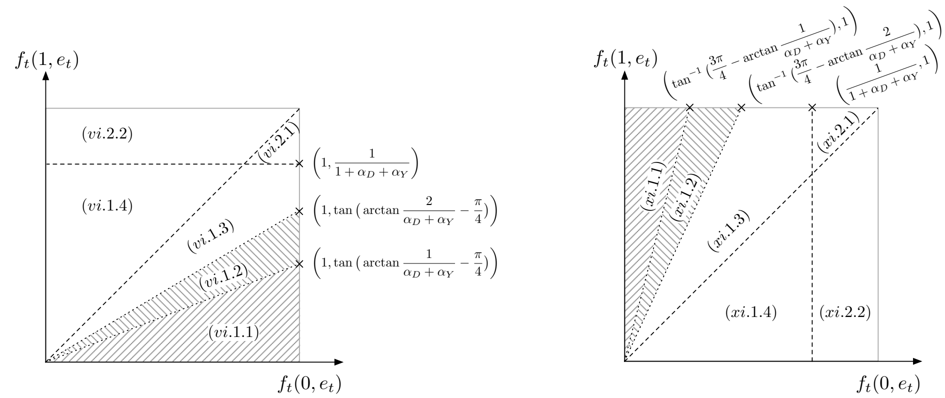
When $(d, d', y, y') = (1, 0, 1, 0)$, i.e., for Case (xi):

(xi.1.1) $|\varphi(e_{t+1})| - |\varphi(e_t)| = -(\alpha_D + \alpha_Y)\big(f_t(0, e_t) + f_t(1, e_t)\big) < 0$

$$\text{if we have } \begin{cases} f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y}), f_t(1, e_t) \in (0, 1] \\ f_t(1, e_t) \geq \tan\big(\frac{3\pi}{4} - \arctan \frac{1}{\alpha_D+\alpha_Y}\big) \end{cases} ;$$

(xi.1.2) $|\varphi(e_{t+1})| - |\varphi(e_t)| = (\alpha_D + \alpha_Y + 2)f_t(0, e_t) + (\alpha_D + \alpha_Y - 2)f_t(1, e_t) < 0$

$$\text{if we have } \begin{cases} f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y}), f_t(1, e_t) \in (0, 1] \\ f_t(1, e_t) < \tan\big(\frac{3\pi}{4} - \arctan \frac{1}{\alpha_D+\alpha_Y}\big) \\ f_t(1, e_t) \geq \tan\big(\frac{3\pi}{4} - \arctan \frac{2}{\alpha_D+\alpha_Y}\big) \end{cases} ;$$



(a) Case (vi), $(d, d', y, y') = (0, 1, 0, 1)$ (b) Case (xi), $(d, d', y, y') = (1, 0, 1, 0)$

Figure 7: Illustration of the sliced squares on the $\big(f_t(0, e_t), f_t(1, e_t)\big)$ plane. Depending on the initial situation, i.e., the slice that the $\big(f_t(0, e_t), f_t(1, e_t)\big)$ pair falls upon, the term $|\varphi(e_{t+1})| - |\varphi(e_t)|$ takes different values. The shaded slices indicate that if the initial situation satisfies the corresponding condition, the calculated $|\varphi(e_{t+1})| - |\varphi(e_t)| < 0$.

(xi.1.3) $|\varphi(e_{t+1})| - |\varphi(e_t)| = (\alpha_D + \alpha_Y + 2) f_t(0, e_t) + (\alpha_D + \alpha_Y - 2) f_t(1, e_t) > 0$

$$\text{if we have} \begin{cases} f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y}), f_t(1, e_t) \in (0, 1] \\ f_t(1, e_t) < \tan\big(\frac{3\pi}{4} - \arctan\frac{2}{\alpha_D+\alpha_Y}\big) \\ f_t(1, e_t) \geq f_t(0, e_t) \end{cases} ;$$

(xi.1.4) $|\varphi(e_{t+1})| - |\varphi(e_t)| = (\alpha_D + \alpha_Y)\big(f_t(0, e_t) + f_t(1, e_t)\big) > 0$

$$\text{if we have} \begin{cases} f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y}), f_t(1, e_t) \in (0, 1] \\ f_t(1, e_t) < f_t(0, e_t) \end{cases} ;$$

(xi.2.1) $|\varphi(e_{t+1})| - |\varphi(e_t)| = 1 + f_t(0, e_t) - (2 - \alpha_D - \alpha_Y) f_t(1, e_t) > 0$

$$\text{if we have} \begin{cases} f_t(0, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1], f_t(1, e_t) \in (0, 1] \\ f_t(1, e_t) \geq f_t(0, e_t) \end{cases} ;$$

(xi.2.2) $|\varphi(e_{t+1})| - |\varphi(e_t)| = 1 - f_t(0, e_t) + (\alpha_D + \alpha_Y) f_t(1, e_t) > 0$

$$\text{if we have} \begin{cases} f_t(0, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1], f_t(1, e_t) \in (0, 1] \\ f_t(1, e_t) < f_t(0, e_t) \end{cases} .$$

When $(d, d', y, y') = (0, 0, 0, 0)$, i.e., for Case (i), or $(d, d', y, y') = (1, 1, 1, 1)$, i.e., for Case (xvi), $|\varphi(e_{t+1})| - |\varphi(e_t)| = 0$.

Now we proceed to the second step and aggregate $|\varphi(e_{t+1})| - |\varphi(e_t)|$ terms. According to Equation 5 and Equation B.5, for the perfect predictor we have:

$$\begin{aligned} \Delta_{\text{STIR}}^{\text{(Perfect Predictor)}}\Big|_t^{t+1} = P_t(0, 1, 0, 1) \cdot \int_{\epsilon \in \mathcal{E}} \int_{\xi \in \mathcal{E}} & \big(|\varphi_{t+1}(\xi)| - |\varphi_t(\epsilon)|\big) \\ & \cdot \mathbb{1}\{\varphi_{t+1}(\xi) = G(f_t, g_t^D, g_t^{Y^{\text{(ori)}}}; 0, 1, 0, 1, \epsilon, \alpha_D, \alpha_Y)\} \\ & \cdot q_t\big(f_t(0, \epsilon), f_t(1, \epsilon) \mid 0, 1, 0, 1\big) d\xi d\epsilon \\ + P_t(1, 0, 1, 0) \cdot \int_{\epsilon \in \mathcal{E}} \int_{\xi \in \mathcal{E}} & \big(|\varphi_{t+1}(\xi)| - |\varphi_t(\epsilon)|\big) \\ & \cdot \mathbb{1}\{\varphi_{t+1}(\xi) = G(f_t, g_t^D, g_t^{Y^{\text{(ori)}}}; 1, 0, 1, 0, \epsilon, \alpha_D, \alpha_Y)\} \\ & \cdot q_t\big(f_t(0, \epsilon), f_t(1, \epsilon) \mid 1, 0, 1, 0\big) d\xi d\epsilon. \end{aligned} \tag{C.2}$$

As we can see from Equation C.2, we need to perform two-dimensional integrations on the $\big(f_t(0,e_t), f_t(1,e_t)\big)$ plane, calculating the expectation of the term $|\varphi(e_{t+1})| - |\varphi(e_t)|$ over the conditional densities $q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid 0,1,0,1\big)$ and $q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid 1,0,1,0\big)$. Since these conditional joint densities could be convoluted in general cases, the calculation of conditional expectations in Equation C.2 could be rather complicated. Therefore, we propose to take advantage of Assumption 3.6 to quantitatively simplify the calculation yet remain consistent with the rather mild qualitative assumption (Assumption 3.5), and derive a result that is numerically clear and informative. For the purpose of better illustrating the connection between (qualitative and quantitative) assumptions on $q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid d, d', y, y'\big)$ and the computation of $\Delta_{\mathrm{STIR}}|_t^{t+1}$, we also provide illustrative figures as shown in Figure 7.

With the help of Assumption 3.6, we convert the conditional expectations in Equation C.2 into calculations of multiple integrals on slices within a 1×1 square on the 2-D plane, where ϕ_0 and ϕ_1 axes correspond to the value of $f_t(0, E_t)$ and $f_t(1, E_t)$ respectively:

$$
\begin{aligned}
&\int_{\epsilon\in\mathcal{E}}\int_{\xi\in\mathcal{E}} \big(|\varphi_{t+1}(\xi)| - |\varphi_t(\epsilon)|\big) \cdot \mathbb{1}\{\varphi_{t+1}(\xi) = G(f_t, g_t^D, g_t^{Y^{(\text{ori})}}; 0,1,0,1,\epsilon,\alpha_D,\alpha_Y)\} \\
&\qquad \cdot q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid 0,1,0,1\big) d\xi d\epsilon \\
&= \gamma_{0101}^{(\text{low})} \cdot \Bigg\{ \int_0^1 \int_0^{\tan\left(\arctan\frac{1}{\alpha_D+\alpha_Y} - \frac{\pi}{4}\right)\phi_0} -(\alpha_D+\alpha_Y)(\phi_0+\phi_1)\, d\phi_1 d\phi_0 \\
&\qquad + \int_0^{\frac{1}{1+\alpha_D+\alpha_Y}} \int_{\tan\left(\arctan\frac{1}{\alpha_D+\alpha_Y} - \frac{\pi}{4}\right)\phi_0}^{\phi_0} (\alpha_D+\alpha_Y-2)\phi_0 + (\alpha_D+\alpha_Y+2)\phi_1\, d\phi_1 d\phi_0 \\
&\qquad + \int_{\frac{1}{1+\alpha_D+\alpha_Y}}^1 \int_{\tan\left(\arctan\frac{1}{\alpha_D+\alpha_Y} - \frac{\pi}{4}\right)\phi_0}^{\frac{1}{1+\alpha_D+\alpha_Y}} (\alpha_D+\alpha_Y-2)\phi_0 + (\alpha_D+\alpha_Y+2)\phi_1\, d\phi_1 d\phi_0 \\
&\qquad + \int_{\frac{1}{1+\alpha_D+\alpha_Y}}^1 \int_{\frac{1}{1+\alpha_D+\alpha_Y}}^{\phi_0} 1 - (2-\alpha_D-\alpha_Y)\phi_0 + \phi_1\, d\phi_1 d\phi_0 \Bigg\} \\
&+ \gamma_{0101}^{(\text{up})} \cdot \Bigg\{ \int_0^{\frac{1}{1+\alpha_D+\alpha_Y}} \int_{\phi_0}^{\frac{1}{1+\alpha_D+\alpha_Y}} (\alpha_D+\alpha_Y)(\phi_0+\phi_1)\, d\phi_1 d\phi_0 \\
&\qquad + \int_0^{\frac{1}{1+\alpha_D+\alpha_Y}} \int_{\frac{1}{1+\alpha_D+\alpha_Y}}^1 1 + (\alpha_D+\alpha_Y)\phi_0 - \phi_1\, d\phi_1 d\phi_0 \\
&\qquad + \int_{\frac{1}{1+\alpha_D+\alpha_Y}}^1 \int_{\phi_0}^1 1 + (\alpha_D+\alpha_Y)\phi_0 - \phi_1\, d\phi_1 d\phi_0 \Bigg\},
\end{aligned}
$$

$$\int_{\epsilon\in\mathcal{E}}\int_{\xi\in\mathcal{E}}\big(|\varphi_{t+1}(\xi)|-|\varphi_t(\epsilon)|\big)\cdot\mathbb{1}\{\varphi_{t+1}(\xi)=G(f_t,g_t^D,g_t^{Y^{\text{(ori)}}};1,0,1,0,\epsilon,\alpha_D,\alpha_Y)\}$$
$$\cdot\, q_t\big(f_t(0,\epsilon),f_t(1,\epsilon)\mid 1,0,1,0\big)d\xi d\epsilon$$
$$=\gamma_{1010}^{\text{(up)}}\cdot\Bigg\{\int_0^1\int_0^{\tan^{-1}\left(\frac{3\pi}{4}-\arctan\frac{1}{\alpha_D+\alpha_Y}\right)\phi_1}-(\alpha_D+\alpha_Y)(\phi_0+\phi_1)\,d\phi_0d\phi_1$$
$$+\int_0^{\frac{1}{1+\alpha_D+\alpha_Y}}\int_{\tan^{-1}\left(\frac{3\pi}{4}-\arctan\frac{1}{\alpha_D+\alpha_Y}\right)\phi_1}^{\phi_1}(\alpha_D+\alpha_Y+2)\phi_0+(\alpha_D+\alpha_Y-2)\phi_1\,d\phi_0d\phi_1$$
$$+\int_{\frac{1}{1+\alpha_D+\alpha_Y}}^1\int_{\tan^{-1}\left(\frac{3\pi}{4}-\arctan\frac{1}{\alpha_D+\alpha_Y}\right)\phi_1}^{\frac{1}{1+\alpha_D+\alpha_Y}}(\alpha_D+\alpha_Y+2)\phi_0+(\alpha_D+\alpha_Y-2)\phi_1\,d\phi_0d\phi_1$$
$$+\int_{\frac{1}{1+\alpha_D+\alpha_Y}}^1\int_{\frac{1}{1+\alpha_D+\alpha_Y}}^{\phi_1}1+\phi_0-(2-\alpha_D-\alpha_Y)\phi_1\,d\phi_0d\phi_1\Bigg\}$$
$$+\gamma_{1010}^{\text{(low)}}\cdot\Bigg\{\int_0^{\frac{1}{1+\alpha_D+\alpha_Y}}\int_{\phi_1}^{\frac{1}{1+\alpha_D+\alpha_Y}}(\alpha_D+\alpha_Y)(\phi_0+\phi_1)\,d\phi_0d\phi_1$$
$$+\int_0^{\frac{1}{1+\alpha_D+\alpha_Y}}\int_{\frac{1}{1+\alpha_D+\alpha_Y}}^1 1-\phi_0+(\alpha_D+\alpha_Y)\phi_1\,d\phi_0d\phi_1$$
$$+\int_{\frac{1}{1+\alpha_D+\alpha_Y}}^1\int_{\phi_1}^1 1-\phi_0+(\alpha_D+\alpha_Y)\phi_1\,d\phi_0d\phi_1\Bigg\}.$$

Since $\gamma_{0101}^{\text{(low)}}+\gamma_{0101}^{\text{(up)}}=2$ and $\gamma_{1010}^{\text{(low)}}+\gamma_{1010}^{\text{(up)}}=2$, we can derive the form of $\Delta_{\text{STIR}}^{\text{(Perfect Predictor)}}\big|_t^{t+1}$:

$$\Delta_{\text{STIR}}^{\text{(Perfect Predictor)}}\big|_t^{t+1}=\big(P_t(0,1,0,1)\cdot\gamma_{0101}^{\text{(low)}}+P_t(1,0,1,0)\cdot\gamma_{1010}^{\text{(up)}}\big)\cdot\Bigg\{$$
$$-\frac{1+\alpha_D+\alpha_Y}{3}\cdot\tan^2\big(\arctan\frac{1}{\alpha_D+\alpha_Y}-\frac{\pi}{4}\big)$$
$$+\frac{2(1-\alpha_D-\alpha_Y)}{3}\cdot\tan\big(\arctan\frac{1}{\alpha_D+\alpha_Y}-\frac{\pi}{4}\big)$$
$$-\frac{1-\alpha_D-\alpha_Y}{6}+\frac{3-\alpha_D-\alpha_Y}{2(1+\alpha_D+\alpha_Y)}$$
$$+\frac{3(\alpha_D+\alpha_Y)^3-6(\alpha_D+\alpha_Y)^2-19(\alpha_D+\alpha_Y)-10}{6(1+\alpha_D+\alpha_Y)^3}\Bigg\}$$
$$+\big(P_t(0,1,0,1)+P_t(1,0,1,0)\big)\cdot\Bigg[$$
$$\frac{\alpha_D+\alpha_Y-2}{3}+\frac{\alpha_D+\alpha_Y}{1+\alpha_D+\alpha_Y}+\frac{3(\alpha_D+\alpha_Y)^2+5(\alpha_D+\alpha_Y)+2}{3(1+\alpha_D+\alpha_Y)^3}\Bigg],$$

where $\gamma_{0101}^{\text{(low)}},\gamma_{1010}^{\text{(up)}}\in(0,1)$ (according to Assumption 3.6), and $\alpha_D,\alpha_Y\in[0,\frac{1}{2})$ (according to Assumption 3.3).

Let us denote $\beta(\alpha_D,\alpha_Y)\coloneqq\tan\big(\arctan\frac{1}{\alpha_D+\alpha_Y}-\frac{\pi}{4}\big)$ to simplify the notation. Without loss of generality let us assume that $P_t(0,1,0,1)\cdot\gamma_{0101}^{\text{(low)}}+P_t(1,0,1,0)\cdot\gamma_{1010}^{\text{(up)}}>0$.

We can further compute the partial derivatives and find out that:

$$
\begin{aligned}
&\frac{\partial \Delta_{\text{STIR}}^{\text{(Perfect Predictor)}}\big|_t^{t+1}}{\partial\big(P_t(0,1,0,1)\cdot\gamma_{0101}^{\text{(low)}} + P_t(1,0,1,0)\cdot\gamma_{1010}^{\text{(up)}}\big)} \\
&= -\frac{1+\alpha_D+\alpha_Y}{3}\cdot\beta^2(\alpha_D,\alpha_Y) \\
&\quad + \frac{2(1-\alpha_D-\alpha_Y)}{3}\cdot\beta(\alpha_D,\alpha_Y) \\
&\quad - \frac{1-\alpha_D-\alpha_Y}{6} + \frac{3-\alpha_D-\alpha_Y}{2(1+\alpha_D+\alpha_Y)} \\
&\quad + \frac{3(\alpha_D+\alpha_Y)^3 - 6(\alpha_D+\alpha_Y)^2 - 19(\alpha_D+\alpha_Y) - 10}{6(1+\alpha_D+\alpha_Y)^3} \\
&< 0,\ \forall\, \alpha_D,\alpha_Y \in [0,\frac{1}{2}),
\end{aligned}
$$

and that:

$$
\begin{aligned}
\frac{\partial \Delta_{\text{STIR}}^{\text{(Perfect Predictor)}}\big|_t^{t+1}}{\partial(\alpha_D+\alpha_Y)} &= \big(P_t(0,1,0,1)+P_t(1,0,1,0)\big)\cdot\left[\frac{1}{3}+\frac{2}{3(1+\alpha_D+\alpha_Y)^3}\right] \\
&\quad + \big(P_t(0,1,0,1)\cdot\gamma_{0101}^{\text{(low)}} + P_t(1,0,1,0)\cdot\gamma_{1010}^{\text{(up)}}\big)\cdot\Bigg[\\
&\qquad - \frac{2(1+\alpha_D+\alpha_Y)}{3}\cdot\beta(\alpha_D,\alpha_Y)\cdot\frac{\partial\beta(\alpha_D,\alpha_Y)}{\partial(\alpha_D+\alpha_Y)} \\
&\qquad + \frac{2(1-\alpha_D-\alpha_Y)}{3}\cdot\frac{\partial\beta(\alpha_D,\alpha_Y)}{\partial(\alpha_D+\alpha_Y)} \\
&\qquad + \frac{1}{6} - \frac{2}{(1+\alpha_D+\alpha_Y)^2} + \frac{15(\alpha_D+\alpha_Y)+11}{6(1+\alpha_D+\alpha_Y)^3}\Bigg] \\
&= \big(P_t(0,1,0,1)+P_t(1,0,1,0)\big)\cdot\left[\frac{1}{3}+\frac{2}{3(1+\alpha_D+\alpha_Y)^3}\right] \\
&\quad + \big(P_t(0,1,0,1)\cdot\gamma_{0101}^{\text{(low)}} + P_t(1,0,1,0)\cdot\gamma_{1010}^{\text{(up)}}\big)\cdot\Bigg[\\
&\qquad \frac{2\big(1+\beta^2(\alpha_D,\alpha_Y)\big)\cdot\big[(1+\alpha_D+\alpha_Y)\beta(\alpha_D,\alpha_Y)+\alpha_D+\alpha_Y-1\big]}{3(1+\alpha_D+\alpha_Y)^3} \\
&\qquad + \frac{(\alpha_D+\alpha_Y)^3+3(\alpha_D+\alpha_Y)^2+6(\alpha_D+\alpha_Y)}{6(1+\alpha_D+\alpha_Y)^3}\Bigg] \\
&> 0,\ \forall\, \gamma_{0101}^{\text{(low)}},\gamma_{1010}^{\text{(up)}} \in (0,1), \alpha_D,\alpha_Y\in[0,\frac{1}{2}),
\end{aligned}
$$

where we utilize the fact that $\frac{\partial\beta(\alpha_D,\alpha_Y)}{\partial(\alpha_D+\alpha_Y)} = \big(1+\beta^2(\alpha_D,\alpha_Y)\big)\cdot\frac{1}{1+(\alpha_D+\alpha_Y)^2}$.

Therefore, we can conclude that

$$\begin{aligned}
\Delta_{\text{STIR}}^{(\text{Perfect Predictor})}\Big|_t^{t+1} > \lim_{\substack{\gamma_{0101}^{(\text{low})}\to 1\\ \gamma_{1010}^{(\text{up})}\to 1\\ \alpha_D+\alpha_Y\to 0}} & \Big(P_t(0,1,0,1)\cdot\gamma_{0101}^{(\text{low})} + P_t(1,0,1,0)\cdot\gamma_{1010}^{(\text{up})}\Big)\cdot\Big\{\\
& -\frac{1+\alpha_D+\alpha_Y}{3}\cdot\tan^2\Big(\arctan\frac{1}{\alpha_D+\alpha_Y}-\frac{\pi}{4}\Big)\\
& +\frac{2(1-\alpha_D-\alpha_Y)}{3}\cdot\tan\Big(\arctan\frac{1}{\alpha_D+\alpha_Y}-\frac{\pi}{4}\Big)\\
& -\frac{1-\alpha_D-\alpha_Y}{6}+\frac{3-\alpha_D-\alpha_Y}{2(1+\alpha_D+\alpha_Y)}\\
& +\frac{3(\alpha_D+\alpha_Y)^3-6(\alpha_D+\alpha_Y)^2-19(\alpha_D+\alpha_Y)-10}{6(1+\alpha_D+\alpha_Y)^3}\Big\}\\
& +\Big(P_t(0,1,0,1)+P_t(1,0,1,0)\Big)\cdot\Big[\\
& \frac{\alpha_D+\alpha_Y-2}{3}+\frac{\alpha_D+\alpha_Y}{1+\alpha_D+\alpha_Y}+\frac{3(\alpha_D+\alpha_Y)^2+5(\alpha_D+\alpha_Y)+2}{3(1+\alpha_D+\alpha_Y)^3}\Big]\\
= 0, &
\end{aligned}$$

i.e., under the specified assumptions and dynamics, we have

$$\forall\ \gamma_{0101}^{(\text{low})}, \gamma_{1010}^{(\text{up})} \in (0,1), \alpha_D, \alpha_Y \in [0, \frac{1}{2}) : \Delta_{\text{STIR}}^{(\text{Perfect Predictor})}\Big|_t^{t+1} > 0. \tag{C.3}$$

□

C.4 Proof for Theorem 4.3

Theorem. *Let us consider the general situation where both D_t and $Y_t^{(ori)}$ are dependent with A_t, i.e., $D_t \not\perp\!\!\!\perp A_t, Y_t^{(ori)} \not\perp\!\!\!\perp A_t$. Let us further assume that the data dynamics satisfies $\alpha_D \in (0, \frac{1}{2}), \alpha_Y = 0$. Then under Fact 3.2, Assumption 3.3, and Assumption 3.4, and Assumption 3.6, as well as the specified dynamics, when $H_t \not\perp\!\!\!\perp A_t$, it is possible for the Counterfactual Fair predictor to get closer to the long-term fairness goal after one-step intervention, if certain properties of the data dynamics and the predictor behavior are satisfied simultaneously, i.e.,*

$$\begin{cases}
g_t^D(0, E_t) = g_t^D(1, E_t)\\
\frac{P_t(1,1,0,1)+P_t(1,1,1,0)}{P_t(0,0,0,1)+P_t(0,0,1,0)} < \frac{27}{8}\\
\alpha_D \in \Big(\big(\frac{P_t(1,1,0,1)+P_t(1,1,1,0)}{P_t(0,0,0,1)+P_t(0,0,1,0)}\big)^{\frac{1}{3}} - 1, \frac{1}{2}\Big)\\
\alpha_Y = 0
\end{cases}$$
$$\Longrightarrow\ \Delta_{\text{STIR}}^{(\textit{Counterfactual Fair})}\Big|_t^{t+1} < 0.$$

Proof (sketch). Similar to proving Theorem 4.2 (proof in Appendix C.3), the goal is to calculate if it is possible for the *Single-step Tier Imbalance Reduction* $\Delta_{\text{STIR}}|_t^{t+1}$ to be smaller than 0 when using Counterfactual Fair predictors.

Since $\Delta_{\text{STIR}}|_t^{t+1}$ is a weighted aggregation of $|\varphi(e_{t+1})| - |\varphi(e_t)|$ (as defined in Equation 5), the quantitative analysis involves three key components: instantiations of $\varphi_{t+1}(e_{t+1})$, the knowledge/assumptions on $q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid d, d', y, y'\big)$, and characteristics of $P_t(d, d', y', y')$.

For the first component, since $\alpha_Y = 0$ is a special case of scenarios where $\alpha_D > \alpha_Y$, we can list all possible instantiations of $\varphi_{t+1}(e_{t+1})$ in Table 4 (when $\alpha_D > \alpha_Y$). For the second component, we can introduce a quantitative assumption on $q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid d, d', y, y'\big)$ (Assumption 3.6). For the third component, we need to exploit the characteristic of the predictor of interest to gain further insight into the joint distribution $P_t(d, d', y, y')$. For Counterfactual Fair predictors, we have $P_t(d, d', y, y')$ satisfies Equation 10 (as we have discussed in Section 4.2.2).

For the purpose of calculating the value of $\Delta_{\text{STIR}}|_t^{t+1}$, the proof contains two steps: (1) exhaustively derive the value of $|\varphi(e_{t+1})| - |\varphi(e_t)|$ after one-step dynamics (finished in Appendix C.3 when proving Theorem 4.2), and (2) aggregate the difference $|\varphi(e_{t+1})| - |\varphi(e_t)|$ with the help of the additional knowledge/assumptions on $q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid d, d', y, y'\big)$ and $P_t(d, d', y, y')$. $\square$

Proof (full). Based on the definition of $\Delta_{\text{STIR}}|_t^{t+1}$, the proof calculates the aggregation (integration followed by summation) of the difference $|\varphi(e_{t+1})| - |\varphi(e_t)|$ with the help of the additional knowledge/assumptions on $q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid d, d', y, y'\big)$ and $P_t(d, d', y, y')$.

Since we assume $\alpha_D \in (0, \frac{1}{2}), \alpha_Y = 0$, we focus on possible instantiations of $\varphi_{t+1}(e_{t+1})$ as listed in Table 4 ($\alpha_D > \alpha_Y$). For the Counterfactual Fair predictor that satisfies $g_t^D(0, E_t) = g_t^D(1, E_t)$, not every case in Table 4 corresponds to a nonzero $P_t(d, d', y, y')$ and therefore may not contribute to the computation of $\Delta_{\text{STIR}}|_t^{t+1}$ as detailed in Equation 5. By applying Equation 10 we need to consider Case (i), Case (ii), Case (iii), Case (iv), Case (xiii), Case (xiv), Case (xv), and Case (xvi) in Table 4.

When (d, d', y, y') satisfies $y = y'$, i.e, for Case (i), Case (iv), Case (xiii), and Case (xvi), we have $|\varphi(e_{t+1})| - |\varphi(e_t)| = 0$. Therefore we only need to calculate $\Delta_{\text{STIR}}^{\text{(Counterfactual Fair)}}|_t^{t+1}$ for Case (ii), Case (iii), Case (xiv), and Case (xv) (although $\alpha_Y = 0$, we explicitly keep the hyperparameter α_Y in the proof for the purpose of notation consistency).

According to Equation 5 and Equation B.5, for the Counterfactual Fair predictor we have:

$$
\begin{aligned}
\Delta_{\text{STIR}}^{\text{(Counterfactual Fair)}}\Big|_t^{t+1} = P_t(0,0,0,1) \cdot \int_{\epsilon \in \mathcal{E}} \int_{\xi \in \mathcal{E}} & \big(|\varphi_{t+1}(\xi)| - |\varphi_t(\epsilon)|\big) \\
& \cdot \mathbb{1}\{\varphi_{t+1}(\xi) = G(f_t, g_t^D, g_t^{Y^{\text{(ori)}}}; 0,0,0,1,\epsilon,\alpha_D,\alpha_Y)\} \\
& \cdot q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid 0,0,0,1\big) d\xi d\epsilon \\
+ P_t(0,0,1,0) \cdot \int_{\epsilon \in \mathcal{E}} \int_{\xi \in \mathcal{E}} & \big(|\varphi_{t+1}(\xi)| - |\varphi_t(\epsilon)|\big) \\
& \cdot \mathbb{1}\{\varphi_{t+1}(\xi) = G(f_t, g_t^D, g_t^{Y^{\text{(ori)}}}; 0,0,1,0,\epsilon,\alpha_D,\alpha_Y)\} \\
& \cdot q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid 0,0,1,0\big) d\xi d\epsilon \\
+ P_t(1,1,0,1) \cdot \int_{\epsilon \in \mathcal{E}} \int_{\xi \in \mathcal{E}} & \big(|\varphi_{t+1}(\xi)| - |\varphi_t(\epsilon)|\big) \\
& \cdot \mathbb{1}\{\varphi_{t+1}(\xi) = G(f_t, g_t^D, g_t^{Y^{\text{(ori)}}}; 1,1,0,1,\epsilon,\alpha_D,\alpha_Y)\} \\
& \cdot q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid 1,1,0,1\big) d\xi d\epsilon \\
+ P_t(1,1,1,0) \cdot \int_{\epsilon \in \mathcal{E}} \int_{\xi \in \mathcal{E}} & \big(|\varphi_{t+1}(\xi)| - |\varphi_t(\epsilon)|\big) \\
& \cdot \mathbb{1}\{\varphi_{t+1}(\xi) = G(f_t, g_t^D, g_t^{Y^{\text{(ori)}}}; 1,1,1,0,\epsilon,\alpha_D,\alpha_Y)\} \\
& \cdot q_t\big(f_t(0,\epsilon), f_t(1,\epsilon) \mid 1,1,1,0\big) d\xi d\epsilon.
\end{aligned}
\tag{C.4}
$$

Similar to the proof of the result for perfect predictors presented in Appendix C.3, with the help of Assumption 3.6, we convert the conditional expectations in Equation C.4 into calculations of multiple integrals on slices within a 1×1 square on the 2-D plane, where ϕ_0 and ϕ_1 axes correspond to the value of $f_t(0, E_t)$ and $f_t(1, E_t)$ respectively:

$$
\begin{aligned}
&\int_{\epsilon\in\mathcal{E}}\int_{\xi\in\mathcal{E}}\big(|\varphi_{t+1}(\xi)|-|\varphi_t(\epsilon)|\big)\cdot\mathbb{1}\{\varphi_{t+1}(\xi)=G(f_t,g_t^D,g_t^{Y^{\text{(ori)}}};0,0,0,1,\epsilon,\alpha_D,\alpha_Y)\}\\
&\qquad\cdot q_t\big(f_t(0,\epsilon),f_t(1,\epsilon)\mid 0,0,0,1\big)d\xi d\epsilon\\
&=\gamma_{0001}^{\text{(low)}}\cdot\Bigg\{\int_0^1\int_0^{\frac{1-\alpha_D-\alpha_Y}{1-\alpha_D+\alpha_Y}\phi_0}-(\alpha_D+\alpha_Y)\phi_0+(\alpha_D-\alpha_Y)\phi_1\;d\phi_1 d\phi_0\\
&\qquad+\int_0^1\int_{\frac{1-\alpha_D-\alpha_Y}{1-\alpha_D+\alpha_Y}\phi_0}^{\phi_0}-(2-\alpha_D-\alpha_Y)\phi_0+(2-\alpha_D+\alpha_Y)\phi_1\;d\phi_1 d\phi_0\Bigg\}\\
&\quad+\gamma_{0001}^{\text{(up)}}\cdot\int_0^1\int_0^{\phi_1}(\alpha_D+\alpha_Y)\phi_0-(\alpha_D-\alpha_Y)\phi_1\;d\phi_0 d\phi_1,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\epsilon\in\mathcal{E}}\int_{\xi\in\mathcal{E}}\big(|\varphi_{t+1}(\xi)|-|\varphi_t(\epsilon)|\big)\cdot\mathbb{1}\{\varphi_{t+1}(\xi)=G(f_t,g_t^D,g_t^{Y^{\text{(ori)}}};0,0,1,0,\epsilon,\alpha_D,\alpha_Y)\}\\
&\qquad\cdot q_t\big(f_t(0,\epsilon),f_t(1,\epsilon)\mid 0,0,1,0\big)d\xi d\epsilon\\
&=\gamma_{0010}^{\text{(up)}}\cdot\Bigg\{\int_0^1\int_0^{\frac{1-\alpha_D-\alpha_Y}{1-\alpha_D+\alpha_Y}\phi_1}(\alpha_D-\alpha_Y)\phi_0-(\alpha_D+\alpha_Y)\phi_1\;d\phi_0 d\phi_1\\
&\qquad+\int_0^1\int_{\frac{1-\alpha_D-\alpha_Y}{1-\alpha_D+\alpha_Y}\phi_1}^{\phi_1}(2-\alpha_D+\alpha_Y)\phi_0-(2-\alpha_D-\alpha_Y)\phi_1\;d\phi_0 d\phi_1\Bigg\}\\
&\quad+\gamma_{0010}^{\text{(low)}}\cdot\int_0^1\int_0^{\phi_0}-(\alpha_D-\alpha_Y)\phi_0+(\alpha_D+\alpha_Y)\phi_1\;d\phi_1 d\phi_0,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\epsilon\in\mathcal{E}}\int_{\xi\in\mathcal{E}}\big(|\varphi_{t+1}(\xi)|-|\varphi_t(\epsilon)|\big)\cdot\mathbb{1}\{\varphi_{t+1}(\xi)=G(f_t,g_t^D,g_t^{Y^{\text{(ori)}}};1,1,0,1,\epsilon,\alpha_D,\alpha_Y)\}\\
&\qquad\cdot q_t\big(f_t(0,\epsilon),f_t(1,\epsilon)\mid 1,1,0,1\big)d\xi d\epsilon\\
&=\gamma_{1101}^{\text{(low)}}\cdot\Bigg\{\int_0^{\frac{1}{1+\alpha_D+\alpha_Y}}\int_{\phi_1}^{\frac{1+\alpha_D+\alpha_Y}{1+\alpha_D-\alpha_Y}\phi_1}-(2+\alpha_D-\alpha_Y)\phi_0+(2+\alpha_D+\alpha_Y)\phi_1\;d\phi_0d\phi_1\\
&\qquad+\int_0^{\frac{1}{1+\alpha_D-\alpha_Y}}\int_0^{\frac{1+\alpha_D-\alpha_Y}{1+\alpha_D+\alpha_Y}\phi_0}(\alpha_D-\alpha_Y)\phi_0-(\alpha_D+\alpha_Y)\phi_1\;d\phi_1d\phi_0\\
&\qquad+\int_{\frac{1}{1+\alpha_D+\alpha_Y}}^{\frac{1}{1+\alpha_D-\alpha_Y}}\int_{\frac{1}{1+\alpha_D+\alpha_Y}}^{\phi_0}1-(2+\alpha_D-\alpha_Y)\phi_0+\phi_1\;d\phi_1d\phi_0\Bigg\}\\
&\quad+\gamma_{1101}^{\text{(up)}}\cdot\Bigg\{\int_0^{\frac{1}{1+\alpha_D+\alpha_Y}}\int_0^{\phi_1}-(\alpha_D-\alpha_Y)\phi_0+(\alpha_D+\alpha_Y)\phi_1\;d\phi_0d\phi_1\\
&\qquad+\int_0^{\frac{1}{1+\alpha_D+\alpha_Y}}\int_{\frac{1}{1+\alpha_D+\alpha_Y}}^{1}1-(\alpha_D-\alpha_Y)\phi_0-\phi_1\;d\phi_1d\phi_0\\
&\qquad+\int_{\frac{1}{1+\alpha_D+\alpha_Y}}^{\frac{1}{1+\alpha_D-\alpha_Y}}\int_{\phi_0}^{1}1-(\alpha_D-\alpha_Y)\phi_0-\phi_1\;d\phi_1d\phi_0\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\epsilon\in\mathcal{E}}\int_{\xi\in\mathcal{E}}\big(|\varphi_{t+1}(\xi)|-|\varphi_t(\epsilon)|\big)\cdot\mathbb{1}\{\varphi_{t+1}(\xi)=G(f_t,g_t^D,g_t^{Y^{\text{(ori)}}};1,1,1,0,\epsilon,\alpha_D,\alpha_Y)\}\\
&\qquad\cdot q_t\big(f_t(0,\epsilon),f_t(1,\epsilon)\mid 1,1,1,0\big)d\xi d\epsilon\\
&=\gamma_{1110}^{\text{(up)}}\cdot\Bigg\{\int_0^{\frac{1}{1+\alpha_D+\alpha_Y}}\int_{\phi_0}^{\frac{1+\alpha_D+\alpha_Y}{1+\alpha_D-\alpha_Y}\phi_0}(2+\alpha_D+\alpha_Y)\phi_0-(2+\alpha_D-\alpha_Y)\phi_1\;d\phi_1d\phi_0\\
&\qquad+\int_0^{\frac{1}{1+\alpha_D-\alpha_Y}}\int_0^{\frac{1+\alpha_D-\alpha_Y}{1+\alpha_D+\alpha_Y}\phi_1}-(\alpha_D+\alpha_Y)\phi_0+(\alpha_D-\alpha_Y)\phi_1\;d\phi_0d\phi_1\\
&\qquad+\int_{\frac{1}{1+\alpha_D-\alpha_Y}}^{1}\int_0^{\frac{1}{1+\alpha_D+\alpha_Y}}1-\phi_0-(\alpha_D+\alpha_Y)\phi_1\;d\phi_1d\phi_0\Bigg\}\\
&\quad+\gamma_{1110}^{\text{(low)}}\cdot\int_0^{\frac{1}{1+\alpha_D+\alpha_Y}}\int_0^{\phi_0}(\alpha_D+\alpha_Y)\phi_0-(\alpha_D-\alpha_Y)\phi_1\;d\phi_1d\phi_0.
\end{aligned}
$$

Since $\gamma_{0001}^{(\text{low})}+\gamma_{0001}^{(\text{up})}=2$, $\gamma_{0010}^{(\text{low})}+\gamma_{0010}^{(\text{up})}=2$, $\gamma_{1101}^{(\text{low})}+\gamma_{1101}^{(\text{up})}=2$, and $\gamma_{1110}^{(\text{low})}+\gamma_{1110}^{(\text{up})}=2$, we can derive the form of the term $\Delta_{\text{STIR}}^{(\text{Counterfactual Fair})}\big|_t^{t+1}$:

$$
\begin{aligned}
\Delta_{\text{STIR}}^{(\text{Counterfactual Fair})}\big|_t^{t+1} &= \big(P_t(0,0,0,1)\cdot\gamma_{0001}^{(\text{low})}+P_t(0,0,1,0)\cdot\gamma_{0010}^{(\text{up})}\big)\cdot\Big\{\\
&\quad \frac{(\alpha_D-\alpha_Y)(1-\alpha_D-\alpha_Y)^2}{6(1-\alpha_D+\alpha_Y)^2}-\frac{(\alpha_D+\alpha_Y)(1-\alpha_D-\alpha_Y)}{3(1-\alpha_D+\alpha_Y)}\\
&\quad +\frac{2\alpha_Y}{3(1-\alpha_D+\alpha_Y)}\Big[-(2-\alpha_D-\alpha_Y)+\frac{(2-\alpha_D+\alpha_Y)(1-\alpha_D)}{1-\alpha_D+\alpha_Y}\Big]+\frac{\alpha_D}{6}-\frac{\alpha_Y}{2}\Big\}\\
&+P_t(1,1,0,1)\cdot\gamma_{1101}^{(\text{low})}\cdot\Big\{\\
&\quad \frac{2\alpha_Y}{3(1+\alpha_D-\alpha_Y)(1+\alpha_D+\alpha_Y)^3}\Big[(2+\alpha_D+\alpha_Y)-\frac{(2+\alpha_D-\alpha_Y)(1+\alpha_D)}{1+\alpha_D-\alpha_Y}\Big]\\
&\quad +\frac{1}{3(1+\alpha_D+\alpha_Y)(1+\alpha_D-\alpha_Y)^3}\Big[\\
&\qquad (\alpha_D-\alpha_Y)(1+\alpha_D-\alpha_Y)-\frac{(\alpha_D+\alpha_Y)(1+\alpha_D-\alpha_Y)^2}{2(1+\alpha_D+\alpha_Y)}\Big]\\
&\quad -\frac{1}{(1+\alpha_D+\alpha_Y)^3}\big(\frac{\alpha_D}{6}+\frac{\alpha_Y}{2}\big)\\
&\quad -\frac{2}{3}(1+\alpha_D-\alpha_Y)\cdot\Big[\frac{1}{(1+\alpha_D-\alpha_Y)^3}-\frac{1}{(1+\alpha_D+\alpha_Y)^3}\Big]\\
&\quad +\Big[\frac{3+2\alpha_D}{2(1+\alpha_D+\alpha_Y)}+\frac{1+\alpha_D-\alpha_Y}{2}\Big]\cdot\Big[\frac{1}{(1+\alpha_D-\alpha_Y)^2}-\frac{1}{(1+\alpha_D+\alpha_Y)^2}\Big]\\
&\quad -\Big[\frac{3+2\alpha_D+2\alpha_Y}{2(1+\alpha_D+\alpha_Y)^2}+\frac{1}{2}\Big]\cdot\Big[\frac{1}{1+\alpha_D-\alpha_Y}-\frac{1}{1+\alpha_D+\alpha_Y}\Big]-\frac{(\alpha_D+\alpha_Y)\alpha_Y}{(1+\alpha_D+\alpha_Y)^3}\Big\}\\
&+P_t(1,1,1,0)\cdot\gamma_{1110}^{(\text{up})}\cdot\Big\{\\
&\quad \frac{2\alpha_Y}{3(1+\alpha_D-\alpha_Y)(1+\alpha_D+\alpha_Y)^3}\Big[(2+\alpha_D+\alpha_Y)-\frac{(2+\alpha_D-\alpha_Y)(1+\alpha_D)}{1+\alpha_D-\alpha_Y}\Big]\\
&\quad +\frac{1}{3(1+\alpha_D+\alpha_Y)(1+\alpha_D-\alpha_Y)^3}\Big[\\
&\qquad (\alpha_D-\alpha_Y)(1+\alpha_D-\alpha_Y)-\frac{(\alpha_D+\alpha_Y)(1+\alpha_D-\alpha_Y)^2}{2(1+\alpha_D+\alpha_Y)}\Big]\\
&\quad +\frac{\alpha_D-\alpha_Y}{(1+\alpha_D+\alpha_Y)(1+\alpha_D-\alpha_Y)}-\frac{(\alpha_D+\alpha_Y)(\alpha_D-\alpha_Y)}{2(1+\alpha_D+\alpha_Y)^2(1+\alpha_D-\alpha_Y)}\\
&\quad -\frac{1}{2(1+\alpha_D+\alpha_Y)}\Big[1-\frac{1}{(1+\alpha_D-\alpha_Y)^2}\Big]-\big(\frac{\alpha_D}{6}+\frac{\alpha_Y}{2}\big)\frac{1}{(1+\alpha_D+\alpha_Y)^3}\Big\}\\
&+\big(P_t(0,0,0,1)+P_t(0,0,1,0)\big)\cdot\big(-\frac{\alpha_D}{3}+\alpha_Y\big)\\
&+P_t(1,1,0,1)\cdot\Big\{\frac{1}{(1+\alpha_D+\alpha_Y)^3}\big(\frac{\alpha_D}{3}+\alpha_Y\big)\\
&\quad +\frac{2(\alpha_D+\alpha_Y)\alpha_Y}{(1+\alpha_D+\alpha_Y)^3}+\frac{1}{1+\alpha_D-\alpha_Y}-\frac{1}{1+\alpha_D+\alpha_Y}\\
&\quad +\big(\frac{1}{3}+\frac{2\alpha_D}{3}-\frac{2\alpha_Y}{3}\big)\cdot\Big[\frac{1}{(1+\alpha_D-\alpha_Y)^3}-\frac{1}{(1+\alpha_D+\alpha_Y)^3}\Big]\\
&\quad -(1+\alpha_D-\alpha_Y)\cdot\Big[\frac{1}{(1+\alpha_D-\alpha_Y)^2}-\frac{1}{(1+\alpha_D+\alpha_Y)^2}\Big]\Big\}\\
&+P_t(1,1,1,0)\cdot\big(\frac{\alpha_D}{3}+\alpha_Y\big)\frac{1}{(1+\alpha_D+\alpha_Y)^3},
\end{aligned}
$$

where $\gamma_{0101}^{\text{(low)}}, \gamma_{1010}^{\text{(up)}} \in (0, 1)$ (according to Assumption 3.6), and $\alpha_D, \alpha_Y \in [0, \frac{1}{2})$ (according to Assumption 3.3).

Now let us consider the data dynamics where $\alpha_Y = 0$ and simplify the form of $\Delta_{\mathrm{STIR}}^{\text{(Counterfactual Fair)}}\big|_t^{t+1}$:

$$\begin{aligned}
&\Delta_{\mathrm{STIR}}^{\text{(Counterfactual Fair)}}\big|_t^{t+1} \\
&= -\frac{\alpha_D}{3} \cdot \big(P_t(0,0,0,1) + P_t(0,0,1,0)\big) + \frac{\alpha_D}{3(1+\alpha_D)^3} \cdot \big(P_t(1,1,0,1) + P_t(1,1,1,0)\big).
\end{aligned}$$

As we can see, as long as we have $\frac{P_t(1,1,0,1)+P_t(1,1,1,0)}{P_t(0,0,0,1)+P_t(0,0,1,0)} < \frac{27}{8}$ and at the same time the parameter satisfies $\alpha_D \in \Big(\big(\frac{P_t(1,1,0,1)+P_t(1,1,1,0)}{P_t(0,0,0,1)+P_t(0,0,1,0)}\big)^{\frac{1}{3}} - 1, \frac{1}{2}\Big)$, it is possible for the counterfactual fair predictor to achieve a negative value for $\Delta_{\mathrm{STIR}}\big|_t^{t+1}$ after a one-step intervention:

$$\begin{cases}
\frac{P_t(1,1,0,1)+P_t(1,1,1,0)}{P_t(0,0,0,1)+P_t(0,0,1,0)} < \frac{27}{8} \\
\alpha_D \in \Big(\big(\frac{P_t(1,1,0,1)+P_t(1,1,1,0)}{P_t(0,0,0,1)+P_t(0,0,1,0)}\big)^{\frac{1}{3}} - 1, \frac{1}{2}\Big) \\
\alpha_Y = 0
\end{cases} \implies \Delta_{\mathrm{STIR}}^{\text{(Counterfactual Fair)}}\big|_t^{t+1} < 0.$$

□

Table 1: When $\alpha_D > \alpha_Y$, compare cases of H_{t+1} with different values of A_t.

Case	D_t		$Y_t^{(\text{ori})}$		H_{t+1}	
	if $A_t = 0$	if $A_t = 1$	if $A_t = 0$	if $A_t = 1$	if $A_t = 0$	if $A_t = 1$
(i)	0	0	0	0	$f_t(0, e_t)$	$f_t(1, e_t)$
(ii)	0	0	0	1	$f_t(0, e_t)(1 - \alpha_D - \alpha_Y)$	$f_t(1, e_t)(1 - \alpha_D + \alpha_Y)$
(iii)	0	0	1	0	$f_t(0, e_t)(1 - \alpha_D + \alpha_Y)$	$f_t(1, e_t)(1 - \alpha_D - \alpha_Y)$
(iv)	0	0	1	1	$f_t(0, e_t)$	$f_t(1, e_t)$
(v)	0	1	0	0	$f_t(0, e_t)(1 - \alpha_D - \alpha_Y)$	$\min\{f_t(1, e_t)(1 + \alpha_D - \alpha_Y), 1\}$
(vi)	0	1	0	1	$f_t(0, e_t)(1 - \alpha_D - \alpha_Y)$	$\min\{f_t(1, e_t)(1 + \alpha_D + \alpha_Y), 1\}$
(vii)	0	1	1	0	$f_t(0, e_t)(1 - \alpha_D + \alpha_Y)$	$\min\{f_t(1, e_t)(1 + \alpha_D - \alpha_Y), 1\}$
(viii)	0	1	1	1	$f_t(0, e_t)(1 - \alpha_D + \alpha_Y)$	$\min\{f_t(1, e_t)(1 + \alpha_D + \alpha_Y), 1\}$
(ix)	1	0	0	0	$\min\{f_t(0, e_t)(1 + \alpha_D - \alpha_Y), 1\}$	$f_t(1, e_t)(1 - \alpha_D - \alpha_Y)$
(x)	1	0	0	1	$\min\{f_t(0, e_t)(1 + \alpha_D - \alpha_Y), 1\}$	$f_t(1, e_t)(1 - \alpha_D + \alpha_Y)$
(xi)	1	0	1	0	$\min\{f_t(0, e_t)(1 + \alpha_D + \alpha_Y), 1\}$	$f_t(1, e_t)(1 - \alpha_D - \alpha_Y)$
(xii)	1	0	1	1	$\min\{f_t(0, e_t)(1 + \alpha_D + \alpha_Y), 1\}$	$f_t(1, e_t)(1 - \alpha_D + \alpha_Y)$
(xiii)	1	1	0	0	$f_t(0, e_t)$	$f_t(1, e_t)$
(xiv)	1	1	0	1	$\min\{f_t(0, e_t)(1 + \alpha_D - \alpha_Y), 1\}$	$\min\{f_t(1, e_t)(1 + \alpha_D + \alpha_Y), 1\}$
(xv)	1	1	1	0	$\min\{f_t(0, e_t)(1 + \alpha_D + \alpha_Y), 1\}$	$\min\{f_t(1, e_t)(1 + \alpha_D - \alpha_Y), 1\}$
(xvi)	1	1	1	1	$f_t(0, e_t)$	$f_t(1, e_t)$

Table 2: When $\alpha_D < \alpha_Y$, compare cases of H_{t+1} with different values of A_t.

Case	D_t		$Y_t^{(\text{ori})}$		H_{t+1}	
	if $A_t = 0$	if $A_t = 1$	if $A_t = 0$	if $A_t = 1$	if $A_t = 0$	if $A_t = 1$
(i)	0	0	0	0	$f_t(0, e_t)$	$f_t(1, e_t)$
(ii)	0	0	0	1	$f_t(0, e_t)(1 - \alpha_D - \alpha_Y)$	$\min\{f_t(1, e_t)(1 - \alpha_D + \alpha_Y), 1\}$
(iii)	0	0	1	0	$\min\{f_t(0, e_t)(1 - \alpha_D + \alpha_Y), 1\}$	$f_t(1, e_t)(1 - \alpha_D - \alpha_Y)$
(iv)	0	0	1	1	$f_t(0, e_t)$	$f_t(1, e_t)$
(v)	0	1	0	0	$f_t(0, e_t)(1 - \alpha_D - \alpha_Y)$	$f_t(1, e_t)(1 + \alpha_D - \alpha_Y)$
(vi)	0	1	0	1	$f_t(0, e_t)(1 - \alpha_D - \alpha_Y)$	$\min\{f_t(1, e_t)(1 + \alpha_D + \alpha_Y), 1\}$
(vii)	0	1	1	0	$\min\{f_t(0, e_t)(1 - \alpha_D + \alpha_Y), 1\}$	$f_t(1, e_t)(1 + \alpha_D - \alpha_Y)$
(viii)	0	1	1	1	$\min\{f_t(0, e_t)(1 - \alpha_D + \alpha_Y), 1\}$	$\min\{f_t(1, e_t)(1 + \alpha_D + \alpha_Y), 1\}$
(ix)	1	0	0	0	$f_t(0, e_t)(1 + \alpha_D - \alpha_Y)$	$f_t(1, e_t)(1 - \alpha_D - \alpha_Y)$
(x)	1	0	0	1	$f_t(0, e_t)(1 + \alpha_D - \alpha_Y)$	$\min\{f_t(1, e_t)(1 - \alpha_D + \alpha_Y), 1\}$
(xi)	1	0	1	0	$\min\{f_t(0, e_t)(1 + \alpha_D + \alpha_Y), 1\}$	$f_t(1, e_t)(1 - \alpha_D - \alpha_Y)$
(xii)	1	0	1	1	$\min\{f_t(0, e_t)(1 + \alpha_D + \alpha_Y), 1\}$	$\min\{f_t(1, e_t)(1 - \alpha_D + \alpha_Y), 1\}$
(xiii)	1	1	0	0	$f_t(0, e_t)$	$f_t(1, e_t)$
(xiv)	1	1	0	1	$f_t(0, e_t)(1 + \alpha_D - \alpha_Y)$	$\min\{f_t(1, e_t)(1 + \alpha_D + \alpha_Y), 1\}$
(xv)	1	1	1	0	$\min\{f_t(0, e_t)(1 + \alpha_D + \alpha_Y), 1\}$	$f_t(1, e_t)(1 + \alpha_D - \alpha_Y)$
(xvi)	1	1	1	1	$f_t(0, e_t)$	$f_t(1, e_t)$

Table 3: When $\alpha_D = \alpha_Y = \alpha$, compare cases of H_{t+1} with different values of A_t.

Case	D_t		$Y_t^{\text{(ori)}}$		H_{t+1}	
	if $A_t = 0$	if $A_t = 1$	if $A_t = 0$	if $A_t = 1$	if $A_t = 0$	if $A_t = 1$
(i)	0	0	0	0	$f_t(0, e_t)$	$f_t(1, e_t)$
(ii)	0	0	0	1	$f_t(0, e_t)(1 - 2\alpha)$	$f_t(1, e_t)$
(iii)	0	0	1	0	$f_t(0, e_t)$	$f_t(1, e_t)(1 - 2\alpha)$
(iv)	0	0	1	1	$f_t(0, e_t)$	$f_t(1, e_t)$
(v)	0	1	0	0	$f_t(0, e_t)(1 - 2\alpha)$	$f_t(1, e_t)$
(vi)	0	1	0	1	$f_t(0, e_t)(1 - 2\alpha)$	$\min\{f_t(1, e_t)(1 + 2\alpha), 1\}$
(vii)	0	1	1	0	$f_t(0, e_t)$	$f_t(1, e_t)$
(viii)	0	1	1	1	$f_t(0, e_t)$	$\min\{f_t(1, e_t)(1 + 2\alpha), 1\}$
(ix)	1	0	0	0	$f_t(0, e_t)$	$f_t(1, e_t)(1 - 2\alpha)$
(x)	1	0	0	1	$f_t(0, e_t)$	$f_t(1, e_t)$
(xi)	1	0	1	0	$\min\{f_t(0, e_t)(1 + 2\alpha), 1\}$	$f_t(1, e_t)(1 - 2\alpha)$
(xii)	1	0	1	1	$\min\{f_t(0, e_t)(1 + 2\alpha), 1\}$	$f_t(1, e_t)$
(xiii)	1	1	0	0	$f_t(0, e_t)$	$f_t(1, e_t)$
(xiv)	1	1	0	1	$f_t(0, e_t)$	$\min\{f_t(1, e_t)(1 + 2\alpha), 1\}$
(xv)	1	1	1	0	$\min\{f_t(0, e_t)(1 + 2\alpha), 1\}$	$f_t(1, e_t)$
(xvi)	1	1	1	1	$f_t(0, e_t)$	$f_t(1, e_t)$

Table 4: When $\alpha_D > \alpha_Y$, list possible instantiations of $\varphi_{t+1}(e_{t+1})$.

Case	D_t		$Y_t^{(\text{ori})}$		$\varphi_{t+1}(e_{t+1}) = f_{t+1}(0, e_{t+1}) - f_{t+1}(1, e_{t+1})$	
	if $A_t = 0$	if $A_t = 1$	if $A_t = 0$	if $A_t = 1$		
(i)	0	0	0	0		$\varphi_t(e_t)$
(ii)	0	0	0	1		$\varphi_t(e_t)(1 - \alpha_D) - \alpha_Y \eta_t(e_t)$
(iii)	0	0	1	0		$\varphi_t(e_t)(1 - \alpha_D) + \alpha_Y \eta_t(e_t)$
(iv)	0	0	1	1		$\varphi_t(e_t)$
(v)	0	1	0	0	(v.1)	$\varphi_t(e_t)(1 - \alpha_Y) - \alpha_D \eta_t(e_t)$, if $f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D-\alpha_Y})$
	-	-	-	-	(v.2)	$f_t(0, e_t)(1 - \alpha_D - \alpha_Y) - 1$, otherwise
(vi)	0	1	0	1	(vi.1)	$\varphi_t(e_t) - (\alpha_D + \alpha_Y)\eta_t(e_t)$, if $f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$
	-	-	-	-	(vi.2)	$f_t(0, e_t)(1 - \alpha_D - \alpha_Y) - 1$, otherwise
(vii)	0	1	1	0	(vii.1)	$\varphi_t(e_t) - (\alpha_D - \alpha_Y)\eta_t(e_t)$, if $f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D-\alpha_Y})$
	-	-	-	-	(vii.2)	$f_t(0, e_t)(1 - \alpha_D + \alpha_Y) - 1$, otherwise
(viii)	0	1	1	1	(viii.1)	$\varphi_t(e_t)(1 + \alpha_Y) - \alpha_D \eta_t(e_t)$, if $f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$
	-	-	-	-	(viii.2)	$f_t(0, e_t)(1 - \alpha_D + \alpha_Y) - 1$, otherwise
(ix)	1	0	0	0	(ix.1)	$\varphi_t(e_t)(1 - \alpha_Y) + \alpha_D \eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D-\alpha_Y})$
	-	-	-	-	(ix.2)	$1 - f_t(1, e_t)(1 - \alpha_D - \alpha_Y)$, otherwise
(x)	1	0	0	1	(x.1)	$\varphi_t(e_t) + (\alpha_D - \alpha_Y)\eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D-\alpha_Y})$
	-	-	-	-	(x.2)	$1 - f_t(1, e_t)(1 - \alpha_D + \alpha_Y)$, otherwise
(xi)	1	0	1	0	(xi.1)	$\varphi_t(e_t) + (\alpha_D + \alpha_Y)\eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$
	-	-	-	-	(xi.2)	$1 - f_t(1, e_t)(1 - \alpha_D - \alpha_Y)$, otherwise
(xii)	1	0	1	1	(xii.1)	$\varphi_t(e_t)(1 + \alpha_Y) + \alpha_D \eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$
					(xii.2)	$1 - f_t(1, e_t)(1 - \alpha_D + \alpha_Y)$, otherwise

Table 4 (continued from the previous page)

Case	D_t		$Y_t^{\text{(ori)}}$			$\varphi_{t+1}(e_{t+1}) = f_{t+1}(0, e_{t+1}) - f_{t+1}(1, e_{t+1})$
	if $A_t = 0$	if $A_t = 1$	if $A_t = 0$	if $A_t = 1$		
(xiii)	1	1	0	0		$\varphi_t(e_t)$
(xiv)	1	1	0	1	(xiv.1)	$\varphi_t(e_t)(1+\alpha_D) - \alpha_Y \eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D-\alpha_Y})$ and $f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$
	-	-	-	-	(xiv.2)	$f_t(0, e_t)(1+\alpha_D-\alpha_Y) - 1$, if $f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D-\alpha_Y})$ and $f_t(1, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1]$
	-	-	-	-	(xiv.3)	$1 - f_t(1, e_t)(1+\alpha_D+\alpha_Y)$, if $f_t(0, e_t) \in [\frac{1}{1+\alpha_D-\alpha_Y}, 1]$ and $f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$
	-	-	-	-	(xiv.4)	0, otherwise
(xv)	1	1	1	0	(xv.1)	$\varphi_t(e_t)(1+\alpha_D) + \alpha_Y \eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$ and $f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D-\alpha_Y})$
	-	-	-	-	(xv.2)	$f_t(0, e_t)(1+\alpha_D+\alpha_Y) - 1$, if $f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$ and $f_t(1, e_t) \in [\frac{1}{1+\alpha_D-\alpha_Y}, 1]$
	-	-	-	-	(xv.3)	$1 - f_t(1, e_t)(1+\alpha_D-\alpha_Y)$, if $f_t(0, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1]$ and $f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D-\alpha_Y})$
	-	-	-	-	(xv.4)	0, otherwise
(xvi)	1	1	1	1		$\varphi_t(e_t)$

Table 5: When $\alpha_D < \alpha_Y$, list possible instantiations of $\varphi_{t+1}(e_{t+1})$.

Case	D_t		$Y_t^{(\text{ori})}$		$\varphi_{t+1}(e_{t+1}) = f_{t+1}(0, e_{t+1}) - f_{t+1}(1, e_{t+1})$	
	if $A_t = 0$	if $A_t = 1$	if $A_t = 0$	if $A_t = 1$		
(i)	0	0	0	0		$\varphi_t(e_t)$
(ii)	0	0	0	1	(ii.1)	$\varphi_t(e_t)(1-\alpha_D) - \alpha_Y \eta_t(e_t)$, if $f_t(1, e_t) \in (0, \frac{1}{1-\alpha_D+\alpha_Y})$
	-	-	-	-	(ii.2)	$f_t(0, e_t)(1-\alpha_D-\alpha_Y) - 1$, otherwise
(iii)	0	0	1	0	(iii.1)	$\varphi_t(e_t)(1-\alpha_D) + \alpha_Y \eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1-\alpha_D+\alpha_Y})$
	-	-	-	-	(iii.2)	$1 - f_t(1, e_t)(1-\alpha_D-\alpha_Y)$, otherwise
(iv)	0	0	1	1		$\varphi_t(e_t)$
(v)	0	1	0	0		$\varphi_t(e_t)(1-\alpha_Y) - \alpha_D \eta_t(e_t)$
(vi)	0	1	0	1	(vi.1)	$\varphi_t(e_t) - (\alpha_D+\alpha_Y)\eta_t(e_t)$, if $f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$
	-	-	-	-	(vi.2)	$f_t(0, e_t)(1-\alpha_D-\alpha_Y) - 1$, otherwise
(vii)	0	1	1	0	(vii.1)	$\varphi_t(e_t) - (\alpha_D-\alpha_Y)\eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1-\alpha_D+\alpha_Y})$
	-	-	-	-	(vii.2)	$1 - f_t(1, e_t)(1+\alpha_D-\alpha_Y)$, otherwise
(viii)	0	1	1	1	(viii.1)	$\varphi_t(e_t)(1+\alpha_Y) - \alpha_D \eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1-\alpha_D+\alpha_Y})$ and $f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$
	-	-	-	-	(viii.2)	$f_t(0, e_t)(1-\alpha_D+\alpha_Y) - 1$, if $f_t(0, e_t) \in (0, \frac{1}{1-\alpha_D+\alpha_Y})$ and $f_t(1, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1]$
	-	-	-	-	(viii.3)	$1 - f_t(1, e_t)(1+\alpha_D+\alpha_Y)$, if $f_t(0, e_t) \in [\frac{1}{1-\alpha_D+\alpha_Y}, 1]$ and $f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$
	-	-	-	-	(viii.4)	0, otherwise
(ix)	1	0	0	0		$\varphi_t(e_t)(1-\alpha_Y) + \alpha_D \eta_t(e_t)$
(x)	1	0	0	1	(x.1)	$\varphi_t(e_t) + (\alpha_D-\alpha_Y)\eta_t(e_t)$, if $f_t(1, e_t) \in (0, \frac{1}{1-\alpha_D+\alpha_Y})$
	-	-	-	-	(x.2)	$f_t(0, e_t)(1+\alpha_D-\alpha_Y) - 1$, otherwise
(xi)	1	0	1	0	(xi.1)	$\varphi_t(e_t) + (\alpha_D+\alpha_Y)\eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$
	-	-	-	-	(xi.2)	$1 - f_t(1, e_t)(1-\alpha_D-\alpha_Y)$, otherwise

Table 5 (continued from the previous page)

Case	D_t		$Y_t^{(\text{ori})}$		$\varphi_{t+1}(e_{t+1}) = f_{t+1}(0, e_{t+1}) - f_{t+1}(1, e_{t+1})$	
	if $A_t = 0$	if $A_t = 1$	if $A_t = 0$	if $A_t = 1$		
(xii)	1	0	1	1	(xii.1)	$\varphi_t(e_t)(1+\alpha_Y) + \alpha_D \eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$ and $f_t(1, e_t) \in (0, \frac{1}{1-\alpha_D+\alpha_Y})$
	-	-	-	-	(xii.2)	$f_t(0, e_t)(1+\alpha_D+\alpha_Y) - 1$, if $f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$ and $f_t(1, e_t) \in [\frac{1}{1-\alpha_D+\alpha_Y}, 1]$
	-	-	-	-	(xii.3)	$1 - f_t(1, e_t)(1-\alpha_D+\alpha_Y)$, if $f_t(0, e_t) \in [\frac{1}{1+\alpha_D+\alpha_Y}, 1]$ and $f_t(1, e_t) \in (0, \frac{1}{1-\alpha_D+\alpha_Y})$
	-	-	-	-	(xii.4)	0, otherwise
(xiii)	1	1	0	0		$\varphi_t(e_t)$
(xiv)	1	1	0	1	(xiv.1)	$\varphi_t(e_t)(1+\alpha_D) - \alpha_Y \eta_t(e_t)$, if $f_t(1, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$
	-	-	-	-	(xiv.2)	$f_t(0, e_t)(1+\alpha_D-\alpha_Y) - 1$, otherwise
(xv)	1	1	1	0	(xv.1)	$\varphi_t(e_t)(1+\alpha_D) + \alpha_Y \eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1+\alpha_D+\alpha_Y})$
	-	-	-	-	(xv.2)	$1 - f_t(1, e_t)(1+\alpha_D-\alpha_Y)$, otherwise
(xvi)	1	1	1	1		$\varphi_t(e_t)$

Table 6: When $\alpha_D = \alpha_Y = \alpha$, list possible instantiations of $\varphi_{t+1}(e_{t+1})$.

Case	D_t		$Y_t^{\text{(ori)}}$			$\varphi_{t+1}(e_{t+1}) = f_{t+1}(0, e_{t+1}) - f_{t+1}(1, e_{t+1})$
	if $A_t = 0$	if $A_t = 1$	if $A_t = 0$	if $A_t = 1$		
(i)	0	0	0	0		$\varphi_t(e_t)$
(ii)	0	0	0	1		$\varphi_t(e_t)(1-\alpha) - \alpha\eta_t(e_t)$
(iii)	0	0	1	0		$\varphi_t(e_t)(1-\alpha) + \alpha\eta_t(e_t)$
(iv)	0	0	1	1		$\varphi_t(e_t)$
(v)	0	1	0	0		$\varphi_t(e_t)(1-\alpha) - \alpha\eta_t(e_t)$
(vi)	0	1	0	1	(vi.1)	$\varphi_t(e_t) - 2\alpha\eta_t(e_t)$, if $f_t(1, e_t) \in (0, \frac{1}{1+2\alpha})$
	-	-	-	-	(vi.2)	$f_t(0, e_t)(1-2\alpha) - 1$, otherwise
(vii)	0	1	1	0		$\varphi_t(e_t)$
(viii)	0	1	1	1	(viii.1)	$\varphi_t(e_t)(1+\alpha) - \alpha\eta_t(e_t)$, if $f_t(1, e_t) \in (0, \frac{1}{1+2\alpha})$
	-	-	-	-	(viii.2)	$f_t(0, e_t) - 1$, otherwise
(ix)	1	0	0	0		$\varphi_t(e_t)(1-\alpha) + \alpha\eta_t(e_t)$
(x)	1	0	0	1		$\varphi_t(e_t)$
(xi)	1	0	1	0	(xi.1)	$\varphi_t(e_t) + 2\alpha\eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1+2\alpha})$
	-	-	-	-	(xi.2)	$1 - f_t(1, e_t)(1-2\alpha)$, otherwise
(xii)	1	0	1	1	(xii.1)	$\varphi_t(e_t)(1+\alpha) + \alpha\eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1+2\alpha})$
	-	-	-	-	(xii.2)	$1 - f_t(1, e_t)$, otherwise
(xiii)	1	1	0	0		$\varphi_t(e_t)$
(xiv)	1	1	0	1	(xiv.1)	$\varphi_t(e_t)(1+\alpha) - \alpha\eta_t(e_t)$, if $f_t(1, e_t) \in (0, \frac{1}{1+2\alpha})$
	-	-	-	-	(xiv.2)	$f_t(0, e_t) - 1$, otherwise
(xv)	1	1	1	0	(xv.1)	$\varphi_t(e_t)(1+\alpha) + \alpha\eta_t(e_t)$, if $f_t(0, e_t) \in (0, \frac{1}{1+2\alpha})$
	-	-	-	-	(xv.2)	$1 - f_t(1, e_t)$, otherwise
(xvi)	1	1	1	1		$\varphi_t(e_t)$

REFERENCES

Jeffrey Adams, Niels Hansen, and Kun Zhang. Identification of partially observed linear causal models: Graphical conditions for the non-gaussian and heterogeneous cases. *Advances in Neural Information Processing Systems*, 34, 2021.

Board of Governors of the Federal Reserve System (US). *Report to the congress on credit scoring and its effects on the availability and affordability of credit*. Board of Governors of the Federal Reserve System, 2007.

Toon Calders, Faisal Kamiran, and Mykola Pechenizkiy. Building classifiers with independency constraints. In *2009 IEEE International Conference on Data Mining Workshops*, pp. 13–18. IEEE, 2009.

Silvia Chiappa. Path-specific counterfactual fairness. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, pp. 7801–7808, 2019.

Amanda Coston, Alan Mishler, Edward H Kennedy, and Alexandra Chouldechova. Counterfactual risk assessments, evaluation, and fairness. In *Proceedings of the 2020 Conference on Fairness, Accountability, and Transparency*, pp. 582–593, 2020.

Alexander D'Amour, Hansa Srinivasan, James Atwood, Pallavi Baljekar, D Sculley, and Yoni Halpern. Fairness is not static: deeper understanding of long term fairness via simulation studies. In *Proceedings of the 2020 Conference on Fairness, Accountability, and Transparency*, pp. 525–534, 2020.

Danielle Ensign, Sorelle A Friedler, Scott Neville, Carlos Scheidegger, and Suresh Venkatasubramanian. Runaway feedback loops in predictive policing. In *Conference on Fairness, Accountability and Transparency*, pp. 160–171. PMLR, 2018.

Yingqiang Ge, Shuchang Liu, Ruoyuan Gao, Yikun Xian, Yunqi Li, Xiangyu Zhao, Changhua Pei, Fei Sun, Junfeng Ge, Wenwu Ou, and Yongfeng Zhang. Towards long-term fairness in recommendation. In *Proceedings of the 14th ACM International Conference on Web Search and Data Mining*, pp. 445–453, 2021.

Moritz Hardt, Eric Price, and Nati Srebro. Equality of opportunity in supervised learning. In *Advances in Neural Information Processing Systems*, pp. 3315–3323, 2016.

Lily Hu and Yiling Chen. A short-term intervention for long-term fairness in the labor market. In *Proceedings of the 2018 World Wide Web Conference*, pp. 1389–1398, 2018.

Kosuke Imai and Zhichao Jiang. Principal fairness for human and algorithmic decision-making. *arXiv preprint arXiv:2005.10400*, 2020.

Shahin Jabbari, Matthew Joseph, Michael Kearns, Jamie Morgenstern, and Aaron Roth. Fairness in reinforcement learning. In *International Conference on Machine Learning*, pp. 1617–1626. PMLR, 2017.

Faisal Kamiran, Indrė Žliobaitė, and Toon Calders. Quantifying explainable discrimination and removing illegal discrimination in automated decision making. *Knowledge and Information Systems*, 35(3):613–644, 2013.

Sampath Kannan, Aaron Roth, and Juba Ziani. Downstream effects of affirmative action. In *Proceedings of the Conference on Fairness, Accountability, and Transparency*, pp. 240–248, 2019.

Michael Kearns and Aaron Roth. *The ethical algorithm: The science of socially aware algorithm design*. Oxford University Press, 2019.

Niki Kilbertus, Mateo Rojas-Carulla, Giambattista Parascandolo, Moritz Hardt, Dominik Janzing, and Bernhard Schölkopf. Avoiding discrimination through causal reasoning. In *Proceedings of the 31st International Conference on Neural Information Processing Systems*, pp. 656–666, 2017.

Bohdan Kivva, Goutham Rajendran, Pradeep Ravikumar, and Bryon Aragam. Learning latent causal graphs via mixture oracles. *Advances in Neural Information Processing Systems*, 34, 2021.

Matt J Kusner, Joshua Loftus, Chris Russell, and Ricardo Silva. Counterfactual fairness. In *Advances in Neural Information Processing Systems*, pp. 4066–4076, 2017.

Lydia T Liu, Sarah Dean, Esther Rolf, Max Simchowitz, and Moritz Hardt. Delayed impact of fair machine learning. In *International Conference on Machine Learning*, pp. 3150–3158. PMLR, 2018.

Vishwali Mhasawade and Rumi Chunara. Causal multi-level fairness. In *Proceedings of the 2021 AAAI/ACM Conference on AI, Ethics, and Society*, pp. 784–794, 2021.

Alan Mishler, Edward H Kennedy, and Alexandra Chouldechova. Fairness in risk assessment instruments: Post-processing to achieve counterfactual equalized odds. In *Proceedings of the 2021 ACM Conference on Fairness, Accountability, and Transparency*, pp. 386–400, 2021.

Hussein Mouzannar, Mesrob I Ohannessian, and Nathan Srebro. From fair decision making to social equality. In *Proceedings of the Conference on Fairness, Accountability, and Transparency*, pp. 359–368, 2019.

Hamed Nilforoshan, Johann D Gaebler, Ravi Shroff, and Sharad Goel. Causal conceptions of fairness and their consequences. In *International Conference on Machine Learning*, pp. 16848–16887. PMLR, 2022.

Judea Pearl. *Causality*. Cambridge University Press, 2009.

Shohei Shimizu, Patrik O Hoyer, Aapo Hyvärinen, Antti Kerminen, and Michael Jordan. A linear non-gaussian acyclic model for causal discovery. *Journal of Machine Learning Research*, 7(10), 2006.

Umer Siddique, Paul Weng, and Matthieu Zimmer. Learning fair policies in multi-objective (deep) reinforcement learning with average and discounted rewards. In *International Conference on Machine Learning*, pp. 8905–8915. PMLR, 2020.

Peter Spirtes, Clark N Glymour, and Richard Scheines. *Causation, prediction, and search*. Springer New York, 1993.

Min Wen, Osbert Bastani, and Ufuk Topcu. Algorithms for fairness in sequential decision making. In *International Conference on Artificial Intelligence and Statistics*, pp. 1144–1152. PMLR, 2021.

Yongkai Wu, Lu Zhang, Xintao Wu, and Hanghang Tong. Pc-fairness: A unified framework for measuring causality-based fairness. In *Advances in Neural Information Processing Systems*, pp. 3399–3409, 2019.

Feng Xie, Ruichu Cai, Biwei Huang, Clark Glymour, Zhifeng Hao, and Kun Zhang. Generalized independent noise condition for estimating latent variable causal graphs. *Advances in Neural Information Processing Systems*, 33:14891–14902, 2020.

Feng Xie, Biwei Huang, Zhengming Chen, Yangbo He, Zhi Geng, and Kun Zhang. Identification of linear non-gaussian latent hierarchical structure. In *International Conference on Machine Learning*, pp. 24370–24387. PMLR, 2022.

Kun Zhang and Aapo Hyvärinen. On the identifiability of the post-nonlinear causal model. In *25th Conference on Uncertainty in Artificial Intelligence (UAI 2009)*, pp. 647–655. AUAI Press, 2009.

Kun Zhang, Jonas Peters, Dominik Janzing, and Bernhard Schölkopf. Kernel-based conditional independence test and application in causal discovery. In *27th Conference on Uncertainty in Artificial Intelligence (UAI 2011)*, pp. 804–813. AUAI Press, 2011.

Lu Zhang, Yongkai Wu, and Xintao Wu. A causal framework for discovering and removing direct and indirect discrimination. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence*, pp. 3929–3935, 2017.

Xueru Zhang, Mohammadmahdi Khaliligarekani, Cem Tekin, and Mingyan Liu. Group retention when using machine learning in sequential decision making: the interplay between user dynamics and fairness. *Advances in Neural Information Processing Systems*, 32:15269–15278, 2019.

Xueru Zhang, Ruibo Tu, Yang Liu, Mingyan Liu, Hedvig Kjellstrom, Kun Zhang, and Cheng Zhang. How do fair decisions fare in long-term qualification? *Advances in Neural Information Processing Systems*, 33:18457–18469, 2020.

Matthieu Zimmer, Claire Glanois, Umer Siddique, and Paul Weng. Learning fair policies in decentralized cooperative multi-agent reinforcement learning. In *International Conference on Machine Learning*, pp. 12967–12978. PMLR, 2021.